STEADY MODES IN RELAY CONTROL SYSTEMS WITH TIME DELAY AND PERIODIC DISTURBANCES

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Abstract— **We study stability of slow oscillatory motions in first order one- and two-dimensional systems with delayed relay control element and periodic disturbances, which serve as models of stabilization of the fingers of an underwater manipulator and of control of fuel injectors in automobile engines. Various types of stability observed are used to design a direct adaptive control of relay type with time delay that extinguishes parasite auto-oscillations in these models.**

Keywords: *Time Delay Systems, Steady Modes, Adaptive Control.*

Introduction

Time delay in relay control systems is usually present and must be taken into account. In practice, time delay is due to:

- *Measuring devices having time delay.* A controller of an exhausted gas in fuel injector automotive control systems is an example of such a system (see [7, 12]).
- *Actuators having time delay.* This can be observed, for instance, in a controller for stabilizing the fingers of an underwater manipulator (see [2]).

Here we will consider a system with time delay in a control element of relay type

$$\dot{x}(t) = -\text{sign}\,[x(t-1)] + F(x(t),t), \quad t \geq 0 \tag{1}$$

$$|F(x,t)| \leq p < 1, \quad F \in C^1(\mathbf{R}^2)\ . \tag{2}$$

This system was studied in [10, 21] with emphasis on the *autonomous* case

$$F(x,t) \equiv F(x)\ . \tag{3}$$

The main observation made in [5], [16] is that only *slowly oscillating solutions*, *SOS* (having a relatively large magnitude), may be stable, and in the autonomous or quasi-autonomous case, all SOS are non-asymptotically stable.

In this paper we focus on the *periodic* case

$$F(x,t+T_0) \equiv F(x,t), \quad T_0 = \text{const} > 0\ . \tag{4}$$

Such a situation naturally appears in the study of multi-dimensional systems. For example, in the two-dimensional triangular system

$$\dot{x}(t) = -\text{sign}\ [x(t-1)] + f(x(t)), \tag{5}$$

$$\dot{y}(t) = -\text{sign}\ [y(t-\gamma)] + g(x(t),y(t)), \tag{6}$$

$$\gamma > 0, \quad |f(x)| \leq p_1 < 1,\ f \in C^1(\mathbf{R})\ ,$$

$$|g(x,y)| \leq p_2 < 1,\ g \in C^1(\mathbf{R}^2)\ ,$$

which describes the behavior of a fuel injector with two relay λ-sensors (see [7, 12]), the first equation produces a periodic disturbance for the second one, which then turns into a system of type (1), (2), (4).

Our main result is that the periodic system (1), (2), (4) reveals a dichotomy in the stability of SOS: either all SOS are non-asymptotically stable, or all but finitely many are asymptotically stable. The latter situation reflects the resonant behavior of the system, when the rotation angle (introduced below) becomes commensurable with T_0, the period of the perturbation $F(x,t)$, and there appear periodic (stable and unstable) oscillations, whose period is a multiple of T_0.

The theoretical conclusions apply to the design of a relay control algorithm exponentially extinguishing oscillations. It is adapted to the autonomous and periodic scalar systems, as well as to system (5), (6). It does not require complete information on the perturbation, is stable with respect to measurement errors, and is based on the variation in magnitude of the relay control element. For a concrete application of such an algorithm we refer the reader to [2], where it was implemented into the stabilizers of the fingers of an underwater manipulator.

I. Frequency of oscillations and steady modes

Under condition (2) any Cauchy problem

$$x(t) = \varphi(t),\ t \in [-1,0], \quad \varphi \in C[-1,0], \tag{7}$$

has a unique continuous solution (see, for example, [6]) $x_\varphi : [-1,\infty) \to \mathbf{R}$. Its zero set $Z_\varphi = (x_\varphi)^{-1}(0)$ is nonempty and unbounded [10, 21], which allows us to define the *frequency of oscillations*

$$\nu_\varphi(t) = \text{card}(Z_\varphi \cap (t^*-1,t^*)),\ t^* = \max(Z_\varphi \cap [0,t]), \quad t \geq 0.$$

The crucial property of this frequency (observed in similar situations in [13-15, 17, 19, 20]) is

Proposition 1: ([10, 21]) For any $\varphi \in C[-1,0]$, the function $\nu_\varphi(t)$ is non-increasing. Consequently, there exists a limit frequency

$$N_\varphi = \lim_{t\to\infty} \nu_\varphi(t)\ ,$$

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which is either infinite, or an even nonnegative number.

