Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Systems & Control Letters 50 (2003) 81–93
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Output regulation of nonlinear systems with delay

E. Fridman*

Department of Electrical Engineering, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel

Received 17 February 2002; received in revised form 25 November 2002; accepted 25 February 2003

Abstract

Output regulation of retarded type nonlinear systems is considered. Regulator equations are derived, which generalize Francis–Byrnes–Isidori equations to the case of systems with delay. It is shown that, under standard assumptions, the regulator problem is solvable if and only if these equations are solvable. In the linear case, the solution of these equations is reduced to linear matrix equations. An example of a delayed Van der Pol equation illustrates the efficiency of the results.

Keywords: Time-delay systems; Nonlinear systems; Output regulation; Regulator equations; Center manifold

1. Introduction

One of the most important problems in control theory is that of controlling the output of the system so as to achieve asymptotic tracking of prescribed trajectories. This problem of output regulation has been studied by many authors (see e.g. a survey paper by Byrnes and Isidori [2] and the references therein). In the linear case, Francis [4] showed that the solvability of a multivariable regulator problem corresponds to the solvability of a system of two linear matrix equations. In the nonlinear case, Isidori and Byrnes [10] proved that the solvability of the output regulation problem is equivalent to the solvability of a set of partial differential and algebraic equations. This set of partial differential and algebraic equations is now known as the *regulator equations* or *Francis–Isidori–Byrnes equations*.

For linear *infinite-dimensional* control systems bounded input and output operators, a solution of the regulator problem was introduced by Schumacher [12] and Byrnes et al. [3], where a *Hilbert* space was used as a state space. The case of the bounded input and output operators was considered. In the case of systems with time-delay it means that there are *no discrete delays* in the *control input, controller output* and *measured output*. The solution was given in terms of the operator regulator equations.

The solution of the output regulation problem is usually based on the application of the center manifold theory. The existence, smoothness and the attractiveness of the center manifold for systems with delay were proved by Hale [7] (see also [8, Chapter 10.2]), where a *Banach* space was used as a state space. A partial differential equation for the function, determining the center manifold for system with delay was derived in [1,5,13].

* Corresponding author. Tel.: +54-947935; fax: +36-40-7095.
E-mail address: emilia@eng.tau.ac.il (E. Fridman).

0167-6911/03/$ - see front matter
doi:10.1016/S0167-6911(03)00131-2

In the present paper, we consider output regulation of *nonlinear* systems with *state*, *controller output* and *measured output delays*, using a *Banach space* formulation. The systems with delay are infinite-dimensional systems, which are important in applications. We generalize the result of [10] to time-delay systems by showing that the problem is solvable iff certain regulator equations are solvable. These equations consist of partial differential equations for a center manifold of the closed-loop system with delay and of an algebraic equation. In the linear case the solution of these equations is reduced to linear matrix equations. We find the relation between our linear equations, derived by using a Banach space approach, and the operator regulator equations of [3,12], obtained in the Hilbert space framework. We analyze the solvability of the linear matrix equations. An example of delayed Van der Pol equation illustrates the developed theory.

Notations. R^m is the Euclidean space with the norm $|\cdot|$ and $C^m[a,b]$ is the Banach space of continuous functions $\phi:[a,b]\to R^m$ with the supremum norm $\|\cdot\|$.

A function $f:X\to Y$, where X and Y are Banach spaces, is a C^k function if it has k continuous Frechet derivatives.

Denote by $x_t(\theta)=x(t+\theta)$ $(\theta\in[-h;0])$.

$L_2([-h,0],R^n)$ is the Hilbert space of square integrable R^n valued functions with the corresponding norm.

$W^{1,2}([-h,0],R^n)$ is the Sobolev space of absolutely continuous R^n valued functions on $[-h,0]$ with square integrable derivatives.

The transpose of a matrix M is written M'.

2. Problem formulation

We consider a nonlinear system modeled by equations of the form

$$\dot{x}(t)=f(x_t,u(t),w(t)),\qquad e(t)=g(x_t,w(t)), \tag{1a,b}$$

where $x(\theta)=\phi(\theta)$, $\theta\in[-h,0]$, with state $x(t)\in R^n$, initial function $\phi\in C^n[-h,0]$, control input $u(t)\in R^m$, exogenous input $w(t)\in R^r$ and tracking error $e(t)\in R^p$. The exogenous input is generated by an autonomous dynamical system of the form

$$\dot{w}(t)=s(w(t)), \tag{2}$$

The functions $f:V\to R^n$, $s:W\to R^r$, $g:Y\to R^p$ are smooth (i.e. C^∞) mappings, where $V\subset C^n[-h,0]\times R^m\times R^r$, $W\subset R^r$, $Y\subset C^n[-h,0]\times R^r$ are some neighborhoods of the origin of the corresponding spaces. We assume that $f(0,0,0)=0$, $s(0)=0$, $g(0,0)=0$. Thus, for $u=0$, system (1a) has an equilibrium state $(x,w)=(0,0)$ with zero error (1b).

We consider both, a state-feedback and an error-feedback regulator problems.

Problem 1 (State-feedback regulator problem). Find a state-feedback control law

$$u(t)=\alpha(x_t,w(t)), \tag{3}$$

where $\alpha:Y\to R^m$ is a C^k $(k\geqslant 2)$ function and $\alpha(0,0)=0$ such that

(1a) the equilibrium $x(t)\equiv 0$ of

$$\dot{x}(t)=f(x_t,\alpha(x_t,0),0)$$

is exponentially stable;

(1b) there exists a neighborhood $Y \subset C^n[-h,0] \times W$ of the origin such that, the solution of the closed-loop system

$$\dot{x}(t) = f(x_t, \alpha(x_t, w(t)), w(t)), \quad \dot{w}(t) = s(w(t)) \tag{4}$$

satisfies

$$\lim_{t\to\infty} g(x_t, w(t)) = 0. \tag{5}$$

Problem 2 (Error-feedback regulator problem). Find an error-feedback controller

$$u = \Theta(z_t), \quad \dot{z}(t) = \eta(z_t, e(t)), \quad z(t) \in R^v \tag{6}$$

with C^k functions $\eta : Z_0 \to R^v$ and $\Theta : Z_1 \to R^m$, where $Z_0 \subset C^v[-h,0] \times R^p$, $Z_1 \subset C^v[-h,0]$ are some neighborhoods of the origin, such that

