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Simple LMIs for stability of stochastic systems with delay term given by Stieltjes integral or with stabilizing delay ☆

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ABSTRACT

This paper introduces stability conditions in the form of linear matrix inequalities (LMIs) for general linear retarded systems with a delay term described by Stieltjes integral. The derived LMIs provide in the unified form conditions for both discrete and distributed delays. Two Lyapunov-based methods for the asymptotic mean square stability of stochastic linear time-invariant systems are presented. The first one employs neutral type model transformation and augmented Lyapunov functionals. Differently from the existing LMI stability conditions based on neutral type transformation, the proposed conditions do not require the stability of the corresponding integral equations. Moreover, it is shown that in the simplest existing LMIs based on non-augmented Lyapunov functionals, the stability analysis of the integral equation can be omitted. The second method is based on a stochastic extension of simple Lyapunov functionals depending on the state derivative. The same two methods are further applied to delay-induced stability analysis of stochastic vector second-order systems, simplifying the recent results via neutral type transformation and leading to new conditions for stochastic systems via the second method. Numerical examples give comparison of results via different methods.

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1. Introduction

Construction of simple Lyapunov functionals that provide delay-dependent conditions for linear time-delay systems originates from the presentation of the delayed state in the form of the non-delayed one plus perturbation [1–3]. Thus, for systems with a constant discrete delay $h > 0$ there are two main presentations. The first one

$$x(t - h) = x(t) - \frac{d}{dt}\int_{t-h}^{t} x(s)ds$$

leading to neutral type model transformation and to Lyapunov functionals that depend on the state x only. However, in the existing results [1,3–6], additionally to the positivity of these functionals and to the negativity of their derivatives along the neutral type system, one has to guarantee the stability of the corresponding integral equation. In the present paper we will show that the LMI conditions for the positivity of these functionals and for the negativity of their derivatives guarantee the asymptotic stability of the system, whereas the condition on the stability of the integral

equation can be omitted. The second presentation

$$x(t - h) = x(t) - \int_{t-h}^{t} \dot{x}(s)ds$$

leads to Lyapunov functionals that depend on the state derivative $\dot{x}$. This approach is applicable to systems with fast-varying delays (without any constraints on the delay derivative) [1].

Differently from the deterministic case, the solution of a stochastic differential equation does not have a derivative. Therefore, in the stochastic case it is not possible to use directly Lyapunov functionals that depend on the state derivative. That is why, for stochastic case, either functionals depending on the state [2,4,7,8] or *stochastic extension* of Lyapunov functionals depending on $\dot{x}$ [9–11] have been used. The stochastic extension of Lyapunov functionals depending on $\dot{x}$ employs Lyapunov functionals that *depend on the deterministic part* of $\dot{x}$ (i.e. on the deterministic part of the right-hand side of differential equation). Also input–output approach to the stability of retarded systems with multiplicative noise has been employed [12].

In this paper a linear time-invariant stochastic system is considered with the delay term described by Stieltjes integral (that can be interpreted either as discrete or distributed delay). We develop two methods for the asymptotic mean square stability analysis of linear stochastic systems. The first one employs a neutral type model transformation and the corresponding augmented Lyapunov functional. Note that augmented Lyapunov functionals have been applied directly to the original system in the case of

deterministic system with distributed or discrete delay in [13] and in the case of stochastic system with discrete delays in [7]. However, our application of augmented Lyapunov functional via neutral type transformation allows to derive the simplest LMI stability conditions. Our second method is based on stochastic extension of simple (non-augmented) Lyapunov functionals depending on $\dot{x}$ and employs novel Lyapunov functionals.

Finally, delay-induced stability of stochastic vector second-order systems is analyzed by using the same two methods. Here the same conclusion on the simplification of the existing LMI conditions based on simple Lyapunov functionals via neutral type transformation [5,14]: the stability condition for the integral equation may be omitted. Moreover, for the first time, conditions for the asymptotic mean square stability are derived via the second method.

The efficiency of the results is illustrated by numerical examples. As expected, augmented Lyapunov functionals improve results based on simple Lyapunov functionals (both, depending on x or stochastic extension of the functionals depending on $\dot{x}$), though for the case of delay-induced stability, the improvement is less essential. Moreover, for large stochastic perturbations, the simplest (minimal-size) LMIs via neutral type transformation lead to slightly more conservative results than the ones via augmented Lyapunov functional and improve the results via the second method.