This suggests a natural classification of the solutions to (1), (2) with respect to the limit oscillation frequency: the set of initial functions $C[-1,0]$ splits into the disjoint union of the sets

$$\mathcal{U}_n = \{\varphi \in C[-1,0] \ : \ N_\varphi = 2n\}, \quad n \geq 0,$$

$$\mathcal{U}_\infty = \{\varphi \in C[-1,0] \ : \ N_\varphi = \infty\}.$$

A solution with a constant frequency $\nu_\varphi \equiv N_\varphi$ is called *steady mode*. Correspondingly, we introduce the sets of steady modes

$$\mathcal{U}_n^{sm} = \{\varphi \in C[-1,0] \ : \ \nu_\varphi \equiv 2n\}, \quad 0 \leq n \leq \infty \ .$$

Solution and steady modes with any finite even limit frequency do exist (for the existence of infinite frequency steady modes we refer the reader to [1, 8, 18, 22]):

Proposition 2: ([10, 21]) For any nonnegative integer n the set $\mathcal{U}_n^{sm}$ is nonempty. Moreover, for each $T \geq 0$ there exists a steady mode $g_n(t) \in \mathcal{U}_n^{sm}$ such that $g_n(T) = 0$, $\dot{g}_n(T) > 0$, which is unique if $n = 0$. For any $x_\varphi \in \mathcal{U}_n$, there exist a steady mode $x(t)$ and $T \geq 0$ such that $x_\varphi(t) = x(t)$ as $t \geq T$.

The limit oscillation frequency basically determines the stability properties of solutions to system (1), (2), which we discuss next.

II. Stability

A. Stability and the limit oscillation frequency

We consider the stable behavior of solutions x_φ to (1), (2) with respect to variation of the initial function φ in the space $C[-1,0]$ equipped with the standard sup-norm.

Proposition 1 indicates that the non-zero limit frequency should be unstable, and the property of zero limit frequency should be stable (cf. [24, 25]). We present here precise statements, which strengthen similar results in [16].

Theorem 1: ([16]) The set $\mathcal{U}_0$ has a nonempty interior. Moreover, $\mathrm{Int}(\mathcal{U}_0)$ contains a non-empty set

$$\mathcal{U}_0 \cap \{\varphi \in C[-1;0] \ : \ \mathrm{mes}(\varphi^{-1}(0)) = 0\}.$$

This implies the stability of the zero limit frequency which holds under some condition (the necessity of that condition is demonstrated in an example in [16]).

To formulate results on higher frequencies, we introduce the functions

$$\mu_1(t) = \max_x \left|\frac{\partial F}{\partial x}(x,t)\right|, \quad \mu_2(t) = \max_x \left|\frac{\partial F}{\partial t}(x,t)\right| \ ,$$

and the quantities

$$\mu_1^{(0)} = \frac{1+p}{1-p} \lim_{T\to\infty} \sup_{t\geq T} \mu_1(t) \ ,$$

$$\mu_2^{(0)} = \frac{1+p}{(1-p)^2} \lim_{T\to\infty} \sup_{t\geq T} \mu_2(t) \ ,$$

$$\Theta = \max\left\{\frac{1}{2}\left(\min\{\mu_1^{(0)}, \mu_2^{(0)}\}\left(\log\frac{2}{1+p}\right)^{-1} - 1\right), \ 0\right\} \ .$$

Theorem 2: (1) The set $\mathcal{U}_\infty$ is nowhere dense in $C[-1,0]$.
(2) If

$$\lim_{t\to\infty} \sup \int_t^{t+\frac{(1+p)^2}{2(1-p)}} \mu_2(\tau)d\tau < 1-p \ , \qquad (8)$$

or

$$\lim_{t\to\infty} \sup \int_t^{t+\frac{(1+p)^2}{2(1-p)}} \mu_1(\tau)d\tau < \frac{1-p}{1+p} \ , \qquad (9)$$

then the set $\bigcup_{n>0}\mathcal{U}_n$ is nowhere dense in $C[-1,0]$, and all the solutions x_φ with positive N_φ are unstable.

(3) The set $\bigcup_{\Theta<n}\mathcal{U}_n$ is nowhere dense in $C[-1,0]$, and all the solutions x_φ with $N_\varphi > \Theta$ are unstable.

In particular, all solutions with nonzero limit frequency are unstable in the autonomous system (3) and in the quasi-autonomous system discussed later. Conjecturally, this is always the case.

Proposition 2 and Theorems 1 and 2, in fact, reduce the study of realistic motions in system (1) to an analysis of the stability of the zero frequency steady modes, on which we concentrate in the next section.

B. Stability of zero frequency steady modes

In the autonomous system (3) all the zero frequency steady modes are periodic and non-asymptotically stable; in fact, they all come from one steady mode by shifts in t. Moreover, the zero frequency steady modes are non-asymptotically stable if the system is quasi-autonomous, i.e., satisfies

$$\int_0^\infty \mu_2(t)dt < \infty$$

(for details see [16]).

Suppose now that the function $F(x,t)$ does depend on t and is periodic in t with period T_0. Let S be a circle of length T_0, and let

$$\mathrm{pr}_{T_0} : \mathbf{R} \to \mathcal{S}, \quad \mathrm{pr}_{T_0}(t) = t - T_0 \cdot \left[\frac{t}{T_0}\right],$$

be a natural projection. By Proposition 2, for an arbitrary $T \in \mathbf{R}$ there exists a unique zero frequency steady mode $g_T(t)$ such that $g_T(T) = 0$ and $\dot{g}_T(T) > 0$. Denote by T' the second zero of g_T in the interval (T, ∞). Thus, we obtain a smooth map

$$\widetilde{f} : \mathbf{R} \to \mathbf{R}, \quad \widetilde{f}(T) = T' \ .$$

Clearly, it factors through pr_{T_0} and gives us a diffeomorphism

$$f : \mathcal{S} \to \mathcal{S} \ ,$$

which is determined by the function $F(x,t)$.

