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Brief Paper

Robust H_∞ minimum entropy static output-feedback control of singularly perturbed systems☆

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Abstract

The problem of designing static output-feedback ε-independent controllers for linear time-invariant singularly perturbed systems is considered. The controller is required to satisfy a prescribed H_∞-norm bound and to minimize the closed-loop entropy (at $s = \infty$) for all small enough ε. The optimal controller gain is designed on the basis of generalized Riccati and Lyapunov equations with symmetric block (2,2), that are coupled via a projection. This gain is either purely fast, purely slow or a composite one, depending on the structure of the output coupling matrix. A well-posed problem with a finite value of entropy for $\varepsilon \to 0$ is obtained by assuming that the entropy of the fast subproblem is zero or by scaling the matrices of the system. In the first case the optimal controller is the one that minimizes the entropy of the corresponding descriptor system. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: H_∞ Control; Static output-feedback; Minimum entropy; Singular perturbations; Descriptor systems

1. Introduction

Robust state-feedback and dynamic output-feedback H_∞ control for singularly perturbed systems have been considered by Khalil and Chen (1992), Pan and Basar (1993, 1994), Dragan (1993), Tuan and Hosoe (1997) in the standard case, and by Xu and Mizukami (1996), Tan, Leung and Tu (1998) in the non-standard case (information on non-standard singularly perturbed systems is found in Khalil, 1989). In the present paper we investigate the problem of achieving minimum entropy by static output-feedback ε-independent H_∞ controller for non-standard singularly perturbed systems. We denote this controller as the 'robust optimal controller'. This controller should minimize, for all small enough values of ε, the closed-loop entropy while ensuring a prescribed H_∞-norm bound. For each $\varepsilon > 0$, the minimizing controller gain can be designed by solving a coupled pair of ε-dependent Riccati and Lyapunov equations (Yaesh & Shaked, 1997). We shall show that the robust optimal controller is the formal first-order approximation to the above minimizing controller. Unlike the conventional approaches we shall prove the optimality of the obtained robust controller directly without considering its closeness to the optimal ε-dependent one (the proof that relies on the closeness arguments requires additional restrictive assumptions).

The present paper is organized as follows. In the next section we formulate the problem and the known results from (Yaesh & Shaked, 1997). In Sections 3.1–3.4 we derive ε-independent generalized Riccati and Lyapunov equations coupled via a projection for design of robust optimal controller. This controller leads to unbounded value of entropy for $\varepsilon \to 0$. A well-posed problem with a finite value of entropy for $\varepsilon \to 0$ is considered in Sections 3.5 and 3.6. Numerical example is given in Section 3.7. The paper ends with Conclusions.

Notations. We denote by $(\cdot)'$ a transpose of a matrix, by $\|\cdot\|_\infty$ the H_∞-norm of the transfer function.

2. Problem formulation

Consider the following linear time-invariant system

$$E_\varepsilon \dot{x} = Ax + B_1 w + B_2 u, \tag{1a}$$

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$$y = C_2 x, \tag{1b}$$

$$z = C_1 x + D_{12} u, \tag{1c}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad E_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix},$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B_i = \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix},$$

$C_i = [C_{i1}\ C_{i2}]$, $i = 1, 2$, $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, and where ε is a small positive scalar parameter. We assume that C_2 is of full row rank, and $D'_{12}[D_{12}\ C_1] = [R\ 0]$, $R > 0$. We note that A_{22} may be singular.

Denoting by T_{zw} the transference from the exogenous input w to the objective vector z, for a given scalar $\gamma > 0$, the problem is to find, of all ε-independent static output-feedback controllers $u = Ky$ that satisfy, for all small enough ε,

$$\|T_{z,w}\|_\infty < \gamma, \tag{2}$$

the one that minimizes, for small enough ε, the entropy of the closed-loop transfer-function matrix

$$T_{z,w}(s) = T^K_{z,w}(s) = (C_1 + D_{12} K C_2) [sE_\varepsilon - A - B_2 K C_2]^{-1} B_1, \tag{3}$$

where the entropy is given by

$$\mathscr{E}(T_{z,w}, \gamma) \triangleq -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln \det [I - \gamma^{-2} T^{\sim}_{z,w}(jw) T_{z,w}(jw)]\, dw, \tag{4}$$

and where $T^{\sim}(s) = T'(-s)$. By minimizing the entropy, we push $T_{z,w}$ away from the upper bound γ in the magnitude Bode plot. We are thus looking for robust optimal controller gain K^* for which there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ (2) is satisfied and

$$K^* = \underset{K}{\operatorname{argmin}}\, \mathscr{E}(T_{z,w}, \gamma). \tag{5}$$

We begin by denoting the following:

$$A_\varepsilon = E_\varepsilon^{-1} A, \quad B_{i\varepsilon} = E_\varepsilon^{-1} B_i, \quad \tilde{A} = A + B_2 K C_2,$$
$$\tilde{A}_\varepsilon = E_\varepsilon^{-1} \tilde{A}, \quad \tilde{C} = C_1 + D_{12} K C_2. \tag{6}$$

