IMA Journal of Mathematical Control and Information Page 1 of 18
doi:10.1093/imamci/dnq023

Stability of linear systems with general sawtooth delay

KUN LIU, VLADIMIR SUPLIN AND EMILIA FRIDMAN*
Department of Electrical Engineering-Systems, School of Electrical Engineering,
Tel-Aviv University, Tel-Aviv 69978, Israel
liukun@eng.tau.ac.il suplin@eng.tau.ac.il *Corresponding author: emilia@eng.tau.ac.il

[Received on 4 February 2010; revised on 2 June 2010; accepted 31 August 2010]

It is well known that in many particular systems, the upper bound on a certain time-varying delay that preserves the stability may be higher than the corresponding bound for the constant delay. Moreover, sometimes oscillating delays improve the performance (Michiels, W., Van Assche, V. & Niculescu, S. (2005) Stabilization of time-delay systems with a controlled time-varying delays and applications. *IEEE Trans. Automat. Control*, **50**, 493–504). Sawtooth delays τ with $\dot{\tau} = 1$ (almost everywhere) can posses this property (Louisell, J. (1999) New examples of quenching in delay differential equations having time-varying delay. *Proceedigns of the 5th ECC*, Karlsruhe, Germany). In this paper, we show that general sawtooth delay, where $\dot{\tau} \neq 0$ is constant (almost everywhere), also can posses this property. By the existing Lyapunov-based methods, the stability analysis of such systems can be performed in the framework of systems with bounded fast-varying delays. Our objective is to develop 'qualitatively new methods' that can guarantee the stability for sawtooth delay which may be not less than the analytical upper bound on the constant delay that preserves the stability. We suggest two methods. One method develops a novel input–output approach via a Wirtinger-type inequality. By this method, we recover the result by Mirkin (2007, Some remarks on the use of time-varying delay to model sample-and-hold circuits. *IEEE Trans. Automat. Control*, **52**, 1109–1112) for $\dot{\tau} = 1$ and we show that for any integer $\dot{\tau}$, the same maximum bound that preserves the stability is achieved. Another method extends piecewise continuous (in time) Lyapunov functionals that have been recently suggested for the case of $\dot{\tau} = 1$ in Fridman (2010, A refined input delay approach to sampled-data control. *Automatica*, **46**, 421–427) to the general sawtooth delay. The time-dependent terms of the functionals improve the results for all values of $\dot{\tau}$, though the most essential improvement corresponds to $\dot{\tau} = 1$.

Keywords: time-varying delay; Lyapunov-based methods; LMI.

1. Introduction

Over the past decades, much effort has been invested in the analysis and design of uncertain systems with time-varying delays (see, e.g. Kolmanovskii & Myshkis, 1999; Niculescu, 2001; Kharitonov & Niculescu, 2002; Fridman & Shaked, 2003; Richard, 2003; Gu *et al.*, 2003; He *et al.*, 2007; Park & Ko, 2007). The delay under consideration has been either differentiable with a known upper bound $0 \leqslant \dot{\tau} \leqslant d < 1$ or piecewise continuous without any constraints on the delay derivative (fast-varying delay) (Fridman & Shaked, 2003). In the existing Lyapunov-based methods, the maximum delay bound that preserves the stability corresponds to $d = 0$ and this bound is usually a decreasing function of d. However, it is well known (see examples in Louisell, 1999 and discussions on 'quenching' in Papachristodoulou *et al.*, 2007 as well as Example 1 below) that in many particular systems, the upper bound on a certain time-varying delay that preserves the stability may be higher than the corresponding bound for the constant delay. Moreover, sometimes oscillating delays improve the performance (Michiels *et al.*, 2005).

Recently, a discontinuous Lyapunov function method was introduced to sampled-data control systems (corresponds to the sawtooth delay with $\dot{\tau} = 1$ (almost everywhere)) in Naghshtabrizi *et al.* (2008). This method improved the existing Lyapunov-based results and it inspired a piecewise continuous (in time) Lyapunov–Krasovskii functional (LKF) approach to sampled-data systems (Fridman, 2010). The LKFs in the latter paper are time dependent and they do not grow after the sampling times. The introduced novel discontinuous terms of Lyapunov functionals in Fridman (2010) lead to qualitatively new results, allowing a superior performance under the sampling, than the one under the constant delay. The input delay approach to sampled-data control was also recently revised by using the scaled small gain theorem and a tighter upper bound on the L_2-induced norm of the uncertain term (Mirkin, 2007).

In the present paper, stability of systems with bounded sawtooth delay τ is analysed, where $\dot{\tau} \neq 0$ is piecewise constant. By the existing Lyapunov-based methods, the stability analysis of such systems can be performed in the framework of systems with bounded fast-varying delays. Our objective is to develop 'qualitatively new methods' that can guarantee the stability for sawtooth delay which may be not less than the analytical upper bound on the constant delay that preserves the stability. We suggest two methods. One method develops a novel input–output (I–O) approach via a Wirtinger-type inequality. By this method, we recover the result by Mirkin (2007) for $\dot{\tau} = 1$ and we show that for any integer $\dot{\tau}$, the same maximum bound that preserves the stability is achieved. Another method extends direct Lyapunov approach to systems with a general form of sawtooth delay. By constructing appropriate discontinuous LKFs, we obtain sufficient delay-dependent conditions that guarantee the exponential stability of systems in terms of linear matrix inequalities (LMIs). The time-dependent terms of LKFs improve the results for all values of $\dot{\tau}$, though the most essential improvement corresponds to $\dot{\tau} = 1$. A conference version of discontinuous LKF approach was presented in Liu & Fridman (2009).

Notations. Throughout the paper, the superscript '$\top$' stands for matrix transposition, $\mathcal{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. L_2 is the space of square integrable functions $v\colon [0,\infty) \to \mathcal{R}^n$ with the norm $\|v\|_{L_2} = [\int_0^\infty \|v(t)\|^2\,\mathrm{d}t]^{1/2}$. The space of functions $\phi\colon [a,b] \to \mathcal{R}^n$, which are absolutely continuous on $[a,b)$, have a finite $\lim_{\theta\to b^-}\phi(\theta)$ and have square integrable first-order derivatives is denoted by $W_n[a,b)$ with the norm

$$\|\phi\|_{W_n[a,b)} = \max_{\theta\in[a,b]} |\phi(\theta)| + \left[\int_a^b |\dot{\phi}(s)|^2\,\mathrm{d}s\right]^{\frac{1}{2}}.$$

We also denote $W = W_n[-h,0)$ and $x_t(\theta) = x(t+\theta)(\theta \in [-h,0])$.

2. Problem formulation

Consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-\tau(t)), \\ x(t) = \phi(t), \quad t \in [-h,0], \end{cases} \tag{2.1}$$

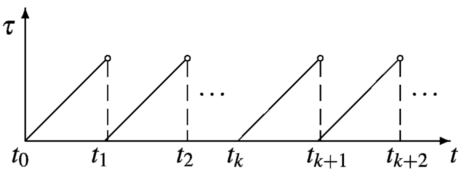

FIG. 1. Case A1.

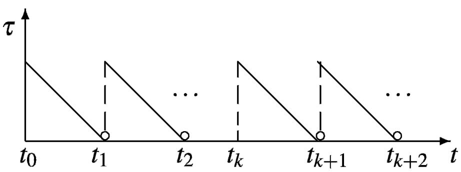

FIG. 2. Case A2.

where $x(t) \in \mathcal{R}^n$ is the state vector, A and A_1 denote the constant matrices, $\phi(t)$ is the initial function, $\tau(t) \in [0, h]$ is the time-varying delay. It is assumed that the delay function has the form of sawtooth (see Figs 1 and 2), satisfying either A1 or A2 below:

$$\text{A1: } \tau(t) = d(t - t_k), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2 \ldots, \tag{2.2}$$

$$\text{A2: } \tau(t) = d(t_{k+1} - t), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2 \ldots, \tag{2.3}$$

where $d > 0$ and $t_k = \frac{kh}{d}$.

It is clear that under A1, we have $\dot{\tau} = d > 0$ and under A2, we have $\dot{\tau} = -d < 0$. Both cases can be analysed by using time-independent Lyapunov functionals corresponding to systems with fast-varying delays. Our objective is to derive delay-dependent stability criteria for system (2.1) that improve the recent results for fast-varying delays (see, e.g. Park & Ko, 2007).