We note that the stability of $g_T(t)$ is equivalent to the stability of the trajectory $T, f(T), f(f(T)), ..., f^n(T), ...$ of point T.

We introduce the parameter

$$\omega(f) = \lim_{n\to\infty} \frac{f^n(t)}{n} \ ,$$

which is called *the rotation angle of* f. This parameter does not depend on t (see [11]).

Finite (or, periodic) orbits of f are called cycles. As stated above, cycles can occur only when $\frac{\omega(f)}{T_0}$ is rational. A cycle $\mathcal{T} = \{T_1, T_2, ..., T_n\}$ such that $T_2 = f(T_1)$, ..., $T_n = f(T_{n-1})$, $T_1 = f(T_n)$, is called *non-degenerate* if

$$\mu(\mathcal{T}) = \prod_{i=1}^{n} f'(T_i) \neq 1 \ .$$

A non-degenerate cycle $\mathcal{T}$ is asymptotically stable if $\mu(\mathcal{T}) < 1$, and is unstable if $\mu(\mathcal{T}) > 1$. The well-known properties of iterates of a circle diffeomorphism (see [11]) translate into the corresponding properties of steady modes:

Theorem 3: (1) If $\frac{\omega(f)}{T_0}$ is irrational then the diffeomorphism f is topologically conjugate to a rotation by angle $\frac{\omega(f)}{T_0}$, and all the zero frequency steady modes are non-asymptotically stable.

(2) If $\frac{\omega(f)}{T_0}$ is rational, then f has periodic orbits (cycles). If, in addition, all the cycles of f are non-degenerate, then there is an even number $2k$ of cycles of the same length, k of them asymptotically stable and k unstable, with the remaining orbits of f being infinite. In the latter situation, system (1) has $2k$ periodic zero frequency steady modes of the same period multiple of T_0, k of them asymptotically stable and k unstable. The other zero frequency steady modes are aperiodic, asymptotically stable and approach the stable periodic steady modes.

Remark 1: According to [11], in a generic one-parametric family of T_0-periodic functions $F(x,t)$, the set of functions with rational $\frac{\omega(f)}{T_0}$ and non-degenerate cycles of f is open dense and the set of functions with irrational $\frac{\omega(f)}{T_0}$ is nowhere dense, but of positive measure. Lastly, the set of functions with rational $\frac{\omega(f)}{T_0}$ and degenerate cycles is nowhere dense and has zero measure.

III. Design of relay type controllers with time delay

A. Statement of the adaptive control problem

Consider the system

$$\dot{x}(t) = F(x,t) + u(t), \quad u(t) = -\alpha(t)\cdot \mathrm{sign}\,[x(t-1)]. \quad (10)$$

A real controller operates with unavoidable time delay. Here we develop a direct adaptive delay control of relay type $u(t) = -\alpha \cdot \mathrm{sign}\,[x(t-1)]$ with a step function α depending on the only information on the time interval $(-1; t-1)$, giving an exponential decay of oscillations even in the presence of disturbances.

Note that for small α we lose restriction (2), and the solutions to system (10) can be unbounded and inextensible to the infinite interval. On the other hand, there are steady modes with sufficiently large frequency and small magnitude. It turns out that the existence of stable *zero frequency* steady modes implies the existence of a wide class of bounded solutions. Namely,

Proposition 3: Let $\alpha = \mathrm{const} > 0$ and

$$F(0,t) \equiv 0 \quad (11)$$

$$\frac{F(x,t)}{x} \leq k < \log 2, \quad x \neq 0, \ t \in \mathbf{R} \ . \quad (12)$$

Then all the solutions of the equation

$$\dot{x}(t) = F(x,t) - \alpha \cdot \mathrm{sign}\,[x(t-1)] \quad (13)$$

with initial condition (7), where

$$|x(0)| = |\varphi(0)| < \alpha \frac{2e^{-k} - 1}{k}, \quad (14)$$

are extensible to the interval $(-1; \infty)$ and satisfy the inequalities

$$|x_\varphi(t)| \leq \frac{\alpha}{k}(e^k - 1), \quad |\dot{x}_\varphi(t)| \leq \alpha e^k. \quad (15)$$

B. Scalar system with known perturbation

Let $F(x,t)$ satisfy (12). Assume that we know the function $F(x,t)$ and have an observer, which indicates zeros of $x(t)$ and signs of $x(t)$ with delay 1. We design the desired control by means of the following algorithm.

