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Networked Control with Stochastic Scheduling

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***Abstract*—This paper develops the time-delay approach to networked control systems with scheduling protocols, variable delays and variable sampling intervals. The scheduling of sensor communication is defined by a stochastic protocol. Two classes of protocols are considered. The first one is defined by an independent and identically-distributed stochastic process. The activation probability of each sensor node for this protocol is a given constant, whereas it is assumed that collisions occur with a certain probability. The resulting closed-loop system is a stochastic impulsive system with delays both in the continuous dynamics and in the reset equations, where the system matrices have stochastic parameters with Bernoulli distributions. The second scheduling protocol is defined by a discrete-time Markov chain with a known transition probability matrix taking into account collisions. The resulting closed-loop system is a Markovian jump impulsive system with delays both in the continuous dynamics and in the reset equations. Sufficient conditions for exponential mean-square stability of the resulting closed-loop system are derived via a Lyapunov-Krasovskii-based method. The efficiency of the method is illustrated on an example of a batch reactor. It is demonstrated how the time-delay approach allows treating network-induced delays larger than the sampling intervals in the presence of collisions.**

Keywords: networked control systems, Lyapunov functional, stochastic protocols, stochastic impulsive system.

I. Introduction

Networked control systems (NCSs) have received considerable attention in recent years (see e.g., [1], [2]). In many such systems, only one node is allowed to use the communication channel at a time. The communication is orchestrated by a scheduling rule called a protocol. The time-delay approach was recently developed for the stabilization of NCSs under the round-robin (RR) protocol [3] and under the try-once-discard (TOD) protocol [4]. The closed-loop system was modeled as a switched system with multiple and ordered time-varying delays under RR scheduling or as a hybrid system with time-varying delays in the dynamics and in the reset equations under the TOD scheduling. Differently from the existing results on NCSs in the presence of scheduling protocols (in the frameworks of hybrid and discrete-time systems), the transmission delay is allowed to be large (larger than the sampling interval), but a crucial point is that data packet dropout is not allowed for large delays in either [3] or [4].

In the framework of hybrid systems, a stochastic protocol was introduced in [5] and analyzed for the input-output stability of NCSs in the presence of data packet dropouts or collisions. An i.i.d (independent and identically-distributed) sequence of Bernoulli random variables is applied to describe the stochastic protocol. Communication delays, however, are not included in the analysis. The stability of NCSs under a stochastic protocol, where the activated node is modeled by a Markov chain, was studied in [6] by applying the discrete-time modeling framework. In [6], data packet dropouts can be regarded as prolongations of the sampling interval for small delays.

In the present note, to overcome the lack of stability analysis of NCS under scheduling protocols with large communication delays and data packet dropouts, we develop a time-delay approach considering multiple sensors under a stochastic scheduling protocol. The resulting closed-loop system is a stochastic impulsive system with delays both in the continuous dynamics and in the reset equations. We treat two classes of stochastic protocols. The first one is defined by an i.i.d. stochastic process. The activation probability of each node for this protocol is a given constant, whereas it is assumed that the collisions occur with a certain probability. The second protocol is defined by a discrete-time Markov chain with a known transition probability matrix taking into account collisions.

By developing appropriate Lyapunov-Krasovskii techniques, we derive linear matrix inequalities (LMIs) conditions for the exponential mean-square stability of the closed-loop system. As in [3] and [4], differently from the hybrid and discrete-time approaches, we allow the transmission delays to be larger than the sampling intervals in the presence of scheduling protocols. The efficiency of the presented approach is illustrated by a batch reactor example.

The rest of this note is organized as follows. Section II presents the model of NCS and the hybrid delayed system model for the closed-loop system. In Section III below, the exponential mean-square stability of the closed-loop system under i.i.d stochastic protocol will be studied. The exponential mean-square stability of the closed-loop system under Markovian stochastic protocol will be presented in Section IV. In Section V, the efficiency and advantages of the presented approach are illustrated by a batch reactor example. Finally, the conclusions and the future work are stated in Section VI. Preliminary results on the stabilization of NCSs with two sensor nodes under i.i.d stochastic protocol have been presented in [7].

Notations: Throughout this note, the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. the space of functions $\phi:[-\tau_M,0]\to\mathbb{R}^n$, which are absolutely continuous on $[-\tau_M,0]$, and have square integrable first-order derivatives is denoted by $W[-\tau_M,0]$ with the norm $\|\phi\|_W = \max_{\theta\in[-\tau_M,0]}|\phi(\theta)| + \left[\int_{-\tau_M}^0 |\dot\phi(s)|^2 ds\right]^{\frac{1}{2}}$. $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers.

II. System Model

A. NCS model

Consider the system architecture in Figure 1 with plant

$$\dot x(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the control input and A, B are system matrices of appropriate dimensions. The initial condition is given by $x(0)=x_0$.

The NCS has N distributed sensors, a controller and an actuator connected via two wireless networks. Their measurements are given by $y_i(t)=C_ix(t)$, $i=1,\dots,N$. Let $C=[C_1^T\ \cdots\ C_N^T]^T$, $y(t)=[y_1^T(t)\ \cdots\ y_N^T(t)]^T\in\mathbb{R}^{n_y}$. We denote by s_k the unbounded and monotonously increasing sequence of sampling instants $0=s_0<s_1<\cdots<s_k<\cdots$, $k\in\mathbb{Z}_{\geq 0}$, $\lim_{k\to\infty}s_k=\infty$, $s_{k+1}-s_k\leq$ MATI, where MATI denotes the maximum allowable transmission interval. At each sampling instant s_k, at most one of the outputs $y_i(s_k)\in\mathbb{R}^{n_i}$, $\sum_{i=1}^N n_i = n_y$, is transmitted over the network.