3. I–O approach via Wirtinger-type inequality

We recall the following Wirtinger-type inequality (Hardy *et al.*, 1934): let $z \in W_1[a, b)$ be a scalar function with $z(a) = 0$. Then,

$$\int_a^b z^2(\xi)\mathrm{d}\xi \leqslant \frac{4(b-a)^2}{\pi^2} \int_a^b \dot{z}^2(\xi)\mathrm{d}\xi. \tag{3.1}$$

This inequality is trivially extended to the vector case.

LEMMA 3.1 Let $z \in W_n[a, b)$. Assume that $z(a) = 0$. Then, for any $n \times n$-matrix $R > 0$, the following inequality holds:

$$\int_a^b z^\top(\xi) R z(\xi)\mathrm{d}\xi \leqslant \frac{4(b-a)^2}{\pi^2} \int_a^b \dot{z}^\top(\xi) R \dot{z}(\xi)\mathrm{d}\xi. \tag{3.2}$$

Proof. We have

$$\int_a^b z^\top(\xi)Rz(\xi)\mathrm{d}\xi \leqslant \frac{4(b-a)^2}{\pi^2}\int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}\xi}\sqrt{z^\top(\xi)Rz(\xi)}\right]^2 \mathrm{d}\xi$$
$$= \frac{4(b-a)^2}{\pi^2}\int_a^b \left(\frac{\dot{z}^\top(\xi)Rz(\xi)}{\sqrt{z^\top(\xi)Rz(\xi)}}\right)^2 \mathrm{d}\xi \leqslant \frac{4(b-a)^2}{\pi^2}\int_a^b \dot{z}^\top(\xi)R\dot{z}(\xi)\mathrm{d}\xi. \quad (3.3)$$

□

System (2.1) can be rewritten as follows:

$$\dot{x}(t) = (A+A_1)x(t) - A_1\int_{t-\tau(t)}^t \dot{x}(s)\mathrm{d}s. \quad (3.4)$$

We present the latter as the following forward system:

$$\begin{cases} \dot{x} = (A+A_1)x(t) + A_1u(t) \\ y(t) = \dot{x}(t), \end{cases} \quad (3.5)$$

with the feedback

$$u(t) = -\int_{t-\tau(t)}^t y(s)\mathrm{d}s. \quad (3.6)$$

Assume that $A+A_1$ is Hurwitz and $y(t)=0$ for $t \leqslant 0$.

LEMMA 3.2 Assume that the time delay is given by (2.2), where $d \in N$. Then the following holds:

$$\|u\|_{L_2} \leqslant \frac{2h}{\pi}\|y\|_{L_2}. \quad (3.7)$$

Proof. Defining

$$u_s(t) = -\int_{t_k}^t y(s)\mathrm{d}s, \quad u_{st}(t) = -\int_{t-\tau}^{t_k} y(s)\mathrm{d}s, \quad t \in [t_k, t_{k+1}),$$

we note that $u(t) = u_s(t) + u_{st}(t)$. We will prove next the following bounds:

$$\|u_s\|_{L_2} \leqslant \frac{2h}{\pi d}\|y\|_{L_2}, \quad (3.8)$$
$$\|u_{st}\|_{L_2} \leqslant (d-1)\frac{2h}{\pi d}\|y\|_{L_2}, \quad (3.9)$$

which imply (3.7) since

$$\|u\|_{L_2} \leqslant \|u_s\|_{L_2} + \|u_{st}\|_{L_2} \leqslant \frac{2h}{\pi}\|y\|_{L_2}.$$

By using (3.3), we obtain

$$\|u_s\|_{L_2}^2 = \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} (x(t_k) - x(t))^\top (x(t_k) - x(t))\mathrm{d}t \leqslant \sum_{k=0}^{\infty} \frac{4(t_{k+1}-t_k)^2}{\pi^2} \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}}{\mathrm{d}t}(x(t_k) - x(t))^\top \frac{\mathrm{d}}{\mathrm{d}t}(x(t_k) - x(t))\mathrm{d}t \leqslant \frac{4h^2}{\pi^2 d^2} \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \dot{x}^{\mathrm{T}}(t)\dot{x}(t)\mathrm{d}t = \frac{4h^2}{\pi^2 d^2} \|\dot{x}\|_{L_2}^2. \tag{3.10}$$

Similarly, we have

$$\|u_{st}\|_{L_2}^2 = \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} (x(t - d(t-t_k)) - x(t_k))^\top (x(t - d(t-t_k)) - x(t_k))\mathrm{d}t$$
$$\leqslant (d-1)^2 \sum_{k=0}^{\infty} \frac{4(t_{k+1}-t_k)^2}{\pi^2} \int_{t_k}^{t_{k+1}} \dot{x}^{\mathrm{T}}(t - d(t-t_k))\dot{x}(t - d(t-t_k))\mathrm{d}t.$$

Using the following change of variables

$$s = t - d(t-t_k) \quad \mathrm{d}s = (1-d)\mathrm{d}t, \quad d = 2, 3, \ldots$$
$$\left\langle \begin{array}{l} t = t_k \\ s = t_k \end{array} \middle| \begin{array}{l} t = t_{k+1} \\ s = (1-d)t_{k+1} + dt_k \end{array} \right\rangle,$$

we arrive to

$$\|u_{st}\|_{L_2}^2 \leqslant (d-1) \sum_{k=0}^{\infty} \frac{4(t_{k+1}-t_k)^2}{\pi^2} \int_{(1-d)t_{k+1}+dt_k}^{t_k} \dot{x}^\top(s)\dot{x}(s)\mathrm{d}s$$
$$\leqslant (d-1)\frac{4h^2}{\pi^2 d^2} \sum_{k=0}^{\infty} \int_{(k+1)h-dh}^{kh} \dot{x}^{\mathrm{T}}(s)\dot{x}(s)\mathrm{d}s = (d-1)^2 \frac{4h^2}{\pi^2 d^2} \|\dot{x}\|_{L_2}^2.$$

□

REMARK 3.1 For $d = 1$, the bound of Lemma 3.2 coincides with the bound of Mirkin (2007), where sampled-data control with variable sampling $t_{k+1} - t_k \leqslant h$ was analysed by using the lifting technique. We note that the bounds in (3.10) are valid also for $t_{k+1} - t_k \leqslant h$, i.e. for $d = 1$, we recover result of Mirkin (2007). Moreover, Lemma 3.2 strengthens the result of Mirkin (2007), showing that the same bound holds for any $d \in N$ if $t_{k+1} - t_k = h$.

It follows that stability of (3.5) can be verified by using the small gain theorem (Gu *et al.*, 2003). Namely, (3.5) is stable if

$$\text{there exists non-singular } M \in \mathcal{R}^{n\times n} \quad \text{such that } \|MGM^{-1}\|_\infty < 1, \tag{3.11}$$

where

$$G(s) = s[sI - (A + A_1)]^{-1} \cdot \frac{2h}{\pi} A_1. \tag{3.12}$$

We can verify the condition (3.11) via the following LMI:

$$\begin{bmatrix} P(A+A_1)+(A+A_1)^\top P & \frac{2h}{\pi}PA_1 & (A+A_1)^\top R \\ * & -R & \frac{2h}{\pi}A_1^\top R \\ * & * & -R \end{bmatrix} < 0 \tag{3.13}$$

for $P > 0$, $R > 0$.

THEOREM 3.1 For $d \in N$, (3.5)–(3.6) is I–O stable (and, thus, (2.1)–(2.2) is asymptotically stable) if one of the following conditions is satisfied:

- Condition (3.11) holds, where G is given by (3.12).
- There exist positive $n \times n$-matrices P, R such that LMI (3.13) is feasible.

4. Lyapunov–Krasovskii approach

4.1 *Lyapunov-based exponential stability*

DEFINITION 4.1 The system (2.1) is said to be exponentially stable if there exists constants $\mu > 0$ and $\delta > 0$ such that $\|x(t)\| \leqslant \mu\, \mathrm{e}^{-\delta(t-t_0)}\|\phi\|_W$ for $t \geqslant t_0$.