Let (14) hold for some constant $\alpha = \alpha_0$. Put $\alpha(t) = \alpha_0$, $t \geq 0$, and consider the equation

$$\dot{x}(t) = -\alpha_0 \cdot \mathrm{sign}\ [x(t-1)] + F(x(t), t), \quad t \geq 0.$$

We fix a time moment $t_1 + 1$, when the observer indicates the first zero t_1 of $x(t)$ greater than 1. Using the distribution of zeros and signs of $x(t)$ on the segment $[0; t_1]$, we extrapolate $x(t)$ on the interval $t > t_1$ and compute the first zero t_2 of $x(t)$ greater than $t_1 + 1$. Now in the ideal situation we can put

$$\alpha(t) = \alpha_1, \quad t \geq t_2 \ ,$$

where α_1 is an arbitrary small positive constant, and, according to (15), we obtain a solution $x(t)$ which lies in a prescribed neighborhood of zero.

Assume now that we compute the zero t_2 with error δ. Let δ satisfy the condition

$$\rho \stackrel{\mathrm{def}}{=} \frac{e^{k\delta} - 1}{2e^{-k} - 1} < 1 \Longleftrightarrow \delta < \frac{\ln 2}{k} - 1. \quad (16)$$

Notice that if T is a zero of some solution $x(t)$ of (13), and $|T^* - T| < \delta$, then

$$|x(T^*)| < \alpha(e^{k\delta} - 1)/k \ .$$

From this it follows immediately that the considered solution satisfies (14) at point t_2 with constant $\alpha = \alpha_0 \rho$. Now we put $\alpha(t) = \alpha_0 \rho$, $t \geq t_2$ and repeat our algorithm from the beginning. After m steps we get from (15)

$$|x(t)| \leq \frac{e^k - 1}{k} \alpha_0 \rho^m \ . \quad (17)$$

The left hand side of (17) tends to zero as $m \to \infty$.

C. Scalar system with unknown perturbation

Having error δ_0 of the observer and property (12) as the only information on $F(x,t)$, we still can apply the previous algorithm, provided, we know how to construct the zero sequence on an interval $(t;\infty)$ having a zero sequence on $(-1;t-1)$.

1) In the autonomous case Theorems 1 and 2 state that almost all bounded solutions of the equation

$$\dot{x}(t) = -\alpha \cdot \text{sign}\,[x(t-1)] + F(x(t))$$

turn into zero frequency steady modes. Assume that by the time moment $t_{2n}+1$ our observer indicated successive zeros $t_0, t_1, \dots, t_{2n}$ such that $t_i + 1 < t_{i+1}$, $i = 0 \cdots 2n-1$. According to the periodicity of steady modes (see Proposition 2), the following zero equals $t_{2n+1} = t_{2n-1} + (t_{2n} - t_0)/n > t_{2n} + 1$ with error $\delta = \delta_0(1 + 2/n)$. If δ satisfies (16), by repeating such steps, we stabilize the zero solution as above.

2) In the periodic case (4), by Theorem 2 almost any bounded solution of (13) turns into some zero frequency steady mode for every $\alpha > 0$, as far as

$$\sup\left|\frac{\partial F}{\partial t}(x,t) \cdot x^{-1}\right| < 2(2e^{-k} - 1)^2 \ .$$

For further estimates we use the following simple consequence of inequality (12)

Lemma 1: Let $F(x,t)$ satisfy (2), (11), (12). If g_1, g_2 are zero frequency steady modes such that

$$g_1(t_1) = g_1(t_2) = 0, \quad g_1(t) > 0, \quad t \in (t_1;t_2),$$

$$g_2(t_1') = g_2(t_2') = 0, \quad g_2(t) > 0, \quad t \in (t_1';t_2')$$

then

$$|t_2' - t_2| \le |t_1' - t_1| \cdot \xi(p,k), \quad \xi(p,k) =$$

$$= 1 + \alpha\left(\frac{2}{\alpha+p} + \frac{2\alpha - 2p}{(\alpha+p)^2}e^k\right) \cdot \exp\left(k\frac{\alpha+p}{\alpha-p}\right) \ .$$

In our situation by (11), (12), (15)

$$|F(x,t)| \le \sup\left|\frac{\partial F}{\partial x}\right| \cdot \sup|x| \le \alpha(e^k - 1).$$

Hence we have $p = \alpha(e^k - 1)$ and, by Lemma 1,

$$\xi = \xi(p,k) = 1 + 2e^{-k}(3 - e^k) \cdot \exp\left(\frac{ke^k}{2 - e^k}\right) \ .$$

Let us fix some integer $n > 0$. Suppose that the observer gave us two successive zeros t_0, t_1 of $x(t)$ such that $t_0 + 1 < t_1$. That means $x(t)$ coincides with some zero frequency steady mode for $t \ge t_0$. We consider the projections $\tilde{t}_0, \tilde{t}_1, \dots$ of t_0, t_1 and the following zeros of $x(t)$ on circle S of length T_0 (see section II). It is easy to see that there are $r < s < n$ such that

$$|\tilde{t}_{2s} - \tilde{t}_{2r}| \le \frac{T_0}{n} \ .$$

According to the periodicity of $F(x,t)$ and Lemma 1 we obtain the following zero $t_{2r+1} = t_{2r} + t_{2s+1} - t_{2s} > t_{2r} + 1$ with error

$$\delta = \delta_0 + \frac{T_0}{n} \cdot \xi \ .$$

If δ satisfies (16) we can realize our algorithm by iterating the step described above.