We suppose that the transmission of the information (between the sensor and the actuator) is subject to a variable delay $\eta_k=\eta_k^{sc}+\eta_k^{ca}+\eta_k^c$, where η_k^{sc} and η_k^{ca} are the network-induced delays (from the sensor to the controller and from the controller to the actuator, respectively), and where η_k^c is the computational delay in the controller node. Denote $s_k+\eta_k$ by t_k. Differently from [8], [9], we

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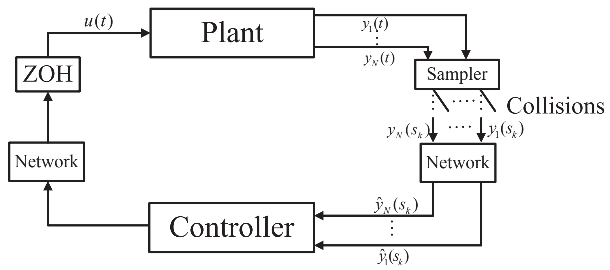

Fig. 1. NCS architecture

do not restrict the network delays to be small with $\eta_k < s_{k+1} - s_k$. Following [3], [4], [10], we allow the delay to be large provided that packet ordering is maintained. Assume that the network-induced delay η_k and the time span between the instant t_{k+1} and the current sampling instant s_k are bounded:

$$t_{k+1} - t_k + \eta_k \leq \tau_M, \; 0 \leq \eta_m \leq \eta_k \leq \text{MAD}, \; k \in \mathbb{Z}_{\geq 0}, \quad (2)$$

where MAD denotes the maximum allowable delay. Here η_m and MAD are known bounds and $\tau_M = \text{MATI} + \text{MAD}$. The inequality $\eta_m > \tau_M/2$ implies the case of large delay. For the given example in Section V, we show that our method is applicable also for $\eta_m > \tau_M/2$.

Remark 1 *Differently from [10], where subscript k in t_k corresponds to the measurements that are not lost, in our paper k corresponds to the sampling time. This is because we consider the probability of collisions or data packet dropouts (see further details below). Therefore, t_k is the actual or the fictitious (when collisions occur or the data packet is lost) updating time instant of the zero-order hold (ZOH) device.*

Remark 2 *We follow a commonly used assumption on the boundedness of the network-induced delays, e.g., [8], [11]. Another possibility is the Markov chain model of the network-induced delays, e.g., [12].*

B. The impulsive model

At each sampling instant s_k, at most one of the system nodes $i \in \{1, \ldots, N\}$ is active. In some cases, collisions may occur when nodes access the network [5]. If this happens, then packet with sensor data is dropped. At the sampling instant s_k, let $\sigma_k \in \mathcal{I} = \{0, 1, \ldots, N\}$ denote the active output node, which will be chosen according to the stochastic protocol. Here $\sigma_k = 0$ means that either collisions occur when nodes access the network or the data packet is lost during the transmission over the network from the sensor to the controller. We suppose data loss is not possible during the transmission from the controller to the actuator.

Denote by $\hat{y}(s_k) = [\hat{y}_1^T(s_k) \; \cdots \; \hat{y}_N^T(s_k)]^T \in \mathbb{R}^{n_y}$ the most recently received output information on the controller side. We consider the error between the system output $y(s_k)$ and the last available information $\hat{y}(s_{k-1})$:

$$\begin{array}{l} e(t) = \text{col}\{e_1(t), \cdots, e_N(t)\} \equiv \hat{y}(s_{k-1}) - y(s_k), \\ t \in [t_k, t_{k+1}), \; k \in \mathbb{Z}_{\geq 0}, \; \hat{y}(s_{-1}) \triangleq 0, \; e(t) \in \mathbb{R}^{n_y}. \end{array} \quad (3)$$

We suppose that the controller and the actuator are event-driven (in the sense that the controller and the ZOH device update their outputs as soon as they receive a new sample).

Static output feedback control: Assume that there exists a matrix $K = [K_1 \; \cdots \; K_N]$, $K_i \in \mathbb{R}^{m \times n_i}$ such that $A + BKC$ is Hurwitz. Then the static output feedback controller has the form

$$\begin{array}{l} u(t) = K_{\sigma_k} y_{\sigma_k}(s_k) + \sum_{i=1, i \neq \sigma_k}^N K_i \hat{y}_i(s_{k-1}), \\ \quad t \in [t_k, t_{k+1}), \; k \in \mathbb{Z}_{\geq 0}, \end{array} \quad (4)$$

where $K_{\sigma_k} y_{\sigma_k}(s_k) = 0$ when $\sigma_k = 0$. Therefore, we obtain the following continuous dynamics:

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + A_1 x(t_k - \eta_k) + \sum_{i=1, i \neq \sigma_k}^N B_i e_i(t), \\ \dot{e}(t) = 0, \quad t \in [t_k, t_{k+1}), \end{array} \right. \quad (5)$$

where $A_1 = BKC, \; B_i = BK_i, \; i = 1, \ldots, N.$

From (3), it follows that

$$\begin{array}{rcl} e_i(t_{k+1}) & = & \hat{y}_i(s_k) - y_i(s_{k+1}) \\ & = & y_i(s_k) - y_i(s_{k+1}), \quad i = \sigma_k \in \mathcal{I} \backslash \{0\}, \\ e_i(t_{k+1}) & = & \hat{y}_i(s_k) - y_i(s_{k+1}) \\ & = & \hat{y}_i(s_{k-1}) - y_i(s_{k+1}), \; i \neq \sigma_k, \; i \in \mathcal{I} \backslash \{0\}. \end{array}$$

Thus, the delayed reset system is given by

$$\left\{ \begin{array}{rcl} x(t_{k+1}) & = & x(t_{k+1}^-), \\ e_i(t_{k+1}) & = & C_i[x(s_k) - x(s_{k+1})], \; i = \sigma_k \in \mathcal{I} \backslash \{0\}, \\ e_i(t_{k+1}) & = & e_i(t_{k+1}^-) + C_i[x(s_k) - x(s_{k+1})], \\ & & i \neq \sigma_k, \; i \in \mathcal{I} \backslash \{0\}. \end{array} \right. \quad (6)$$

Applying the time-delay approach to sampled-data control [13], denote $\tau(t) = t - t_k + \eta_k$, then $\tau(t) \in [\eta_m, \tau_M]$ (cf., (2)) and $x(t_k - \eta_k) = x(t - \tau(t))$ for $t \in [t_k, t_{k+1})$. Therefore, the impulsive system model (5)–(6) contains the piecewise-continuous delay $\tau(t)$ in the continuous-time dynamics (5). Even for $\eta_k = 0$, we have the delayed state $x(t_k) = x(t - \tau(t))$ with $\tau(t) = t - t_k$. The initial condition for (5)–(6) has the form of $x(t) = \phi(t), \; t \in [t_0 - \tau_M, t_0]$, $\phi(0) = x_0$ and $e(t_0) = -Cx(t_0 - \eta_0) = -Cx_0$, where $\phi(t)$ is a continuous function on $[t_0 - \tau_M, t_0]$.