It is known (Doyle, Glover, Khargonekar & Francis, 1989) that for each $\varepsilon > 0$ the matrix $\tilde{A}_\varepsilon$ is stable and the transfer-function matrix $T_{z,w} = \tilde{C}(sI - \tilde{A}_\varepsilon)^{-1} B_{1\varepsilon}$ satisfies (2) iff there exists a matrix $X_\varepsilon \geq 0$ that satisfies the following Riccati equation:

$$\tilde{A}'_\varepsilon X_\varepsilon + X_\varepsilon \tilde{A}_\varepsilon + \gamma^{-2} X_\varepsilon B_{1\varepsilon} B'_{1\varepsilon} X_\varepsilon + \tilde{C}'\tilde{C} = 0, \tag{7}$$

so that $\tilde{A}_\varepsilon + \gamma^{-2} B_{1\varepsilon} B'_{1\varepsilon} X_\varepsilon$ is asymptotically stable. If such X_ε exists, then $\mathscr{E}(T_{z,w}, \gamma)$ is given by Stoorvogel (1991)

$$\mathscr{E}(T_{z,w}, \gamma) = Tr\{B'_{1\varepsilon} X_\varepsilon B_{1\varepsilon}\}. \tag{8}$$

Considering next the following Lyapunov (with respect to Y_ε) equation

$$(A_\varepsilon + B_{2\varepsilon} K C_2 + \gamma^{-2} B_{1\varepsilon} B'_{1\varepsilon} X_\varepsilon) Y_\varepsilon + Y_\varepsilon (A'_\varepsilon + C'_2 K' B'_{2\varepsilon} + \gamma^{-2} X_\varepsilon B_{1\varepsilon} B'_{1\varepsilon}) + B_{1\varepsilon} B'_{1\varepsilon} = 0. \tag{9}$$

It has been shown (Yaesh & Shaked, 1997) that for each $\varepsilon > 0$, the following gain matrix:

$$K = K_\varepsilon = -R^{-1} B'_{2\varepsilon} X_\varepsilon Y_\varepsilon C'_2 (C_2 Y_\varepsilon C'_2)^{-1}, \tag{10}$$

solves the H_∞ minimum entropy static output-feedback control problem. Denote

$$v_\varepsilon = Y_\varepsilon C'_2 (C_2 Y_\varepsilon C'_2)^{-1} C_2 = C_2^\dagger C_2, \quad v_{\varepsilon\perp} = I - v_\varepsilon, \tag{11}$$

where $C_2^\dagger$ is the right inverse of C_2 (i.e. $C_2 C_2^\dagger = I$). It has been also found that $v_\varepsilon^2 = v_\varepsilon$, $KC_2 = -R^{-1} B'_{2\varepsilon} X_\varepsilon v_\varepsilon$, and that (7) can be written in the form

$$A'_\varepsilon X_\varepsilon + X_\varepsilon A_\varepsilon + \gamma^{-2} X_\varepsilon B_{1\varepsilon} B'_{1\varepsilon} X_\varepsilon + C'_1 C_1 - X_\varepsilon B_{2\varepsilon} R^{-1} B'_{2\varepsilon} X_\varepsilon + v'_{\varepsilon\perp} X_\varepsilon B_{2\varepsilon} R^{-1} B'_{2\varepsilon} X_\varepsilon v_{\varepsilon\perp} = 0. \tag{12}$$

The above are summarized in the following lemma (Yaesh & Shaked, 1997):

Lemma 2.1. *For each* $\varepsilon > 0$, *if there exist* $X_\varepsilon, Y_\varepsilon$ *and* K_ε *that satisfy* (9)–(12) *with the following properties*:
(**a**) $X_\varepsilon \geq 0$, $C_2 Y_\varepsilon C'_2 > 0$ *and* $A_\varepsilon + \gamma^{-2} B_{1\varepsilon} B'_{1\varepsilon} X_\varepsilon - B_{2\varepsilon} R^{-1} B'_{2\varepsilon} X_\varepsilon v_\varepsilon$ *is stable*, *then* (2) *holds and the gain* K_ε *achieves* (5).

Note that for $v_{\varepsilon\perp} = 0$ (this corresponds to the state-feedback case) (12) and (9) are decoupled Riccati and Lyapunov equations. For $v_{\varepsilon\perp} \neq 0$ (9)–(12) constitutes a system of modified Riccati and Lyapunov equations coupled via projection that are highly nonlinear in X_ε, Y_ε and K_ε. For each ε, this system has been successfully solved in (Yaesh & Shaked, 1997) by applying the homotopy method (Richter, Hodel & Pruet, 1993).

3. Main results

3.1. System transformation (diagonalization of C_2)

Since C_2 is of full row rank, we assume, without loss of generality, that C_2 possesses one of the following three forms:

(i) $C_2 = [\begin{smallmatrix} \bar{C}_{21} & 0 \\ \bar{C}_3 & \bar{C}_{22} \end{smallmatrix}]$, where $\bar{C}_{2i}$, $i = 1,2$ are of full row rank,
(ii) $C_2 = [\bar{C}_3\ \ \bar{C}_{22}]$, where $\bar{C}_{22}$ is of full row rank,
(iii) $C_2 = [\bar{C}_{21}\ \ 0]$, where $\bar{C}_{21}$ is of full row rank.

Cases (ii) and (iii) are degenerate cases of (i), where (ii) corresponds to $\bar{C}_{21}$ with zero number of rows, and (iii) corresponds to $\bar{C}_{22}$ with zero number of rows. Cases (ii) and (iii) physically mean that linear independent combinations of the fast or the slow variables are, respectively, observed.

