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Robustness assessment via stability radii in delay parameters

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SUMMARY

This paper focuses on the robust stability analysis of a class of linear systems including multiple delays subjected to constant or time-varying perturbations. The approach considered makes use of appropriate stability radius concepts (dynamic, static) and relies on a feedback interconnection interpretation of the uncertain system. Various computable bounds on stability radii are obtained that exploit the structure of the problem. Systems including perturbations on both system matrices and delays are also dealt with. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

It is well known that the characterization of stability regions of time-delay systems in the delay parameter space is a difficult problem (see, e.g. [1, 2] and the references therein). Stability radii are well known in the context of matrix distance problems, see [3] and the references therein. Recently, such concepts have been used to assess or optimize the robustness of stability of linear time-delay systems subjected to structured uncertainty on the corresponding system matrices in [4–7]. In this context, stability radii correspond to the size of the smallest perturbations that render the system unstable.

The aim of this paper is to adopt the concept of stability radii to linear systems including multiple delays subject to constant or time-varying perturbations on the *delay* parameters and to

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derive computable expressions. The case of constant perturbations for a class of quasi-polynomials including two delays was addressed in a geometrical setting in [8], where the authors introduced the notion of *delay deviation*. The idea can be resumed in computing the distance between the 'nominal' point in the delay parameter space and the 'closest' curve for which there exists at least one characteristic root on the imaginary axis. Such a delay deviation characterization is nothing else than a characterization of a *stability radius* in the delay parameter space.

The approach considered in this paper is quite different from the one mentioned above. First, we introduce two appropriate notions of stability radii: *static* and *dynamic*, in order to characterize constant and time-varying perturbations on the nominal system's parameters. These stability radii are scalar robustness measures based on a *a priori* chosen weighting of the perturbations of delays and system parameters, as we shall discuss at the end. Secondly, we will employ a feedback interconnection point of view of the uncertain system in order to derive estimates for the stability radii. Note that a similar point of view was taken in [9, Chapter 3; 10–13] ($\mathscr{L}_2$ gain analysis applied to systems with time-varying delay perturbations), [14] (μ analysis applied to systems with constant delay perturbations) and [4, 15] (pseudospectra and stability radii for nonlinear eigenvalue problems), and some of the references therein. In the present paper, the robust stability characterizations in these references are combined and further developed in a unifying framework, and the results are formulated in terms of appropriately defined stability radii. Finally, in [16] and the references therein, the IQC approach is applied to deal with time-varying delays, leading to easy-to-check stability conditions expressed as linear matrix inequalities, under the assumption that the corresponding delay-free system is asymptotically stable. Such an assumption will not be made in this paper. In [17] frequency sweeping tests are presented to assess the effect on stability of coefficient perturbations of quasi-polynomials, but in the context of delay-independent stability. Finally, in [18] a $\mathscr{L}_2$ gain analysis approach is employed to analyze the stability of nonlinear time-delay systems with parametric uncertainty; however, it is assumed that the uncertainty does not affect the delay parameters.

Although in the case of uncertainty on the delays, the feedback interconnection point of view and the adopted tools from robust control will typically lead to expressions for lower bounds on stability radii (corresponding to sufficient yet not necessary robust stability conditions), they offer several advantages. Explicit computable expressions for bounds are namely obtained that impose no limitations on the number of delays and the dimension of the problem. In addition, *time-varying* perturbations and *combined* perturbation on delays and system matrices (matrix-valued perturbations) can be easily dealt with, as we shall demonstrate. Finally, the interconnection framework is appropriate for solving associated synthesis problems. The latter issue will however not be further addressed in this paper.

The paper adopts a step-by-step approach by imposing more conditions on the perturbations and exploiting this information accordingly. More precisely, first time-varying perturbations are considered in a $\mathscr{L}_2$ analysis framework. Next, it is shown how the derived explicit bounds on the stability can be improved for the special case of constant perturbations, where besides the inherent increase of the stability radii (due to the restriction of the allowable perturbations), the structure of the interconnection can be better exploited by using frequency domain techniques. Finally, implicit expressions are given, which rely on exploiting all structure of the problem and leave conservatism only in the fact that the phase information is not fully exploited in the feedback loop (inherent to the adopted approach). For reasons of simplicity and clarity of the presentation, the cases of uncertainty on delays only and of uncertainty on both delays and system matrices are treated separately.

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2. UNCERTAINTY ON THE DELAYS

2.1. Concept

We address the uncertain system as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i - w_i \delta\tau_i(t)) \\ x(\theta) &= \phi(\theta), \quad -\eta \leqslant \theta \leqslant 0, \quad \phi \in \mathscr{C}([-\eta, 0], \mathbb{R}^n) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $\eta > 0$, $A_i \in \mathbb{R}^{n\times n}$ and $\tau_i \geqslant 0$. The uncertainty on the delays is modeled by uniformly bounded scalar functions $[0, \infty] \ni t \to \delta\tau_i(t)$ and scalar weights $w_i > 0$, which are such that

$$w_i \delta\tau_i(t) \geqslant -\tau_i \quad \forall t \geqslant 0 \;\; \forall i \in \{1, \ldots, m\} \tag{2}$$

We assume that the zero solution of the corresponding unperturbed system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) \\ x(\theta) &= \phi(\theta), \quad -\eta \leqslant \theta \leqslant 0 \end{aligned} \tag{3}$$

is asymptotically stable.

The dynamic *stability radius* $r_{\boldsymbol{\tau}}^{\mathrm{d}}$ of the system (3) w.r.t. the delays is defined as

$$\begin{aligned} r_{\boldsymbol{\tau}}^{\mathrm{d}} := \sup \Bigg\{ &\gamma \geqslant 0 \colon \text{ the zero solution of (1) is asymptotically stable for all functions} \\ &\delta\boldsymbol{\tau}(t) = (\delta\tau_1(t), \ldots, \delta\tau_m(t)) \text{ satisfying (2) and ess } \sup_{t \geqslant 0} |\delta\tau_i(t)| \leqslant \gamma, \;\; i = 1, \ldots, m \Bigg\} \end{aligned} \tag{4}$$

Note that, although $r_{\boldsymbol{\tau}}^{\mathrm{d}}$ explicitly depends on the weights w_i, this dependence is suppressed in the notation for reasons of simplicity.

Similarly, if the uncertainty on the delay is assumed as time invariant, then the static *stability radius* w.r.t. the delays is defined as

$$\begin{aligned} r_{\boldsymbol{\tau}}^{\mathrm{s}} := \sup \{ &\gamma \geqslant 0 \colon \text{ the zero solution of (1) is asymptotically stable for all constant} \\ &\delta\boldsymbol{\tau} = (\delta\tau_1, \ldots, \delta\tau_m) \text{ satisfying } w_i \delta\tau_i \geqslant -\tau_i \text{ and } |\delta\tau_i| \leqslant \gamma, \;\; i = 1, \ldots, m \} \end{aligned} \tag{5}$$

Remark 1

The weights w_i are useful because in practical problems the time delays may have a large variation in size and the amount of uncertainty may differ from one delay parameter to another [5]. A special choice is given by $w_i = \tau_i$, $1 \leqslant i \leqslant m$, where the stability radii correspond to the maximal *relative* error on the delays such that stability is preserved. If there is no uncertainty at all on some delay parameters, then one can incorporate this information by setting the corresponding weights to zero.

Remark 2

It may happen that a stability radius is larger than the nominal delay values. For example, $r_{\boldsymbol{\tau}}^{\mathrm{s}} > \alpha$, $\alpha > 0$, means that the stability of the system with delays $\boldsymbol{\tau}$ guarantees the stability of the system with delays $\mathbf{v} \geqslant 0$ whenever $|\tau_i - v_i| \leqslant w_i \alpha$, $1 \leqslant i \leqslant m$. In the special case, where $\tau_i - w_i r_{\boldsymbol{\tau}}^{\mathrm{s}} < 0$ for some

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$i\in\{1,\dots,m\}$ a stability notion in terms of the maximum allowable delay deviation could be more accurate that the generally accepted term 'stability radius'.