Dynamic output feedback: Assume that the controller is directly connected to the actuator. Consider a dynamic output feedback controller of the form

$$\begin{array}{l} \dot{x}_c(t) = A_c x_c(t) + B_c \hat{y}(s_k), \\ u(t) = C_c x_c(t) + D_c \hat{y}(s_k), \quad t \in [t_k, t_{k+1}), \; k \in \mathbb{Z}_{\geq 0}, \end{array}$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the state of the controller, A_c, B_c, C_c and D_c are matrices of appropriate dimensions. Let $e_i(t), \; i = 1, \ldots, N$, be defined by (3). The closed-loop system can be presented in the form of (5)–(6), where x, e and matrices are replaced by the ones with bars as follows:

$$\begin{array}{l} \bar{x} = [x^T \; x_c^T]^T, \; \bar{A} = \begin{bmatrix} A & BC_c \\ 0_{n_c \times n} & A_c \end{bmatrix}, \bar{B}_i = \begin{bmatrix} BD_c \\ B_c \end{bmatrix}, \\ \bar{A}_1 = \begin{bmatrix} BD_c C & 0_{n \times n_c} \\ B_c C & 0_{n_c \times n_c} \end{bmatrix}, \bar{C} = \begin{bmatrix} \bar{C}_1^T & \cdots & \bar{C}_N^T \end{bmatrix}^T, \\ \bar{C}_1 = \begin{bmatrix} C_1^T & 0 \\ 0 & 0 \end{bmatrix}^T, \; \bar{C}_2 = \begin{bmatrix} 0_{n \times n_1} & C_2^T & 0 \\ 0_{n_c \times n_1} & 0 & 0 \end{bmatrix}^T, \; \cdots, \\ \bar{C}_N = \begin{bmatrix} 0 & C_N^T \\ 0 & 0 \end{bmatrix}^T, \; \bar{e}(t) = [\bar{e}_1^T(t) \; \cdots \; \bar{e}_N^T(t)]^T, \\ \bar{e}_1(t) = [e_1^T(t) \; 0]^T, \; \bar{e}_2(t) = [0_{1 \times n_1} \; e_2^T(t) \;\; 0]^T, \cdots, \\ \bar{e}_N(t) = [0 \; e_N^T(t)]^T, \bar{C}_i \in \mathbb{R}^{n_y \times (n + n_c)}, \bar{e}_i(t) \in \mathbb{R}^{n_y}, \; i = 1, \ldots, N. \end{array}$$

C. Stochastic scheduling protocols

In the following, we will consider two classes of stochastic protocols, which are defined by i.i.d and Markovian process, respectively.

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1) I.i.d scheduling: The choice of σ_k is assumed to be i.i.d with the probabilities given by

$$\text{Prob}\{\sigma_k = i\} = \beta_i,\ i \in \mathcal{I}, \tag{7}$$

where $\beta_i, i = 0, 1, \ldots, N$ are non-negative scalars and $\sum_{i=0}^{N} \beta_i = 1$. Here $\beta_j, j = 1, ..., N$ are the probabilities of the measurement $y_j(s_k)$ to be transmitted at s_k, whereas β_0 is the probability of collision.

2) Markovian scheduling: The protocol determines σ_k through a Markov Chain. The conditional probability that node $j \in \mathcal{I}$ gets access to the network at time s_k, given the values of $\sigma_{k-1} \in \mathcal{I}$, is defined by

$$\text{Prob}\{\sigma_k = j | \sigma_{k-1} = i\} = \pi_{ij}, \tag{8}$$

where $0 \le \pi_{ij} \le 1$ for all $i, j \in \mathcal{I}$, $\sum_{j=0}^{N} \pi_{ij} = 1$ for all $i \in \mathcal{I}$ and $\sigma_0 \in \mathcal{I}$ is assumed to be given. The transition probability matrix is denoted by $\Pi = \{\pi_{ij}\} \in \mathbb{R}^{(N+1)\times(N+1)}$.

Remark 3 *The i.i.d scheduling is a special case of the Markovian scheduling. For instance, assume that there are $N = 2$ sensor nodes and collisions do not occur, the Markovian scheduling with $\Pi = \begin{bmatrix} p & 1-p \\ p & 1-p \end{bmatrix}$, $0 \le p \le 1$, is an i.i.d. scheduling with $\beta_1 = p$, $\beta_2 = 1 - p$.*

Definition 1 *The hybrid system (5)–(6) is said to be exponentially mean-square stable with respect to x if there exist constants $b > 0$, $\alpha > 0$ such that the following bound holds*

$$\mathbb{E}\{|x(t)|^2\} \le be^{-2\alpha(t-t_0)}\mathbb{E}\{\|x_{t_0}\|_W^2 + |e(t_0)|^2\},\ t \ge t_0$$

for the solutions of the stochastic impulsive system (5)–(6) initialized with $e(t_0) \in \mathbb{R}^{n_y}$ and $x(t) = \phi(t),\ t \in [t_0 - \tau_M, t_0]$. The hybrid system (5)–(6) is exponentially mean-square stable if additionally the following bound is valid

$$\mathbb{E}\{|e(t)|^2\} \le be^{-2\alpha(t-t_0)}\mathbb{E}\{\|x_{t_0}\|_W^2 + |e(t_0)|^2\},\ t \ge t_0.$$

III. NCSs under i.i.d stochastic scheduling protocol

A. Stochastic impulsive time-delay model with Bernoulli distributed parameters

Following [14], we introduce the indicator functions

$$\pi_{\{\sigma_k=i\}} = \begin{cases} 1, & \sigma_k = i \\ 0, & \sigma_k \ne i, \end{cases} \quad i \in \mathcal{I},\ k \in \mathbb{Z}_{\ge 0}.$$

Thus, from (7) it follows that

$$\begin{array}{l} \mathbb{E}\{\pi_{\{\sigma_k=i\}}\} = \mathbb{E}\{[\pi_{\{\sigma_k=i\}}]^2\} = \text{Prob}\{\sigma_k = i\} = \beta_i, \\ \mathbb{E}\{[\pi_{\{\sigma_k=i\}} - \beta_i][\pi_{\{\sigma_k=j\}} - \beta_j]\} = \begin{cases} -\beta_i\beta_j,\ i \ne j, \\ \beta_i(1-\beta_i),\ i = j. \end{cases} \end{array} \tag{9}$$

Therefore, the stochastic impulsive system model (5)–(7) can be rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t_k - \eta_k) + \sum_{i=1}^{N}(1 - \pi_{\{\sigma_k=i\}})B_i e_i(t), \\ \dot{e}(t) = 0, \quad t \in [t_k, t_{k+1}) \end{cases} \tag{10}$$

with the delayed reset system

$$\begin{cases} x(t_{k+1}) = x(t_{k+1}^-), \\ e_i(t_{k+1}) = (1 - \pi_{\{\sigma_k=i\}})e_i(t_{k+1}^-) \\ \qquad + C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})],\ i = 1, \ldots, N. \end{cases} \tag{11}$$