(2a) the equilibrium $(x(t), z(t)) \equiv 0$ of

$$\dot{x}(t) = f(x_t, \Theta(z_t), 0), \quad \dot{z}(t) = \eta(z_t, h(x_t, 0))$$

is exponentially stable;

(2b) there exists a neighborhood $Z \subset C^n[-h,0] \times C^v[-h,0] \times W$ of the origin such that, the solution of the closed-loop system

$$\dot{x}(t) = f(x_t, \Theta(z_t), w(t)), \quad \dot{z}(t) = \eta(z_t, h(x_t, w(t))), \quad \dot{w}(t) = s(w(t)) \tag{7}$$

satisfies (5).

3. Linearized problem and assumptions

Using Taylor expansion in the neighborhood of the origin of the Banach space $C^n[-h,0] \times R^m \times R^r$, we obtain the following approximation of the smooth function f:

$$f(x_0, u, w) = Ax_0 + Bu + Pw + O(x_0, u, w)^2,$$

where the linear bounded operator $[A, B, P] : C^n[-h,0] \times R^m \times R^r \to R^n$ is a Frechet derivative of f at the origin. The function $O(\cdot)^2$ vanishes at the origin with its first-order Frechet derivative. Similarly, smooth functions h, α, Θ and η can be represented in the form

$$h(x_0, w) = Cx_0 + Qw + O(x_0, w)^2, \quad \alpha(x_0, w) = Kx_0 + Lw(t) + O(x_0, w)^2,$$

$$\Theta(z_0) = Hz_0 + O(z_0)^2, \quad \eta(z_0, e) = Fz_0 + Ge + O(z_0, e)^2,$$

where the functions $O(\cdot)^2$ vanish at the origin with their first-order Frechet derivatives and where Q, L and G are the matrices of the appropriate dimensions. The linear bounded operators $A : C^n[-h,0] \to R^n$ and $C : C^n[-h,0] \to R^p$ by Riesz theorem can be represented in the form of Stieltjes integrals [8]

$$A\phi = \int_{-h}^{0} \mathrm{d}[\mu(\theta)]\phi(\theta), \quad C\phi = \int_{-h}^{0} \mathrm{d}[\zeta(\theta)]\phi(\theta), \tag{8}$$

with $n \times n$ and $p \times n$-matrix functions μ and ζ of bounded variations. A similar representation can be written for the linear bounded operators $K : C^n[-h,0] \to R^m$, $H : C^v[-h,0] \to R^m$ and $F : C^v[-h,0] \to R^v$.

The linearized system is given by

$$\dot{x}(t) = Ax_t + Bu(t) + Pw(t), \quad \dot{w}(t) = Sw(t), \quad e(t) = Cx_t + Qw(t). \tag{9a–c}$$

The linearized state- and error-feedback controllers have the form

$$u(t) = Kx_t + Lw(t) \quad \text{and} \quad u(t) = Hz_t, \quad \dot{z}(t) = Fz_t + Ge(t), \tag{10a,b}$$

respectively.

Similarly to the case without delay [10] we assume the following:

H1. Exosystem (2) is neutrally stable (i.e. Lyapunov stable in forward and backward time, and thus S has all its eigenvalues on the imaginary axis).

H2. The pair $\{A, B\}$ is stabilizable, i.e. there exists a linear bounded operator $K : C^n[-h, 0] \to R^m$ such that the system

$$\dot{x}(t) = (A + BK)x_t \tag{11}$$

is asymptotically stable.

H3. The pair

$$\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \quad [C \quad Q]$$

is detectable, i.e. there exists a $(n + r) \times p$-matrix G such that the system

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \left\{ \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + G[C \quad Q] \right\} \begin{bmatrix} \xi_{1t} \\ \xi_2(t) \end{bmatrix}, \tag{12}$$

where $\xi_1(t) \in R^n$, $\xi_2(t) \in R^r$, is asymptotically stable.

We note that H2 is equivalent to the following condition [9]:

H2′. $rank[\lambda I - \int_{-h}^{0} \mathrm{d}[\mu(\theta)]\mathrm{e}^{\lambda\theta}, \ B] = n$ for all $\lambda \in C$ with $\mathrm{Re}\,\lambda \geqslant 0$.

Similar condition equivalent to H3 can be written for the case of $Cx_t = C_0 x(t)$, where C_0 is a constant matrix. Some sufficient conditions for H2 and for finding a stabilizing controller $u(t) = K_0 x(t)$ or $u(t) = K_1 x(t - h)$ may be found e.g. in [6] (see also references therein) in terms of linear matrix inequalities. Similar sufficient conditions may be derived for H3.

4. Solution of the regulator problems

4.1. Center manifold of the closed-loop system

The solution of the output regulation problem is based on the center manifold theory [7,8].

Lemma 1. *Assume that all eigenvalues of S are on the imaginary axis and that for some $\alpha(x_t, w)$ condition* (1a) *holds. Then the closed-loop system* (4) *has a local center manifold $x_t(\theta) = \pi(w(t))(\theta)$, $\theta \in [-h, 0]$, where $\pi : W_0 \to C^n[-h, 0]$ $(0 \in W_0 \subset W \subset R^r)$ is a C^k mapping with $\pi(0)(\theta) \equiv 0$. The center manifold is locally attractive, i.e. satisfies*

$$\|x_t - \pi(w(t))\| \leqslant M\mathrm{e}^{-at}\|x_0 - \pi(w(0))\|, \quad M > 0, \ a > 0 \tag{13}$$

for all $x_0, w(0)$ sufficiently close to 0 and all $t \geqslant 0$.