In the remainder of this section several lower bounds on the above stability radii are derived. Such lower bounds correspond to *robust stability conditions*.

2.2. *Feedback interconnection point of view*

We factorize

$$A_i = B_i C_i, \quad B_i \in \mathbb{R}^{n\times n_i}, \quad C_i \in \mathbb{R}^{n_i\times n}, \;\; i=1,\dots,m \tag{6}$$

where all B_i have full column rank, all C_i have full row rank, and we let $\hat{n}=\sum_{i=1}^{m} n_i$.

For $u\in\mathscr{L}_2([0,\infty],\mathbb{R}^{\hat{n}})$, $y=\mathscr{G}u$ be defined by

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + \sum_{i=1}^{m} [B_1\cdots B_m]u(t), \quad x(\theta)=0, \;\; \theta\leqslant 0\\ y(t) &= [w_1 C_1^{\mathrm{T}}\cdots w_m C_m^{\mathrm{T}}]^{\mathrm{T}}\dot{x}(t)\end{aligned}$$

Clearly, $y\in\mathscr{L}_2([0,\infty],\mathbb{R}^{\hat{n}})$. By the asymptotic stability of the unperturbed system and Parseval's theorem the $\mathscr{L}_2$-induced norm of $\mathscr{G}$ satisfies

$$\|\mathscr{G}\|_{\mathscr{L}_2} = \|G(\lambda)\|_{\mathscr{H}_\infty} = \max_{\omega\geqslant 0}\sigma_1(G(\mathrm{j}\omega))$$

where

$$G(\lambda) = \lambda \begin{bmatrix} w_1 C_1\\ \vdots \\ w_m C_m\end{bmatrix}\left(\lambda I - A_0 - \sum_{i=1}^{m} A_i \mathrm{e}^{-\lambda\tau_i}\right)^{-1}[B_1\cdots B_m] \tag{7}$$

Next, we let

$$\mathscr{S}_i^{\nu}:\mathscr{L}_2([0,\infty),\mathbb{R}^{\nu})\rightarrow\mathscr{L}_2([0,\infty),\mathbb{R}^{\nu})$$

$$(\mathscr{S}_i^{\nu}\xi)(t) = \frac{1}{w_i}\int_{t-\tau_i-w_i\delta\tau_i(t)}^{t-\tau_i}\tilde{\xi}(s)\,\mathrm{d}s, \quad t\geqslant 0$$

where $\nu\in\mathbb{N}, i\in\{1,\dots,m\}$ and $\tilde{\xi}\in\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\nu})$ satisfies

$$\tilde{\xi}(t) = \begin{cases}\xi(t), & t\geqslant 0\\ 0, & t<0\end{cases} \tag{8}$$

By defining

$$\begin{aligned}\mathscr{D} &: \mathscr{L}_2([0,\infty),\mathbb{R}^{\hat{n}})\rightarrow\mathscr{L}_2([0,\infty),\mathbb{R}^{\hat{n}})\\ &(\mathscr{D}\xi)(t) = \mathrm{diag}((\mathscr{S}_1^{n_1}\xi_1)(t),\dots,(S_m^{n_m}\xi_m)(t))\end{aligned} \tag{9}$$

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where $\xi(t)=[\xi_1^{\mathrm{T}}(t)\cdots\xi_m^{\mathrm{T}}(t)]^{\mathrm{T}}$, with $\xi_i(t)\in\mathbb{R}^{n_i}, i=1,\ldots,m$, we can interpret the system (1) as a *feedback interconnection* of $\mathscr{G}$ and $\mathscr{D}$.

Remark 3
If some of the matrices $A_i,\ i=1,\ldots,m$, have low rank, then $\mathscr{G}$ and $\mathscr{D}$ have $\hat{n}<nm$ inputs and outputs. This is due to the factorization (6).

2.3. Time-varying perturbations

As a first step we characterize the induced $\mathscr{L}_2$ gain of $\mathscr{D}$. We need the following result:

Lemma 1
Assume that $\tau_i+w_i\delta\tau_i(t)\geqslant 0$ and that $|\delta\tau_i(t)|\leqslant\mu_i$ for all $t\geqslant 0$. Then, the induced $\mathscr{L}_2$ norm of $\mathscr{S}_i^v$ is bounded by $\sqrt{7/4}\mu_i$.

Proof
This result corresponds to Lemma 2 of [13], to which we refer for a detailed proof. To make the paper self contained we outline the main steps, which are as follows. First, one extends the operator $\mathscr{S}_i^v$ to an operator $\hat{\mathscr{S}}_i^v$ on $\mathscr{L}_2(\mathbb{R},\mathbb{R}^v)$, defined by:

$$\hat{\mathscr{S}}_i^v:\mathscr{L}_2(\mathbb{R},\mathbb{R}^v)\rightarrow\mathscr{L}_2(\mathbb{R},\mathbb{R}^v)$$

$$(\hat{\mathscr{S}}_i^v\xi)(t)=\frac{1}{w_i}\int_{t-\tau_i-w_i\delta\tau_i(t)}^{t-\tau_i}\xi(s)\,\mathrm{d}s,\quad t\in\mathbb{R}$$

Clearly, we have $\|\hat{\mathscr{S}}_i^v\|_{\mathscr{L}_2}=\|\mathscr{S}_i^v\|_{\mathscr{L}_2}$. Next, one estimates $\|\hat{\mathscr{S}}_i^{v*}\|_{\mathscr{L}_2}$, where $\hat{\mathscr{S}}_i^{v*}$ is the adjoint of $\hat{\mathscr{S}}_i^v$, using the definition of the induced $\mathscr{L}_2$ norm and a geometric interpretation of the integrals. This leads to

$$\|\hat{\mathscr{S}}_i^{v*}\|_{\mathscr{L}_2}\leqslant\sqrt{7/4}\mu_i$$

Finally, the assertion follows from:

$$\|\hat{\mathscr{S}}_i^{v*}\|_{\mathscr{L}_2}=\|\hat{\mathscr{S}}_i^v\|_{\mathscr{L}_2}=\|\mathscr{S}_i^v\|_{\mathscr{L}_2}$$ □

Lemma 2
Assume that $\tau_i+w_i\delta\tau_i(t)\geqslant 0$ and that $|\delta\tau_i(t)|\leqslant\mu_i$, for all $t\geqslant 0$ and $1\leqslant i\leqslant m$. Then

$$\|\mathscr{D}\|_{\mathscr{L}_2}\leqslant\sqrt{7/4}\|\boldsymbol{\mu}\|_\infty \tag{10}$$

Proof
Expression (10) follows from:

$$\|\mathscr{D}\|_{\mathscr{L}_2}=\max_{1\leqslant i\leqslant m}\|S_i^{n_i}\|_{\mathscr{L}_2}$$

and Lemma 1. □

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By combining the above lemmas we arrive at the following result:

Proposition 1
We have the following estimate:

$$r_{\boldsymbol{\tau}}^{\mathrm{d}} \geqslant \frac{1}{\sqrt{7/4}} (\|G(\lambda)\|_{\mathscr{H}_\infty})^{-1} \tag{11}$$

Proof
From the small gain theorem we have that if

$$\|\mathscr{G}\|_{\mathscr{L}_2} \|\mathscr{D}\|_{\mathscr{L}_2} = \|G(\lambda)\|_{\mathscr{H}_\infty} \|\mathscr{D}\|_{\mathscr{L}_2} < 1 \tag{12}$$

then the feedback interconnection of $\mathscr{G}$ and $\mathscr{D}$ is $\mathscr{L}_2$ stable, which induces the asymptotic stability of the zero solution of (1). Under the assumptions of Lemma 2 the condition (12) is fulfilled if

$$\sqrt{7/4} \|G(\lambda)\|_{\mathscr{H}_\infty} \|\boldsymbol{\mu}\|_\infty < 1 \tag{13}$$