D. Two-dimensional triangular system

The above control algorithm applies to quench oscillations in system (5), (6), provided

$$f(0) = g(0,0) = 0, \ \frac{f(x)}{x} < \log 2, \ \frac{g(0,y)}{y} < \frac{\log 2}{\gamma} \ .$$

By means of control elements $-\alpha(t)\text{sign}[x(t-1)]$ and $-\beta(t)\text{sign}[y(t-\gamma)]$ in the right-hand side of (5), (6), first, we quench oscillations in equation (5) using the process described in sections III-B, III-C, second, we quench oscillations in equation (6) which then becomes close to an autonomous with respect to y.

Conclusions

1. The steady modes studied in this paper have similar properties to those of sliding modes [18]:

- the set of switches for any steady mode is unbounded and thus, a steady mode is not equivalent to any solution of one of the continuous parts of the given equation;
- for any solution there exists a finite time input into a steady mode;
- the shift operator is not invertible;
- the previous three properties are invariant with respect to bounded perturbations which satisfy conditions (2).

2. The instability of steady modes with non-zero frequency is established for a wide class of systems (1).

3. Two types of stability in periodic systems (1) are observed.

4. A direct adaptive control of relay type with time delay that extinguishes parasite auto-oscillations is designed.

IV. Appendix. Proofs

Proof of Theorem 2. We shall show that the set $\mathcal{U}_0$ is dense, and, thus, by Theorem 1 we obtain the nowhere density of $\bigcup_{0<n\le\infty} \mathcal{U}_n$ in $C[-1,0]$.

Fix even $N > 0$. Put

$$\Sigma = \{(a_0, \dots, a_N) \in R^{N+1} \ : \ a_0 \ge 0, \dots, a_N \ge 0$$

$$a_0 + \ldots + a_N = 1\} \ .$$

Let $Z_\varphi \cap [T;+\infty)$ be locally finite, and

$$T = t_1 < t_2 < t_3 < \ldots$$

be all zeros of $x_\varphi(t)$ in $[T;+\infty)$. We define the operators of "step forward" and "step backward". Assume that $\nu_\varphi(t_k) = \nu_\varphi(t_{k+1}) = N$. Define the following vectors of sign changes: $\overline{a} = (a_0, \dots, a_N), \overline{b} = (b_0, \dots, b_N) \in \Sigma$, where

$$a_0 = t_k - t_{k-1}, \quad a_1 = t_{k-1} - t_{k-2}, \dots,$$

$$a_{N-1}=t_{k-N+1}-t_{k-N},\, a_N=t_{k-N}-(t_k-1)$$
$$b_0=t_{k+1}-t_k, b_1=t_k-t_{k-1},\dots,b_{N-1}=t_{k-N+2}-t_{k-N+1},$$
$$b_N=t_{k-N+1}-(t_{k+1}-1).$$

Thus we obtain a correspondence

$$\Gamma:(\overline{a},\alpha,\varepsilon)\to(\overline{b},\beta,-\varepsilon)\ ,$$

where $\alpha=t_k,\ \beta=t_{k+1},\varepsilon=\text{sign}\ \dot{x}_\varphi(t_k)$.

Lemma 2: For a fixed ε, the correspondence inverse to Γ, is a smooth map

$$M_\varepsilon:\Sigma\times R\to\Sigma\times R.$$

Proof. Denote by $x_\varepsilon(t_0,x_0,a)$, $\varepsilon=\pm1$, the solution of the Cauchy problem

$$\frac{dx}{da}=\varepsilon+F(x,t_0+a),x(0)=x_0.$$

Define functions $T=\lambda_\varepsilon(t,a),\varepsilon=\pm1$, by the equations

$$x_{-\varepsilon}(t+a,x_\varepsilon(t,0,a),b)=0,\quad T=t+a+b. \tag{18}$$

It is easy to see that for a fixed t_0, the function $\lambda_\pm(t_0,a)$ strongly increases, and $\lambda_\pm(t_0,a)>a$ if $a>0$. Therefore, for a fixed t_0, we can define positive functions of $b>0$:

- $\rho_\varepsilon(t_0,b)$ inverse to $b=\lambda_\varepsilon(t_0,\rho_\varepsilon)$;
- $\sigma_\varepsilon(t_0,b)=b-\rho_\varepsilon(t_0,b)$.