Proof. The closed-loop system (4) has the form

$$\dot{w}(t) = Sw(t) + O(w(t))^2, \quad \dot{x}(t) = (A + BK)x_t + (P + BL)w(t) + O(x_t, w(t))^2. \tag{14a,b}$$

By assumption, the zeros of the characteristic equation corresponding to (11) are in C^-, and the eigenvalues of the matrix S are on the imaginary axis. Then

$$|X(t)| \leqslant K\mathrm{e}^{-at}, \quad |\mathrm{e}^{St}| \leqslant K\mathrm{e}^{at/2}, \quad K \geqslant 1,\ a > 0, \tag{15}$$

where $X(t),\ t \in [-h, \infty)$ is a fundamental matrix of (11). Moreover, it is well-known (see e.g. [8, p. 312]) that according to this dichotomy, the space $R^r \times C^n[-h, 0]$ of the initial values of the linear system

$$\dot{w}(t) = Sw(t), \quad \dot{x}(t) = (A + BK)x_t + (P + BL)w(t) \tag{16}$$

can be decomposed as a direct sum $R^r \times C^n[-h, 0] = \mathscr{P} \oplus \mathscr{Q}$, where $\mathscr{P}$ and $\mathscr{Q}$ are invariant sub-spaces of the solutions of (16), in the sense that for all initial conditions from $\mathscr{P}$ ($\mathscr{Q}$), solutions of (16) satisfy $\{w(t), x_t\} \in \mathscr{P}$ ($\{w(t), x_t\} \in \mathscr{Q}$) for all $t \geqslant 0$. Moreover, $\mathscr{P}$ is an r-dimensional and corresponds to solutions of (16) of the form $p(t)\mathrm{e}^{\lambda t}$, where $p(t)$ is a polynomial in t and λ is an eigenvalue of S. The space $\mathscr{Q}$ corresponds to exponentially decaying solutions of (16).

We will determine the projections of $R^r \times C^n[-h, 0]$ onto $\mathscr{P}$ and $\mathscr{Q}$ following [8, p. 314] and will show that by appropriate choice of the basis Φ for $\mathscr{P}$, the projection of (14) onto $\mathscr{P}$ is governed by (14a). Consider for each $\theta \in [-h, 0]$

$$\Pi(\theta) = \int_{-\infty}^{0} X(-s + \theta)(P + BL)\mathrm{e}^{Ss}\,\mathrm{d}s = \int_{-\infty}^{0} X(-\tau)(P + BL)\mathrm{e}^{S(\tau+\theta)}\,\mathrm{d}\tau. \tag{17}$$

From (15) it follows that the integrals in (17) converge uniformly in $\theta \in [-h, 0]$ and, since the function inside the last integral is continuous in (τ, θ), the matrix function $\Pi(\theta)$ is continuous. Moreover, $x_0 = \Pi w$ is a center manifold of (16) (see e.g. (2.7) in p. 315 of [8]). Then $\Phi = \mathrm{col}\{I_r, \Pi\}$ may be chosen as a basis for $\mathscr{P}$. Let $\Psi = \mathrm{col}\{I_r, 0\}$ be a corresponding basis for the adjoint linear system

$$\dot{w}(t) = -S'w(t), \quad \dot{x}(t) = -A' - (BK)'x_t - (P + BL)'w(t),$$

where $A'x_t = \int_{-h}^{0} \mathrm{d}[\mu'(\theta)]x(t + \theta)$ and $(BK)'x_t$ is defined similarly. Let $(A + BK)x_t = \int_{-h}^{0} \mathrm{d}[\bar{\mu}(\theta)x(t + \theta)$, where $\bar{\mu}$ is an $n \times n$-matrix function of the bounded variation. We have $(\Psi, \Phi) = I_r$, where (ψ, ϕ) for $\psi = \mathrm{col}\{\psi_1, \psi_2\} \in R^r \times C^n[-h, 0]$, $\phi = \mathrm{col}\{\phi_1, \phi_2\} \in R^r \times C^n[-h, 0]$ is defined by (see (2.23) in p. 268 of [8]):

$$(\psi, \phi) = \psi'(0)\phi(0) - \int_{-h}^{0} \int_{0}^{\theta} \psi_2'(s - \theta)[\mathrm{d}\bar{\mu}(\theta)]\phi_2(s)\,\mathrm{d}s.$$

For a solution col $\{w(t), x_t\}$ of (16) starting from Φ, i.e. col $\{w(0), x_0\} = \Phi$, the following holds: col $\{w(t), x_t\}$ $= \Phi \exp St$. Then (see (2.3) in p. 314 of [8]) the solution col $\{w(t), x_t\} = \Phi w(t) + z_t$ ($z_t \in \mathscr{Q}$) of (14) is a solution of the system for $w(t)$ and z_t, where $w(t)$ satisfies (14a) and z_t satisfies some integral equation, and conversely. By Theorem 2.1 of [8, p. 314] system (14) has a local smooth center manifold $x_0 = \pi(w)$. The flow on this manifold is governed by (14a). By Theorem 2.2 of [8, p. 216] this manifold is locally attractive. □

Remark 1. The existence and smoothness of π may be proved also by applying the standard arguments of the center manifolds theory directly to Eq. (14), where the functions $O(w)^2$ and $O(x_0, w)^2$ are extended to all values of $w \in R^r$ as described in [8] (see (2.4) in p. 315). In this case π is a solution of the integral equation

$$\pi(w_0)(\theta) = \int_{-\infty}^{0} X(-s + \theta)[(P + BL)w(s) + O(\pi(w(s)), w(s))^2]\,\mathrm{d}s,$$

where $w(t)$ is a solution of (14a) with the initial condition $w(0) = w_0$.

The function π which determines a center manifold of (4) can be considered as a function of one variable $\pi: W_0 \to C^n[-h,0]$ in the Banach space or a function of two variables $\pi: W_0 \times [-h,0] \to R^n$ in the Euclidean space. Further, we find relation between the smoothness properties in both considerations by introducing two classes of functions:

Class $\mathscr{M}_1$ of C^1 functions $\pi: W_0 \to C^n[-h,0] (W_0 \subset R^r)$, satisfying the following conditions:

(i) For each $w \in W_0$ there exists a continuous in $\theta \in [-h,0]$ partial derivative $\partial\pi(w)(\theta)/\partial\theta \triangleq \gamma(w)(\theta)$;
(ii) The function $\gamma: W_0 \to C^n[-h,0]$ is continuous.