Remark 4 *Applying the Bernoulli-distributed stochastic variables $\pi_{\{\sigma_k=i\}}$, $i = 0, 1, \ldots, N$, the closed-loop system (10)–(11) is presented as an impulsive time-delay system with stochastic parameters in the system matrices. Note that the Bernoulli distribution has previously been applied to NCS with probabilistic measurements missing [15], stochastic sampling intervals [16], time-delay system with stochastic interval delays [14], output tracking control under unreliable communication [17] and fuzzy control for nonlinear NCSs [18].*

B. Exponential mean-square stability of stochastic impulsive delayed system

Our objective of this section is to derive LMI conditions for exponential mean-square stability of the stochastic impulsive system (10)–(11). Consider Lyapunov-Krasovskii functional (LKF):

$$\begin{array}{l} V_e(t) = V(t, x_t, \dot{x}_t) + \sum_{i=1}^{N} e_i^T(t)Q_i e_i(t), \\ V(t, x_t, \dot{x}_t) = \tilde{V}(t, x_t, \dot{x}_t) + V_G, \\ V_G = \sum_{i=1}^{N}(\tau_M - \eta_m)\int_{s_k}^{t} e^{2\alpha(s-t)}|\sqrt{G_i}C_i\dot{x}(s)|^2 ds, \\ \tilde{V}(t, x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{t-\eta_m}^{t} e^{2\alpha(s-t)}x^T(s)S_0 x(s)ds \\ \quad + \int_{t-\tau_M}^{t-\eta_m} e^{2\alpha(s-t)}x^T(s)S_1 x(s)ds \\ \quad + \eta_m \int_{-\eta_m}^{0}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)R_0\dot{x}(s)dsd\theta \\ \quad + (\tau_M - \eta_m)\int_{-\tau_M}^{-\eta_m}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)R_1\dot{x}(s)dsd\theta, \\ P > 0, S_j > 0, R_j > 0, G_i > 0, Q_i > 0, \alpha > 0, \\ j = 0, 1, i = 1, \ldots, N,\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_{\ge 0}, \end{array} \tag{12}$$

where $x_t(\theta) \triangleq x(t + \theta),\ \theta \in [-\tau_M, 0]$. Here the term

$$e_i^T(t)Q_i e_i(t) \equiv e_i^T(t_k)Q_i e_i(t_k),\ t \in [t_k, t_{k+1}),\ i = 1, \ldots, N,$$

is piecewise-constant. The term $\tilde{V}(t, x_t, \dot{x}_t)$ represents the standard Lyapunov functional for systems with a time-varying delay $\tau(t) \in [\eta_m, \tau_M]$. The novel piecewise-continuous in time term V_G is inserted to cope with the delays in the reset conditions. It is continuous on $[t_k, t_{k+1})$ and does not grow at the jumps $t = t_{k+1}$, since

$$\begin{array}{l} \mathbb{E}\{V_{G|t=t_{k+1}} - V_{G|t=t_{k+1}^-}\} \\ \le -(\tau_M - \eta_m)e^{-2\alpha\tau_M}\sum_{i=1}^{N}\int_{t_k-\eta_k}^{t_{k+1}-\eta_{k+1}}\mathbb{E}\{|\sqrt{G_i}C_i\dot{x}(s)|^2\}ds \\ \le -e^{-2\alpha\tau_M}\sum_{i=1}^{N}\mathbb{E}\{|\sqrt{G_i}C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2\}, \end{array} \tag{13}$$

where we applied Jensen's inequality. The infinitesimal operator $\mathcal{L}$ of $V_e(t)$ is defined as

$$\mathcal{L}V_e(t) = \lim_{\Delta\to 0^+}\frac{1}{\Delta}\{\mathbb{E}\{V_e(t + \Delta)|t\} - V_e(t)\}. \tag{14}$$

The following lemma gives sufficient conditions for exponential stability of (10)–(11) in the mean-square sense:

Lemma 1 *If there exist positive constant α, $0 < Q_i \in \mathbb{R}^{n_i\times n_i}$, $0 < U_i \in \mathbb{R}^{n_i\times n_i}$, $0 < G_i \in \mathbb{R}^{n_i\times n_i}$, $i = 1, \ldots, N$, and $V_e(t)$ of (12) such that along (10) for $t \in [t_k, t_{k+1})$*

$$\mathbb{E}\{\mathcal{L}V_e(t) + 2\alpha V_e(t) - \tfrac{1}{\tau_M - \eta_m}\sum_{i=1}^{N} e_i^T(t)U_i e_i(t)\} \le 0, \tag{15}$$

with

$$\Omega_i = \begin{bmatrix} -\beta_i Q_i + U_i & (1-\beta_i)Q_i \\ * & Q_i - G_i e^{-2\alpha\tau_M} \end{bmatrix} \le 0,\ i = 1, \ldots, N. \tag{16}$$

Then $V_e(t)$ does not grow in the jumps along (10)–(11)

$$\Theta = \mathbb{E}\{V_e(t_{k+1}) - V_e(t_{k+1}^-) + \sum_{i=1}^{N} e_i^T(t_k)U_i e_i(t_k)\} \le 0. \tag{17}$$

Moreover, the following bounds hold for the solutions of (10)–(11) with the initial condition x_{t_0}, $e(t_0)$:

$$\begin{array}{l} \mathbb{E}\{V(t, x_t, \dot{x}_t)\} \le e^{-2\alpha(t-t_0)}\mathbb{E}\{V_e(t_0)\},\ t \ge t_0, \\ V_e(t_0) = V(t_0, x_{t_0}, \dot{x}_{t_0}) + \sum_{i=1}^{N} e_i^T(t_0)Q_i e_i(t_0), \end{array} \tag{18}$$

and

$$\sum_{i=1}^{N}\mathbb{E}\{|\sqrt{Q_i}e_i(t)|^2\} \le \tilde{c}e^{-2\alpha(t-t_0)}\mathbb{E}\{V_e(t_0)\}, \tag{19}$$

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where $\tilde{c} = e^{2\alpha(\tau_M - \eta_m)}$, implying exponential mean-square stability of (10)–(11).