Let C_2 be in the form of (i) and let L be a matrix that transforms C_2 to block-diagonal form $\bar{C}_2$ as follows:

$$\bar{C}_2 = \begin{bmatrix} \bar{C}_{21} & 0 \\ 0 & \bar{C}_{22} \end{bmatrix} = \begin{bmatrix} \bar{C}_{21} & 0 \\ \bar{C}_3 & \bar{C}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ L & I \end{bmatrix}. \tag{13}$$

We introduce the following nonsingular transformation of the state variables:

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2 - Lx_1, \quad \bar{x} = col\{\bar{x}_1, \bar{x}_2\}. \tag{14}$$

Then, from (1a)–(1c), (14) and (13), we obtain the following system for $\bar{x}$:

$$\dot{\bar{x}} = \bar{A}_\varepsilon \bar{x} + \bar{B}_{1\varepsilon} w + \bar{B}_{2\varepsilon} u, \tag{15a}$$

$$y = \bar{C}_2 \bar{x}, \tag{15b}$$

$$z = \bar{C}_1 \bar{x} + D_{12} u, \tag{15c}$$

where $\bar{A}_{i1} = A_{i1} + A_{i2}L$, $i = 1,2$ and

$$\bar{A}_\varepsilon = \begin{bmatrix} \bar{A}_{11} & A_{12} \\ \varepsilon^{-1}\bar{A}_{21} - L\bar{A}_{11} & \varepsilon^{-1}A_{22} - LA_{12} \end{bmatrix},$$

$$\bar{B}_{i\varepsilon} = \begin{bmatrix} B_{i1} \\ \varepsilon^{-1}B_{i2} - LB_{i1} \end{bmatrix},$$

$$\bar{C}_1 = [\bar{C}_{11} \quad \bar{C}_{12}] = C_1 \begin{bmatrix} I & 0 \\ L & I \end{bmatrix}.$$

Since the closed-loop transfer-function matrix T_{zw} of the new system of (15a)–(15c) is identical to the one defined by (3), the robust optimal control law $u = Ky$ for the H_∞ minimum entropy control of (15a)–(15c) is also optimal for the original system. For the system of (15a)–(15c) the optimal controller is derived by solving the coupled equations of (12), (9) and (10), where

$$A_\varepsilon = \bar{A}_\varepsilon, \quad B_{i\varepsilon} = \bar{B}_{i\varepsilon}, \quad C_i = \bar{C}_i, \quad i = 1,2. \tag{16}$$

3.2. Generalized ε-independent Riccati and Lyapunov equations

Denote

$$E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{17a}$$

$$\bar{A} = A \begin{bmatrix} I & 0 \\ L & I \end{bmatrix}, \tag{17b}$$

$$\tilde{A} = [\bar{A} + B_2 K \bar{C}_2], \tag{17c}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \tag{17d}$$

$$\tilde{C} = [\bar{C}_1 + D_{12} K \bar{C}_2] = [\tilde{C}_1 \quad \tilde{C}_2]. \tag{17e}$$

With Riccati and Lyapunov equations (7) and (9) with (16) we associate (similarly to Tan et al., 1998) the following generalized Riccati and Lyapunov equations:

$$\tilde{A}'X + X'\tilde{A} + \gamma^{-2}X'B_1B_1'X + \tilde{C}'\tilde{C} = 0, \tag{18}$$

$$(\bar{A} + B_2K\bar{C}_2 + \gamma^{-2}B_1B_1'X)Y + Y'(\bar{A}' + \bar{C}_2'K'B'_2 + \gamma^{-2}X'B_1B_1') + B_1B_1' = 0, \tag{19}$$

where

$$X = \begin{bmatrix} X_1^{(0)} & 0 \\ X_2'^{(0)} & X_3^{(0)} \end{bmatrix}, \tag{20a}$$

$$Y = \begin{bmatrix} Y_1^{(0)} & 0 \\ Y_2'^{(0)} & Y_3^{(0)} \end{bmatrix}, \tag{20b}$$

$$K = [K_1^{(0)} \quad K_2^{(0)}], \tag{20c}$$

$$X_1^{(0)} = X_1'^{(0)} \geq 0, \tag{20d}$$

$$X_3^{(0)} = X_3'^{(0)} \geq 0, \tag{20e}$$

$$Y_1^{(0)} = Y_1'^{(0)}, \tag{20f}$$

$$Y_3^{(0)} = Y_3'^{(0)}. \tag{20g}$$

Consider

$$K = -R^{-1}B_2'XY\bar{C}_2'(\bar{C}_2Y\bar{C}_2')^{-1}. \tag{21}$$

Denoting

$$v = Y\bar{C}_2'(\bar{C}_2Y\bar{C}_2')^{-1}\bar{C}_2, \quad v_\perp = I - v. \tag{22}$$

we find that $v^2 = v$ and (18) can be written in the form

$$\bar{A}'X + X'\bar{A} + \gamma^{-2}X'B_1B_1'X + \bar{C}_1'\bar{C}_1 - X'B_2R^{-1}B'_2X + v_\perp'X'B_2R^{-1}B_2'Xv_\perp = 0. \tag{23}$$

3.3. Fast and slow subsystems

Expanding (21) and noting that

$$(\bar{C}_2Y\bar{C}_2')^{-1} = \begin{bmatrix} (\bar{C}_{21}Y_1^{(0)}\bar{C}_{21}')^{-1} & 0 \\ M' & (\bar{C}_{22}Y_3^{(0)}\bar{C}_{22}')^{-1} \end{bmatrix}, \tag{24a}$$

$$M = -(\bar{C}_{21}Y_1^{(0)}\bar{C}_{21}')^{-1}\bar{C}_{21}Y_2^{(0)}\bar{C}_{22}'(\bar{C}_{22}Y_3^{(0)}\bar{C}_{22}')^{-1}, \tag{24b}$$

we find

$$K_1^{(0)} = -\{R^{-1}[B_{21}'X_1^{(0)}Y_1^{(0)} + B_{22}'(X_2'^{(0)}Y_1^{(0)} + X_3^{(0)}Y_2'^{(0)})] + K_2^{(0)}\bar{C}_{22}Y_2'^{(0)}\}\bar{C}_{21}'(\bar{C}_{21}Y_1^{(0)}\bar{C}_{21}')^{-1}, \tag{25}$$

$$K_2^{(0)} = -R^{-1}B_{22}'X_3^{(0)}Y_3^{(0)}\bar{C}_{22}'(\bar{C}_{22}Y_3^{(0)}\bar{C}_{22}')^{-1}. \tag{26}$$