Class $\mathscr{M}_2$ of functions $\psi: W_0 \to C^n[-h,0]$ such that the functions $\bar\psi(w,\theta) \triangleq \psi(w)(\theta)$, $\bar\psi: W_0 \times [-h,0] \to R^n$ are continuously differentiable.

Proposition 1. $\mathscr{M}_1 = \mathscr{M}_2$.

Proof. Let $\pi \in \mathscr{M}_1$, then $\partial\pi(w)(\theta)/\partial\theta$ and $\partial\pi(w)(\theta)/\partial w$ (the Frechet derivative of π) are continuous in w uniformly in $\theta \in [-h,0]$ and for each $w \in W_0$ they are continuous in θ. Therefore, these partial derivatives are continuous in (w,θ) and thus $\bar\pi(w,\theta) \triangleq \pi(w)(\theta)$, $\bar\pi: W_0 \times [-h,0] \to R^n$ is continuously differentiable, i.e. $\pi \in \mathscr{M}_2$. Hence $\mathscr{M}_1 \subset \mathscr{M}_2$.

To prove that $\mathscr{M}_2 \subset \mathscr{M}_1$, consider $\psi \in \mathscr{M}_2$, $\bar\psi$ as defined above, $w \in W_0$ and the closed bounded set $\bar W$ such that $w \in \bar W \subset W_0$. Functions $\bar\psi, \partial\bar\psi/\partial w$ and $\partial\bar\psi/\partial\theta$ are uniformly continuous on the compact set $\bar W \times [-h,0]$. Hence, these functions are continuous as functions from W to $C^n[-h,0]$. From the relation

$$\psi(w+\Delta w)(\theta) - \psi(w)(\theta) - \frac{\partial\bar\psi(w,\theta)}{\partial w}\Delta w = \int_0^1 \left[\frac{\partial\bar\psi(w+s\Delta w,\theta)}{\partial w} - \frac{\partial\bar\psi(w,\theta)}{\partial w}\right] \mathrm{d}s\,\Delta w \triangleq \varepsilon(\Delta w)\Delta w$$

and the fact that $\partial\bar\psi(w,\theta)/\partial w$ is continuous in w uniformly in θ it follows that $\lim_{\Delta w \to 0} \|\varepsilon(\Delta w)\| = 0$ and thus $\partial\bar\psi(w,\theta)/\partial w$ is a continuous Frechet derivative of ψ. Hence, $\psi \in \mathscr{M}_1$. $\square$

Lemma 2. *A C^1 mapping $\pi: W_0 \to C^n[-h,0]$, $\pi(0)=0$ defines a center manifold $x_t(\theta)=\pi(w(t))(\theta)$, $\theta \in [-h,0]$ of (4) if and only if $\pi \in \mathscr{M}_1$ and $\forall w \in W_0$, $\forall\theta \in [-h,0]$ it satisfies the following system of partial differential equations*:

$$\frac{\partial\pi(w)(\theta)}{\partial w}s(w) = \frac{\partial\pi(w)(\theta)}{\partial\theta}, \qquad \frac{\partial\pi(w)(0)}{\partial w}s(w) = f(\pi(w), \alpha(\pi(w),w), w). \tag{18a,b}$$

Proof. Note that for a C^1 mapping $\pi: W_0 \to C^n[-h,0]$ and for $w(t)$, satisfying (2), we find that for each $\theta \in [-h,0]$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\pi(w(t))(\theta)] = \frac{\partial\pi(w(t))(\theta)}{\partial w}s(w(t)). \tag{19}$$

Necessity: Let a C^1 mapping $\pi: W_0 \to C^n[-h,0]$ determine a center manifold of (14). Then there exists $\delta > 0$ such that $x_t(\theta) = \pi(w(t))(\theta)$ satisfies (4) for $t \in [-\delta,\delta]$ and, hence

$$\begin{aligned}
&\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial\theta}, \quad x_0 = \phi,\ \theta \in [-h,0],\ t \in [-\delta,\delta],\\
&\frac{\partial x_t(0)}{\partial t} = f(x_t, \alpha(x_t, w(t)), w(t)), \quad \dot w(t) = s(w(t)).
\end{aligned} \tag{20}$$

Substituting $x_t = \pi(w(t))$, $w(0) = w$, $t \in [-\delta, \delta]$ into (21) and setting further $t = 0$, we obtain that for all $w \in W_0$, $\pi(w)(\theta)$ is differentiable in $\theta \in [-h, 0]$ and π satisfies (18). The function $\partial\pi/\partial\theta : W_0 \to C^n[-h, 0]$ is continuous since the left-hand side of (18a) has the same property.

Sufficiency: let a C^1 mapping $\pi : W_0 \to C^n[-h, 0]$ satisfy (18). Substitute $w = w(t)$ into (18), where $w(t)$ is a solution of (2), then $x_t = \pi(w(t))$ satisfies (20) (and thus (4)) and therefore π determines the invariant manifold of (4). □

Remark 2. Approximate solution to (18) can be found in a form of series expansions in the powers of w (similarly to [1,7,13]).

4.2. State-feedback regulator problem

Applying Lemmas 1 and 2, we obtain regulator equations by using arguments of [10].