The assertion of the proposition follows: □

Proposition 1 can be strengthened by an appropriate scaling in the feedback loop. More precisely, with the set $\mathscr{T}$ defined as

$$\mathscr{T} := \{\mathrm{diag}(T_1, \ldots, T_m) : T_i \in \mathbb{C}^{n_i \times n_i}, \det T_i \neq 0, \quad i = 1, \ldots, m\} \tag{14}$$

we get:

Proposition 2
We have the following estimate:

$$r_{\boldsymbol{\tau}}^{\mathrm{d}} \geqslant \frac{1}{\sqrt{7/4}} \left(\min_{T \in \mathscr{T}} \|T G(\lambda) T^{-1}\|_{\mathscr{H}_\infty} \right)^{-1} \tag{15}$$

Remark 4
The optimization problem

$$\min_{T \in \mathscr{T}} \|T G(\lambda) T^{-1}\|_{\mathscr{H}_\infty}$$

can be reformulated as

$$\begin{aligned} \min_{U,\gamma} \quad & \gamma \\ \text{s.t.} \quad & \gamma > 0, \quad U \in \mathscr{T}, \quad U = U^* > 0 \\ & G(\mathrm{j}\omega)^* U G(\mathrm{j}\omega) - \gamma^2 U < 0 \quad \forall \omega \geqslant 0 \end{aligned} \tag{16}$$

where T can be computed from $U = T^* T$. If γ is fixed, then the resulting feasibility problem is convex.

Remark 5
If the delay perturbations are such that the functions

$$t \mapsto t - \tau_i - w_i \delta\tau_i(t), \quad i = 1, \ldots, m$$

DOI: 10.1002/rnc

are non-decreasing, then the factor $\sqrt{7/4}$ in (11) and (15) can be replaced with 1. This follows from the fact that in such case $\|S_i^v\|_{\mathscr{L}_2} \leqslant \mu_i$ if $|\delta\tau_i(t)| \leqslant \mu_i$, for all $t \geqslant 0$, see [12].

For improvements of Lemma 1 for the case where the delays are differential functions with a given upper bound on their derivatives, we refer to [13].

2.4. *Time-invariant perturbations*

We reconsider the estimates for the stability radii under the additional assumption of constant delay perturbations. Then, improvements can be made by decoupling signals in the frequency domain, and by further exploiting the structure of the problem under consideration.

Let the entire functions s_i be defined as

$$s_i(\lambda;\chi) := \begin{cases} e^{-\lambda\tau_i} \dfrac{1-e^{-\lambda(w_i\chi)}}{w_i\lambda}, & \lambda \neq 0, \\ 1, & \lambda = 0, \end{cases} \qquad i = 1, \ldots, m$$

As they satisfy

$$|s_i(j\omega;\delta\tau_i)| \leqslant \left| \frac{1-e^{-j\omega(w_i\delta\tau_i)}}{w_i\omega} \right| \leqslant \left| \frac{\sin \dfrac{w_i\delta\tau_i}{2}\omega}{\dfrac{w_i}{2}\omega} \right| \leqslant \delta\tau_i \quad \forall\omega \geqslant 0 \;\; \forall i \in \{1,\ldots,m\} \tag{17}$$

we obtain

$$\|S_i^v\|_{\mathscr{L}_2} = \|s_i(j\omega;\delta\tau_i)I_v\|_{\mathscr{H}_\infty} \leqslant \delta\tau_i \tag{18}$$

This result can also be derived in the time domain, see [12]. Denote with

$$D(\lambda;\delta\boldsymbol{\tau}) := \mathrm{diag}(s_1(\lambda;\delta\tau_1)I_{n_1},\ldots,s_m(\lambda;\delta\tau_m)I_{n_m})$$

the transfer function associated with the operator $\mathscr{D}$, defined in (9). From (17) it follows that

$$\|D(j\omega;\delta\boldsymbol{\tau})\|_{\mathscr{H}_\infty} \leqslant \|\delta\boldsymbol{\tau}\|_\infty$$

The characteristic equation of (1) can be written on the imaginary axis as

$$\det\left(j\omega I - A_0 - \sum_{i=1}^{m} A_i e^{-j\omega\tau_i}\right)\det(I - G(j\omega)D(j\omega;\delta\boldsymbol{\tau})) = 0 \tag{19}$$

where the first factor is non-zero for all $\omega \geqslant 0$ because the unperturbed system is assumed to be asymptotically stable. The perturbed system is asymptotically stable if the perturbations cannot shift characteristic roots to the imaginary axis, that is, if (19) has no solutions. Based on this observation we have the following result, which makes use of structured singular values (see the appendix for a short introduction):

Proposition 3

Define the uncertainty set

$$\boldsymbol{\Delta} := \{\mathrm{diag}(d_1 I_{n_1},\ldots,d_m I_{n_m}) : d_i \in \mathbb{C},\, 1 \leqslant i \leqslant m\} \tag{20}$$

DOI: 10.1002/rnc

We have:

$$r_{\boldsymbol{\tau}}^{\mathrm{s}} \geqslant \left(\sup_{\omega\geqslant 0} \mu_{\boldsymbol{\Delta}} G(\mathrm{j}\omega)\right)^{-1} \tag{21}$$

where $\mu_{\boldsymbol{\Delta}}(\cdot)$ is the structured singular value with respect to (20).

Proof
From (19) and the fact that $D(\mathrm{j}\omega;\delta\boldsymbol{\tau})\in\boldsymbol{\Delta}$ for all $\omega\geqslant 0$, a sufficient stability condition is given by

$$\|D(\mathrm{j}\omega;\delta\boldsymbol{\tau})\|_2 < (\mu_{\boldsymbol{\Delta}}(G(\mathrm{j}\omega)))^{-1} \quad \forall\omega\geqslant 0$$

This condition is satisfied if

$$\|\boldsymbol{\delta}\tau\|_\infty < (\mu_{\boldsymbol{\Delta}}(G(\mathrm{j}\omega)))^{-1} \quad \forall\omega\geqslant 0$$

which leads to the statement of the proposition. □

Because the exact computation of the structured singular of a complex $\hat{n}\times\hat{n}$ matrix M with respect to the uncertainty structure (20) is a hard problem if m is large [19], the available numerical algorithms typically compute lower and upper bounds, see the appendix. We have for instance

$$\mu_{\boldsymbol{\Delta}}(M) \leqslant \min_{T\in\mathscr{T}} \sigma_1(TMT^{-1}) \tag{22}$$

where $\mathscr{T}$ is given by (14) and $\sigma_1(\cdot) := \|\cdot\|_2$. The computation of the upper bound in (22) can be formulated as a *convex* optimization problem, using the arguments spelled out in Remark 4.

From Proposition 3 and the estimate (22) we obtain:

$$r_{\boldsymbol{\tau}}^{\mathrm{s}} \geqslant \left(\sup_{\omega\geqslant 0} \min_{T\in\mathscr{T}} \|T^{-1}G(\mathrm{j}\omega)T\|_2\right)^{-1} \tag{23}$$

It is instructive to compare expressions (23) and (15), the latter corresponding to

$$r_{\boldsymbol{\tau}}^{\mathrm{d}} \geqslant \left(\sqrt{7/4}\, \min_{T\in\mathscr{T}} \sup_{\omega\geqslant 0} \|T^{-1}G(\mathrm{j}\omega)T\|_2\right)^{-1}$$

Besides the factor $\sqrt{7/4}$ (due to the better estimate of $\|S_i^y\|_{\mathscr{L}_2}$ in the time-invariant case), the outer and inner optimization have been interchanged, that is, the scaling has become frequency dependent in (23).