LEMMA 4.1 (Fridman, 2010). Let there exist positive numbers β, δ and a functional $V\colon \mathcal{R} \times W \times L_2[-h,0] \to \mathscr{R}$ such that

$$\beta|\phi(0)|^2 \leqslant V(t,\phi,\dot{\phi}) \leqslant \delta\|\phi\|_W^2. \tag{4.1}$$

Let the function $\bar{V}(t) = V(t, x_t, \dot{x}_t)$ is continuous from the right for $x(t)$ satisfying (2.1) absolutely continuous for $t \neq t_k$ and satisfies

$$\lim_{t\to t_k^-} \bar{V}(t) \geqslant \bar{V}(t_k). \tag{4.2}$$

Given $\alpha > 0$, if along (2.1)

$$\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leqslant 0, \quad \text{almost for all } t, \tag{4.3}$$

then (2.1) is exponentially stable with the decay rate α.

4.2 *Exponential stability: Case A1*

We start with the case, where τ satisfies $\dot{\tau} = d > 0$. We consider separately $0 < d \leqslant 1$ and $d > 1$.

(i) When $0 < d \leqslant 1$, we are looking for the functional of the form

$$V(t,x_t,\dot{x}_t) = \bar{V}(t) = V_0(x_t,\dot{x}_t) + \sum_{i=1}^{2} V_i(t,x_t,\dot{x}_t) + V_3(x_t), \tag{4.4}$$

where

$$V_0(x_t,\dot{x}_t) = x^\top(t)Px(t) + \int_{t-h}^{t} \mathrm{e}^{2\alpha(s-t)}x^\top(s)Sx(s)\mathrm{d}s + \frac{1}{h}\int_{-h}^{0}\int_{t+\theta}^{t} \mathrm{e}^{2\alpha(s-t)}\dot{x}^\top(s)R\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta, \tag{4.5}$$

$$V_1(t, x_t, \dot{x}_t) = \frac{h-\tau}{h}\xi_1^\top(t)\begin{bmatrix} \frac{X+X^\top}{2} & -X+X_1 \\ * & -X_1-X_1^\top+\frac{X+X^\top}{2}\end{bmatrix}\xi_1(t), \tag{4.6}$$

$$V_2(t, x_t, \dot{x}_t) = \frac{h-\tau}{h}\int_{t-\tau}^{t} \mathrm{e}^{2\alpha(s-t)}\dot{x}^\top(s)U\dot{x}(s)\mathrm{d}s, \tag{4.7}$$

$$V_3(x_t) = \int_{t-\tau}^{t} \mathrm{e}^{2\alpha(s-t)}x^\top(s)Qx(s)\mathrm{d}s, \tag{4.8}$$

with $\xi_1(t) = \mathrm{col}\{x(t), x(t-\tau)\}$, $\alpha \geqslant 0$, $P > 0$, $S > 0$, $R > 0$, $U > 0$, $Q > 0$.

The above functional coincides with the one introduced in Fridman (2010) for $S = R = Q = 0$, where $\dot{\tau} = 1$ was considered. The positive terms, depending on S, R, Q, guarantee that the results will be not worse than for the case of time-varying delays, where the above functional with $U = X = X_1$ can be applied. We note that the term V_3 is non-negative before the jumps at $t = t_k$ and it becomes zero just after the jumps (because $t_{|t=t_k} = (t-\tau)_{|t=t_k}$). The time-dependent terms V_1 and V_2 vanish before the jumps (because $\tau = h$) and after the jumps (because $\tau = 0$ and thus $x(t-\tau) = x(t)$). Thus, $\bar{V}$ does not increase after the jumps and the condition $\lim_{t\to t_k^-}\bar{V}(t) \geqslant \bar{V}(t_k)$ holds.

To guarantee that $V > 0$ in the sense that V satisfies (4.1), we assume that

$$\begin{bmatrix} P+\frac{X+X^\top}{2} & X_1-X \\ * & -X_1-X_1^\top+\frac{X+X^\top}{2}\end{bmatrix} > 0. \tag{4.9}$$

Differentiating $\bar{V}$, we find along (2.1)

$$\begin{aligned}
\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leqslant{}& 2x^\top(t)P\dot{x}(t) + \dot{x}^\top(t)\left[R + \frac{h-\tau}{h}U\right]\dot{x}(t) + 2\alpha[x^\top(t)Px(t)] \\
&- \frac{1}{h}\mathrm{e}^{-2\alpha h}\int_{t-h}^{t}\dot{x}^\top(s)R\dot{x}(s)\mathrm{d}s - \frac{d}{h}\mathrm{e}^{-2\alpha h}\int_{t-\tau}^{t}\dot{x}^\top(s)U\dot{x}(s)\mathrm{d}s \\
&+ x^\top(t)[S+Q]x(t) - x^\top(t-h)\mathrm{e}^{-2\alpha h}Sx(t-h) - (1-d)\dot{x}^\top(t-\tau) \\
&\times \frac{h-\tau}{h}U\,\mathrm{e}^{-2\alpha h}\dot{x}(t-\tau) - (1-d)x^\top(t-\tau)\mathrm{e}^{-2\alpha h}Qx(t-\tau) \\
&- \frac{d-2\alpha(h-\tau)}{h}\xi_1^\top(t)\begin{bmatrix} \frac{X+X^\top}{2} & -X+X_1 \\ * & -X_1-X_1^\top+\frac{X+X^\top}{2}\end{bmatrix}\xi_1(t) \\
&+ \frac{h-\tau}{h}\Big[\dot{x}^\top(t)(X+X^\top)x(t) + 2\dot{x}^\top(t)(-X+X_1)x(t-\tau) \\
&\qquad +2(1-d)x^\top(t)(-X+X_1)\dot{x}(t-\tau) \\
&\qquad +2(1-d)\dot{x}^\top(t-\tau)\left(-X_1-X_1^\top+\frac{X+X^\top}{2}\right)x(t-\tau)\Big].
\end{aligned}$$

Following He *et al.* (2007), we employ the representation

$$-\int_{t-h}^{t}\dot{x}^\top(s)R\dot{x}(s)\mathrm{d}s = -\int_{t-h}^{t-\tau}\dot{x}^\top(s)R\dot{x}(s)\mathrm{d}s - \int_{t-\tau}^{t}\dot{x}^\top(s)R\dot{x}(s)\mathrm{d}s. \tag{4.10}$$

We apply the Jensen's inequality (Gu *et al.*, 2003)

$$\int_{t-\tau}^{t} \dot{x}^{\top}(s)[R+U]\dot{x}(s)\mathrm{d}s \geqslant \frac{1}{\tau}\int_{t-\tau}^{t} \dot{x}^{\top}(s)\mathrm{d}s[R+U]\int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s,$$
$$\int_{t-h}^{t-\tau} \dot{x}^{\top}(s)R\dot{x}(s)\mathrm{d}s \geqslant \frac{1}{h-\tau}\int_{t-h}^{t-\tau} \dot{x}^{\top}(s)\mathrm{d}sR\int_{t-h}^{t-\tau} \dot{x}(s)\mathrm{d}s. \tag{4.11}$$

Here, for $\tau = 0$, we understand by

$$\frac{1}{\tau}\int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s = \lim_{\tau\to 0}\frac{1}{\tau}\int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s = \dot{x}(t).$$