The presented stability conditions are the first LMIs for systems with Stieltjes integral (even in the deterministic case). They provide in the unified form conditions for the discrete and distributed delay, which is different from the existing results that derive separately conditions for discrete and for distributed delays (see e.g. [7,10,15–18]). Some preliminary results on stability of stochastic systems with a delay term described by Stieltjes integral were presented in [19].

1.1. Necessary notations, definitions and statements

Throughout the paper the superscript "'" stands for matrix transposition, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive definite, I_n is the identity $n \times n$-matrix. The symmetric elements of the symmetric matrix are denoted by $*$.

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a probability space, $\{\mathfrak{F}_t, t \geq 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e., $\mathfrak{F}_s \subset \mathfrak{F}_t$ for $s < t$, $\mathbf{P}\{\cdot\}$ be the probability of an event enclosed in the braces.

The mathematical expectation $\mathbb{E}$ of a random variable $\xi = \xi(\omega)$ on the probability space $\{\Omega, \mathfrak{F}, \mathbb{P}\}$ is defined as $\mathbb{E}\xi = \int_{\Omega} \xi(\omega) \mathbf{P}(d\omega)$. If a random variable $X \in \mathbb{R}^n$ is defined by a density of distribution $f(x)$ then the mathematical expectation $\mathbb{E}$ of a random variable X is defined as $\mathbb{E}X = \int_{\mathbb{R}^n} x f(x) dx$.

The standard (one-dimensional) Wiener process (also called Brownian motion) is a stochastic Gaussian process $w(t)$ with the density of distribution $f(x) = \frac{1}{\sqrt{2\pi t}} \exp\{-x^2/2t\}, t > 0, w(0) = 0$, $\mathbb{E}w(t) = 0, \mathbb{E}w^2(t) = t$. A n-dimensional Wiener process is a vector-valued stochastic process $w(t) = (w_1(t), \ldots, w_n(t))$ whose components are independent, standard one-dimensional Wiener processes.

Let H_2 be the space of $\mathfrak{F}_0$-adapted stochastic processes $\varphi(s), s \leq 0, \|\varphi\|_0 = \sup_{s \leq 0} |\varphi(s)|, \|\varphi\|^2 = \sup_{s \leq 0} \mathbb{E}|\varphi(s)|^2$. Following [20,21], we will consider the Ito stochastic functional differential equation

$$dx(t) = \alpha(t, x_t)dt + \beta(t, x_t)dw(t), \quad t \geq 0,$$
$$x_0 = \phi \in H_2. \tag{1.1}$$

Here $x(t) \in \mathbb{R}^n$ is a value of the solution of Eq. (1.1) in the time moment t, $x_t = x(t + s), s < 0$, is a trajectory of the solution of Eq. (1.1) until the time moment t, $w(t) \in \mathbb{R}^m$ is the $\mathfrak{F}_t$-adapted Wiener process, the continuous functionals $\alpha(t, \varphi) \in \mathbb{R}^n$,

$\beta(t, \varphi) \in \mathbb{R}^{n \times m}$. It is supposed also that Eq. (1.1) has the zero solution, i.e., $\alpha(t, 0) \equiv 0, \beta(t, 0) \equiv 0$.

A solution of (1.1) with the initial condition $x(s) = \phi(s)$ for $s \leq 0$ is a stochastic process $x(t), t \in \mathbb{R}^n$ that satisfies the initial condition for $t \leq 0$ and for $t \geq 0$ with probability 1 satisfies the equation

$$x(t) = \phi(0) \quad + \int_0^t \alpha(s, x_s)ds + \int_0^t \beta(s, x_s)dw(s).$$

Consider a functional $V(t, \varphi) : [0, \infty) \times H_2 \to \mathbb{R}_+$ that can be presented in the form $V(t, \varphi) = V(t, \varphi(0), \varphi(s)), s < 0$, and for $\varphi = x_t$ put

$$V_\varphi(t, x) = V(t, \varphi) = V(t, x_t) = V(t, x, x(t + s)),$$
$$x = \varphi(0) = x(t), \quad s < 0. \tag{1.2}$$