Proof: Since $\int_{t_k}^{t} e^{-2\alpha(t-s)}ds \le \tau_M - \eta_m,\ t \in [t_k, t_{k+1})$ and $\mathcal{L}[e^{2\alpha t}V_e(t)] = e^{2\alpha t}[2\alpha V_e(t) + \mathcal{L}V_e(t)]$, $\alpha > 0$, then (15) implies

$$\begin{aligned}\mathbb{E}\{V_e(t)\} &\le e^{-2\alpha(t-t_k)}\mathbb{E}\{V_e(t_k)\}\\ &\quad + \textstyle\sum_{i=1}^N \mathbb{E}\{e_i^T(t_k)U_i e_i(t_k)\},\ t \in [t_k, t_{k+1}).\end{aligned} \tag{20}$$

Because (16) yields $U_i \le \beta_i Q_i < Q_i,\ i = 1, \dots, N$, we have

$$\mathbb{E}\{V(t, x_t, \dot{x}_t)\} \le e^{-2\alpha(t-t_k)}\mathbb{E}\{V_e(t_k)\},\ t \in [t_k, t_{k+1}). \tag{21}$$

Note that

$$\mathbb{E}\{V_e(t_{k+1})\} = \mathbb{E}\{\tilde{V}_{|t=t_{k+1}} + V_{G|t=t_{k+1}} + \sum_{i=1}^N e_i^T(t_{k+1})Q_i e_i(t_{k+1})\}$$

and

$$\begin{aligned}&\mathbb{E}\{e_i^T(t_{k+1})Q_i e_i(t_{k+1})\}\\ &= \mathbb{E}\{|\sqrt{Q_i}[(1 - \pi_{\{\sigma_k = i\}})e_i(t_k)\\ &\qquad + C_i x(t_k - \eta_k) - C_i x(t_{k+1} - \eta_{k+1})]|^2\}\\ &= \mathbb{E}\{(1-\beta_i)e_i^T(t_k)Q_i e_i(t_k)\\ &\qquad + 2(1-\beta_i)e_i^T(t_k)Q_i C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]\\ &\qquad + |\sqrt{Q_i}C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2\},\ i = 1, \dots, N.\end{aligned}$$

Taking (13) and (16) into account, we obtain

$$\begin{aligned}\Theta &= \mathbb{E}\{\textstyle\sum_{i=1}^N[|\sqrt{Q_i}e_i(t_{k+1})|^2 - |\sqrt{Q_i}e_i(t_k)|^2]\\ &\qquad + V_{G|t=t_{k+1}} - V_{G|t=t_{k+1}^-}\}\\ &\le \mathbb{E}\{\textstyle\sum_{i=1}^N[|\sqrt{Q_i}e_i(t_{k+1})|^2 - |\sqrt{Q_i}e_i(t_k)|^2]\}\\ &\qquad - e^{-2\alpha\tau_M}\textstyle\sum_{i=1}^N \mathbb{E}\{|\sqrt{G_i}C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2\}\\ &\le \textstyle\sum_{i=1}^N \mathbb{E}\{\zeta_i(t)^T\Omega_i\zeta_i(t)\} \le 0,\end{aligned}$$

where $\zeta_i(t) = \mathrm{col}\{e_i(t_k), C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]\}$ and Ω_i is given by (16).

Therefore, the inequalities (17) and (20) with $t = t_{k+1}^-$ imply

$$\begin{aligned}\mathbb{E}\{V_e(t_{k+1})\} &\le e^{-2\alpha(t_{k+1}-t_k)}\mathbb{E}\{V_e(t_k)\}\\ &\le e^{-2\alpha(t_{k+1}-t_{k-1})}\mathbb{E}\{V_e(t_{k-1})\}\\ &\le e^{-2\alpha(t_{k+1}-t_0)}\mathbb{E}\{V_e(t_0)\}.\end{aligned} \tag{22}$$

The latter inequality, with $k+1$ replaced by k and (21) give (18). The inequality (18) implies exponential mean-square stability of (10)–(11) with respect to x because

$$\begin{aligned}\lambda_{min}(P)\mathbb{E}\{|x(t)|^2\} &\le \mathbb{E}\{V(t, x_t, \dot{x}_t)\},\\ \mathbb{E}\{V(t_0, x_{t_0}, \dot{x}_{t_0})\} &\le v\mathbb{E}\{\|x_{t_0}\|_W^2\}\end{aligned}$$

for some scalar $v > 0$. Moreover, the inequality (22) with $k+1$ replaced by k implies (19) since for $t \in [t_k, t_{k+1})$,

$$e^{-2\alpha(t_k - t_0)} = e^{-2\alpha(t-t_0)}e^{-2\alpha(t_k - t)} \le \tilde{c}e^{-2\alpha(t-t_0)}. \qquad \square$$

By using Lemma 1 and the standard arguments for the delay-dependent analysis, we derive LMI conditions for the exponential mean-square stability of (10)–(11):

Theorem 1 *Given $0 \le \eta_m < \tau_M$, $\alpha > 0$, $\beta_0 \ge 0$, $\beta_i \ge 0$, $\sum_{i=0}^N \beta_i = 1$ and K_i, $i = 1, \dots, N$. Suppose there exist $n \times n$ matrices $P > 0$, $S_j > 0$, $R_j > 0$, $j = 0, 1$, S_{12} and $n_i \times n_i$ matrices $Q_i > 0, U_i > 0, G_i > 0, i = 1, \dots, N$, such that (16) and*

$$\Phi = \begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \ge 0, \tag{23}$$

$$\Sigma + \Xi^T H\Xi + \textstyle\sum_{i=1}^N \beta_i\Xi_i^T H\Xi_i < 0 \tag{24}$$

are feasible, where

$$\begin{aligned}&H = \eta_m^2 R_0 + (\tau_M - \eta_m)^2 R_1 + (\tau_M - \eta_m)\textstyle\sum_{l=1}^N C_l^T G_l C_l,\\ &\Sigma = F_1^T P\Xi + \Xi^T P F_1 + \Upsilon - F_2^T R_0 F_2 e^{-2\alpha\eta_m} - F^T\Phi F e^{-2\alpha\tau_M},\\ &F_1 = [I_n\ 0_{n\times(3n+n_y)}],\ F_2 = [I_n\ -I_n\ 0_{n\times(2n+n_y)}],\\ &F = \begin{bmatrix} 0_{n\times n} & I_n & -I_n & 0_{n\times n} & 0_{n\times n_y} \\ 0_{n\times n} & 0_{n\times n} & I_n & -I_n & 0_{n\times n_y}\end{bmatrix},\\ &\Xi = [A\ 0_{n\times n}\ A_1\ 0_{n\times n}\ (1-\beta_1)B_1\ \cdots\ (1-\beta_N)B_N],\\ &\Xi_1 = [0_{n\times 4n}\ -B_1\ 0],\ \Xi_2 = [0_{n\times(4n+n_1)}\ -B_2\ 0], \cdots,\\ &\Xi_N = [0\ -B_N],\ \Xi_j \in \mathbb{R}^{n\times(4n+n_y)},\\ &\Upsilon = \mathrm{diag}\{S_0 + 2\alpha P, -(S_0 - S_1)e^{-2\alpha\eta_m}, 0, -S_1 e^{-2\alpha\tau_M}, \psi_1, \cdots, \psi_N\},\\ &\psi_j = -\tfrac{1}{\tau_M - \eta_m}U_j + 2\alpha Q_j,\ j = 1, \dots, N.\end{aligned} \tag{25}$$