From (18) and (19) we obtain the following fast equations (Tan et al., 1998):

$$\tilde{A}'_{22}X_3^{(0)} + X_3^{(0)}\tilde{A}_{22} + \tilde{C}'_2\tilde{C}_2 + \gamma^{-2}X_3^{(0)}B_{12}B'_{12}X_3^{(0)} = 0, \quad (27a)$$

$$F_{22}Y_3^{(0)} + Y_3^{(0)}F'_{22} + B_{12}B'_{12} = 0. \quad (27b)$$

where

$$F_{22} = A_{22} + \gamma^{-2}B_{12}B'_{12}X_3^{(0)} + B_{22}K_2^{(0)}\bar{C}_{22}.$$

We write (27a) in the form

$$A'_{22}X_3^{(0)} + X_3^{(0)}A_{22} + \gamma^{-2}X_3^{(0)}B_{12}B'_{12}X_3^{(0)} - X_3^{(0)}B_{22}R^{-1}B'_{22}X_3^{(0)} + v'_{f\perp}X_3^{(0)}B_{22}R^{-1}B'_{22}X_3^{(0)}v_{f\perp} + \bar{C}'_{12}\bar{C}_{12} = 0,$$

$$v_f = Y_3^{(0)}\bar{C}'_{22}(\bar{C}_{22}Y_3^{(0)}\bar{C}'_{22})^{-1}\bar{C}_{22}, \quad v_{f\perp} = I - v_f. \quad (28)$$

Assume that

A1. The system of coupled equations (19), (21)–(23) has a solution X, Y, K of (20) with the following properties:

(**b**) $X_3^{(0)} \geq 0$, $\bar{C}_{22}Y_3^{(0)}\bar{C}'_{22} > 0$ and F_{22}, where $K_2^{(0)}\bar{C}_{22} = -R^{-1}B'_{22}X_3^{(0)}v_{f\perp}$, is stable,

(**c**) $X_1^{(0)} \geq 0$, $\bar{C}_{21}Y_1^{(0)}\bar{C}'_{21} > 0$ and $[E, \bar{A} + \gamma^{-2}B_1B'_1X - B_2R^{-1}B'_2Xv_\perp]$ is stable.

Remark 3.1. The second property of (**b**) implies that $B_{12} \neq 0$ (otherwise $Y_3^{(0)} = 0$).

The system of coupled equations (26), (27b) and (28) provides a solution to the H_∞ mininimum entropy static output-feedback control problem for the *fast subsystem*:

$$\dot{\bar{x}}_2 = A_{22}\bar{x}_2 + B_{12}w + B_{22}u, \quad y_2 = \bar{C}_{22}\bar{x}_2,$$
$$z_2 = \bar{C}_{12}\bar{x}_2 + D_{12}u. \quad (29)$$

The optimal controller for (29) $u_f = K_2^{(0)}\bar{C}_{22}\bar{x}_2$ leads to the stable matrix $\tilde{A}_{22}$ and to the minimum value of entropy

$$\mathscr{E}_f = Tr\{B'_{12}X_3^{(0)}B_{12}\}. \quad (30)$$

The *slow subsystem* is the descriptor one

$$E\dot{\bar{x}} = \bar{A}\bar{x} + B_1w + B_2u, \quad y = \bar{C}_2\bar{x},$$
$$z = \bar{C}_1\bar{x} + D_{12}u. \quad (31)$$

It will be shown in Section 3.5 that the system of generalized Riccati and Lyapunov equations coupled by projection (19), (21)–(23) provide the optimal solution to (31), if $\mathscr{E}_f = 0$.

3.4. Robust optimal controller design

We now are in a position to state our main result — the design of ε-independent robust optimal controller gain K^*:

Theorem 3.1. *Given* $\gamma > 0$, *for* C_2 *of the forms* (i)–(iii) *we have correspondingly the following results*:

(i) *Under* **A1** $K^* = K$ *is the robust optimal gain. The robust optimal controller* $u^* = K^*C_2x$ $(u^* = K^*\bar{C}_2\bar{x})$ *leads* (1) (15) *for all small enough values of* ε, *to the* H_∞*-norm bound of* γ *and to the following minimum value of entropy*:

$$\mathscr{E}^* = \varepsilon^{-1}\mathscr{E}_f + Tr\{B'_{11}X_1^{(0)}B_{11} + 2B'_{12}X_2'^{(0)}B_{11} + B'_{12}X_3^{(1)}B_{12}\} + O(\varepsilon), \quad (32)$$

where $\mathscr{E}_f$ *is defined in* (30), *and* $X_3^{(1)}$ *is a solution of the following Lyapunov equation*:

$$F'_{22}X_3^{(1)} + X_3^{(1)}F_{22} + \tilde{A}'_{12}X_2^{(0)} + X_2'^{(0)}\tilde{A}_{12} + \gamma^{-2}X_3^{(0)}B_{12}B'_{11}X_2^{(0)} + \gamma^{-2}X_2'^{(0)}B_{11}B'_{12}X_3^{(0)} = 0. \quad (33)$$