Denote by D the set of the functionals, for which the function $V_\varphi(t, x)$ defined in (1.2) has a continuous derivative with respect to t and two continuous derivatives with respect to x. Let ∇ and ∇^2 be respectively the first and the second derivatives of the function $V_\varphi(t, x)$. For the functionals from D the generator L of Eq. (1.1) has the form [8,20]

$$LV(t, x_t) = \frac{\partial}{\partial t} V_\varphi(t, x(t)) + \nabla V'_\varphi(t, x(t))\alpha(t, x_t)$$
$$+ \frac{1}{2} Tr[\beta'(t, x_t)\nabla^2 V_\varphi(t, x(t))\beta(t, x_t)]. \tag{1.3}$$

It is known that the trajectories of the Wiener process are continuous with probability 1, but not differentiable at any point functions. So, solutions of the stochastic differential equation (1.1) are also continuous with probability 1 but not differentiable functions. This is the reason why stochastic differential equations even if sometimes written in the form of derivatives always are understood in the form of differentials. This is also the reason why differently from the deterministic case it is impossible to use directly Lyapunov functionals that depend on the derivative of the solution of the considered stochastic differential equation.

2. Problem formulation

Consider the Ito stochastic differential equation [20,21]

$$\dot{x}(t) = Ax(t) + A_1 \int_0^\infty x(t - s)dK(s)$$
$$+ Cx(t)\dot{w}(t), \quad t \geq 0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $A, A_1, C \in \mathbb{R}^{n \times n}$, $w(t)$ is the scalar standard Wiener process, $K(s)$ is a scalar right-continuous function of bounded variation on $[0, \infty)$ such that

$$k_i = \int_0^\infty s^i dK(s) < \infty, \quad i = 0, 1, \tag{2.2}$$

and the integral is understood in the Stieltjes sense. For an arbitrary continuous with probability 1 initial function ϕ there exists a unique solution of the linear equation (2.1) subject to (2.2), which is continuous with probability 1 [2,20,21]. Classical stability theory for systems of such type is represented in [2,8,22].

Definition 2.1. System (2.1) is mean square stable if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all initial functions with $\sup_{s \leq 0} \mathbb{E}|\phi(s)|^2 < \delta$ the solutions of (2.1) satisfy $\mathbb{E}|x(t)|^2 < \varepsilon$ for all $t \geq 0$. System (2.1) is asymptotically mean square stable if it is mean square stable and $\lim_{t \to \infty} \mathbb{E}|x(t)|^2 = 0$ for each initial function $\phi(s)$.

Our objective is to derive sufficient LMI conditions for the asymptotic mean square stability of (2.1). The general form of delay as in (2.1) includes the cases of discrete and distributed delays (depending on the concrete choice of the kernel $K(s)$). Put for example

$$dK(s) = \delta(s - h)ds,$$

where $\delta(s)$ is the Dirac function, $h > 0$. In this case (2.1) has the discrete delay

$$\dot{x}(t) = Ax(t) + A_1 x(t - h) + Cx(t)\dot{w}(t),$$
$$k_0 = 1, \ k_1 = h. \tag{2.3}$$

Putting

$$dK(s) = K_0(s)ds,$$

where $K_0(s)$ is an integrable function, satisfying the condition (2.2), we obtain the distributed delay

$$\dot{x}(t) = Ax(t) + A_1 \int_0^\infty x(t - s)K_0(s)ds$$
$$+Cx(t)\dot{w}(t), \quad k_i = \int_0^\infty s^i K_0(s)ds, \quad i = 0, 1. \tag{2.4}$$

In particular, $K_0(s) = 1$ for $s \in [0, h]$ and $K_0(s) = 0$ for $s > h$ then $k_0 = h, k_1 = \frac{1}{2}h^2$.

Note that (2.1) includes the case of several discrete and distributed delays multiplied from the left by the same matrix A_1. Our results can be easily extended to the case of several Stieltjes integrals

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^k A_j \int_0^\infty x(t - s)dK_j(s) + Cx(t)\dot{w}(t)$$

with different matrices $A_j \in \mathbb{R}^{n \times n}$ and different scalar kernels K_j.

We will derive stability conditions by using neutral type transformation via augmented Lyapunov functional and without model transformation via stochastic extensions of simple (non-augmented) Lyapunov functionals depending on $\dot{x}$. We will consider separately the case of Hurwitz $A + k_0 A_1$ (see Section 3) and the case of delay-induced stability, where $A + k_0 A_1$ is not Hurwitz (see Section 4).