Further improvements of the estimate (21) can be obtained by, instead of (17), using the smallest possible upper bound on $|s_i(\mathrm{j}\omega;\delta\tau_i)|$, given the bound μ_i on $|\delta\tau_i|$. The price to be paid is that the expression for the stability radius is no longer explicit. The following result generalizes Theorem 3 of [14]:

Proposition 4
Let $s:\mathbb{R}_+\rightarrow\mathbb{R}_+$

$$\omega\mapsto s(\omega) := \begin{cases} \sin(\omega), & \omega\leqslant \dfrac{\pi}{2} \\ 1, & \omega\geqslant\dfrac{\pi}{2} \end{cases} \tag{24}$$

DOI: 10.1002/rnc

Furthermore, let $F:\mathbb{R}_+\setminus\{0\}\rightarrow\mathbb{R}_+,\ \alpha\mapsto F(\alpha)$, where

$$F(\alpha):=\sup_{\omega\geqslant 0}\mu_{\mathbf{\Delta}}\left(\begin{bmatrix} 2s\left(\dfrac{w_1\alpha\omega}{2}\right)C_1\\ \vdots \\ 2s\left(\dfrac{w_m\alpha\omega}{2}\right)C_m\end{bmatrix}\left(\mathrm{j}\omega I-A_0-\sum_{i=1}^{m}A_i\mathrm{e}^{-\mathrm{j}\omega\tau_i}\right)^{-1}[B_1\ldots B_m]\right)$$

and $\mu_{\mathbf{\Delta}}(\cdot)$ is the structured singular value with respect to the uncertainty set (20). Then, we have the following estimate:

$$r_{\boldsymbol{\tau}}^{\mathrm{s}}\geqslant\sup\{\alpha>0: F(\alpha)<1\}\tag{25}$$

Proof

The proof is based on an additional scaling within the feedback loop. Equation (19) is equivalent with

$$\det(I-\Lambda^{-1}(\omega;\alpha)G(\mathrm{j}\omega)D(\mathrm{j}\omega;\delta\boldsymbol{\tau})\Lambda(\omega;\alpha))=0$$

where

$$\Lambda(\omega;\alpha):=\mathrm{diag}\left(\frac{\mathrm{j}\omega w_1}{2s\left(\dfrac{w_1\alpha\omega}{2}\right)}I_{n_1},\ldots,\frac{\mathrm{j}\omega w_m}{2s\left(\dfrac{w_1\alpha\omega}{2}\right)}I_{n_m}\right)$$

By construction, we have

$$F(\alpha)=\sup_{\omega\geqslant 0}\mu_{\mathbf{\Delta}}(\Lambda^{-1}(\omega;\alpha)G(\mathrm{j}\omega))$$

Furthermore, the structure of $D(\mathrm{j}\omega;\delta\boldsymbol{\tau})$ is not affected by the post-multiplication with $\Lambda(\omega;\alpha)$. Hence, under the assumption $F(\alpha)<1$, the system is stable if

$$\begin{aligned}\|\Lambda(\omega;\alpha)D(\mathrm{j}\omega;\delta\boldsymbol{\tau})\|_2<1\quad\forall\omega\geqslant 0 &\Leftrightarrow\left|\frac{1-\mathrm{e}^{-\mathrm{j}\omega w_i\delta\tau_i}}{2s\left(\dfrac{w_i\alpha\omega}{2}\right)}\right|<1\quad\forall\omega\geqslant 0,\ \ i=1,\ldots,m\\ &\Leftrightarrow\left|\frac{\sin\left(\dfrac{w_i\delta\tau_i\omega}{2}\right)}{s\left(\dfrac{w_i\alpha\omega}{2}\right)}\right|<1\quad\forall\omega\geqslant 0,\ \ i=1,\ldots,m\\ &\Leftrightarrow|\delta\tau_i|<\alpha,\quad i=1,\ldots,m\end{aligned}$$

The following implication can be concluded:

$$F(\alpha)<1\Rightarrow r_{\boldsymbol{\tau}}^{\mathrm{s}}\geqslant\alpha$$

and the assertion of the proposition follows. □

DOI: 10.1002/rnc

Remark 6
Since for all $\omega\geqslant 1$ and $i=1,\ldots,m$, we have

$$\sup_{|\delta\tau_i|<\alpha}\left|\frac{\sin\left(\frac{w_i\delta\tau_i\omega}{2}\right)}{s\left(\frac{w_i\alpha\omega}{2}\right)}\right|=1\quad\forall\omega\geqslant 0$$

a further improvement of the estimate (25) can only be achieved by exploiting phase information in the feedback loop, which is not possible with the adopted μ approach.

Remark 7
Using (22) we can relax (25) to the slightly more conservative, but computationally more tractable expression

$$r_{\boldsymbol{\tau}}^{s}\geqslant\sup\{\alpha>0:\tilde{F}(\alpha)<1\}\tag{26}$$

where

$$\tilde{F}(\alpha):=\sup_{\omega\geqslant 0}\min_{T\in\mathscr{T}}\sigma_1\left\{T^{-1}\left(\begin{bmatrix}2s\left(\frac{w_1\alpha\omega}{2}\right)C_1\\ \vdots\\ 2s\left(\frac{w_m\alpha\omega}{2}\right)C_m\end{bmatrix}\times\left(\mathrm{j}\omega I-A_0-\sum_{i=1}^{m}A_i\mathrm{e}^{-\mathrm{j}\omega\tau_i}\right)^{-1}[B_1\cdots B_m]\right)T\right\}\tag{27}$$

and $\mathscr{T}$ is defined in (14).

3. UNCERTAINTY IN COEFFICIENT MATRICES AND DELAYS

We consider the uncertain system

$$\dot{x}(t)=(A_0+D_0\delta A_0(t)E_0)x(t)+\sum_{i=1}^{m}(A_i+D_i\delta A_i(t)E_i)x(t-\tau_i-w_i\delta\tau_i(t))\tag{28}$$

under appropriate initial conditions. The uncertainty is expressed by the piece-wise continuous functions as

$$\begin{aligned}\delta A_i&\in\mathscr{L}_\infty([0,\infty),\mathbb{R}^{n_i\times n_i}),\quad i=0,\ldots,m\\ \delta\tau_i&\in\mathscr{L}_\infty([0,\infty),[-\tau_i,\infty)),\quad i=1,\ldots,m\end{aligned}\tag{29}$$

while $D_i\in\mathbb{R}^{n\times n_i}$ and $E_i\in\mathbb{R}^{n_i\times n}$ are weight matrices, and $w_i>0$ are scalar weights.

DOI: 10.1002/rnc

The dynamic stability radius of the unperturbed system (3) w.r.t. the *combined* uncertainty in (28) is defined as

$$r_{\mathrm{c}}^{\mathrm{d}} =: \sup \Big\{ \gamma \geqslant 0 : \text{ the zero solution of (28) is asymptotically stable for all functions}$$

$$\delta A_i(t) \text{ and } \delta\boldsymbol{\tau}(t) \text{ satisfying } \operatorname*{ess\,sup}_{t\geqslant 0} \|\delta A_i(t)\|_2 \leqslant \gamma,\ i=0,\dots,m,$$

$$\min_{t\geqslant 0} \tau_i + w_i \delta\tau_i(t) \geqslant 0 \text{ and } \operatorname{ess}\sup_{t\geqslant 0} |\delta\tau_i(t)| \leqslant \gamma,\ i=1,\dots,m \Big\} \tag{30}$$

The corresponding static stability radius $r_{\mathrm{c}}^{\mathrm{s}}$ is defined in a similar way by assuming time-invariant perturbations.

Remark 8

The weights w_i and the weight matrices are not only useful to express the amount of uncertainty on the different delays and matrices relative to each other. In addition, the weight matrices D_i and E_i allow to introduce additional structure on the perturbations of the system matrix A_i, which appears for instance if only one element or a particular block is uncertain.