Lemma 3. *Under* H1 *assume that for some* $\alpha(x_t, w)$ *condition* (1a) *holds. Then, condition* (1b) *is also fulfilled iff there exists a* C^k $(k \geqslant 2)$ *mapping* $\pi : W_0 \to C^n[-h, 0]$, $\pi(0)=0$ *satisfying* (18) *and the algebraic equation*

$$h(\pi(w), w) = 0. \tag{21}$$

Proof. The proof is similar to Lemma 1 of [10] and it is based on Lemmas 1 and 2 above. The closed-loop system (4) has a center manifold. By H1 no trajectory on this manifold converges to zero. Then (1b) holds only if this manifold is annihilated by the error map e, i.e. only if (21) holds. On the other hand, since the center manifold is locally attractive, (21) guarantees that (1b) is satisfied. □

Theorem 1. *Under* H1 *and* H2, *the state-feedback regulator problem is solvable if and only if there exist* C^k $(k \geqslant 2)$ *mappings* $x_0(\theta) = \pi(w)(\theta)$, *with* $\pi \in \mathscr{M}_1$, $\pi(0)(\theta) = 0$, *and* $u = c(w)$, *with* $c(0) = 0$, *both defined in a neighborhood* $W \subset R^r$ *of the origin, satisfying the conditions* $\forall w \in W_0$, $\forall \theta \in [-h, 0]$

$$\frac{\partial\pi(w)(\theta)}{\partial w} s(w) = \frac{\partial\pi(w)(\theta)}{\partial\theta}, \qquad \frac{\partial\pi(w)(0)}{\partial w} s(w) = f(\pi(w), c(w), w), \qquad h(\pi(w), w) = 0. \tag{22a–c}$$

Suppose that π *and* c *satisfy* (22), *then the state-feedback*

$$u = \alpha(x_t, w(t)) = c(w(t)) + K[x_t - \pi(w(t))], \tag{23}$$

where K *is a stabilizing gain which is defined in* H2, *solves the state-feedback regulator problem.*

Proof. The necessity follows immediately from Lemma 3. For the sufficiency consider the state-feedback (23). This choice satisfies (1a), since

$$f(x_t, \alpha(x_t, 0), 0) = (A + BK)x_t + O(x_t)^2.$$

Moreover, by construction

$$\alpha(\pi(w), w) = c(w)$$

and therefore, (22a), (22b) reduce to (18). From (22c) by Lemma 2 it follows that condition (1b) is also fulfilled. □

4.3. Error-feedback regulator problem

Applying Lemmas 1 and 2 to system (7), we obtain the following:

Lemma 4. *Assume that all eigenvalues of S are on the imaginary axis and that for some $\theta(z_t)$ and $\eta(z_t,e)$ condition* (2a) *holds. Then*

(i) *the closed-loop system* (7) *has a local center manifold $x_t(\theta)=\pi(w(t))(\theta)$, $z_t(\theta)=\sigma(w(t))(\theta)$, where $\pi: W_0 \to C^n[-h,0]$, $\sigma: W_0 \to C^v[-h,0]$ $(0\in W_0 \subset W \subset R^r)$ are C^k mappings with $\pi(0)(\theta)\equiv 0$, $\sigma(0)(\theta)\equiv 0$;*

(ii) *the center manifold is locally attractive, i.e. satisfies*

$$\|x_t-\pi(w(t))\|+\|z_t-\sigma(w(t))\| \leqslant M\mathrm{e}^{-at}(\|x_0-\pi(w(0))\|+\|z_0-\sigma(w(0))\|), \quad M>0,\ a>0 \tag{24}$$

for all $x_0, z_0, w(0)$ sufficiently close to 0 and all $t \geqslant 0$.

(iii) *C^1 mappings $\pi: W_0 \to C^n[-h,0]$, $\pi(0)(\theta)=0$, $\sigma: W_0\to C^v[-h,0]$, $\sigma(0)(\theta)=0$ define a center manifold $x_t(\theta)=\pi(w(t))(\theta)$, $z_t(\theta)=\sigma(w(t))(\theta)$, $\theta\in[-h,0]$ of* (7) *if and only if $\pi: W_0\times[-h,0]\to R^n$, $\sigma: W_0\times[-h,0]\to R^v$ are continuously differentiable functions and $\forall w\in W_0$, $\forall\theta\in[-h,0]$ they satisfy the following system of partial differential equations*:

$$\frac{\partial\pi(w)(\theta)}{\partial w}s(w)=\frac{\partial\pi(w)(\theta)}{\partial\theta}, \qquad \frac{\partial\sigma(w)(\theta)}{\partial w}s(w)=\frac{\partial\sigma(w)(\theta)}{\partial\theta},$$

$$\frac{\partial\pi(w)(0)}{\partial w}s(w)=f(\pi(w),\theta(\sigma(w)),w), \qquad \frac{\partial\sigma(w)(0)}{\partial w}s(w)=\eta(\sigma(w),0). \tag{25a–d}$$

Remark 3. In the case when $z(t)=\mathrm{col}\{z_1(t),z_2(t)\}$, where z_2 appears in (7) without delay and thus $\mathrm{col}\{z_{1t}(\theta), z_2(t)\}=\mathrm{col}\{\sigma_1(w(t))(\theta),\ \sigma_2(w(t))\}$, (25b) holds only for $\sigma=\sigma_1$.

Similarly to Lemma 3, the following lemma can be proved

Lemma 5. *Under* H1, *assume that for some $\Theta(z_t)$ and $\eta(z_t,e)$ condition* (2a) *holds. Then, condition* (2b) *is also fulfilled iff there exist C^k $(k\geqslant 2)$ mappings $\pi: W_0\to C^n[-h,0]$, $\pi(0)=0$, $\sigma: W_0\to C^v[-h,0]$, $\sigma(0)=0$ satisfying* (25) *and the algebraic equation* (21).

From the latter lemmas we deduce a necessary and sufficient condition for the solvability of the error-feedback regulator problem

Theorem 2. *Under* H1, H2 *and* H3, *the error-feedback regulator problem is solvable if and only if there exist C^k $(k\geqslant 2)$ mappings $x_0(\theta)=\pi(w)(\theta)$, with $\pi\in\mathscr{M}_1$, $\pi(0)(\theta)=0$, and $u=c(w)$, with $c(0)=0$, both defined in a neighborhood $W\subset R^r$ of the origin, satisfying conditions* (22) $\forall w\in W$, $\forall\theta\in[-h,0]$.