For $h-\tau = 0$, the vector $\frac{1}{h-\tau}\int_{t-h}^{t-\tau} \dot{x}(s)\mathrm{d}s$ is defined similarly as $\dot{x}(t-h)$. Then, denoting

$$\upsilon_{11} = \frac{1}{\tau}\int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s, \quad \upsilon_{12} = \frac{1}{h-\tau}\int_{t-h}^{t-\tau} \dot{x}(s)\mathrm{d}s,$$

we obtain

$$\begin{aligned}
\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leqslant{}& 2x^{\top}(t)P\dot{x}(t) + \dot{x}^{\top}(t)\left[R + \frac{h-\tau}{h}U\right]\dot{x}(t) + 2\alpha[x^{\top}(t)Px(t)]\\
&- \mathrm{e}^{-2\alpha h}\frac{\tau}{h}\upsilon_{11}^{\top}[R+dU]\upsilon_{11} - \mathrm{e}^{-2\alpha h}\frac{h-\tau}{h}\upsilon_{12}^{\top}R\upsilon_{12} + x^{\top}(t)[S+Q]x(t)\\
&- x^{\top}(t-h)\mathrm{e}^{-2\alpha h}Sx(t-h) - (1-d)\dot{x}^{\top}(t-\tau)\frac{h-\tau}{h}U\,\mathrm{e}^{-2\alpha h}\dot{x}(t-\tau)\\
&- (1-d)x^{\top}(t-\tau)\mathrm{e}^{-2\alpha h}Qx(t-\tau)\\
&- \frac{d-2\alpha(h-\tau)}{h}\xi_1^{\top}(t)\begin{bmatrix}\frac{X+X^{\top}}{2} & -X+X_1\\ * & -X_1-X_1^{\top}+\frac{X+X^{\top}}{2}\end{bmatrix}\xi_1(t)\\
&+ \frac{h-\tau}{h}\Big[\dot{x}^{\top}(t)(X+X^{\top})x(t) + 2\dot{x}^{\top}(t)(-X+X_1)x(t-\tau)\\
&\qquad +2(1-d)x^{\top}(t)(-X+X_1)\dot{x}(t-\tau)\\
&\qquad +2(1-d)\dot{x}^{\top}(t-\tau)\left(-X_1-X_1^{\top}+\frac{X+X^{\top}}{2}\right)x(t-\tau)\Big].
\end{aligned} \tag{4.12}$$

Following He *et al.* (2004), we insert free-weighting $n\times n$-matrices by adding the following expressions to $\dot{\bar{V}}$:

$$0 = 2[x^{\top}(t)Y_1^{\top} + \dot{x}^{\top}(t)Y_2^{\top} + x^{\top}(t-\tau)T^{\top}][-x(t) + x(t-\tau) + \tau\upsilon_{11}],$$
$$0 = 2[x^{\top}(t)Z_1^{\top} + \dot{x}^{\top}(t)Z_2^{\top}][-x(t-\tau) + x(t-h) + (h-\tau)\upsilon_{12}]. \tag{4.13}$$

We use further the descriptor method (Fridman, 2001), where the right-hand side (RHS) of the expression

$$0 = 2[x^{\top}(t)P_2^{\top} + \dot{x}^{\top}(t)P_3^{\top}][Ax(t) + A_1x(t-\tau) - \dot{x}(t)], \tag{4.14}$$

with some $n\times n$-matrices, P_2, P_3 is added into the RHS of (4.12).

Setting

$$\eta_1(t) = \mathrm{col}\{x(t), \dot{x}(t), x(t-h), \upsilon_{11}, \upsilon_{12}, x(t-\tau), \dot{x}(t-\tau)\},$$

we obtain that

$$\dot{\bar{V}}(t) + 2\alpha \bar{V}(t) \leqslant \eta_1^\top(t)\Psi_1\eta_1(t) < 0, \tag{4.15}$$

if the following inequality

$$\Psi_1 = \begin{bmatrix} \Phi_{11} - \frac{d-2\alpha(h-\tau)}{2h}(X+X^\top) & \Phi_{12} + \frac{h-\tau}{2h}(X+X^\top) & Z_1^\top & \tau Y_1^\top & (h-\tau)Z_1^\top & \Phi_{16} & \Phi_{17} \\ * & \Phi_{22} + \frac{h-\tau}{h}U & Z_2^\top & \tau Y_2^\top & (h-\tau)Z_2^\top & \Phi_{26} & 0 \\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 & 0 \\ * & * & * & -\frac{\tau}{h}[R+dU]\,\mathrm{e}^{-2\alpha h} & 0 & \tau T & 0 \\ * & * & * & * & -\frac{h-\tau}{h}R\,\mathrm{e}^{-2\alpha h} & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & \Phi_{67} \\ * & * & * & * & * & * & \Phi_{77} \end{bmatrix} < 0 \tag{4.16}$$

holds, where

$$\begin{aligned}
\Phi_{11} &= A^\top P_2 + P_2^\top A + 2\alpha P + S + Q - Y_1 - Y_1^\top, \\
\Phi_{12} &= P - P_2^\top + A^\top P_3 - Y_2, \\
\Phi_{16} &= Y_1^\top - Z_1^\top + P_2^\top A_1 - T + \frac{d-2\alpha(h-\tau)}{h}(X - X_1), \\
\Phi_{17} &= (1-d)\frac{h-\tau}{h}(-X + X_1), \\
\Phi_{22} &= -P_3 - P_3^\top + R, \\
\Phi_{26} &= Y_2^\top - Z_2^\top + P_3^\top A_1 - \frac{h-\tau}{h}(X - X_1), \\
\Phi_{66} &= -(1-d)Q\,\mathrm{e}^{-2\alpha h} + T + T^\top - [d - 2\alpha(h-\tau)]\frac{X + X^\top - 2X_1 - 2X_1^\top}{2h}, \\
\Phi_{67} &= (1-d)\frac{h-\tau}{2h}(X + X^\top - 2X_1 - 2X_1^\top), \\
\Phi_{77} &= -(1-d)\frac{h-\tau}{h}U\,\mathrm{e}^{-2\alpha h}.
\end{aligned} \tag{4.17}$$

The latter inequality for $\tau \to 0$ and $\tau \to h$ leads to the following LMIs:

$$\Psi_{11} = \begin{bmatrix} \Phi_{11} - \frac{d-2\alpha h}{2h}(X+X^\top) & \Phi_{12} + \frac{X+X^\top}{2} & Z_1^\top & hZ_1^\top & \Phi_{16}|_{\tau=0} & \Phi_{17}|_{\tau=0} \\ * & \Phi_{22} + U & Z_2^\top & hZ_2^\top & \Phi_{26}|_{\tau=0} & 0 \\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 \\ * & * & * & -R\,\mathrm{e}^{-2\alpha h} & 0 & 0 \\ * & * & * & * & \Phi_{66}|_{\tau=0} & \Phi_{67}|_{\tau=0} \\ * & * & * & * & * & \Phi_{77}|_{\tau=0} \end{bmatrix} < 0, \tag{4.18}$$

$$\Psi_{12} = \begin{bmatrix} \Phi_{11} - \frac{d}{2h}(X+X^\top) & \Phi_{12} & Z_1^\top & hY_1^\top & \Phi_{16}|_{\tau=h} \\ * & \Phi_{22} & Z_2^\top & hY_2^\top & \Phi_{26}|_{\tau=h} \\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 \\ * & * & * & -[R+dU]\,\mathrm{e}^{-2\alpha h} & hT \\ * & * & * & * & \Phi_{66}|_{\tau=h} \end{bmatrix} < 0. \tag{4.19}$$

Denoting: $\eta_{1i}(t) = \mathrm{col}\{x(t), \dot{x}(t), x(t-h), \upsilon_{1i}, x(t-\tau), \dot{x}(t-\tau)\}(i = 1, 2)$, the latter two LMIs imply (4.16) because

$$\frac{h-\tau}{h}\eta_{12}^{\top}(t)\Psi_{11}\eta_{12}(t) + \frac{\tau}{h}\eta_{11}^{\top}(t)\Psi_{12}\eta_{11}(t) = \eta_1^{\top}(t)\Psi_1\eta_1(t) < 0,$$

and Ψ_1 is thus convex in $\tau \in [0, h]$.

(ii) For $d > 1$, we consider the following LKF:

$$V(t, x_t, \dot{x}_t) = \bar{V}(t) = V_0(x_t, \dot{x}_t) + \sum_{i=1}^{2} V_i(t, x_t, \dot{x}_t), \tag{4.20}$$

where V_0 is given by (4.5) and

$$V_1(t, x_t, \dot{x}_t) = \frac{h-\tau}{h}\xi_2^{\top}(t)\begin{bmatrix} \frac{X+X^{\top}}{2} & -X + X_1 \\ * & -X_1 - X_1^{\top} + \frac{X+X^{\top}}{2} \end{bmatrix}\xi_2(t), \tag{4.21}$$

$$V_2(t, x_t, \dot{x}_t) = \frac{h-\tau}{h}\int_{t-\frac{\tau}{d}}^{t} \mathrm{e}^{2\alpha(s-t)}\dot{x}^{\top}(s)U\dot{x}(s)\mathrm{d}s, \tag{4.22}$$

with $\xi_2(t) = \mathrm{col}\left\{x(t), x\left(t - \frac{\tau}{d}\right)\right\}, \alpha \geqslant 0, U > 0$. To guarantee that $V > 0$, we assume (4.9). Since $t_{|t=t_k} = \left(t - \frac{\tau}{d}\right)_{|t=t_k}$, we see that $\bar{V}$ does not grow after the jumps.