(ii) *Assume that there exists a solution to the fast equations* (26), (27b) *and* (28), *with the properties of* (**b**), *and there exists a solution to the generalized Riccati equation* (18), *where* $K = K_2^{(0)}$ *of* (26), *such that* $[E, \tilde{A} + \gamma^{-2}B_1B'_1X]$ *is stable. Then,* $K^* = K_2^{(0)}$ *and the robust optimal controller* $u^* = K_2^{(0)}[\bar{C}_3x_1 + \bar{C}_{22}x_2]$ $(u^* = K_2^{(0)}\bar{C}_{22}\bar{x}_2)$ *leads* (1) (15), *for all small enough* ε, *to the* H_∞ *norm bound* γ *and to the minimum value of entropy given by* (32).

(iii) *Assume that* (27a) *with* $\tilde{A}_{22} = A_{22}$ *has a solution* $X_3^{(0)} \geq 0$ *such that* $A_{22} + \gamma^{-2}B_{12}B'_{12}X_3^{(0)}$ *is stable and assume that there exists a solution to the Eqs.* (19), (21)–(23) *with the properties of* (**c**). *Then,* $K^* = K_1^{(0)}$ *and the slow controller* $u^* = K_1^{(0)}\bar{C}_{21}x_1$ *leads* (1) *to the* H_∞*-norm bound of* γ *and to the minimum entropy of* (32).

Proof. Similarly to Yaesh and Shaked (1997), minimizing (8) with respect to K can be performed by forming the following Lagrangian:

$$\mathscr{L}(K, X_\varepsilon, Y_\varepsilon) \triangleq Tr\{\bar{B}'_{1\varepsilon}X_\varepsilon\bar{B}_{1\varepsilon} + [\tilde{A}'_\varepsilon X_\varepsilon + X_\varepsilon\tilde{A}_\varepsilon + X_\varepsilon\bar{B}_{1\varepsilon}\bar{B}'_{1\varepsilon}X_\varepsilon + \bar{C}'_1\bar{C}_1 + \bar{C}'_2K'RK\bar{C}_2]Y_\varepsilon\}. \quad (34)$$

The stationarity of (8) with respect to X_ε requires that $\partial\mathscr{L}/\partial X_\varepsilon = 0$ that implies (9). As in Yaesh and Shaked (1997) we obtain that the part of $\mathscr{L}$ that depends on K is given by $Tr\{T_3\}$, where

$$T_3 = [K(\bar{C}_2Y_\varepsilon\bar{C}'_2)^{1/2} + R^{-1}\bar{B}'_{2\varepsilon}X_\varepsilon Y_\varepsilon\bar{C}'_2(\bar{C}_2Y_\varepsilon\bar{C}'_2)^{-1/2}]' R[K(\bar{C}_2Y_\varepsilon\bar{C}'_2)^{1/2} + R^{-1}\bar{B}'_{2\varepsilon}X_\varepsilon Y_\varepsilon\bar{C}'_2(\bar{C}_2Y_\varepsilon\bar{C}'_2)^{-1/2}].$$

Note that $T_3 \geq 0$. Since the generalized Riccati equations (18) has a stabilizing solutions (in the sense of Tan et al., 1998), then by implicit function theorem the full-order Riccati equation (7) with $K=[K_1^{(0)}\ K_2^{(0)}]$ has a stabilizing solution and by standard arguments

$$X_\varepsilon=\begin{bmatrix} X_1^{(0)}+O(\varepsilon) & \varepsilon X_2^{(0)}+O(\varepsilon^2)\\ \varepsilon X_2'^{(0)}+O(\varepsilon^2) & \varepsilon X_3^{(0)}+\varepsilon^2 X_3^{(1)}+O(\varepsilon^3)\end{bmatrix}, \quad (35a)$$

$$Y_\varepsilon=\begin{bmatrix} Y_1^{(0)}+O(\varepsilon) & Y_2^{(0)}+O(\varepsilon)\\ Y_2'^{(0)}+O(\varepsilon) & \varepsilon^{-1} Y_3^{(0)}+O(1)\end{bmatrix}. \quad (35b)$$

Substituting (35b) in $\bar{C}_2 Y_\varepsilon \bar{C}_2'$ we obtain that for all small enough ε the matrix $\bar{C}_2 Y_\varepsilon \bar{C}_2'$ is invertible and

$$(\bar{C}_2 Y_\varepsilon \bar{C}_2')^{-1} =\begin{bmatrix} (\bar{C}_{21} Y_1^{(0)}\bar{C}_{21}')^{-1}+O(\varepsilon) & \varepsilon M+O(\varepsilon^2)\\ \varepsilon M'+O(\varepsilon^2) & \varepsilon(\bar{C}_{22} Y_3^{(0)}\bar{C}_{22}')^{-1}+O(\varepsilon^2)\end{bmatrix}, \quad (36)$$

where M is given by (24b). In cases (ii) and (iii) we find, that $\bar{C}_2 Y_\varepsilon \bar{C}_2'$ is invertible and

$$(\bar{C}_2 Y_\varepsilon \bar{C}_2')^{-1}=\varepsilon(\bar{C}_{22} Y_3^{(0)}\bar{C}_{22}')^{-1}+O(\varepsilon^2)$$

and

$$(\bar{C}_2 Y_\varepsilon \bar{C}_2')^{-1}=(\bar{C}_{21} Y_1^{(0)}\bar{C}_{21}')^{-1}+O(\varepsilon),$$

correspondingly. We obtain that $K^*=K$ of (21) leads (15) (and thus (1)) to H_∞-norm bound of γ and minimizes to $O(\varepsilon)$ the value of $Tr\{T_3\}$, for all small enough ε. Expanding (8) in the powers of ε and applying (35) we obtain (32). □

We summarize in the *algorithm* the design of the robust optimal controller gain in the general case (i): find X, Y and K of (20) with properties **(b)** and **(c)** by solving the coupled system of equations (19), (21)–(23). Then $K^*=K$.