Suppose that π and c satisfy (22), *and that a linear bounded operator $H: C^n[-h,0]\to R^m$ is such that the system*

$$\dot{x}(t)=(A+BH)x_t \tag{26}$$

is asymptotically stable. Then the error-feedback (6), *where*

$$z(t)=\mathrm{col}\{z_1(t),z_2(t)\}, \qquad \eta=\mathrm{col}\{\eta_1,\eta_2\}, \qquad u=\Theta(z_t)=c(z_2(t))+H[z_{1t}-\pi(z_2(t))],$$

$$\eta_1(z_{1t},z_2(t),e(t))=f(z_{1t},\Theta(z_t),z_2(t))-G_1(h(z_{1t},z_2(t))-e(t)),$$

$$\eta_2(z_{1t},z_2(t),e(t))=s(z_2(t))-G_2(h(z_{1t},z_2(t))-e(t)), \tag{27}$$

and where $G=\mathrm{col}\{G_1,G_2\}$ is defined in H3, *solves the regulator problem.*

Proof. The necessity follows immediately from Lemma 5. For the sufficiency we note, that there exist a linear bounded operator $H: C^v[-h,0]\to R^m$ and a matrix $G=\mathrm{col}\{G_1,G_2\}$ such that (26) and (12) are asymptotically stable. A standard calculation shows that for any $m \times r$-matrix K, the characteristic quasipolynomial that corresponds to the system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} A & BH & BK \\ G_1C & A+BH-G_1C & P+BK-G_1Q \\ G_2C & -G_2C & S-G_2Q \end{bmatrix} \begin{bmatrix} x_t \\ z_{1t} \\ z_2(t) \end{bmatrix} \tag{28}$$

is equal to the product of the characteristic quasipolynomials that correspond to (26) and (12) respectively. Therefore, (28) is asymptotically stable.

Consider the error-feedback controller of (6), (27). The linearized system corresponding to the closed-loop system (7) has exactly the form of (28), where

$$K = \left[\frac{\partial c}{\partial w}\right]_{w=0} - H \left[\frac{\partial \pi}{\partial w}\right]_{w=0}.$$

Thus requirement (2a) is satisfied. By construction $z_2(t)$ appears in (7) without delay and thus (22a)–(22b) imply (26) with $\sigma(w) = \mathrm{col}\{\sigma_1(w), \sigma_2(w)\} = \mathrm{col}\{\pi(w), w\}$, where in (25b) $\sigma = \sigma_1$. Thus requirement (2b) follows from Lemma 5. $\square$

5. Linear case

5.1. Linear regulator equations

Consider the linear regulator problem (9). In the linear case the center manifold has a form $x_t = \Pi(\theta)w(t)$, where Π is an $n \times r$ matrix function continuously differentiable in $\theta \in [-h, 0]$. Note that Π satisfies (17). From Theorems 1 and 2 it follows, that the linear problem (9) is solvable iff there exists Π and an $m \times r$-matrix Γ, that satisfy the following system:

$$\dot{\Pi}(\theta) = \Pi(\theta)S, \quad \theta \in [-h, 0], \quad \Pi(0)S = \int_{-h}^{0} \mathrm{d}[\mu(\theta)]\Pi(\theta) + B\Gamma + P, \quad \int_{-h}^{0} \mathrm{d}[\zeta(\theta)]\Pi(\theta) + Q = 0. \tag{29a–c}$$

Eq. (29a) yields $\Pi(\theta) = \Pi(0)\exp S\theta$. Substituting the latter into (29b) and (29c), we obtain the following linear algebraic system for initial value $\Pi(0)$:

$$\Pi(0)S = \int_{-h}^{0} \mathrm{d}[\mu(\theta)]\Pi(0)\mathrm{e}^{S\theta} + B\Gamma + P, \quad \int_{-h}^{0} \mathrm{d}[\zeta(\theta)]\Pi(0)\mathrm{e}^{S\theta} + Q = 0. \tag{30}$$

The latter system is a generalization of Francis equations [4] to the case of retarded systems.

We consider now a particular, but important in applications case of (9) with

$$Ax_t = \sum_{i=0}^{k} A_i x(t-h_i) + \int_{-h}^{0} A_d(\theta)x(t+\theta)\,\mathrm{d}\theta, \quad Cx_t = \sum_{i=0}^{k} C_i x(t-h_i) + \int_{-h}^{0} C_d(\theta)x(t+\theta)\,\mathrm{d}\theta, \tag{31}$$

where $0 = h_0 < h_1 < \cdots < h_k \leqslant h$, A_d and C_d are piecewise continuous matrix functions and where A_i and C_i are constant matrices of the appropriate dimensions. In this case (30) has the form:

$$\Pi(0)S = \sum_{i=0}^{k} A_i\Pi(0)\mathrm{e}^{-Sh_i} + \int_{-h}^{0} A_d(\theta)\Pi(0)\mathrm{e}^{S\theta}\,\mathrm{d}\theta + B\Gamma + P,$$

$$\sum_{i=0}^{k} C_i\Pi(0)\mathrm{e}^{-Sh_i} + \int_{-h}^{0} C_d(\theta)\Pi(0)\mathrm{e}^{S\theta}\,\mathrm{d}\theta + Q = 0. \tag{32}$$

Theorem 3. *Under* H1 *and* H2, *the linear state-feedback regulator problem* (9) ((9) *and* (31)) *is solvable if and only if there exist* $n \times r$ *and* $m \times r$*-matrices* $\Pi(0)$ *and* Γ *which solve the linear matrix equations* (30) ((32)).

In the case of error-feedback regulator problem, the similar result holds under H1, H2 and H3.

5.2. Relation to the operator regulator equations

We consider linear state-feedback regulator problem for the case of (31), where there is no discrete delay in the controller output and in the equation for the error e, i.e.

$$Kx_t = K_0 x(t) + \int_{-h}^{0} K_d(s)x(t+s)\,\mathrm{d}s, \quad Cx_t = C_0 x(t) + \int_{-h}^{0} C_d(s)x(t+s)\,\mathrm{d}s, \tag{33a,b}$$

where K_d and C_d are piecewise continuous matrix functions and where K_0 and C_0 are constant matrices. We show that in this case the linear problem may be formulated in the form of an infinite-dimensional system, defined on a Hilbert space with the bounded input and output operators, and that regulator equations (29) are equivalent to the operator regulator equations of [3,12].