Hence $(\overline{a},\alpha)=M_\varepsilon(\overline{b},\beta)$ can be defined as

$$a_0=b_1,\quad a_1=b_2,\dots,a_{N-2}=b_{N-1}\ ,$$
$$a_{N-1}=b_N+\sigma_\varepsilon(\beta-b_0,\ b_0),\ a_N=\rho_\varepsilon(\beta-b_0,b_0),$$
$$\alpha=\beta-b_0\ . \tag{19}$$

Lemma 2 defines the operator of step backward with a constant frequency (in fact, independently of the initial assumption $\nu_\varphi(t_k)=\nu_\varphi(t_{k+1})=N$).

We shall also use the following two auxiliary claims.

Lemma 3: If

$$a\le(1+p)/2 \tag{20}$$

and either (8), or (9) is fulfilled, then

$$\frac{\partial\lambda_{\pm1}}{\partial a}(t,a)\ge q,\quad q=\text{const}>1 \tag{21}$$

for sufficiently large t.

Proof. We start with the formula

$$\frac{\partial\lambda_\varepsilon}{\partial a}(t,a)=1+(1-\varepsilon F(0,T))^{-1}\exp\left(\int_{t+a}^{T}\frac{\partial F}{\partial x}(x_{-\varepsilon},t)dt\right)$$
$$\times\left(1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt\right)\ ,$$

where $T=\lambda_\varepsilon(t,a)$. Since $1-p\le|\dot{x}_{-\varepsilon}|\le1+p$, we have $|x_{-\varepsilon}(t,0,t+a)|\le a(1+p)\le(1+p)^2/2$. Hence

$$T-(t+a)\le\frac{(1+p)^2}{2(1-p)}\ , \tag{22}$$

and by (8)

$$\left|\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt\right|\le1-p-c_1$$

where $c_1=\text{const}>0$, as $t>>0$. Therefore

$$1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt$$
$$\ge c_1>0\ .$$

In view of

$$\frac{dF(x_\varepsilon,t)}{dt}=\frac{\partial F(x_\varepsilon,t)}{\partial t}+\frac{\partial F(x_\varepsilon,t)}{\partial x}\dot{x}_\varepsilon$$

and (8), (22) we have for $t>>0$,

$$\int_{t+a}^{T}\frac{\partial F}{\partial x}(x_{-\varepsilon},t)dt=\int_{t+a}^{T}\left(\frac{dF}{dt}-\frac{\partial F}{\partial t}\right)\cdot(\dot{x}_{-\varepsilon})^{-1}dt$$
$$=\int_{t+a}^{T}\frac{dF/dt}{-\varepsilon+F(x_{-\varepsilon},t)}dt-\int_{t+a}^{T}\frac{\partial F/\partial t}{-\varepsilon+F(x_{-\varepsilon},t)}dt$$
$$\ge-\log\frac{1+p}{1-p}-\frac{1}{1-p}\int_{t+a}^{T}\frac{\partial F}{\partial t}dt\ge-\log\frac{1+p}{1-p}-1\ .$$

This altogether implies (21).

Similarly, assuming (9), one derives for $t>>0$ that

$$\left|\int_{t+a}^{T}\frac{\partial F}{\partial x}\cdot\dot{x}_{-\varepsilon}dt\right|\le1-p-c_2,$$

where $c_2=\text{const}>0$. Hence

$$1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt$$
$$=1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{dF}{dt}dt-\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial x}\cdot\dot{x}_{-\varepsilon}dt$$
$$\ge1+\varepsilon\cdot F(0,T)-(1-p-c_2)\ge c_2>0\ ,$$

in view of (8) and (22), which as before gives (21).

Lemma 4: Under the conditions of Theorem 2 the measure of the set $\Pi=\Pi_0\cap\Pi_1\cap\Pi_2\cap\dots$, where

$$\Pi_0=\Sigma\times R,\quad\Pi_{n+1}=(M_-\circ M_+)(\Pi_n),n\ge0,$$

is zero.

Proof. First we show that any $\overline{a}=(a_0,\dots,a_N)=M_\varepsilon(\overline{b}),\overline{b}\in\Sigma$, satisfies $a_N\le(1+p)/2$. Indeed, we have $a_N\le a_{N-1}(1+p)/(1-p)$, which implies the above inequality.

By (19) the Jacobian $|M'_\varepsilon|$ of the map M_ε is equal to

$$\left.\frac{\partial\rho_\varepsilon}{\partial b}(t,b)\right|_{t=\alpha,b=b_0}=\left(\left.\frac{\partial\lambda_\varepsilon}{\partial a}(t,a)\right|_{t=\alpha,a=a_N}\right)^{-1}\le\frac{1}{q}<1$$

according to Lemma 3. Then

$$|(M_-\circ M_+)'|\le q^{-2}<1. \tag{23}$$

Fix $A \in R$ and $T > A$. Then

$$\Pi \cap (\Sigma \times (-\infty; A]) \subset \bigcup_{k \geq n} (M_- \circ M_+)^k (\Sigma \times [T; T+1]) ,$$

where n might be chosen large enough, because $T > A$ is arbitrary. Thus, we obtain from (23)

$$\text{mes}(\Pi \cap (\Sigma \times (-\infty; A])) \leq q^{-2(n-1)} \frac{\text{mes}(\Sigma)}{q^2 - 1} \stackrel{n \to \infty}{\longrightarrow} 0,$$

which completes the proof.