Then the solutions of (10)–(11) satisfy the bounds (18) and (19). Hence, the closed-loop system (10)–(11) with initial condition x_{t_0}, $e(t_0)$ is exponentially mean-square stable. If the aforementioned matrix inequalities are feasible with $\alpha = 0$, then the bounds (18) and (19) hold also for a sufficiently small $\alpha_0 > 0$.

IV. NCSs under Markovian stochastic scheduling protocol

In this section, we will derive LMI conditions for exponential mean-square stability of the stochastic Markovian jump impulsive system (5), (6), (8) with respect to x. Note that the differential equation for x given by (5) depends on $e_j(t) = e_j(t_k),\ t \in [t_k, t_{k+1})$ with $j \ne \sigma_k,\ j \in \mathcal{I}\backslash\{0\}$ only. Consider LKF:

$$\begin{aligned}&V_e(t) = V(t, x_t, \dot{x}_t) + \textstyle\sum_{j=1, j\ne\sigma_k}^N e_j^T(t)Q_j e_j(t),\ \sigma_k \in \mathcal{I},\\ &V(t, x_t, \dot{x}_t) = \tilde{V}(t, x_t, \dot{x}_t) + V_Q,\\ &V_Q = (\tau_M - \eta_m)\textstyle\int_{s_k}^t e^{2\alpha(s-t)}|\sqrt{Q}\dot{x}(s)|^2 ds,\\ &t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_{\ge 0},\ Q > 0,\ Q_j > 0,\ j = 1, \dots, N,\end{aligned} \tag{26}$$

where $\tilde{V}(t, x_t, \dot{x}_t)$ is given by (12). The following statement holds:

Lemma 2 *If there exist positive constant α, matrices $0 < Q \in \mathbb{R}^{n\times n}$, $0 < Q_j \in \mathbb{R}^{n_i\times n_i}$, $0 < U_j \in \mathbb{R}^{n_i\times n_i}$, $j = 1, \dots, N$, and $V_e(t)$ of (26) such that for any $i \in \mathcal{I}$ along (5)*

$$\begin{aligned}&\mathbb{E}\{\mathcal{L}V_e(t) + 2\alpha V_e(t) - \tfrac{1}{\tau_M - \eta_m}\times\\ &\quad \textstyle\sum_{j=1, j\ne i}^N e_j^T(t)(Q_j - U_j)e_j(t)\} \le 0,\ t \in [t_k, t_{k+1}),\end{aligned} \tag{27}$$

with

$$\tilde{\Omega}_i = \begin{bmatrix} \Phi_{11}^i & \Phi_{12}^i \\ * & \Phi_{22}^i \end{bmatrix} \le 0, \tag{28}$$

holds, where

$$\begin{aligned}&\Phi_{11}^i = \textstyle\sum_{l=1}^N\sum_{j=0, j\ne l}^N \pi_{ij}C_l^T Q_l C_l - e^{-2\alpha\tau_M}Q,\\ &\Phi_{12}^i = [\textstyle\sum_{l=2}^N(\pi_{i0} + \pi_{il})C_1^T Q_1\ \cdots \sum_{l=0, l\ne j}^N \pi_{il}(C_j^T Q_j)_{|j\ne i}\ \cdots\\ &\qquad \textstyle\sum_{l=0}^{N-1}\pi_{il}C_N^T Q_N],\\ &\Phi_{22}^i = \mathrm{diag}\{\textstyle\sum_{l=2}^N(\pi_{i0} + \pi_{il})Q_1 - U_1, \cdots,\\ &\qquad \textstyle\sum_{l=0, l\ne j}^N \pi_{il}Q_{j|j\ne i} - U_{j|j\ne i}, \cdots, \sum_{l=0}^{N-1}\pi_{il}Q_N - U_N\}.\end{aligned}$$

Then $V_e(t)$ satisfies

$$\begin{aligned}&\mathbb{E}\{V_e(t_{k+1}) - V_e(t_{k+1}^-)\\ &\qquad + \textstyle\sum_{j=1, j\ne i}^N e_j^T(t_k)(Q_j - U_j)e_j(t_k)\} \le 0,\ i \in \mathcal{I}.\end{aligned} \tag{29}$$

The bound (18) is valid for the solutions of (5), (6), (8) with the initial condition x_{t_0}, $e(t_0)$, implying exponential mean-square stability of (5), (6), (8) with respect to x.

Proof: Consider $t \in [t_k, t_{k+1})$ and assume that $\sigma_k = i \in \mathcal{I}$. Following the proof of Lemma 1, we have from (27)

$$\begin{aligned}\mathbb{E}\{V_e(t)\} &\le e^{-2\alpha(t-t_k)}\mathbb{E}\{V_e(t_k)\}\\ &\quad + \textstyle\sum_{j=1, j\ne i}^N \mathbb{E}\{e_j^T(t_k)(Q_j - U_j)e_j(t_k)\},\ t \in [t_k, t_{k+1}).\end{aligned} \tag{30}$$

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Therefore,

$$\mathbb{E}\{V(t,x_t,\dot{x}_t)\} \leq e^{-2\alpha(t-t_k)}\mathbb{E}\{V_e(t_k)\},\ t\in[t_k,t_{k+1}).$$

Note that

$$\mathbb{E}\{V_e(t_{k+1})\} = \mathbb{E}\{\tilde{V}_{|t=t_{k+1}} + V_{Q\,|t=t_{k+1}} + \sum_{j=1,j\neq\sigma_{k+1}}^N e_j^T(t_{k+1})Q_je_j(t_{k+1})\}$$

and

$$\sum_{j=1,j\neq\sigma_{k+1}}^N \mathbb{E}\{e_j^T(t_{k+1})Q_je_j(t_{k+1})|\sigma_k=i\} = \sum_{l=1}^N\sum_{j=0,j\neq l}^N \pi_{ij}\mathbb{E}\{e_l^T(t_{k+1})Q_le_l(t_{k+1})\}.$$