From an analysis point of view the main difference with respect to the case discussed in the previous section is the *nonlinear* dependence of the right-hand side of (28) on the uncertainty, in particular, on the products of δA_i and $x(t-\tau_i - w_i\delta\tau_i)$. Inspired by [12], this problem can be overcome by introducing additional inputs and outputs. First, let $B_i, C_i, \tilde{A}_i, \tilde{D}_i, \tilde{E}_i$ be such that

$$A_i + D_i \delta A_i(t) E_i = B_i(\tilde{A}_i + \tilde{D}_i \delta A_i(t) \tilde{E}_i) C_i, \quad i=1,\dots,m$$

where each $C_i \in \mathbb{R}^{\tilde{n}_i \times n}$ has full row rank. A trivial choice is given by

$$B_i = C_i = I, \quad \tilde{A}_i = A_i, \quad \tilde{D}_i = D_i, \quad \tilde{E}_i = E_i, \ \ i=1,\dots,m$$

yet it is beneficial if a decomposition can be chosen where $\operatorname{rank}(C_i) = \tilde{n}_i < n$ (as this leads to *smaller* block sizes in the uncertainty structure). Next, we interpret (28) as the feedback interconnection of

$$\begin{aligned}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + D_0 \tilde{u}_0(t) + \sum_{i=1}^{m} B_i \tilde{D}_i \tilde{u}_i(t) + \sum_{i=1}^{m} B_i \tilde{A}_i u_i(t) \\
\tilde{y}_0(t) &= E_0 x(t) \\
\tilde{y}_i(t) &= \tilde{E}_i C_i x(t-\tau_i) + \tilde{E}_i u_i, \quad i=1,\dots,m \\
y_i(t) &= -\zeta w_i C_i \dot{x}(t), \quad i=1,\dots,m
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
\tilde{u}_i(t) &= \delta A_i(t) \tilde{y}_i(t), \quad i=0,\dots,m \\
u_i(t) &= \frac{1}{\zeta} (\mathscr{S}_I^{\tilde{n}_i} y_i)(t), \quad i=1,\dots,m
\end{aligned} \tag{32}$$

where $\zeta > 0$ is a parameter.

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Using this feedback interconnection point of view, lower bounds on the stability radii can be derived analogously as in the case where only the delays are uncertain, which we have discussed in the previous section. In what follows, we therefore restrict ourselves to formulating the main results.

3.1. Time-varying perturbations

Let G be the transfer function of (31), that is,

$$G(\lambda;\zeta)=\begin{bmatrix} E_0 \\ e^{-\lambda\tau_1}\tilde{E}_1 C_1 \\ \vdots \\ e^{-\lambda\tau_m}\tilde{E}_m C_m \\ -\zeta w_1 \lambda C_1 \\ \vdots \\ -\zeta w_m \lambda C_m \end{bmatrix} \left[\lambda I - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda\tau_i}\right]^{-1}$$

$$\times [D_0 \;\; B_1\tilde{D}_1 \ldots B_m\tilde{D}_m \;\; B_1\tilde{A}_1 \ldots B_m\tilde{A}_m] + \begin{bmatrix} 0 & \ldots & 0 & & & 0 \\ & & 0 & \tilde{E}_1 & & \\ & & & & \ddots & \\ & & & & & \tilde{E}_m \\ 0 & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & \cdots & 0 \end{bmatrix} \tag{33}$$

Proposition 5

We have the following estimate:

$$r_{\mathrm{c}}^{\mathrm{d}} \geqslant (\|G(\lambda;\sqrt{7/4})\|_{\mathscr{H}_\infty})^{-1}$$

This proposition can again be strengthened by an appropriate scaling in the feedback loop. With the set $\mathscr{T}$ defined as

$$\mathscr{T}=\{\mathrm{diag}(t_0 I_{n_0},\ldots,t_m I_{n_m},T_1,\ldots,T_m): t_i>0,\ i=0,\ldots,m,\ T_i\in\mathbb{C}^{\tilde{n}_i\times\tilde{n}_i},$$
$$\det T_i\neq 0,\ i=1,\ldots,m\} \tag{34}$$

we obtain:

$$r_{\mathrm{c}}^{\mathrm{d}} \geqslant \left(\min_{T\in\mathscr{T}} \|T^{-1}G(\lambda;\sqrt{7/4})T\|_{\mathscr{H}_\infty}\right)^{-1}$$

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3.2. Time-invariant perturbations

Taking into account the structure of the feedback path (32) and the estimate (18), we arrive at:

Proposition 6
Define the uncertainty set as

$$\mathbf{\Delta} := \{\mathrm{diag}(\Delta_0, \ldots, \Delta_m, d_1 I_{\tilde{n}_1}, \ldots, d_m I_{\tilde{n}_m}) : \Delta_i \in \mathbb{C}^{n_i \times n_i}, d_j \in \mathbb{C},\ i = 0, \ldots, m,\ 1 \leqslant j \leqslant m\} \tag{35}$$

Then

$$r_{\mathrm{c}}^{\mathrm{s}} \geqslant \left(\sup_{\omega \geqslant 0} \mu_{\mathbf{\Delta}}(G(\mathrm{j}\omega;\ 1)) \right)^{-1}$$

Using the scaling-based upper bound on the structured singular value described in the appendix, we arrive at:

$$r_{\mathrm{c}}^{\mathrm{s}} \geqslant \left(\sup_{\omega \geqslant 0} \min_{T \in \mathscr{T}} \|T^{-1} G(\mathrm{j}\omega; 1) T\|_2 \right)^{-1}$$

where $\mathscr{T}$ is given by (34). An improvement of the estimate (18) finally leads to:

Proposition 7
Let the function $s : \mathbb{R}_+ \to \mathbb{R}_+$ be given by (24). Define $F : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+$,

$$\alpha \mapsto F(\alpha) := \sup_{\omega \geqslant 0} \mu_{\mathbf{\Delta}}(G_2(\mathrm{j}\omega; \alpha)) \tag{36}$$

where

$$G_2(\lambda; \alpha) = \begin{bmatrix} \alpha E_0 \\ \mathrm{e}^{-\lambda\tau_1} \alpha \tilde{E}_1 C_1 \\ \vdots \\ \mathrm{e}^{-\lambda\tau_m} \alpha \tilde{E}_m C_m \\ -2s\left(\dfrac{w_1 \alpha \omega}{2}\right) C_1 \\ \vdots \\ -2s\left(\dfrac{w_m \alpha \omega}{2}\right) C_m \end{bmatrix} \left[\lambda I - A_0 - \sum_{i=1}^{m} A_i \mathrm{e}^{-\lambda\tau_i} \right]^{-1}$$

$$\times [D_0\ \ B_1 \tilde{D}_1 \cdots B_m \tilde{D}_m\ \ B_1 \tilde{A}_1 \cdots B_m \tilde{A}_m] + \begin{bmatrix} 0 & \cdots & 0 & & & 0 \\ & & 0 & \alpha \tilde{E}_1 & & \\ & & & & \ddots & \\ & & & & & \alpha \tilde{E}_m \\ 0 & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & \cdots & 0 \end{bmatrix} \tag{37}$$

DOI: 10.1002/rnc

Then, we have the following estimate:

$$r_c^s \geqslant \sup\{\alpha>0 : F(\alpha)<1\} \tag{38}$$

4. EXAMPLES

4.1. Third-order system with two delays

We consider the following system from [20]:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2) \tag{39}$$

where

$$A_0 = \begin{bmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -5.9 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 7.1 & -70.3 \\ 0 & -1 & 5 \\ 0 & 0 & 6 \end{bmatrix} \tag{40}$$

For the delay parameters

$$(\tau_1, \tau_2) = (0.05, 0.2) \tag{41}$$

the zero solution is asymptotically stable with the spectral abscissa equal to $-6.10646\mathtt{E-6}$. Next, we assume perturbations on the nominal delay parameters (41), which are weighted according to

$$(w_1, w_2) = (1, 2)$$

When factorizing $A_i = B_i C_i,\ i = 1, 2$, with

$$B_1 = \begin{bmatrix} -5.9 \\ 2 \\ 2 \end{bmatrix}, \quad C_1 = [1\ \ 0\ \ 0], \quad B_2 = \begin{bmatrix} 7.1 & -70.3 \\ -1 & 5 \\ 0 & 6 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

an application of the results of the previous sections yields the following lower bounds on stability radii in the delay parameters:

	dynamic (r_τ^d)		static (r_τ^s)	
Estimate	(11)	(15)	(23)	(26)
Lower bound	3.09920E−3	7.35772E−3	9.73349E−3	9.79730E−3

Because there are only two delays in this example an exact characterization of stability regions of the zero solution in the delay parameter space can be made, under the assumption that the delays are time invariant, and the static stability radius r_τ^s can be computed subsequently. The results are shown in Figure 1. The nominal delay values (41) are indicated with the plus symbol.