Differentiating $\bar{V}$, we have along (2.1)

$$\begin{aligned}\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leqslant{} & 2x^{\top}(t)P\dot{x}(t) + \dot{x}^{\top}(t)\left[R + \frac{h-\tau}{h}U\right]\dot{x}(t) + 2\alpha[x^{\top}(t)Px(t)] \\ & - \frac{1}{h}\mathrm{e}^{-2\alpha h}\int_{t-h}^{t}\dot{x}^{\top}(s)R\dot{x}(s)\mathrm{d}s - \frac{d}{h}\mathrm{e}^{-\frac{2\alpha h}{d}}\int_{t-\frac{\tau}{d}}^{t}\dot{x}^{\top}(s)U\dot{x}(s)\mathrm{d}s \\ & + x^{\top}(t)Sx(t) - x^{\top}(t-h)\mathrm{e}^{-2\alpha h}Sx(t-h) \\ & - \frac{d - 2\alpha(h-\tau)}{h}\xi_2^{\top}(t)\begin{bmatrix} \frac{X+X^{\top}}{2} & -X + X_1 \\ * & -X_1 - X_1^{\top} + \frac{X+X^{\top}}{2} \end{bmatrix}\xi_2(t) \\ & + \frac{h-\tau}{h}\left[\dot{x}^{\top}(t)(X + X^{\top})x(t) + 2\dot{x}^{\top}(t)(-X + X_1)x\left(t - \frac{\tau}{d}\right)\right].\end{aligned}$$

We employ the representation

$$-\int_{t-h}^{t}\dot{x}^{\top}(s)R\dot{x}(s)\mathrm{d}s = -\int_{t-h}^{t-\tau}\dot{x}^{\top}(s)R\dot{x}(s)\mathrm{d}s - \int_{t-\tau}^{t-\frac{\tau}{d}}\dot{x}^{\top}(s)R\dot{x}(s)\mathrm{d}s - \int_{t-\frac{\tau}{d}}^{t}\dot{x}^{\top}(s)R\dot{x}(s)\mathrm{d}s.$$

Similar to (4.11), applying the Jensen's inequality and then denoting

$$\upsilon_{21} = \frac{d}{\tau}\int_{t-\frac{\tau}{d}}^{t}\dot{x}(s)\mathrm{d}s, \quad \upsilon_{22} = \frac{1}{h-\tau}\int_{t-h}^{t-\tau}\dot{x}(s)\mathrm{d}s, \quad \upsilon_{23} = \frac{d}{h(d-1)}\int_{t-\tau}^{t-\frac{\tau}{d}}\dot{x}(s)\mathrm{d}s,$$

we obtain

$$\begin{aligned}\dot{\bar{V}}(t)+2\alpha\bar{V}(t)\leqslant{}& 2x^{\top}(t)P\dot{x}(t)+\dot{x}^{\top}(t)\left[R+\frac{h-\tau}{h}U\right]\dot{x}(t)+2\alpha[x^{\top}(t)Px(t)]\\ &-\frac{\tau}{h}\upsilon_{21}^{\top}\left[\frac{R}{d}\mathrm{e}^{-2\alpha h}+U\,\mathrm{e}^{-\frac{2\alpha h}{d}}\right]\upsilon_{21}-\mathrm{e}^{-2\alpha h}\frac{h-\tau}{h}\upsilon_{22}^{\top}R\upsilon_{22}-\mathrm{e}^{-2\alpha h}\frac{d-1}{d}\upsilon_{23}^{\top}R\upsilon_{23}\\ &+x^{\top}(t)Sx(t)-x^{\top}(t-h)\mathrm{e}^{-2\alpha h}Sx(t-h)\\ &-\frac{d-2\alpha(h-\tau)}{h}\xi_2^{\top}(t)\begin{bmatrix}\frac{X+X^{\top}}{2} & -X+X_1\\ * & -X_1-X_1^{\top}+\frac{X+X^{\top}}{2}\end{bmatrix}\xi_2(t)\\ &+\frac{h-\tau}{h}\left[\dot{x}^{\top}(t)(X+X^{\top})x(t)+2\dot{x}^{\top}(t)(-X+X_1)x\left(t-\tfrac{\tau}{d}\right)\right].\end{aligned}$$

Similar to (4.13), we insert free-weighting $n\times n$-matrices by adding the following expressions to $\dot{\bar{V}}$:

$$\begin{aligned}0&=2\left[x^{\top}(t)Y_1^{\top}+\dot{x}^{\top}(t)Y_2^{\top}+x^{\top}\left(t-\frac{\tau}{d}\right)T_1^{\top}\right]\left[-x(t)+x\left(t-\frac{\tau}{d}\right)+\frac{\tau}{d}\upsilon_{21}\right],\\ 0&=2[x^{\top}(t)Z_1^{\top}+\dot{x}^{\top}(t)Z_2^{\top}+x^{\top}(t-\tau)T_2^{\top}][-x(t-\tau)+x(t-h)+(h-\tau)\upsilon_{22}],\\ 0&=2[x^{\top}(t)M_1^{\top}+\dot{x}^{\top}(t)M_2^{\top}]\left[-x\left(t-\frac{\tau}{d}\right)+x(t-\tau)+\frac{h(d-1)}{d}\upsilon_{23}\right].\end{aligned}$$

Similar to (4.14), the same expression is added into the RHS of (4.2).
Setting $\eta_2(t)=\mathrm{col}\left\{x(t),\dot{x}(t),x(t-h),\upsilon_{21},\upsilon_{22},\upsilon_{23},x\left(t-\frac{\tau}{d}\right),x(t-\tau)\right\}$, we obtain that

$$\dot{\bar{V}}(t)+2\alpha\bar{V}(t)\leqslant\eta_2^{\top}(t)\Psi_2\eta_2(t)<0, \tag{4.23}$$

if the inequality

$$\Psi_2=\begin{bmatrix}\Omega_{11}-\frac{d-2\alpha(h-\tau)}{2h}(X+X^{\top}) & \Phi_{12}+\frac{h-\tau}{2h}(X+X^{\top}) & Z_1^{\top} & \frac{\tau}{d}Y_1^{\top} & (h-\tau)Z_1^{\top} & \frac{h(d-1)}{d}M_1^{\top} & \Omega_{17} & \Omega_{18}\\ * & \Phi_{22}+\frac{h-\tau}{h}U & Z_2^{\top} & \frac{\tau}{d}Y_2^{\top} & (h-\tau)Z_2^{\top} & \frac{h(d-1)}{d}M_2^{\top} & \Omega_{27} & \Omega_{28}\\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 & 0 & T_2\\ * & * & * & \Omega_{44} & 0 & 0 & \frac{\tau}{d}T_1 & 0\\ * & * & * & * & -\frac{h-\tau}{h}R\,\mathrm{e}^{-2\alpha h} & 0 & 0 & (h-\tau)T_2\\ * & * & * & * & * & -\frac{d-1}{d}R\,\mathrm{e}^{-2\alpha h} & 0 & 0\\ * & * & * & * & * & * & \Omega_{77} & 0\\ * & * & * & * & * & * & * & T_2+T_2^{\top}\end{bmatrix}<0 \tag{4.24}$$

holds, where

$$\begin{aligned}\Omega_{11}&=A^{\top}P_2+P_2^{\top}A+2\alpha P+S-Y_1-Y_1^{\top},\\ \Omega_{17}&=Y_1^{\top}-M_1^{\top}-T_1+\frac{d-2\alpha(h-\tau)}{h}(X-X_1),\\ \Omega_{18}&=M_1^{\top}-Z_1^{\top}+P_2^{\top}A_1,\\ \Omega_{27}&=Y_2^{\top}-M_2^{\top}-\frac{h-\tau}{h}(X-X_1),\\ \Omega_{28}&=M_2^{\top}-Z_2^{\top}+P_3^{\top}A_1,\\ \Omega_{44}&=-\frac{\tau}{h}\left[\frac{R}{d}\mathrm{e}^{-2\alpha h}+U\,\mathrm{e}^{-\frac{2\alpha h}{d}}\right],\\ \Omega_{77}&=T_1+T_1^{\top}-[d-2\alpha(h-\tau)]\frac{X+X^{\top}-2X_1-2X_1^{\top}}{2h}.\end{aligned} \tag{4.25}$$