Taking (28) into account, we obtain

$$\begin{aligned}&\mathbb{E}\{V_e(t_{k+1}|\sigma_k=i) - V_e(t_{k+1}^-)_{|\sigma_k=i} + \textstyle\sum_{j=1,j\neq i}^N e_j^T(t_k)(Q_j-U_j)e_j(t_k)\}\\ &\leq \mathbb{E}\Big\{\textstyle\sum_{l=1}^N\sum_{j=0,j\neq l}^N \pi_{ij}e_l^T(t_{k+1})Q_le_l(t_{k+1})\\ &\quad -\textstyle\sum_{j=1,j\neq i}^N e_j^T(t_k)U_je_j(t_k)\\ &\quad -e^{-2\alpha\tau_M}|\sqrt{Q}[x(t_k-\eta_k)-x(t_{k+1}-\eta_{k+1})]|^2\Big\}\\ &= \mathbb{E}\Big\{\tilde{\zeta}_i^T(t)\tilde{\Omega}_i\tilde{\zeta}_i(t)\Big\}\leq 0,\end{aligned}$$

where $\tilde{\zeta}_i(t) = \mathrm{col}\{x(t_k-\eta_k)-x(t_{k+1}-\eta_{k+1}), e_1(t_k),\cdots, e_{j|j\neq i}(t_k),\cdots,e_N(t_k)\}$ and $\tilde{\Omega}_i$ is given by (28). Therefore, the inequalities (29) and (30) with $t=t_{k+1}^-$ imply $\mathbb{E}\{V_e(t_{k+1})\}\leq e^{-2\alpha(t_{k+1}-t_k)}\mathbb{E}\{V_e(t_k)\} \leq e^{-2\alpha(t_{k+1}-t_0)}\mathbb{E}\{V_e(t_0)\}$. The latter inequality, with $k+1$ replaced by k and (30) give (18), which implies exponential mean-square stability of (5), (6), (8) with respect to x. □

Remark 5 *Differently from Lemma 1, in Lemma 2 the inequality $\mathbb{E}\{V_e(t_{k+1})\}\leq e^{-2\alpha(t_{k+1}-t_0)}\mathbb{E}\{V_e(t_0)\}$ does not give a bound on $e_{\sigma_k}(t_k)$ since $V_e(t)$ of (26) for $t\in[t_k,t_{k+1})$ does not depend on $e_{\sigma_k}(t_k)$. That is why Lemma 2 guarantees only mean-square stablility with respect to x.*

By using the above lemma and the arguments of Theorem 1, we arrive at the following result:

Theorem 2 *Given $0\leq\eta_m<\tau_M$, $\alpha>0$, $0\leq\pi_{ij}\leq 1$, $\sum_{j=0}^N\pi_{ij}=1$, $i,j\in\mathcal{I}$ and K_l, $l=1,\ldots,N$. Suppose there exist $n\times n$ matrices $P>0$, $Q>0$, $S_j>0$, $R_j>0$, $j=0,1$, S_{12} and $n_l\times n_l$ matrices $Q_l>0$, $U_l>0$, $l=1,\ldots,N$, such that for any $i\in\mathcal{I}$, the matrix inequalities (23), (28) and $\tilde{\Sigma}_i+\tilde{\Xi}_i^T\tilde{H}\tilde{\Xi}_i<0$ are feasible, where the notation Φ is given by (23), and where*

$$\begin{aligned}&\tilde{H}=\eta_m^2R_0+(\tau_M-\eta_m)^2R_1+(\tau_M-\eta_m)Q,\\ &\tilde{\Sigma}_i=\tilde{F}_1^{iT}P\tilde{\Xi}_i+\tilde{\Xi}_i^TP\tilde{F}_1^i+\tilde{\Upsilon}_i-(\tilde{F}_2^i)^TR_0\tilde{F}_2^ie^{-2\alpha\eta_m}\\ &\qquad -(\tilde{F}^i)^T\Phi\tilde{F}^ie^{-2\alpha\tau_M},\\ &\tilde{\Xi}_i=[A\ 0_{n\times n}\ A_1\ 0_{n\times n}\ B_1\ \cdots\ B_{j|j\neq i}\ \cdots\ B_N],\\ &\tilde{F}_1^i=[I_n\ 0_{n\times(3n+n_y-n_i)}],\\ &\tilde{F}_2^i=[I_n\ \ -I_n\ \ 0_{n\times(2n+n_y-n_i)}],\\ &\tilde{F}^i=\begin{bmatrix}0_{n\times n} & I_n & -I_n & 0_{n\times n} & 0_{n\times(n_y-n_i)}\\ 0_{n\times n} & 0_{n\times n} & I_n & -I_n & 0_{n\times(n_y-n_i)}\end{bmatrix},\\ &\tilde{\Upsilon}_i=\mathrm{diag}\{S_0+2\alpha P, -(S_0-S_1)e^{-2\alpha\eta_m}, 0, -S_1e^{-2\alpha\tau_M},\\ &\qquad \tilde{\psi}_1,\cdots,\tilde{\psi}_{j|j\neq i},\cdots,\tilde{\psi}_N\},\\ &\tilde{\psi}_j=-\tfrac{1}{\tau_M-\eta_m}(Q_j-U_j)+2\alpha Q_j,\ j=1,\ldots,N.\end{aligned}$$

Then the solutions of (5), (6), (8) satisfy the bound (18), implying exponential mean-square stability with respect to x. If the aforementioned matrix inequalities are feasible with $\alpha=0$, then the solution bound holds also for a sufficiently small $\alpha_0>0$.