Now we can finish the proof of Theorem 2. Fix $\varphi \in \mathcal{U}_n$ and a neighborhood V of φ in $C[-1; 0]$. Introduce the following dense subset in $C[-1; 0]$:

$$\mathcal{F} = \{\varphi \in C[0, 1] \ : \ \text{card}(\varphi^{-1}(0)) < \infty\} .$$

Put

$$m = \min\{k \ : \ \mathcal{F} \cap \mathcal{U}_k \cap V \neq \emptyset\}.$$

Assume $m \geq 1$, and $\psi \in \mathcal{F} \cap \mathcal{U}_m \cap V$. Then there exists $\xi \in \mathcal{U}_m^{sm}$ such that $x_\psi(t) = \xi(t)$, $t \geq T$, $\xi(T) = 0$. Let $2k$ be the number of sign changes of ψ in $[-1; 0]$, and let $\overline{a} \in \Sigma_k \subset R^{2k+1}$ be a vector of the sign changes of ψ, constructed as above, and $\overline{b} \in \Sigma_m \subset R^{2m+1}$ be a vector of the sign changes of ξ in $(T-1; T)$. Suppose $\overline{c} \in \Sigma_t, \overline{d} \in \Sigma_s$ are vectors of the sign changes of $x_\psi(t)$ in the intervals $(t_n - 1; t_n)$ and $(t_{n+1} - 1; t_{n+1})$, respectively. If $r = s$ then, according to Lemma 2, equation (1) generates a diffeomorphism of neighborhoods of $(\overline{c}, t_n)$, $(\overline{d}, t_{n+1})$ in $\Sigma_r \times R$. If $r < s$, it is possible to deduce, following arguments from the proof of Lemma 2, that

$$c_0 = d_1, \ldots, c_{2s-1} = d_{2s}, \ c_{2r} = \Lambda\left(d_0, c_{2s}, \ldots, c_{2r-2}, t_{n+1}\right),$$

$$c_{2r-1} = 1 - c_0 - \ldots - c_{2r-2} - c_{2r}, \ t_n = t_{n+1} - d_0,$$

where Λ is some smooth function. Hence an inverse image of $\left(\overline{d}, t_{n+1}\right)$ in a neighborhood of $(\overline{c}, t_n)$ in $\Sigma_r \times R$ has codimension $2s+1$. This implies that the measure of the inverse image of $\Pi \cap (\Sigma_m \times R)$ in $\Sigma_k \times R$ is zero. Therefore, after a suitable small variation of $(\overline{a}, 0)$ in $\Sigma_k \times R$, the image of $(\overline{a}, 0)$ in $\Sigma_m \times R$ leaves Π, i.e., the limit frequency of the changed solution is less than $2m$, which contradicts the definition of m, and hence our assumption $m > 0$.

Thus, we get that $\mathcal{U}_0 \cap \mathcal{F}$ is dense in $\mathcal{F}$, and also in $C[-1; 0]$, because $\mathcal{F}$ is dense in $C[-1; 0]$. According to Theorem 1, this means that $\mathcal{U}_\infty \cup \bigcup\limits_{k \geq 1} \mathcal{U}_k$ is nowhere dense in $C[-1; 0]$.

The first two statements of Theorem 2 are done.

For the third statement of Theorem 2 we modify the previous argument as follows.

Lemma 5: The function $\lambda_\varepsilon(t, a)$, defined above, satisfies

$$\frac{\partial \lambda_\varepsilon}{\partial a} \geq \frac{2}{1+p} \exp\left(-a \cdot \min\{\mu_1^{(0)}; \mu_2^{(0)}\}\right) .$$

The Proof is based on the following well-known formula: If $w(z, w_0, z_0)$ is the solution of the Cauchy problem

$$\frac{dw}{dz} = \Phi(w, z), \quad w(z_0) = w_0 ,$$

where

$$\left|\frac{\partial \Phi}{\partial w}\right| \leq \beta ,$$

then

$$\frac{\partial w(z, w_0, z_0)}{\partial w_0} \geq \exp(-\beta \cdot |z - z_0|) . \tag{24}$$

Now from (24) and (2) it is not difficult to derive that

$$\frac{\partial \lambda_\varepsilon(t_0, a)}{\partial a} \geq \frac{2}{1+p} \cdot \frac{\partial w\left(0, a + t_0, x_\varepsilon(a, 0, t_0)\right)}{\partial w_0} , \tag{25}$$

where $w = w(x, w_0, x_0)$ is the solution of the Cauchy problem

$$\frac{dw}{dx} = \frac{1}{-\varepsilon + F(x, w)}, \quad w(x_0) = w_0 ,$$

and then, using (24) and the inequality

$$|x_\varepsilon(a, 0, t_0)| \leq a(1+p) ,$$

we obtain from (25) that

$$\frac{\partial \lambda_\varepsilon}{\partial a} \geq \frac{2}{1+p} \exp(-a\mu_2^{(0)}) .$$