The latter inequality for $\tau \to 0$ and $\tau \to h$ leads to the following LMIs:

$$\Psi_{21} = \begin{bmatrix} \Omega_{11} - \frac{d-2\alpha h}{2h}(X+X^\top) & \Phi_{12} + \frac{X+X^\top}{2} & Z_1^\top & hZ_1^\top & \frac{h(d-1)}{d}M_1^\top & \Omega_{17}|_{\tau=0} & \Omega_{18} \\ * & \Phi_{22}+U & Z_2^\top & hZ_2^\top & \frac{h(d-1)}{d}M_2^\top & \Omega_{27}|_{\tau=0} & \Omega_{28} \\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 & T_2 \\ * & * & * & -R\,\mathrm{e}^{-2\alpha h} & 0 & 0 & hT_2 \\ * & * & * & * & -\frac{d-1}{d}R\,\mathrm{e}^{-2\alpha h} & 0 & 0 \\ * & * & * & * & * & \Omega_{77}|_{\tau=0} & 0 \\ * & * & * & * & * & * & T_2+T_2^\top \end{bmatrix} < 0 \text{ and} \tag{4.26}$$

$$\Psi_{22} = \begin{bmatrix} \Omega_{11} - \frac{d}{2h}(X+X^\top) & \Phi_{12} & Z_1^\top & \frac{h}{d}Y_1^\top & \frac{h(d-1)}{d}M_1^\top & \Omega_{17}|_{\tau=h} & \Omega_{18} \\ * & \Phi_{22} & Z_2^\top & \frac{h}{d}Y_2^\top & \frac{h(d-1)}{d}M_2^\top & \Omega_{27}|_{\tau=h} & \Omega_{28} \\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 & T_2 \\ * & * & * & \Omega_{44}|_{\tau=h} & 0 & \frac{h}{d}T_1 & 0 \\ * & * & * & * & -\frac{d-1}{d}R\,\mathrm{e}^{-2\alpha h} & 0 & 0 \\ * & * & * & * & * & \Omega_{77}|_{\tau=h} & 0 \\ * & * & * & * & * & * & T_2+T_2^\top \end{bmatrix} < 0. \tag{4.27}$$

Similarly, we can obtain that Ψ_2 is also convex in $\tau \in [0, h]$.
We summarize the results in the following theorem.

THEOREM 4.2 (i) Given $\alpha > 0$, $0 < d \leqslant 1$, let there exist $n \times n$-matrices $P > 0$, $R > 0$, $U > 0$, $S > 0$, $Q > 0$, X, X_1, T, P_2, P_3, Y_i and $Z_i (i = 1, 2)$ such that the LMIs (4.9), (4.18), (4.19) with notations given in (4.17) are feasible. Then, (2.1) is exponentially stable with the decay rate α for all delays $0 \leqslant \tau \leqslant h$ satisfying A1.

(ii) Given $\alpha > 0$, $d > 1$, let there exist $n \times n$-matrices $P > 0$, $R > 0$, $U > 0$, $S > 0$, X, X_1, P_2, P_3, T_i, Y_i, Z_i and $M_i (i = 1, 2)$ such that the LMIs (4.9), (4.26), (4.27) with notations given in (4.25) are feasible. Then, (2.1) is exponentially stable with the decay rate α for all delays $0 \leqslant \tau \leqslant h$ satisfying A1.

REMARK 4.1 LMIs of Theorem 4.2 with $X = X_1 = U = 0$ give sufficient conditions for exponential stability of (2.1) with $\tau(t) \in [0, h]$: (i) for all slowly varying delays and (ii) for fast-varying delays. In the numerical examples, these conditions lead to the same results as the results of Park & Ko (2007), however, they posses a fewer number of slack matrices.

4.3 *Exponential stability: Case A2*

For $\dot{\tau} = -d < 0$, we also differ between $0 < d \leqslant 1$ and $d > 1$.

(i) When $0 < d \leqslant 1$, we are looking for the functional of the form:

$$V(t, x_t, \dot{x}_t) = \bar{V}(t) = V_0(x_t, \dot{x}_t) + \sum_{i=1}^{2} V_i(t, x_t, \dot{x}_t) + V_3(x_t), \tag{4.28}$$

where V_0 is given by (4.5) and

$$V_1(t, x_t, \dot{x}_t) = \frac{\tau}{h}\xi_3^\top(t) \begin{bmatrix} \frac{X+X^\top}{2} & -X + X_1 \\ * & -X_1 - X_1^\top + \frac{X+X^\top}{2} \end{bmatrix} \xi_3(t),$$

$$V_2(t, x_t, \dot{x}_t) = \frac{\tau}{h} \int_{t-(h-\tau)}^{t} \mathrm{e}^{2\alpha(s-t)} \dot{x}^\top(s) U \dot{x}(s) \mathrm{d}s,$$

$$V_3(x_t) = \int_{t-(h-\tau)}^{t} \mathrm{e}^{2\alpha(s-t)} x^\top(s) Q x(s) \mathrm{d}s,$$

with $\xi_3(t) = \mathrm{col}\{x(t), x(t-(h-\tau))\}$, $\alpha \geqslant 0$, $U > 0$, $Q > 0$ and (4.9) also satisfied to guarantee that $V > 0$.

We note that V_3 is non-negative before the jump at $t = t_k$ and it becomes zero just after the jump (because $t_{|t=t_k} = (t-(h-\tau))_{|t=t_k}$). The time-dependent terms V_1 and V_2 vanish before the jumps (because $\tau = 0$) and after the jumps (because $\tau = h$ and thus $x(t-(h-\tau)) = x(t)$). Hence, the condition $\lim_{t\to t_k^-} \bar{V}(t) \geqslant \bar{V}(t_k)$ holds.

(ii) For $d > 1$, the discontinuous Lyapunov functional is modified as follows:

$$V(t, x_t, \dot{x}_t) = \bar{V}(t) = V_0(x_t, \dot{x}_t) + \sum_{i=1}^{2} V_i(t, x_t, \dot{x}_t), \tag{4.29}$$

where V_0 is also given by (4.5) and

$$V_1(t, x_t, \dot{x}_t) = \frac{\tau}{h} \xi_4^\top(t) \begin{bmatrix} \frac{X+X^\top}{2} & -X + X_1 \\ * & -X_1 - X_1^\top + \frac{X+X^\top}{2} \end{bmatrix} \xi_4(t),$$

$$V_2(t, x_t, \dot{x}_t) = \frac{\tau}{h} \int_{t-\frac{h-\tau}{d}}^{t} \mathrm{e}^{2\alpha(s-t)} \dot{x}^\top(s) U \dot{x}(s) \mathrm{d}s,$$

with $\xi_4(t) = \mathrm{col}\left\{x(t), x\left(t - \frac{h-\tau}{d}\right)\right\}$, $\alpha \geqslant 0$, $U > 0$ and (4.9) also satisfied to guarantee that $V > 0$. Also in this case, $\bar{V}$ does not grow after the jumps.

Similar to Theorem 4.2, we obtain the following theorem.