Our results employ Jensen's inequalities that extend the inequalities of [23] to the case of Stieltjes integrals:

Lemma 2.1 (*Jensen's Inequality*). *For any $n \times n$ matrix $R > 0$, vector function $x : (-\infty, t)$ and scalar function of bounded variation $K : [0, \infty) \to \mathbb{R}$ such that the integrations concerned are well defined, the following inequalities hold with k_0 and k_1 defined by (2.2):*

$$z_0'(t)Rz_0(t) \leq k_0 \int_0^\infty x'(t - s)Rx(t - s)dK(s),$$
$$\text{for} \quad z_0(t) = \int_0^\infty x(t - s)dK(s) \tag{2.5}$$

and

$$z'(t)Rz(t) \leq k_1 \int_0^\infty \int_{t-s}^t x'(\tau)Rx(\tau)d\tau \, dK(s)$$
$$\text{for} \quad z(t) = \int_0^\infty \int_{t-s}^t x(\tau)d\tau \, dK(s). \tag{2.6}$$

Proof. The inequality (2.5) follows from the Cauchy–Schwarz inequality and (2.2):

$$z_0(t)'Rz_0(t) = |R^{1/2}z_0(t)|^2 = \left|\int_0^\infty R^{1/2}x(t - s)dK(s)\right|^2$$
$$\leq k_0 \int_0^\infty x'(t - s)Rx(t - s)dK(s).$$

Similarly for (2.6) we have

$$z(t)'Rz(t) = |R^{1/2}z(t)|^2 = \left|\int_0^\infty \int_{t-s}^t R^{1/2}x(\tau)d\tau \, dK(s)\right|^2$$
$$\leq k_1 \int_0^\infty \int_{t-s}^t x'(\tau)Rx(\tau)d\tau \, dK(s). \quad \square$$

3. Stability in the case of Hurwitz $A + k_0 A_1$

3.1. Stability via neutral type model transformation

Denote

$$z(t) = x(t) + G(t),$$
$$G(t) = \int_0^\infty \int_{t-s}^t A_1 x(\tau)d\tau \, dK(s). \tag{3.1}$$

Then

$$\dot{G}(t) = k_0 A_1 x(t) - G_0(t),$$
$$G_0(t) = \int_0^\infty A_1 x(t - s)dK(s). \tag{3.2}$$

By virtue of (3.1), (3.2) we use a neutral type model transformation of (2.1)

$$\dot{z}(t) = (A + k_0 A_1)x(t) + Cx(t)\dot{w}(t). \tag{3.3}$$

We assume that $A + k_0 A_1$ is Hurwitz. The standard approach to stability analysis of (3.3) includes construction of a Lyapunov functional $V(x_t)$ with the conditions

$$\mathbb{E}V(x_t) \geq c_1\mathbb{E}|z(t)|^2, \quad \mathbb{E}LV(x_t) \leq -c_2\mathbb{E}|x(t)|^2, \quad t \geq 0$$

that hold for some positive constants c_1 and c_2 provided the integral equation $z(t) = 0$ is asymptotically stable [2,8,22]. In the novel approach that we present in this paper, an appropriate augmented Lyapunov functional is constructed in the form $V(x_t) = V(x_t, G(t))$ subject to the conditions

$$\mathbb{E}V(x_t) \geq c_1\mathbb{E}|x(t)|^2, \quad \mathbb{E}LV(x_t) \leq -c_2\mathbb{E}|x(t)|^2,$$
$$t \geq 0 \tag{3.4}$$

for some positive c_1 and c_2. In this case, due to classical Lyapunov–Krasovskii theorem (see e.g. Theorem 2.1 of [8]), there is no need to verify the stability of $z(t) = 0$.