Eqs. (9) may be represented in the form of an evolution equation (see e.g. [14]) by introducing a Hilbert space $M_2 = R^n \times L_2([-h,0];R^n)$ endowed with the inner product

$$\langle \phi, \psi \rangle = \phi^{0'}\psi^0 + \int_{-h}^{0} \phi^{1'}(\theta)\psi(\theta)\,\mathrm{d}\theta, \quad \phi = (\phi^0,\phi^1) \in M^2, \ \psi = (\psi^0,\psi^1) \in M^2.$$

The infinitesimal generator corresponding to the system $\dot{x}(t) = Ax_t$ is characterized by

$$\mathscr{A}(\phi^0,\phi^1) = (A\phi^1, \dot{\phi}^1), \quad (\phi^0,\phi^1) \in D(\mathscr{A}),$$
$$D(\mathscr{A}) = \{(\phi^0,\phi^1) \in M^2\colon\ \phi^0 = \phi^1(0),\ \phi^1 \in W^{1,2}(-h,0;R^n)\}. \tag{34a,b}$$

Stabilizability in M^2 is equivalent to H2′ (see e.g. [14]). Note that in the case of nonzero C_i for some $i > 0$, the linear operator $C : M^2 \to R^n$ is unbounded, while (33) is bounded. Eqs. (9), (31), (33) can be written in the form of the evolution equation (cf. [14])

$$\dot{\bar{x}}(t) = \mathscr{A}\bar{x}(t) + (Bu(t),0) + (Pw(t),0), \quad \dot{w}(t) = Sw(t), \quad C\bar{x}(t) + Qw(t) = 0, \tag{35}$$

where $\bar{x}(t) = \mathrm{col}\{x(t), x_t\} \in M^2$ and where $C\bar{x}(t)$ is defined by the right-hand side of (33b).

In [3] the following regulator equations were derived in the case of bounded input and bounded output operators:

$$\Pi S = \mathscr{A}\Pi + (B,0)\Gamma + (P,0), \quad C\Pi + Q = 0, \tag{36}$$

where $\Pi : R^r \to M^2$ is a linear bounded operator, $\mathrm{Ran}(\Pi) \subset D(\mathscr{A})$, Γ is an $m \times r$-matrix.

Proposition 2. *Eqs.* (36) *are equivalent to Eqs.* (29).

Proof. From (34a) it follows that Eqs. (36) imply (29). Conversely, if $\Pi(\theta)$, $\theta \in [-h,0]$ is a solution to (29), then the bounded continuously differentiable matrix-function Π defines a linear bounded operator $\Pi : R^r \to M^2$ with $\Pi w \in D(\mathscr{A})$ $\forall w \in R^r$ and this operator satisfies (36). □

5.3. On the solvability of the linear regulator equations

As in [3], we assume that $p = m$. First we consider the case of (31), (33), where the results of [3] (on solvability of the regulator equations for infinite-dimensional linear systems with bounded input and output operators) may be applied. Consider the transfer function

$$\mathscr{G}(s) = \left(C_0 + \int_{-h}^{0} C_d(\theta)\mathrm{e}^{s\theta}\,\mathrm{d}\theta\right)\left(sI - \int_{-h}^{0} \mathrm{d}[\mu(\theta)]\mathrm{e}^{s\theta}\right)^{-1} B, \tag{37}$$

which corresponds to the linear system (9a), (9c), (33) with $P=0$ and $Q=0$. A transmission zero of this linear system is such $\lambda \in C$ that $\det \mathscr{G}(\lambda)=0$. Note that there is a finite number of the roots of the characteristic equation corresponding to the retarded type system $\dot{x}(t)=Ax_t$ in the closed right-half plane $\bar{C}^+$. From Corollary V.2 of [3] it follows

Proposition 3. *Under* H1 *and* H2 *the output regulation of* (9) *with* (31) *and* (33) *via state-feedback of* (10a) *is achievable and the regulator equations* (36) (*and thus* (29) *and* (32)) *are solvable for all choice P and Q if and only if* $\det \mathscr{G}(\lambda) \neq 0$ *for all eigenvalues λ of S.*

Consider next more general case of (31) with the general controller output. We assume that the regulator problem for (9) without delay, i.e. for

$$\dot{x}(t) = \left(\sum_{i=0}^{k} A_i\right) x(t) + Bu(t) + Pw(t), \quad \dot{w}(t) = Sw(t), \quad e(t) = \left(\sum_{i=0}^{k} C_i\right) x(t) + Qw(t)$$

is solvable for all P and Q. This is equivalent (see e.g. [4]) to the following assumption:

A1. *det* $\mathscr{G}_0(\lambda) \neq 0$ for all eigenvalues λ of S, where $\mathscr{G}_0(\lambda) = (\sum_{i=0}^{k} C_i)(\lambda I - \sum_{i=0}^{k} A_i)^{-1}B$.

Under A1 the linear regulator equations

$$\Pi_0 S = \left(\sum_{i=0}^{k} A_i\right)\Pi_0 + B\Gamma + P, \quad \left(\sum_{i=0}^{k} C_i\right)\Pi_0 + Q = 0,$$

where Π_0 and Γ are constant matrices, are solvable for all P and Q. Then, by the implicit function theorem for all small enough $h>0$ (32) is solvable. We have:

Proposition 4. *Under* H1, H2 *and* A1, *the output regulation of* (9) *with* (31) *via state-feedback of* (10a) *is achievable and the regulator equations* (32) *are solvable for all small enough h.*

6. Example

Consider the forced delayed Van der Pol Equation

$$\dot{x}_1(t) = -x_2(t-h), \quad \dot{x}_2(t) = x_1(t-h) + ax_2(t-h) + bx_2^3(t-h) + u(t), \quad e(t) = x_1 - w_1 \tag{38a–c}$$

with the exosystem

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \Omega \in [0, 2\pi]. \tag{39}$$

The unforced equation (38) was studied by Murakami [11]. It was shown that for $a>0$, $b<0$, while the system without delay has a stable limit cycle, delayed Van der Pol Equation may exhibit a chaotic behavior. In the case of $a<0$, $b<0$, the equation without delay is asymptotically stable, whereas for $h>0$ there may appear a periodic solution. Output regulation of (38), (39) without delay was considered in [2].