Remark 3.2. For $v_\perp=0$ (23) and (19) are the well-known generalized Riccati and Lyapunov equations (for their solution see, e.g., Tan et al., 1998). For $v_\perp\neq 0$ the system of (19), (21)–(23) constitutes a set of highly nonlinear coupled equations with respect to X, Y and K. One way of solving this system is to use the homotopy method (Richter et al., 1993).

Remark 3.3. It follows from the proof of the theorem that K^* defined by (20c), (25) and (26) is the formal $O(\varepsilon)$-approximation to the optimal gain K_ε of (10).

Remark 3.4. In the case (ii) (in the case (iii)), where the linear independent combinations of the fast (slow) variables are observed, the robust optimal gain is purely fast (slow). Note that unlike Kokotovic, Khalil & O'Reilly (1986, Chapter 3), our static output-feedback control in the case, which is based on the slow model, is robust, in the sense that it cannot destabilize the original system. This is due to the structure (iii) of C_2 with $C_2x=\bar{C}_{21}x_1$.

Remark 3.5. Note that if $\mathscr{E}_f\neq 0$ the value of $\mathscr{E}^*$ approaches infinity for $\varepsilon\to 0$ (see (32)), but still for all small values of ε the controller gain K^* minimizes the value of entropy among all static output-feedback controllers satisfying (2). A well-posed problem is obtained by assuming $\mathscr{E}_f=0$ or by scaling. We treat these cases in the next subsections.

3.5. Descriptor system approach to a well-posed problem

In this subsection we assume that $\mathscr{E}_f$ given by (30) equals zero. A zero $\mathscr{E}_f$ is encountered, for example when $B_{12}=0$ (no disturbances in the fast equation) or when $C_{12}=0$ with stable matrix A_{22} (no fast variables in the objective vector z).

Lemma 3.1. *If $\mathscr{E}_f=0$, then the following relations hold*:

$$B_{12}'X_3^{(0)}=0, \quad (37a)$$

$$\tilde{C}_2\tilde{A}_{22}^{-1}B_{12}=0, \quad (37b)$$

$$\tilde{C}_2B_{12}=0, \quad (37c)$$

$$X_2^{(0)}B_{12}=-X_1^{(0)}\tilde{A}_{12}\tilde{A}_{22}^{-1}B_{12}, \quad (37d)$$

$$X_3^{(0)}\tilde{A}_{22}^{-1}B_{12}=0, \quad (37e)$$

and $X_1^{(0)}$ satisfies the following Riccati equation:

$$A_s'X_1^{(0)}+X_1^{(0)}A_s+\gamma^{-2}X_1^{(0)}B_{1s}B_{1s}'X_1^{(0)}+C_s'C_s=0, \quad (38)$$

where

$$A_s=\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21},\quad B_{1s}=B_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}B_{12},$$
$$C_s=\tilde{C}_1-\tilde{C}_2\tilde{A}_{22}^{-1}\tilde{A}_{21}. \quad (39)$$

Proof. A zero $\mathscr{E}_f$ implies (37a). It also means that the transfer function matrix of the fast subsystem (29) with $u=K_2^{(0)}\bar{C}_{22}\bar{x}_2$, is equal to zero and, hence, (37b) holds. Matrix $X_2^{(0)}$ satisfies the following equation (see e.g. Tan et al., 1998):

$$\tilde{A}_{21}'X_3^{(0)}+X_1^{(0)}\tilde{A}_{12}+X_2^{(0)}\tilde{A}_{22}+\tilde{C}_1'\tilde{C}_2 +\gamma^{-2}X_1^{(0)}B_{11}B_{12}'X_3^{(0)}+\gamma^{-2}X_2^{(0)}B_{12}B_{12}'X_3^{(0)}=0. \quad (40)$$

Eqs. (37d) and (37e) follow from (40) and (27) multiplying, from the right, by $\tilde{A}_{22}^{-1}B_{12}$, while (37c) follows from (27a) multiplying it, from the left, by B_{12}' and, from the right, by B_{12}. Eq. (38) follows from (2.5)–(2.8) of Dragan (1993) and (37b). □

We consider the descriptor system (31) that corresponds to (15). The transfer-function matrix T_d of (31), with $u = K\bar{C}_2\bar{x}$, is given by

$$T_d = (\bar{C}_1 + D_{12}K\bar{C}_2)[sE - \bar{A} - B_2K\bar{C}_2]^{-1}B_1 = \tilde{C}(sE - \tilde{A})^{-1}B_1. \quad (41)$$

We want to choose of all the K that satisfy both $\|T_d\|_\infty < \gamma$ and $\mathscr{E}_f = 0$, the one that minimizes the entropy of T_d given by (4), where $T_{zw} = T_d$.