Proposition 3.1. *Given matrices $A, A_1, C \in \mathbb{R}^{n \times n}$ and a right-continuous scalar function $K(s)$ of the bounded variation on $[0, \infty)$ that satisfies (2.2).*

(i) Let there exist $n \times n$ matrices $P_1, P_2, P_3, R > 0$ and $S > 0$ that satisfy the following LMIs:

$$\Psi_1 = \begin{bmatrix} P_1 & P_1 + P_2 \\ * & P_1 + P_2 + P_2' + P_3 + \frac{k_0}{k_1}S \end{bmatrix} > 0 \tag{3.5}$$

and

$$\Phi_{1aug} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & P_2 \\ * & -R & P_2 + P_3 \\ * & * & -S \end{bmatrix} < 0,$$

$$\Phi_{11} = P_1(A + k_0 A_1) + (A + k_0 A_1)'P_1$$
$$+k_0(P_2 A_1 + A_1'P_2') + A_1'(k_0^2 S + k_1^2 R)A_1 + C'P_1 C,$$

$$\Phi_{12} = A'(P_1 + P_2) + k_0 A_1'(P_1 + P_2 + P_2' + P_3). \tag{3.6}$$

Then system (2.1) is asymptotically mean square stable.

(ii) If there exist $n \times n$ matrices $P_1 > 0$ and $R > 0$ that satisfy the LMI

$$\Phi_{1sim} = \begin{bmatrix} \phi_{1s} & (A + k_0 A_1)'P_1 \\ * & -R \end{bmatrix} < 0,$$
$$\phi_{1s} = P_1(A + k_0 A_1) + (A + k_0 A_1)'P_1$$
$$+k_1^2 A_1'RA_1 + C'P_1 C, \tag{3.7}$$

then system (2.1) is asymptotically mean square stable.

(iii) LMIs of items (i) and (ii) are feasible for small enough k_1 and C provided $A + k_0 A_1$ is Hurwitz.

(iv) The feasibility of (3.7) with $C = 0$ implies that the eigenvalues of $k_1 A_1$ are inside of the unit circle.

Proof (*i*). Let L be the generator of Eq. (3.3) [8,20,21]. Via (3.1), (3.2) for the functional

$$V_1(x_t) = \begin{bmatrix} z(t) \\ G(t) \end{bmatrix}' \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix} \begin{bmatrix} z(t) \\ G(t) \end{bmatrix} \tag{3.8}$$

we have

$$LV_1(x_t) = 2\begin{bmatrix} z(t) \\ G(t) \end{bmatrix}' \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix} \begin{bmatrix} (A + k_0 A_1)x(t) \\ k_0 A_1 x(t) - G_0(t) \end{bmatrix}$$
$$+x'(t)C'P_1 Cx(t). \tag{3.9}$$

Using the additional functional

$$V_2(x_t) = k_0 \int_0^\infty \int_{t-s}^t x'(\tau) A_1' S A_1 x(\tau) d\tau \, dK(s) \\ + k_1 \int_0^\infty \int_{t-s}^t (\tau - t + s) x'(\tau) A_1' R A_1 x(\tau) d\tau \, dK(s)$$

(3.10)

with $S, R > 0$ and Jensen's inequalities (2.5), (2.6)

$$G_0'(t) S G_0(t) \le k_0 \int_0^\infty x'(t-s) A_1' S A_1 x(t-s) dK(s),$$
$$G'(t) R G(t) \le k_1 \int_0^\infty \int_{t-s}^t x'(\tau) A_1' R A_1 x(\tau) d\tau \, dK(s)$$

we have

$$LV_2(x_t) \le x'(t) A_1' (k_0^2 S + k_1^2 R) A_1 x(t) \\ - G_0'(t) S G_0(t) - G'(t) R G(t).$$

(3.11)

Choose Lyapunov functional

$$V(x_t) = V(x_t, G(t)) = V_1(x_t) + V_2(x_t).$$

From (3.9), (3.11) it follows that $LV(x_t) \le \eta'(t) \Phi_{1aug} \eta(t)$, where $\eta(t) = col\{x(t), G(t), -G_0(t)\}$ and $\Phi_{1aug} < 0$ is defined in (3.6). Moreover, by Jensen's inequality

$$V_2(x_t) \ge k_0 \int_0^\infty \int_{t-s}^t x'(\tau) A_1' S A_1 x(\tau) d\tau \, dK(s) \ge \frac{k_0}{k_1} G'(t) S G(t)$$

that implies due to (3.6)

$$V(x_t) \ge \begin{bmatrix} x(t) \\ G(t) \end{bmatrix}' \Psi_1 \begin{bmatrix} x(t) \\ G(t) \end{bmatrix} \ge c_1 |x(t)|^2$$

with $c_1 > 0$ since $\Psi_1 > 0$. Therefore, conditions (3.4) hold and (2.1) is asymptotically mean square stable.