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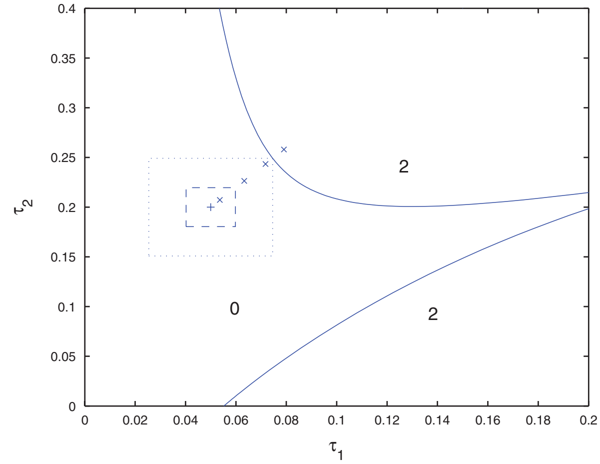

Figure 1. Stability regions of (39)–(40) in the (τ_1, τ_2)-parameter space. For the delay values corresponding to the crosses solutions are shown in Figure 2.

The numbers correspond to the number of characteristic roots in the open right half plane. The full curves bound the stability region, and the dashed lines bound the set as

$$\max\left(\left|\frac{\tau_1-0.05}{1}\right|, \left|\frac{\tau_2-0.2}{2}\right|\right) \leqslant 9.79730\text{E}-3$$

that is, the set of delays for which the stability is guaranteed by the criterion (26). The dotted box corresponds to the region

$$\max\left(\left|\frac{\tau_1-0.05}{1}\right|, \left|\frac{\tau_2-0.2}{2}\right|\right) = r_{\boldsymbol{\tau}}^{\mathrm{s}} = 2.45769\text{E}-2$$

In Figure 2 we show some solutions of (39)–(40) for the delay values given in Table I, which correspond to the crosses in Figure 1. The initial conditions are

$$x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^{\mathrm{T}} = [1\ 1\ 1]^{\mathrm{T}} \quad \forall t \in [-\tau_2,\ 0] \tag{42}$$

Note that the behavior of the solutions is consistent with the stability region shown in Figure 1.

4.2. *Multiple integrator with delayed output feedback*

As a second example, we consider the function as

$$p(\lambda; \boldsymbol{\tau}) := \lambda^4 + 0.24719\mathrm{e}^{-\lambda\tau_1} - 0.7.2479\mathrm{e}^{-\lambda\tau_2} + 0.70852\mathrm{e}^{-\lambda\tau_3} - 0.23091\mathrm{e}^{-\lambda\tau_4} \tag{43}$$

This quasi-polynomial is stable for the nominal parameter values

$$\boldsymbol{\tau} = (1, 2, 3, 4)$$

with rightmost zeros equal to $-1.45364\text{E}-2 \pm \mathrm{j}2.96287\text{E}-2$. It can be interpreted as the characteristic function of a quadruple integrator controlled with delayed output feedback. It is constructed

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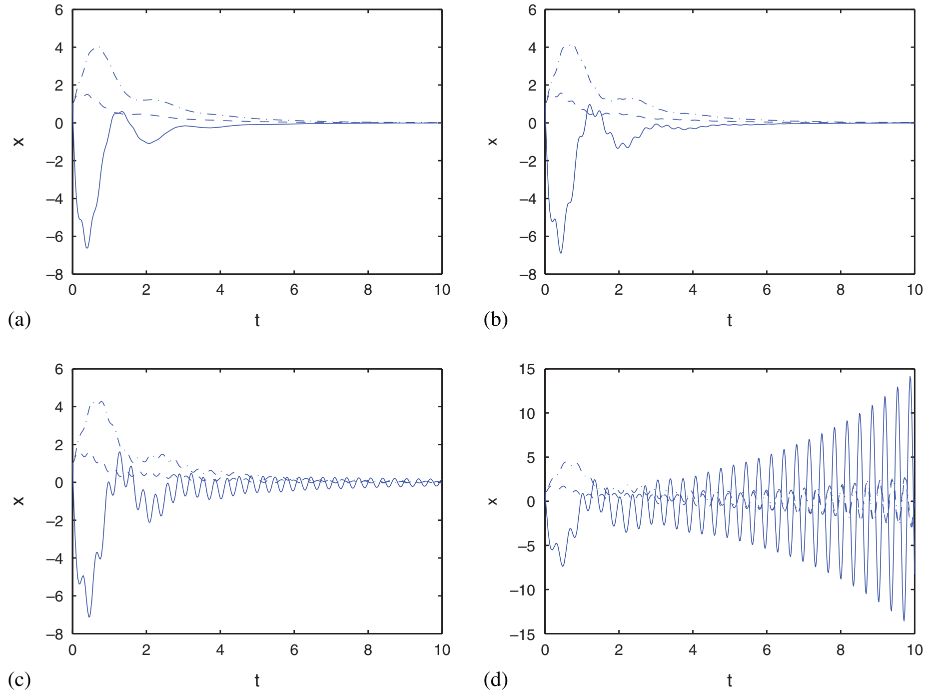

Figure 2. Solutions of (39)–(40) for the delay values shown in Table I. The initial conditions are given by (42). The full curve corresponds to $x_1(t)$, the dash-dotted curve to $x_2(t)$ and the dashed one to $x_3(t)$.

Table I. Parameter values corresponding the simulations shown in Figure 2.

	τ_1	τ_2
(a)	0.0536	0.207
(b)	0.0633	0.226
(c)	0.0718	0.243
(d)	0.0791	0.258

by using the algorithm of Theorem 1 of [21], which leads to the following important properties:

1. for $\boldsymbol{\tau} = (0, 0, 0, 0)$ the function p is *unstable*, with two zeros confined to the open right half plane;
2. the stability of p is extremely sensitive to parametric uncertainty. This is a consequence of ill-conditioned zeros, which is due to the fact that $p(\lambda; \tau)$ is a perturbation of the polynomial λ^4, exhibiting a zero with multiplicity four. This sensitivity is illustrated with the following (exact) stability margins, which consider the effect of a time-invariant perturbation on one delay parameter, while keeping the other delay parameters fixed to the nominal values:

i	1	2	3	4
$\max \lvert\delta\tau_i\rvert$	1.3684E−3	4.7114E−4	4.8683E−4	1.5092E−3

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These margins correspond to the static stability radii for weight vectors $(w_1, \dots, w_4)$ equal to one of the four unit vectors.