Lemma 3.2. *For the descriptor system of* (31), *with* $u = B_2K\bar{C}_2\bar{x}$ *and* $\mathscr{E}_f = 0$, *where* $\mathscr{E}_f$ *is given by* (30), *the following holds*:

(i) *The transfer-function matrix is given by*

$$T_d = C_s(sI - A_s)^{-1}B_{1s}. \quad (42)$$

(ii) A_s *is stable and* $\|T_d\|_\infty < \gamma$ *iff there exists a solution* $X_1^{(0)} \geq 0$ *to the Riccati equation* (38) *such that* $A_s + \gamma^{-2}B_{1s}B'_{1s}X_1^{(0)}$ *is stable.*

(iii) *The entropy of the system is given by*

$$\mathscr{E}_d = Tr\{B'_{1s}X_1^{(0)}B_{1s}\}. \quad (43)$$

This entropy $\mathscr{E}_d$ *is* $O(\varepsilon)$*-close to the entropy of* (15), *where* $u = K\bar{C}_2\bar{x}$.

Proof. Denote

$$N_1 = \begin{bmatrix} I & -\tilde{A}_{12}\tilde{A}_{22}^{-1} \\ 0 & \tilde{A}_{22}^{-1} \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} I & 0 \\ -\tilde{A}_{22}^{-1}\tilde{A}_{21} & I \end{bmatrix}.$$

Then,

$$N_1\tilde{A}N_2 = \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad N_1EN_2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

and thus

$$T_d = C_s(sI - A_s)^{-1}B_{1s} - \tilde{C}_2\tilde{A}_{22}^{-1}B_{12}. \quad (44)$$

Relation (42) follows from (44) and (37b).

Item (i) implies (ii) and (43). It follows from (33) using $F_{22} = \tilde{A}_{22}$, that $X_3^{(1)} = -X_2^{(0)\prime}\tilde{A}_{12}\tilde{A}_{22}^{-1}$. The $O(\varepsilon)$-closeness of $\mathscr{E}_d$ to the entropy of (15) follows from (32), (43) and (37d). □

Remark 3.6. Note that if $\tilde{C}_2\tilde{A}_{22}^{-1}B_{12} \neq 0$ (and thus $\mathscr{E}_f \neq 0$), then (44) implies $\mathscr{E}_d = \infty$ since $T_d(\infty) \neq 0$ (Mustafa and Glover, 1990).

We obtain the following from the last lemma and Tan et al. (1998):

Theorem 3.2. *The controller* $u = K^*\bar{C}_2\bar{x}$ *that leads to* $\mathscr{E}_f = 0$ *achieves a* H_∞*-norm bound of* γ *and minimizes the entropy of* (31) *iff it is the robust optimal controller for the singularly perturbed system* (15). *This controller is given by Theorem* 3.1.

Remark 3.7. If $B_{12} = 0$, then only (iii) of Theorem 3.1 may hold (here $Y_3^{(0)} = 0$) and there is a purely-slow gain. If $C_{12} = 0$ and A_{22} is stable, then from (i) we obtain $K_2^{(0)} = 0$ and thus also in this case the gain is purely slow.

3.6. Scaled well-posed problems

A well-posed problem with a finite entropy for $\varepsilon \to 0$ can be also obtained by scaling the matrices of the system:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \varepsilon^\alpha B_{11}w + B_{21}u, \quad (45a)$$

$$\varepsilon\dot{x}_2 = \varepsilon^\beta A_{21}x_1 + A_{22}x_2 + \varepsilon^{1/2}B_{12}w + \varepsilon^\delta B_{22}u. \quad (45b)$$

The parameters α, β, δ represent the relative size of the small parameters within the system, with respect to the small time constants of the fast subsystem. We multiply B_{12} by $\varepsilon^{1/2}$ to obtain a finite and positive value of entropy for $\varepsilon \to 0$. For information on scaled LQG problem refer to Kokotovic, Khalil and Reilly (1986) and Saskena and Basar (1986). The results below are obtained by using arguments similar to Theorem 3.1.

In the case of 'uniform scaling', when $\alpha = \frac{1}{2}$, $\beta = \delta = 0$ and y and z are non-scaled, we choose the H_∞-norm bound of $\varepsilon^{1/2}\gamma$. For each ε we obtain the same equations (7), (9) and (10) as in the case without scaling. Then the optimal robust controller of Theorem 3.1 achieves the H_∞-norm bound of $\varepsilon^{1/2}\gamma$ and the minimum value of entropy $\mathscr{E}_f + O(\varepsilon)$ for all small enough ε.

Given $\gamma > 0$, consider next the case of $\alpha = 0$. The entropy of (45) with $u = KC_2x$ satisfies relation

$$\lim_{\varepsilon \to 0} \mathscr{E} = Tr\{B'_{11}X_1^{(0)}B_{11} + B'_{12}X_3^{(0)}B_{12}\}.$$

A solution to Lyapunov equation (9) has the following form:

$$Y_\varepsilon = \begin{bmatrix} Y_1 & Y_2 \\ Y'_2 & Y_3 \end{bmatrix},$$

where since $B_{11}B'_{12} = 0$ and $B_{12}B'_{12} = 0$

$$\tilde{A}_{11}Y_1 + \tilde{A}_{12}Y'_2 + Y_1\tilde{A}'_{11} + Y_2\tilde{A}'_{12} + B_{11}B'_{11} = 0,$$
$$Y_2 = -Y_1(\tilde{A}'_{21}\tilde{A}'_{22}{}^{-1}) + O(\varepsilon^{1/2}), \quad (46)$$
$$\tilde{A}_{21}Y_2 + \tilde{A}_{22}Y_3 + Y'_2\tilde{A}'_{21} + Y_3\tilde{A}'_{22} = 0$$

with

$$\tilde{A}_{11} = A_{11} + A_{12}L + B_{21}K_1^{(0)}\bar{C}_{21},$$
$$\tilde{A}_{12} = A_{12} + \varepsilon^\delta B_{21}K_2^{(0)}\bar{C}_{22},$$
$$\tilde{A}_{21} = \varepsilon^\beta A_{21} + A_{22}L + \varepsilon^\delta B_{22}K_1^{(0)}\bar{C}_{21},$$
$$\tilde{A}_{22} = A_{22} + \varepsilon^\delta B_{22}K_2^{(0)}\bar{C}_{22}.$$