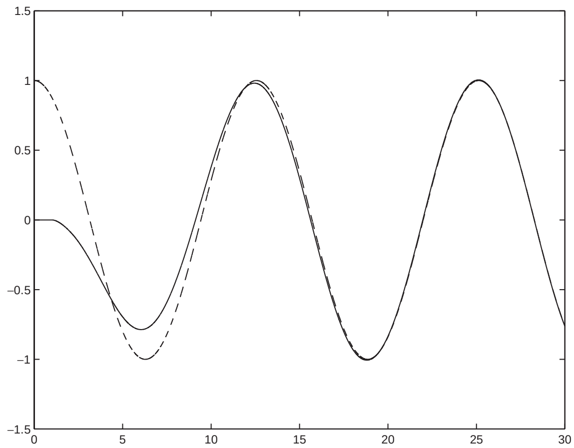

Fig. 1. $x_1(t)$—solid line, $w_1(t)$—dashed line.

Regulator equations for (38), (39) with $w = \mathrm{col}\{w_1, w_2\}$, $\pi = \mathrm{col}\{\pi_1, \pi_2\}$ have the form

$$\frac{\partial \pi(w)(\theta)}{\partial w}\begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} w = \frac{\partial \pi(w)(\theta)}{\partial \theta}, \quad \theta \in [-h, 0],$$

$$\frac{\partial \pi(w)(0)}{\partial w}\begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} w = \begin{bmatrix} -\pi_2(w)(-h) \\ \pi_1(w)(-h) + a\pi_2(w)(-h) + b\pi_2^3(w)(-h) + c(w) \end{bmatrix},$$

$$\pi_1(w)(0) = w_1. \tag{40a–c}$$

Substituting (40c) into the first row of (40b) we find

$$\pi_2(w)(-h) = -\Omega w_2. \tag{41}$$

Solving the boundary value problem (40a), (40c) and (41) we obtain

$$\pi(w)(\theta) = \begin{bmatrix} \cos \Omega\theta & \sin \Omega\theta \\ \Omega \sin(\Omega(h+\theta)) & -\Omega\cos(\Omega(h+\theta)) \end{bmatrix} w. \tag{42}$$

Finally, from the second row of (40b) and from (42) we derive

$$c(w) = (\Omega^2 - 1)\cos \Omega h \cdot w_1 + [(\Omega^2 + 1)\sin \Omega h + a\Omega]w_2 + b\Omega^3 w_2^3. \tag{43}$$

For $h = 0$ the memoryless controller $u(t) = -(3 + a)x_2(t)$ stabilisizes the linearized system (38a,b) (where $b = 0$). Then for all small enough $h > 0$ this controller is stabilizing for the linearized system (38a,b) and thus H2 holds. Moreover, the linearized problem (38), (39) is solvable by Proposition 3 since $\mathscr{G}(s) = \mathrm{e}^{-sh} \neq 0$ for $s = \pm\Omega j$. The corresponding state-feedback for nonlinear problem may be chosen as follows:

$$u = c(w) - (3 + a)[x_2(t) - \Omega(\sin \Omega h \cdot w_1 - \cos \Omega h \cdot w_2)]. \tag{44}$$

We made numerical simulations of (38), (44) for $a = 1$, $b = -1$, $\Omega = 0.5$, $w_1 = \cos \Omega t$, $h = 1$ and $x_0 = 0$. Note that by the stability condition of [6], this state-feedback stabilizes the linearized system. Plots of the output $x_1(t)$ and of the reference signal $w_1(t)$ are given in Fig. 1. It is clear that $x_1(t)$ asymptotically approaches $w_1(t)$.

7. Conclusions

The geometric theory of output regulation is generalized to nonlinear systems with delay. It is shown that the state-feedback and the error-feedback regulator problems are solvable, under the standard assumptions on stabilizability and detectability of the linearized system, if and only if a set of regulator equations is solvable. This set consists of partial differential and algebraic equations. In the linear case these equations are reduced to the linear matrix equations.

The issues of the solvability of the nonlinear regulator equations and of approximate solutions to these equations are currently under study.

References

[1] M. Ait Babram, O. Arino, M. Hbid, Computational scheme of a center manifold for neutral functional differential equations, J. Math. Anal. Appl. 258 (2001) 396–414.
[2] C.I. Byrnes, A. Isidori, Output regulation for nonlinear systems: an overview, Internat. J. Robotics Nonlinear Control 10 (2000) 323–337.
[3] C.I. Byrnes, I.G. Lauko, D.S. Gilliam, V.I. Shubov, Output regulation for linear distributed parameter systems, IEEE Trans. Automat. Control 45 (2000) 2236–2252.
[4] B.A. Francis, The linear multivariable regulator problem, SIAM J. Control Optim. 15 (1977) 486–505.
[5] E. Fridman, Asymptotics of integral manifolds and decomposition of singularly perturbed systems of neutral type, Differential Equations 26 (1990) 457–467.
[6] E. Fridman, U. Shaked, A descriptor system approach to H_∞ control of time-delay systems, IEEE Trans. Automat. Control 47 (2002).
[7] J. Hale, Critical cases for neutral functional differential equations, J. Differential Equations 10 (1971) 59–82.
[8] J. Hale, S. Verduyn Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.
[9] J. Hale, S. Verduyn Lunel, Strong stabilization of neutral functional differential equations, IMA J. Math. Control Inform. 19 (2002) 5–23.
[10] A. Isidori, C.I. Byrnes, Output regulation of nonlinear systems, IEEE Trans. Automat. Control 35 (1990) 131–140.
[11] K. Murakami, Bifurcated periodic solutions for delayed Van der Pol equation, Neural Parallel Sci. Comput. 7 (1999) 1–16.
[12] J. Schumacher, Finite-dimensional regulators for a class of infinite-dimensional systems, Systems Control Lett. 3 (1983) 7–12.
[13] V. Strygin, E. Fridman, Asymptotics of integral manifolds of singularly perturbed differential equations with retarded argument, Mathematische Nachrichten 117 (1984) 83–109.
[14] R. Vinter, R. Kwong, The infinite time quadratic control proble m for linear systems with state and control delays: an evolution equation approach, SIAM J. Control 19 (1981) 139–153.