In order to assess the effect on stability of *combined* perturbations on the four delays, we apply the results of Section 2. If we assume that all weights w_i are unity, then an application of formula (26), respectively (15), yields:

$$r_{\boldsymbol{\tau}}^{s} \geqslant 1.6098\mathrm{E}-4, \quad r_{\boldsymbol{\tau}}^{d} \geqslant 1.2169\mathrm{E}-4 \tag{44}$$

As a further illustration of the sensitivity of the problem we have used the package DDE-BIFTOOL [22] to compute the stability region in the (τ_1, τ_2) parameter space for $(\tau_3, \tau_4) = (3, 4)$, under the assumption of time-invariant delays. The result are shown in Figure 3. The nominal delays are indicated with the plus sign The dashed box bounds the region as

$$\max(|\tau_1 - 0.05|, |\tau_2 - 0.2|) \leqslant 1.6098\mathrm{E}-4$$

Here, it is important to note that the actual conservatism of the estimates (44) is much less than the distance from the dashed box to the stability boundary suggests. The bounds (44) namely concern combined perturbations on *all* four delays, while the figure shows stability margins in the parameters (τ_1, τ_2) for particular *fixed* values of (τ_3, τ_4). Finally, for the delay values given in Table II and indicated with crosses in Figure 3, we show in Figure 4 the solutions of the following state-space realization of (43) as:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
\dot{x}_2(t) &= x_3(t)\\
\dot{x}_3(t) &= x_4(t)\\
\dot{x}_4(t) &= -0.24719\, x_1(t-\tau_1) + 0.7.2479\, x_1(t-\tau_2) - 0.70852\, x_1(t-\tau_3)\\
&\qquad + 0.23091\, x_1(t-\tau_4)
\end{aligned} \tag{45}$$

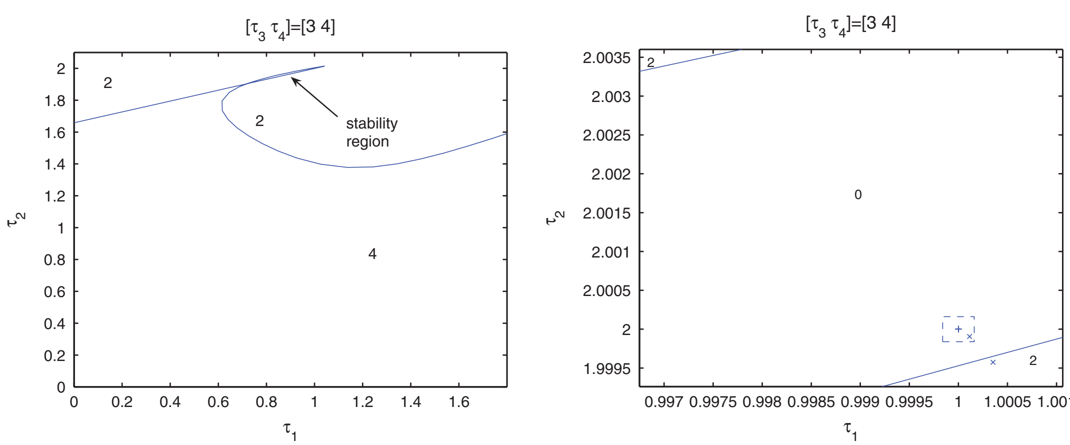

Figure 3. Stability regions of the quasi-polynomial (43) in the (τ_1, τ_2) parameter space for $(\tau_3, \tau_4) = (3, 4)$ on two different scales. The numbers refer to the number of zeros in the open right half plane. For parameter values corresponding to the crosses, some solutions of (45) are shown in Figure 4.

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Table II. Parameter values corresponding to the simulations shown in Figure 4.

	τ_1	τ_2	τ_3	τ_4
(a)	1.000114	1.9999062	3	4
(b)	1.000352	1.9995744	3	4

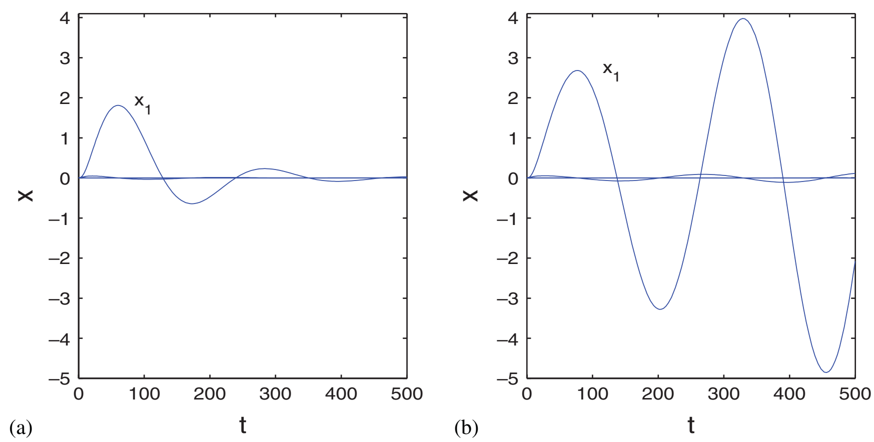

Figure 4. Solutions of (45), for initial conditions (46) and delay parameters given in Table II.

with initial conditions

$$x(t)=[x_1(t)^{\mathrm{T}} \;\; \dots \;\; x_4(t)^{\mathrm{T}}]^{\mathrm{T}}=0.001\;[1\;\;1\;\;1\;\;1]^{\mathrm{T}} \quad \forall t\in[-\tau_4,0] \tag{46}$$

Note that one component of x, x_1, dominates the others. This is a consequence of the slow peaking phenomenon inherent to the control of integrator chains with low gain feedback [23].

5. CONCLUDING REMARKS

Stability radii of linear systems with uncertain delays were defined and lower bounds were derived using a feedback interconnection point of view. The main advantage of the approach is that explicit, easy-to-check expressions are derived, which are generally applicable in the following sense. First, there are no restrictions or assumptions on the system's dimensions and on the number of delays. Second, perturbations on the delays and the system matrices can be dealt with at the same time, with the perturbations on the system matrices being vector valued and possibly structured. Third, both constant and time-varying perturbations can be considered. Finally, in the analysis the stability of the corresponding delay-free system is not required. Inherent to the robust control approach is that the bounds may be conservative [5, 24]. It should be noted that conservatism does not occur when using direct methods based on an explicit characterization or computation of stability regions in parameter spaces (e.g. the τ or D-subdivision method [5] or special methods for computing stability regions in delay spaces [20, 25]). These methods are however only applicable to special situations, where all uncertainty is time invariant and only concerns a very small number of scalar

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parameters, and from a computational point of view more information than necessary is computed (partition of parameter space in regions featuring the same number of characteristic roots instead of the distance to instability).

Finally, we note that if information on the delays' variation and/or derivatives is available, then the derived estimates for the dynamic stability radii may be further improved, as we indicated in Remark 5.

APPENDIX A: THE STRUCTURED SINGULAR VALUE

We introduce the concept of structured singular values of matrices and outline the main principles behind the standard computational schemes. A more elaborate introduction can be found in the review paper [26], Chapter 11 of [24] and Chapter 4 of [3].

A classical result from linear algebra and robust control theory, which lays the basis for the celebrated small gain theorem, relates the largest singular value of a matrix $G \in \mathbb{C}^{N\times M}$ to the solutions of the equation

$$\det(I+G\Delta)=0 \tag{A1}$$

in the following way:

$$\sigma_1(G)=\begin{cases} 0 & \text{if } \det(I+G\Delta)\neq 0 \ \ \forall \Delta\in\mathbb{C}^{M\times N} \\ m_u^{-1} & \text{otherwise}\end{cases} \tag{A2}$$

where

$$m_u := \min\{\sigma_1(\Delta) : \Delta\in\mathbb{C}^{M\times N} \text{ and } \det(I+G\Delta)=0\}$$

We refer to Δ as the 'uncertainty' as in a robust control framework, (A1) typically originates from a feedback interconnection of a nominal transfer function and an uncertainty block.