THEOREM 4.3 (i) Given $\alpha > 0$, $0 < d \leqslant 1$, let there exist $n \times n$-matrices $P > 0$, $R > 0$, $U > 0$, $S > 0$, $Q > 0$, X, X_1, P_{21}, P_{31}, P_{22}, P_{32}, Y_{ij}, M_{ij}, Z_{ij} and $T_{ij} (i, j = 1, 2)$ such that (4.9) and the following four LMIs: (4.30), for $\tau \to 0$ and $\tau \to \frac{h}{2}$, and (4.31), for $\tau \to \frac{h}{2}$ and $\tau \to h$,

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} + \frac{\tau}{2h}(X + X^\top) & M_{11}^\top & \tau Y_{11}^\top & (h-2\tau) Z_{11}^\top & \tau M_{11}^\top & \Theta_{17} & \Theta_{18} + Z_{11}^\top - M_{11}^\top & \Theta_{19} \\ * & \Theta_{22} + \frac{\tau}{h} U & M_{12}^\top & \tau Y_{12}^\top & (h-2\tau) Z_{12}^\top & \tau M_{12}^\top & \Theta_{27} & \Theta_{28} + Z_{12}^\top - M_{12}^\top & 0 \\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\frac{\tau}{h}[R + dU]\mathrm{e}^{-2\alpha h} & 0 & 0 & \tau T_{11} & 0 & 0 \\ * & * & * & * & \Theta_{55} & 0 & 0 & (h-2\tau) T_{12} & 0 \\ * & * & * & * & * & -\frac{\tau}{h} R\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 \\ * & * & * & * & * & * & T_{11} + T_{11}^\top & -T_{12} & 0 \\ * & * & * & * & * & * & * & \Theta_{88} + T_{12} + T_{12}^\top & \Theta_{89} \\ * & * & * & * & * & * & * & * & \Theta_{99} \end{bmatrix} < 0, \tag{4.30}$$

$$\begin{bmatrix}
\Xi_{11} & \Xi_{12}+\frac{\tau}{2h}(X+X^{\top}) & Y_{21}^{\top} & (h-\tau)Y_{21}^{\top} & (2\tau-h)Z_{21}^{\top} & (h-\tau)M_{21}^{\top} & \Xi_{17} & \Theta_{18}+M_{21}^{\top}-Z_{21}^{\top} & \Theta_{19} \\
* & \Xi_{22}+\frac{\tau}{h}U & Y_{22}^{\top} & (h-\tau)Y_{22}^{\top} & (2\tau-h)Z_{22}^{\top} & (h-\tau)M_{22}^{\top} & \Xi_{27} & \Theta_{28}+M_{22}^{\top}-Z_{22}^{\top} & 0 \\
* & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 & T_{21} & 0 & 0 \\
* & * & * & -\frac{h-\tau}{h}R\,\mathrm{e}^{-2\alpha h} & 0 & 0 & (h-\tau)T_{21} & 0 & 0 \\
* & * & * & * & -\frac{2\tau-h}{h}R\,\mathrm{e}^{-2\alpha h} & 0 & 0 & (2\tau-h)T_{22} & 0 \\
* & * & * & * & * & \Xi_{66} & 0 & 0 & 0 \\
* & * & * & * & * & * & -T_{21}-T_{21}^{\top} & T_{22} & 0 \\
* & * & * & * & * & * & * & \Theta_{88}-T_{22}-T_{22}^{\top} & \Theta_{89} \\
* & * & * & * & * & * & * & * & \Theta_{99}
\end{bmatrix} < 0, \tag{4.31}$$

where

$$\begin{aligned}
\Theta_{11} &= A^{\top}P_{21} + P_{21}^{\top}A + 2\alpha P + S + Q - Y_{11} - Y_{11}^{\top} - \frac{d-2\alpha\tau}{2h}(X+X^{\top}), \\
\Theta_{12} &= P - P_{21}^{\top} + A^{\top}P_{31} - Y_{12}, \\
\Theta_{17} &= Y_{11}^{\top} - Z_{11}^{\top} - T_{11} + P_{21}^{\top}A_1, \\
\Theta_{18} &= \frac{d-2\alpha\tau}{h}(X - X_1), \\
\Theta_{19} &= (1-d)\frac{\tau}{h}(-X + X_1), \\
\Theta_{22} &= -P_{31} - P_{31}^{\top} + R, \\
\Theta_{27} &= Y_{12}^{\top} - Z_{12}^{\top} + P_{31}^{\top}A_1, \\
\Theta_{28} &= -\frac{\tau}{h}(X - X_1), \\
\Theta_{55} &= -\frac{h-2\tau}{h}[R + dU]\mathrm{e}^{-2\alpha h}, \\
\Theta_{88} &= -(1-d)Q\,\mathrm{e}^{-2\alpha h} - (d-2\alpha\tau)\frac{X + X^{\top} - 2X_1 - 2X_1^{\top}}{2h}, \\
\Theta_{89} &= (1-d)\frac{\tau}{2h}(X + X^{\top} - 2X_1 - 2X_1^{\top}), \\
\Theta_{99} &= -(1-d)\frac{\tau}{h}U\,\mathrm{e}^{-2\alpha h}, \\
\Xi_{11} &= A^{\top}P_{22} + P_{22}^{\top}A + 2\alpha P + S + Q - M_{21} - M_{21}^{\top} - \frac{d-2\alpha\tau}{2h}(X+X^{\top}), \\
\Xi_{12} &= P - P_{22}^{\top} + A^{\top}P_{32} - M_{22}, \\
\Xi_{17} &= Z_{21}^{\top} - Y_{21}^{\top} + P_{22}^{\top}A_1, \\
\Xi_{22} &= -P_{32} - P_{32}^{\top} + R, \\
\Xi_{27} &= Z_{22}^{\top} - Y_{22}^{\top} + P_{32}^{\top}A_1, \\
\Xi_{66} &= -\frac{h-\tau}{h}[R + dU]\mathrm{e}^{-2\alpha h},
\end{aligned}$$

are feasible. Then (2.1) is exponentially stable with the decay rate α for all delays $0 \leqslant \tau \leqslant h$ satisfying A2.

(ii) Given $\alpha > 0$, $d > 1$, let there exist $n \times n$-matrices $P > 0$, $R > 0$, $U > 0$, $S > 0$, X, X_1, P_{21}, P_{31}, P_{22}, P_{32}, Y_{ij}, M_{ij}, Z_{ij} and $T_{ij}(i, j = 1, 2)$ such that (4.9) and the following four LMIs: (4.32), for $\tau \to 0$ and $\tau \to \frac{h}{1+d}$, and (4.33), for $\tau \to \frac{h}{1+d}$ and $\tau \to h$,

$$\begin{bmatrix} \Sigma_{11} & \Theta_{12}+\frac{\tau}{2h}(X+X^\top) & M_{11}^\top & \tau Y_{11}^\top & \frac{h-(1+d)\tau}{d}Z_{11}^\top & (h-\frac{h-\tau}{d})M_{11}^\top & \Theta_{17} & \Theta_{18}+Z_{11}^\top-M_{11}^\top \\ * & \Theta_{22}+\frac{\tau}{h}U & M_{12}^\top & \tau Y_{12}^\top & \frac{h-(1+d)\tau}{d}Z_{12}^\top & (h-\frac{h-\tau}{d})M_{12}^\top & \Theta_{27} & \Theta_{28}+Z_{12}^\top-M_{12}^\top \\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{44} & 0 & 0 & \tau T_{11} & 0 \\ * & * & * & * & \Sigma_{55} & 0 & 0 & \frac{h-(1+d)\tau}{d}T_{12} \\ * & * & * & * & * & \Sigma_{66} & 0 & 0 \\ * & * & * & * & * & * & T_{11}+T_{11}^\top & -T_{12} \\ * & * & * & * & * & * & * & \Sigma_{88}+T_{12}+T_{12}^\top \end{bmatrix} < 0, \qquad (4.32)$$

$$\begin{bmatrix} \Upsilon_{11} & \Xi_{12}+\frac{\tau}{2h}(X+X^\top) & Y_{21}^\top & (h-\tau)Y_{21}^\top & \frac{(1+d)\tau-h}{d}Z_{21}^\top & \frac{h-\tau}{d}M_{21}^\top & \Xi_{17} & \Theta_{18}+M_{21}^\top-Z_{21}^\top \\ * & \Xi_{22}+\frac{\tau}{h}U & Y_{22}^\top & (h-\tau)Y_{22}^\top & \frac{(1+d)\tau-h}{d}Z_{22}^\top & \frac{h-\tau}{d}M_{22}^\top & \Xi_{27} & \Theta_{28}+M_{22}^\top-Z_{22}^\top \\ * & * & -S\,\mathrm{e}^{-2\alpha h} & 0 & 0 & 0 & T_{21} & 0 \\ * & * & * & -\frac{h-\tau}{h}R\,\mathrm{e}^{-2\alpha h} & 0 & 0 & (h-\tau)T_{21} & 0 \\ * & * & * & * & -\frac{1}{h}\frac{(1+d)\tau-h}{d}R\,\mathrm{e}^{-2\alpha h} & 0 & 0 & \frac{(1+d)\tau-h}{d}T_{22} \\ * & * & * & * & * & \Upsilon_{66} & 0 & 0 \\ * & * & * & * & * & * & -T_{21}-T_{21}^\top & T_{22} \\ * & * & * & * & * & * & * & \Sigma_{88}-T_{22}-T_{22}^\top \end{bmatrix} < 0, \qquad (4.33)$$