On the other hand, (24) and (2) imply that

$$\frac{\partial \lambda_\varepsilon}{\partial a} \geq \frac{2}{1+p} \frac{\partial x_{-\varepsilon}(t_0 + \lambda_\varepsilon, x_\varepsilon(a, 0, t_0), t_0 + a)}{\partial x_0}$$

$$\geq \frac{2}{1+p} \exp\left(-\sup\left|\frac{\partial F}{\partial x}\right| \cdot |\lambda_\varepsilon - a|\right) \geq \frac{2}{1+p} \cdot \exp\left(-a\mu_1^{(0)}\right) .$$

Lemma 6: Under the conditions of Theorem 2(3), if $n > \theta$, then the Jacobian M' of the map $M = (M_+ \circ M_-)^{N+1}$, $N = 2n$, defined above, satisfies the inequality

$$|M'| \leq q < 1, \quad q = \text{const} . \tag{26}$$

Proof. Let $\overline{a} = (a_0, ..., a_N) = M_\varepsilon(\overline{b})$, $\overline{b} \in \Sigma$. Then

$$|M'_\varepsilon| = \frac{\partial \rho_\varepsilon}{\partial b}(t, b)\Big|_{t=\tau_0, b=b_0} = \left(\frac{\partial \lambda_\varepsilon}{\partial a}(t, a)\Big|_{t=\tau_0, a=a_N}\right)^{-1}$$

$$\leq \frac{1+p}{2} \exp\left(a_N \cdot \min\{\mu_1^{(0)}; \mu_2^{(0)}\}\right) . \tag{27}$$

Hence

$$|\left((M_+ \circ M_-)^n \circ M_+\right)'| = \left(\frac{\partial \lambda_\pm}{\partial a}(t, a)\Big|_{t=\tau_0, a=a_N}\right)^{-1}$$

$$\times \prod_{i=0}^{N-1} \left(\frac{\partial \lambda_\pm}{\partial a}(t, a)\Big|_{t=t_i, a=a'_i}\right)^{-1} ,$$

where $0 \leq a'_i < a_i$, $i = 0, ..., N-1$. Finally, this implies, according to (27), that

$$|\left((M_+ \circ M_-)^n \circ M_+\right)'| \leq \left(\frac{1+p}{2}\right)^{2n+1}$$

$$\times \exp\left((a_N + a_{N-1} + \ldots + a_0) \cdot \min\{\mu_1^{(0)}; \mu_2^{(0)}\}\right)$$

$$= \left(\frac{1+p}{2}\right)^{N+1} \exp(\min\{\mu_1^{(0)}; \mu_2^{(0)}\}) \ ,$$

$$|M'| \leq \left(\frac{1+p}{2}\right)^{2N+2} \exp(2\min\{\mu_1^{(0)}; \mu_2^{(0)}\}) = q < 1 \ ,$$

since the last inequality is equivalent to $n > \theta$.

Lemma 7: Under the conditions of Theorem 2(3), if $n > \theta$, then the measure of the set Π, defined in Lemma 4, is zero.

Proof. Fix $A \in \mathbf{R}$ and $T > A$. Then

$$\Pi \cap (\Sigma \times (-\infty; A]) \subset \bigcup_{k \geq s} M^k(\Sigma \times [T; T+1]) \ ,$$

where s can be chosen large enough, because $T > A$ is arbitrary. Thus, we obtain from (26) that

$$\text{mes}\,(\Pi \cap (-\infty; A])) \leq \frac{q^s}{1-q} \cdot \text{mes}(\Sigma) \longrightarrow 0$$

as $s \to \infty$, which completes the proof of the Lemma.

Now one can finish the proof of the third statement of Theorem 2 as was done above for the second statement.

Proof of Proposition 3. Condition (12) means that if $x(t)$ is a solution of (1) then, for $x(T) \geq 0, x(t) \leq \omega(t), t \geq T$, where $\omega(t) = (\alpha + kx(T))\exp(k(t-T)) - \alpha)/k$ is the solution of the Cauchy problem,

$$\dot{\omega}(t) = \alpha + k\omega(t), \quad \omega(T) = x(T),$$

and, for $x(T) \leq 0, x(t) \geq \omega(t), t \geq T$, where $\omega(t) = (-\alpha + kx(T))\exp(k(t-T)) + \alpha)/k$ is the solution of the Cauchy problem,

$$\dot{\omega}(t) = -\alpha + k\omega(t), \quad t \geq T \ .$$

These inequalities and (12) imply that $|F(x,t),t)| < \alpha$ as $t \in [0,1]$, and $x(0) = \varphi(0)$ satisfies (14), and that $x(t)$ satisfies (15) as $t \in [T, T+1], x(T) = 0$. Hence $x(t)$ does not leave the strip $|x| \leq \alpha(e^k - 1)/k$ for $t \leq T$.

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