(ii) If LMI (3.7) holds with $P_1 > 0$ and $R > 0$, then (3.5) and (3.6) hold with the same P_1, R and $P_2 = P_3 = 0$ and any $S > 0$. Thus, the result follows from (i).

(iii) For Hurwitz $A + k_0 A_1$, let $P_1 > 0$ be such that

$$P_1(A + k_0 A_1) + (A + k_0 A_1)' P_1 < 0.$$

Then, by Schur complements, (3.7) with $k_1 = 0$ is feasible with this P_1 and $R = \rho I$, where the scalar $\rho > 0$ is large enough. Hence, (3.7) holds also for small enough $k_1 > 0$.

(iv) The proof follows arguments of Remark 5 from [24]. Denote $\bar{P} = -P_1(A + k_0 A_1)$. By Schur complements, (3.7) implies

$$-\bar{P} - \bar{P}' + k_1^2 A_1' R A_1 + \bar{P}' R^{-1} \bar{P} < 0$$

(3.12)

that can be presented as

$$-R + k_1^2 A_1' R A_1 + (\bar{P} - R)' R^{-1} (\bar{P} - R) < 0.$$

From the latter inequality it follows that $-R + k_1^2 A_1' R A_1 < 0$, i.e. that eigenvalues of $k_1 A_1$ are inside of the unit circle. $\square$

Remark 3.1. Differently from the existing LMI stability conditions via neutral model transformation and simple Lyapunov functional $V(x_t) = x'(t) P_1 x(t) + V_2(x_t)|_{S=0}$ (see e.g. [1,6]), the conditions of (ii) are simplified, where the additional condition on the stability of the integral equation $z(t) = 0$ is omitted. Note that for the discrete delay case, the LMI of (ii) implies due to (iv) that eigenvalues of hA_1 are inside of the unit circle. The latter guarantees the stability of $z(t) = 0$ (see e.g. Lemma 4 of [5]). The same implication was obtained for the simplest LMIs derived via the simplest Lyapunov functional with R-term depending on $\dot{x}$ (see Remark 3.4 in [16]). For the general case of Stieltjes integral, the condition of (iv) is less conservative than the classical condition $k_1 |A_1| < 1$ that guarantees the stability of the integral equation (see e.g. (2.10) in [8]).

Remark 3.2. Augmented Lyapunov functionals have been applied directly to the original system in the deterministic case with distributed or discrete delay in [13] and in stochastic case with

discrete delays in [7]. Augmented Lyapunov functional $V_{aug}(x_t) = \bar{V}_1(x_t) + V_2(x_t)$ with V_2 defined by (3.10) and

$$\bar{V}_1(x_t) = \begin{bmatrix} x(t) \\ G(t) \end{bmatrix}' \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix} \begin{bmatrix} x(t) \\ G(t) \end{bmatrix},$$

where $G(t)$ is defined in (3.1), can be directly applied to the initial system (2.1). By arguments of Proposition 3.1 this leads to the following equivalent to (3.5), (3.6) LMI stability conditions:

$$\begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 + \frac{k_0}{k_1} S \end{bmatrix} > 0$$

(3.13)

and

$$\begin{bmatrix} \Phi_{11} & A'P_2 + k_0 A_1' P_3 & P_1 - P_2 \\ * & -R & P_2' - P_3 \\ * & * & -S \end{bmatrix} < 0,$$

(3.14)

$$\Phi_{11} = P_1 A + A' P_1 + k_0 (P_2 A_1 + A_1' P_2') \\ + A_1' (k_0^2 S + k_1^2 R) A_1 + C' P_1 C.$$

The equivalence of LMIs (3.13), (3.14) and (3.5), (3.6) follows from the following: substitution into (3.13) and (3.14) $P_2 \to P_1 + P_2$, $P_3 \to P_1 + P_2 + P_2' + P_3$ leads to (3.5) and (3.6). Note that (3.13), (3.14) do not lead to feasible reduced-order LMIs (with $P_2 = P_3 = 0$) if A is not Hurwitz (it is seen from (3.14)). Thus, the LMI conditions of (ii) are the simplest that are applicable to the important case, where A is not Hurwitz, but $A + k_0 A_1$ is Hurwitz.