where

$$\begin{aligned}
\Sigma_{11} &= A^\top P_{21} + P_{21}^\top A + 2\alpha P + S - Y_{11} - Y_{11}^\top - \frac{d-2\alpha\tau}{2h}(X+X^\top),\\
\Sigma_{44} &= -\frac{\tau}{h}\left[R\,\mathrm{e}^{-2\alpha h} + dUe^{-\frac{2\alpha h}{d}}\right],\\
\Sigma_{55} &= -\frac{1}{h}\frac{h-(1+d)\tau}{d}\left[R\,\mathrm{e}^{-2\alpha h} + dU\mathrm{e}^{-\frac{2\alpha h}{d}}\right],\\
\Sigma_{66} &= -\frac{1}{h}\left(h - \frac{h-\tau}{d}\right)R\,\mathrm{e}^{-2\alpha h},\\
\Sigma_{88} &= -(d-2\alpha\tau)\frac{X+X^\top-2X_1-2X_1^\top}{2h},\\
\Upsilon_{11} &= A^\top P_{22} + P_{22}^\top A + 2\alpha P + S - M_{21} - M_{21}^\top - \frac{d-2\alpha\tau}{2h}(X+X^\top),\\
\Upsilon_{66} &= -\frac{1}{h}\frac{h-\tau}{d}\left[R\,\mathrm{e}^{-2\alpha h} + dU\,\mathrm{e}^{-\frac{2\alpha h}{d}}\right],
\end{aligned}$$

are feasible. Then (2.1) is exponentially stable with the decay rate α for all delays $0 \leqslant \tau \leqslant h$ satisfying A2.

5. Examples

EXAMPLE 5.1 Consider the system from Yue *et al.* (2005)

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} Kx(t-\tau(t)),\\
K &= -[3.75\ 11.5].
\end{aligned}$$

The stability of this system was studied by many authors (see Naghshtabrizi *et al.*, 2008 and the references therein). For the constant sampling, it was found in Naghshtabrizi *et al.* (2008) that the system remains stable for all constant samplings <1.7 and becomes unstable for samplings >1.7. Moreover, the above system with constant delay τ is asymptotically stable for $\tau \leqslant 1.16$ and becomes unstable for $\tau > 1.17$. The latter means that all the existing methods, that are based on time-independent Lyapunov functionals, corresponding to stability analysis of systems with fast-varying delays, cannot guarantee the stability for the samplings with the upper bound >1.17. In Naghshtabrizi *et al.* (2008), the upper bound on the constant sampling interval that preserves the stability is found to be $h = 1.3277$, improving all the existing LMI-based results.

For different $\dot{\tau}$, by applying Lyapunov–Krasovskii(L–K) approach with $\alpha = 0$ and I–O approach via Wirtinger-type inequality, we obtain the maximum value of h given in Table 1. Our results for $X = X_1 = U = 0$ coincide with the ones of Park & Ko (2007). We see that discontinuous terms of LKFs improve the performance. Moreover, the I–O approach via Wirtinger-type inequality improves the result for $d > 1, d \in N$ compared to time-dependent L–K approach.

EXAMPLE 5.2 We consider the following simple and much-studied problem (Papachristodoulou *et al.*, 2007; Fridman, 2010):

$$\dot{x}(t) = -x(t - \tau(t)). \tag{5.1}$$

It is well known that the equation $\dot{x}(t) = -x(t - \tau)$ with constant delay τ is asymptotically stable for $\tau < \pi/2$ and unstable for $\tau > \pi/2$, whereas for the fast-varying delay, it is stable for $\tau < 1.5$ and there exists a destabilizing delay with an upper bound >1.5. The latter means that all the existing methods, that are based on time-independent Lyapunov functionals, corresponding to stability analysis of systems with fast-varying delays, cannot guarantee the stability for the samplings, which may be >1.5.

It is easy to check, that the system remains stable for all constant samplings <2 and becomes unstable for samplings >2. Conditions of Naghshtabrizi *et al.* (2008) and of Mirkin (2007) guarantee asymptotic stability for all variable samplings up to 1.28 and 1.57, respectively. For different $\dot{\tau}$, by applying our methods, we obtain the maximum value of h given in Table 2.We can also see that discontinuous terms of LKFs improve the performance and the I–O approach via Wirtinger-type inequality improves the result for $d > 1, d \in N$ compared to time-dependent L–K approach.

EXAMPLE 5.3 Consider the system (De Souza & Li, 1999)

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - \tau(t)).$$

TABLE 1 *Example 1: maximum value of h for different $\dot{\tau}$*

$h\backslash\dot{\tau}$	-1.2	-1	-0.9	0.5	0.9	1	1.1	$h\backslash\dot{\tau}$	$\dot{\tau} \in N$
L–K approach	1.14	1.14	1.11	1.24	1.61	1.69	1.34	I–O approach	1.3659
$X = X_1 = U = 0$	1.04	1.04	1.04	1.04	1.04	1.04	1.04		

TABLE 2 *Example 2: maximum value of h for different $\dot{\tau}$*

$h\backslash\dot{\tau}$	-1.2	-1	-0.9	0.5	0.9	1	1.1	1.2	$h\backslash\dot{\tau}$	$\dot{\tau} \in N$
L–K approach	1.41	1.41	1.36	1.47	1.89	1.99	1.61	1.54	I–O approach	1.57
$X = X_1 = U = 0$	1.33	1.33	1.33	1.33	1.33	1.33	1.33	1.33		

TABLE 3 *Example 3: maximum value of h for different $\dot{\tau}$*

$h \backslash \dot{\tau}$	-1.2	-1	-0.9	0.5	0.9	1	1.1	$h \backslash \dot{\tau}$	$\dot{\tau} \in N$
L–K approach	1.95	1.98	1.90	2.34	2.06	2.53	1.91	I–O approach	1.57
$X = X_1 = U = 0$	1.86	1.86	1.86	2.33	1.87	1.86	1.86		

TABLE 4 *Example 4: maximum value of h for different $\dot{\tau}$*

$h \backslash \dot{\tau}$	-1.2	-1	-0.9	0.5	0.9	1	1.1	$h \backslash \dot{\tau}$	$\dot{\tau} \in N$
Our method	1.10	1.11	1.10	1.44	1.48	1.64	1.26	I–O approach	0.8797
$X = X_1 = U = 0$	1.06	1.06	1.06	1.26	1.06	1.06	1.06		

It is well known that this system is stable for constant delay $\tau \leqslant 6.17$, whereas in Park & Ko (2007), it was found that the system is stable for all fast-varying delays $\tau \leqslant 1.86$. For different $\dot{\tau}$, by applying our methods, we obtain the maximum value of h given in Table 3. Our results for $X = X_1 = U = 0$ coincide with the ones of Park & Ko (2007). Also in this Example, the time-dependent terms of LKFs improve the performance. However, the I–O approach via Wirtinger-type inequality has not improved the results for $d > 1, d \in N$ and the result is worse than the result for the fast-varying delays.

EXAMPLE 5.4 Consider the system (Kharitonov & Niculescu, 2002):

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - \tau(t)).$$

For different $\dot{\tau}$, by applying our methods, we obtain the maximum value of h given in Table 4.

REMARK 5.1 Simulations in all the examples above show that our results are conservative (at least for $d \neq 1$) and that the value of maximum h that preserves the stability grows for growing $d > 1$.

6. Conclusions

In this paper, two methods have been introduced to investigate delay-dependent stability problem for systems with sawtooth delay with constant $\dot{\tau} \neq 0$. One method develops a novel I–O approach via a Wirtinger-type inequality. The result by Mirkin (2007) is recovered for $\dot{\tau} = 1$ and for any integer $\dot{\tau}$, the same maximum bound that preserves the stability is achieved. Another method improves stability criteria by constructing piecewise continuous (in time) Lyapunov functionals. Though the most essential improvement corresponds to $\dot{\tau} = 1$, the time-dependent terms improve the results for all values of $\dot{\tau}$.

Funding

Israel Science Foundation (754/10); China Scholarship Council.

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