If $Y_{2|_{\varepsilon=0}} \neq 0$ (it means that $\beta = 0$ or $\delta = 0$ or $\bar{C}_3 \neq 0$), then in cases (i) and (ii) all the equations for the zero-approximations are coupled and there is no slow-fast decomposition of the problem. Thus, in case (ii)

$$K^* = K_2^{(0)} = -R^{-1}\{[B'_{21}X_1^{(0)} + B'_{22}X_2^{(0)'}]Y_2^{(0)} + B'_{22}X_3^{(0)}Y_3^{(0)}\}\bar{C}'_{22}(\bar{C}_{22}Y_3^{(0)}\bar{C}'_{22})^{-1},$$

where $Y_i^{(0)}$ satisfy (46a)–(46c). In case (iii) the optimal controller is the purely slow one of Theorem 3.1, where $Y_1^{(0)}$ and $Y_2^{(0)}$ satisfy (46a) and (46b) with $\varepsilon = 0$.

If $\beta = \frac{1}{2}$, $\delta = \frac{1}{2}$ and $\bar{C}_3 = 0$, then both $X_3^{(0)}$ and $X_1^{(0)}$ do not depend on $K_2^{(0)}$. Assuming that A_{22} is stable we find that the robust optimal controller gain is purely slow $K^* = [K_1^{(0)}\ 0]$, where $K_1^{(0)}$ is the gain of the minimizing controller $u = K_1^{(0)}y_s$ for the slow problem

$$\dot{x}_s = A_{11}x_s + B_{11}w + B_{21}u, \quad y_s = \bar{C}_{21}x_s,$$
$$z_s = \bar{C}_{11}x_s + D_{12}u.$$

3.7. Example

Consider (1a)–(1c) with the following matrices:

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $D_{12} = [0\ 1]'$. This is the case of (i), where both $\bar{C}_{21}$ and $\bar{C}_{22}$ are full-rank, with singular A_{22}. From (13) we find that $L = [0\ -1]$. The fast subproblem in this example is a state-feedback-type H_∞-control. Choosing $\gamma = 8.5$, which is close to the minimum possible value of γ for small values of ε, we obtain from (19), (21)–(23) $K^* = [-1.0688, -0.7153]$. Applying robust optimal controller $u = K^*y$, to (1a)–(1c) and choosing the values of ε that are given in Table 1, we find that (7) has a nonnegative stabilizing solution and thus (2) is satisfied. Using the solution of (7) we compute by (8) the resulting values of entropy $\mathscr{E}^*$ and bring them in Table 1.

For the same value of $\gamma = 8.5$, and for each value of ε under consideration, we obtain the values of the optimal ε-dependent gain K_ε by solving the full-order Eqs. (9)–(12). We see that for small ε the resulting $K_\varepsilon = [K_1\ \ K_2]$ is close to K^* (see Table 1). We also compute the corresponding values of $\mathscr{E}_\varepsilon$ using (8). It is seen from Table 1 that $\mathscr{E}_\varepsilon$ is close to $\mathscr{E}^*$.

In this example $\mathscr{E}_f = 9.619 \neq 0$ and therefore $\mathscr{E}^*$ and $\mathscr{E}_\varepsilon$ are unbounded for small ε. In order to obtain a finite entropy for $\varepsilon \to 0$ we apply a uniform scaling on the system, where instead of B_1 we take $\sqrt{\varepsilon}\ B_1$. Then choosing an H_∞-norm bound of $\sqrt{\varepsilon}$ 8.5, we find the same values of K^* and K_ε. The resulting values of entropy are those given in the Table 1, multiplied now by ε. We see that the resulting values of entropy are bounded for small values of ε and tend to $\mathscr{E}_f = 9.619$.

Table 1

ε	0.1	0.01	0.001	0.0001
$\mathscr{E}^*$	164.3	1031.6	9689	96260
$\mathscr{E}_\varepsilon$	133.2	1021.5	9688	96260
K_1	− 1.0946	− 0.5138	− 0.6808	− 0.7122
K_2	− 1.6563	− 1.1985	− 1.0859	− 1.0706

4. Conclusions

In the present note we have designed ε-independent robust optimal static output-feedback controllers for non-standard singularly perturbed systems that satisfy given H_∞-norm performance bounds and also minimize the entropy at $s = \infty$. Our method is based on solving a generalized Riccati equation coupled via a projection with a generalized Lyapunov equation. Our solution yields a minimum value of the entropy which becomes unbounded when ε tends to zero. A well-posed problem with a finite value of entropy for $\varepsilon \to 0$ is obtained either when the entropy of the fast subproblem is zero or when the matrices of the system are scaled.

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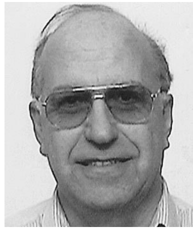

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