Next, we reconsider the solutions of equation (A1), where Δ is restricted to having a particular structure by imposing $\Delta\in\mathbf{\Delta}$, with $\mathbf{\Delta}$ a closed subset of $\mathbb{C}^{M\times N}$. In analogy with (A2) one defines the *structured singular value* of the matrix G with respect to the uncertainty set $\mathbf{\Delta}$ as

$$\mu_{\mathbf{\Delta}}(G) := \begin{cases} 0 & \text{if } \det(I+G\Delta)\neq 0 \ \ \forall \Delta\in\mathbf{\Delta} \\ m_s^{-1} & \text{otherwise}\end{cases} \tag{A3}$$

where

$$m_s := \min\{\sigma_1(\Delta) : \Delta\in\mathbf{\Delta} \text{ and } \det(I+G\Delta)=0\}$$

It directly follows from the definition that:

$$\mu_{\mathbf{\Delta}}(G)\leqslant\sigma_1(G) \tag{A4}$$

Furthermore, if $\mathbb{C}\mathbf{\Delta}=\mathbf{\Delta}$, then

$$\mu_{\mathbf{\Delta}}(G)=\max_{\Delta\in\mathbf{\Delta},\,\sigma_1(\Delta)=1} r_\sigma(G\Delta) \tag{A5}$$

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with $r_\sigma(\cdot)$ the spectral radius.

In what follows we restrict ourselves for simplicity to an uncertainty set $\mathbf{\Delta}$ of the form as

$$\mathbf{\Delta} := \{\mathrm{diag}(\Delta_0, \ldots, \Delta_f, d_0 I_{m_0}, \ldots, d_s I_{m_s}) : \Delta_i \in \mathbb{C}^{k_i \times l_i}, d_j \in \mathbb{C},\ 0 \leqslant i \leqslant f,\ 0 \leqslant j \leqslant s\} \tag{A6}$$

where $\sum_{i=0}^{f} k_i + \sum_{i=0}^{s} m_i = M$ and $\sum_{i=0}^{f} l_i + \sum_{i=0}^{s} m_i = N$. Such a set satisfies $\mathbb{C}\mathbf{\Delta} = \mathbf{\Delta}$. Furthermore, based on a slight generalization of [26, Lemma 6.3] to non-square block diagonal perturbations, the search space of the optimization in the right-hand side of (A5) can be restricted. This results in

$$\mu_{\mathbf{\Delta}}(G) = \max_{U \in \mathscr{U}} r_\sigma(GU) \tag{A7}$$

where $\mathscr{U} \subseteq \mathbf{\Delta}$ is defined as

$$\begin{aligned} \mathscr{U} := \{&\mathrm{diag}(U_0, \ldots, U_f, u_0 I_{m_0}, \ldots, u_s I_{m_s}) : U_i \in \mathbb{C}^{k_i \times l_i},\ u_j \in \mathbb{C},\ \sigma_k(U_i) = 1, \\ &1 \leqslant k \leqslant \min(k_i, l_i),\ \ |u_j| = 1,\ 0 \leqslant i \leqslant f,\ 0 \leqslant j \leqslant s\} \end{aligned} \tag{A8}$$

Note that the elements of $\mathscr{U}$ are *unitary* matrices if the uncertainty structure only involves square blocks, that is, $k_i = l_i,\ i = 1, \ldots, f$.

Next, the following invariance property can easily be checked:

$$\mu_{\mathbf{\Delta}}(G) = \mu_{\mathbf{\Delta}}(D_2 G D_1^{-1}) \quad \forall (D_1, D_2) \in \mathscr{D} \tag{A9}$$

where

$$\begin{aligned} \mathscr{D} := \{&(D_1, D_2) : D_1 = \mathrm{diag}(a_0 I_{k_1}, \ldots, a_f I_{k_f}, D_0, \ldots, D_s), \\ &D_2 = \mathrm{diag}(a_0 I_{l_1}, \ldots, a_f I_{l_f}, D_0, \ldots, D_s) : a_i > 0, D_i \in \mathbb{C}^{m_i \times m_i}, D_i^* = D_i > 0\} \end{aligned}$$

From (A7) and the combination of (A9) and (A4) we finally obtain

$$\max_{U \in \mathscr{U}} r_\sigma(GU) = \mu_{\mathbf{\Delta}}(G) \leqslant \min_{(D_1, D_2) \in \mathscr{D}} \sigma_1(D_2 G D_1^{-1}) \tag{A10}$$

Therefore, optimization algorithms are typically used to compute estimates for $\mu_{\mathbf{\Delta}}(G)$. The function $U \in \mathscr{U} \rightarrow r_\sigma(GU)$ may have several local maxima and, for this, a local search for a maximum is not guaranteed to lead to $\mu_{\mathbf{\Delta}}(G)$, but to lower bounds. An appropriate formulation of the optimality condition enables algorithms, which resemble power algorithms for computing eigenvalues and singular values, see Reference [27] for an example. Although the convergence of such algorithms to $\mu_{\mathbf{\Delta}}(G)$ is not guaranteed either and they may converge to values corresponding to lower bounds on $\mu_{\mathbf{\Delta}}(G)$, they have proven their effectiveness in practise. The computation of the upper bound in (A10) can be recast into a standard *convex* optimization problem. However, in general $\mu_{\mathbf{\Delta}}(G)$ is not equal to the upper bound. An exception to this holds if the number of blocks in the matrices belonging to the uncertainty set $\mathbf{\Delta}$ satisfies $f + 2s \leqslant 3$ and, in addition, all blocks are square, that is, $k_i = l_i,\ i = 0, \ldots, f$.

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NOTATION

$\mathbb{C}$	set of complex numbers
$\mathbb{C}_-, \mathbb{C}_+$	open left half plane, open right half plane
$\mathscr{C}(\mathscr{I}, \mathbb{R}^p), I \subseteq \mathbb{R}$	space of continuous functions from $\mathscr{I}$ to $\mathbb{R}^p$
$\mathrm{diag}(M_1, \ldots, M_r)$	block diagonal matrix with diagonal blocks $M_1, \ldots, M_r$
$\bar{E}$	closure of the set E
$\mathscr{H}_\infty$	space of functions G, analytic in $\mathbb{C}_+$ and satisfying $\sup_{\lambda \in \overline{\mathbb{C}}_+} \sigma_1(G(\lambda)) < \infty$
$I_p, \ p \in \mathbb{N}$	identity matrix in $\mathbb{C}^{p \times p}$
j	imaginary unit
$\mathscr{L}_2(\mathscr{I}, \mathbb{R}^r), \mathscr{I} \subseteq \mathbb{R}$	$\{f : \mathscr{I} \to \mathbb{R}^r \mid (\int_{\mathscr{I}} \|f(s)\|_2^2)^{1/2} \, \mathrm{d}s < \infty\}$
$\mathscr{L}_\infty(\mathscr{I}, \mathbb{R}^r), \mathscr{I} \subseteq \mathbb{R}$	$\{f : \mathscr{I} \to \mathbb{R}^r \mid \text{ess sup}_{s \in \mathscr{I}} \|f(s)\|_2 < \infty\}$
$\mathbb{N}$	set of natural numbers, includes zero
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	$\{r \in \mathbb{R} : \ r \geqslant 0\}$
$\mathbf{r} \in \mathbb{R}^m$	short notation for $(r_1, \ldots, r_m)$
$\mathbf{r} \geqslant 0, \mathbf{r} \in \mathbb{R}^m$	short notation for $r_i \geqslant 0, 1 \leqslant i \leqslant m$
$r_\sigma(A)$	spectral radius of matrix or operator A
$\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots$	singular values of matrix A
$\|x\|_p, x \in \mathbb{C}^n, p \in \mathbb{R}_+ \setminus \{0\} \cup \{\infty\}$	Hölder p norm of x
$\|\mathscr{G}\|_{\mathscr{L}_2}, \mathscr{G} : \mathscr{L}_2(\mathscr{I}_1, \mathbb{R}^{r_1}) \to \mathscr{L}_2(\mathscr{I}_2, \mathbb{R}^{r_2})$	induced $\mathscr{L}_2$ norm of $\mathscr{G}$
$\|G(\lambda)\|_{\mathscr{H}_\infty}, G \in \mathscr{H}_\infty$	H-infinity norm of G, $\|G(\lambda)\|_{\mathscr{H}_\infty} = \sup_{\lambda \in \overline{\mathbb{C}}_+} \sigma_1(G(\lambda))$

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