Remark 6 *Note that Theorem 1 under i.i.d. scheduling protocol guarantees exponential mean-square stability with respect to the full state $\mathrm{col}\{x,e\}$, while Theorem 2 under Markovian scheduling protocol only guarantees exponential mean-square stability with respect to x. The LMI conditions in Theorems 1 and 2 are different. In the special case when the Markovian scheduling protocol is i.i.d, the conditions in Theorem 2 give more conservative results* (MATI *and* MAD) *than those in Theorem 1.*

TABLE I
COMPLEXITY OF STABILITY CONDITIONS UNDER DIFFERENT PROTOCOLS (FOR $y_1,y_2\in\mathbb{R}^{n/2}$)

Method	Decision variables	Number and order of LMIs
[3] (RR)	$8.5n^2+2.5n$	two of $6n\times 6n$, two of $3n\times 3n$
[4] (TOD/RR)	$3.75n^2+3n$	two of $5.5n\times 5.5n$, one of $2n\times 2n$
Theorem 1 (i.i.d.)	$4.25n^2+4n$	one of $8n\times 8n$, two of $2n\times 2n$
Theorem2 (Markovian)	$4.5n^2+4n$	two of $5.5n\times 5.5n$, one of $2n\times 2n$, two of $1.5n\times 1.5n$

Remark 7 *The time-delay approach was developed in [3] and [4] for the stability analysis of NCSs under RR protocol and under TOD protocol, respectively. Assume that collisions do not occur. Let $N=2$ and compare the number of scalar decision variables and the resulting LMIs (application of Schur complement) under different protocols. See Table I for the complexity of the LMI conditions for different protocols. Note that Theorem 2 achieves less conservative results than Theorem 1 at the price of more LMIs (see example in the next section).*

V. EXAMPLE: BATCH REACTOR

We illustrate the efficiency of the given conditions on a benchmark example of a batch reactor under the dynamic output feedback (see e.g., [8], [9], [19]), where $N=2$ and

$$A=\begin{bmatrix}1.380 & -0.208 & 6.715 & -5.676\\ -0.581 & -4.2902 & 0 & 0.675\\ 1.067 & 4.273 & -6.654 & 5.893\\ 0.048 & 4.273 & 1.343 & -2.104\end{bmatrix},$$

$$B=\begin{bmatrix}0 & 0\\ 5.679 & 0\\ 1.136 & -3.146\\ 1.136 & 0\end{bmatrix},\ C=\left[\begin{array}{c}C_1\\ \hline C_2\end{array}\right]=\left[\begin{array}{cccc}1 & 0 & 1 & -1\\ \hline 0 & 1 & 0 & 0\end{array}\right],$$

$$\left[\begin{array}{c|c}A_c & B_c\\ \hline C_c & D_c\end{array}\right]=\left[\begin{array}{cc|cc}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ \hline -2 & 0 & 0 & -2\\ 0 & 8 & 5 & 0\end{array}\right].$$

Assume that $\beta_0=0$, $\pi_{0i}=\pi_{i0}=0$, $i=0,1,2$, which means that collisions do not occur. Let $\beta_1=0.6$ and the transition matrix of Markov chain $\sigma_k\in\{1,2\}$ as $\Pi_1=\begin{bmatrix}0.4 & 0.6\\ 0.9 & 0.1\end{bmatrix}$. For the values of η_m given in Table II, by applying Theorems 1 and 2 with $\alpha=0$, we obtain the maximum values of $\tau_M=\mathrm{MATI}+\mathrm{MAD}$ that preserve mean-square stability of the impulsive system (5)–(6) (see Table II). From Table II it is seen that for small transmission delays, our method essentially improves the results of [9], but is more conservative than the results obtained via the discrete-time approach. However, the latter approach becomes complicated for uncertain systems. Polytopic uncertainties in the system model can be easily included in our analysis [3], [4]. When $\eta_m>\tau_M/2$ $(\eta_m=0.03,0.04)$, note that our method is still applicable.

Choosing $\eta_m=0.02$, by Theorem 1 with $\alpha=0$, we obtain the corresponding maximum values of τ_M shown in Figure 2(a) for

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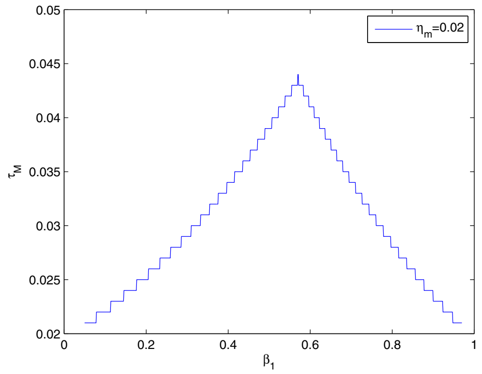
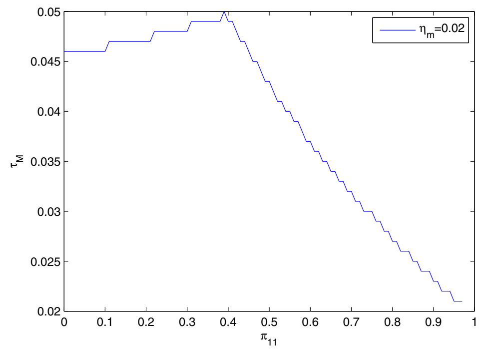
TABLE II
ESTIMATED MAXIMUM VALUES OF $\tau_M = \mathrm{MATI} + \mathrm{MAD}$ FOR DIFFERENT η_m

$\tau_M \setminus \eta_m$	0	0.004	0.02	0.03	0.04
[9](MAD = 0.004, TOD)	0.0108	0.0133	-	-	-
[9] (MAD = 0.004, RR)	0.0088	0.0088	-	-	-
[8](MAD = 0.03, TOD)	0.069	0.069	0.069	0.069	-
[8] (MAD = 0.03, RR)	0.068	0.068	0.068	0.068	-
[4](TOD/RR)	0.035	0.037	0.047	0.053	0.059
[3] (RR)	0.042	0.044	0.053	0.058	0.063
Theorem 1 ($\beta_1 = 0.6$)	0.022	0.025	0.039	0.048	0.056
Theorem 2 (Π_1)	0.035	0.038	0.049	0.055	0.061



(a)



(b)

Fig. 2. (a) Estimated maximum values of $\tau_M(\beta_1)$ by Theorem 1 with $\alpha = 0$; (b) Estimated maximum values of $\tau_M(\pi_{11})$ by Theorem 2 with $\alpha = 0$.

different β_1. Choosing $\eta_m = 0.02$ and $\pi_{21} = 0.9$, by Theorem 2 with $\alpha = 0$, we obtain the corresponding maximum values of τ_M shown in Figure 2(b) for different π_{11}.

VI. CONCLUSIONS

In this note, a time-delay approach has been developed for the stabilization of NCSs under stochastic protocol. Two types of stochastic protocols, which are defined by the i.i.d and Markovian processes are proposed. By developing appropriate Lyapunov methods, the exponential mean-square stability conditions for the delayed stochastic impulsive system were derived in terms of LMIs. Future work will involve the optimization of $\beta_i, i = 0, 1, \ldots, N$ and Π to obtain less conservative results, the implementation aspects of the stochastic protocol in a real wireless network and the consideration of more general NCS models, including stochastic communication delays and scheduling protocols for the actuator nodes.

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