3.2. Stability via stochastic extension of simple Lyapunov functionals depending on $\dot{x}(t)$

In this section we assume that the initial function $x(t) = \phi(t)$, $t < 0$ is continuously differentiable. In the deterministic case, this is a standard assumption for application of Lyapunov functionals depending on $\dot{x}(t)$ (see e.g. [23]).

Proposition 3.2. *Given matrices $A, A_1, C \in \mathbb{R}^{n \times n}$ and K subject to (2.2), assume that*

$$k_2 = \int_0^\infty \left(\int_\theta^\infty dK(s) \right)^2 d\theta < \infty.$$

(3.15)

Let there exist positive definite $n \times n$ matrices P, S, R and F such that the LMI

$$\Phi_{1\dot{x}} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & k_0 R \\ * & \Phi_{22} & -R \\ * & * & -(R+F) \end{bmatrix} < 0$$

(3.16)

holds, where

$$\Phi_{11} = PA + A'P + k_0^2 (S - R) + k_1^2 A' R A + C' (P + k_2 F) C,$$
$$\Phi_{12} = PA_1 + k_0 R + k_1^2 A' R A_1,$$
$$\Phi_{22} = k_1^2 A_1' R A_1 - R - S.$$

(3.17)

Then (2.1) is asymptotically mean square stable. Here $k_2 = h$ in the case of system (2.3) with discrete delay (because $\int_\theta^h dK(s) = 1$), whereas $k_2 = \frac{1}{3} h^3$ in the case of system (2.4) with distributed delay and kernel $K_0(s) = 1$, $s \in [0, h]$, $K_0(s) = 0$ $s > h$.

Proof. Let L be the generator of (2.1). Extending the idea of [11] to general delay, put

$$y(t) = Ax(t) + A_1 y_0(t), \quad t \ge 0,$$
$$y(t) = \dot{x}(t), \quad t < 0,$$
$$y_0(t) = \int_0^\infty x(t-s) dK(s),$$
$$y_1(t) = \int_0^\infty \int_{t-s}^t y(\tau) d\tau \, dK(s).$$

(3.18)

From (2.1) we have

$$dx(t) = y(t) dt + \chi(t) C x(t) dw(t), \quad t \in \mathbb{R},$$

(3.19)

where $\chi(t) = 1$ for $t \geq 0$ and $\chi(t) = 0$ for $t < 0$. Via (3.19), (3.18) the function $V_1(x(t)) = x'(t)Px(t)$ with $P > 0$ satisfies the condition

$$LV_1(x_t) = 2x'(t)P[Ax(t) + A_1y_0(t)]$$
$$+x'(t)C'PCx(t), \quad t \geq 0. \tag{3.20}$$

For the functional

$$V_2(x_t) = k_0 \int_0^\infty \int_{t-s}^t x'(\tau)Sx(\tau)d\tau dK(s)$$
$$+k_1 \int_0^\infty \int_{t-s}^t (\tau - t + s)y'(\tau)Ry(\tau)d\tau dK(s),$$

using (3.18), Jensen's inequalities (2.5), (2.6)

$$y_0'(t)Sy_0(t) \leq k_0 \int_0^\infty x'(t-s)Sx(t-s)dK(s),$$
$$y_1'(t)Ry_1(t) \leq k_1 \int_0^\infty \int_{t-s}^t y'(\tau)Ry(\tau)d\tau dK(s),$$

and via (3.18) the equality

$$y'(t)Ry(t) = x'(t)A'RAx(t) + 2x'(t)A'RA_1y_0(t)$$
$$+xsy_0'(t)A_1'RA_1y_0(t)$$

we obtain

$$LV_2(x_t) = k_0^2x'(t)Sx(t) + k_1^2y'(t)Ry(t)$$
$$-k_0 \int_0^\infty x'(t-s)Sx(t-s)dK(s)$$
$$-k_1 \int_0^\infty \int_{t-s}^t y'(\tau)Ry(\tau)d\tau dK(s)$$
$$\leq x'(t)(k_0^2S + k_1^2A'RA)x(t) \tag{3.21}$$
$$+2k_1^2x'(t)A'RA_1y_0(t)$$
$$+y_0'(t)(k_1^2A_1'RA_1 - S)y_0(t) - y_1'(t)Ry_1(t).$$

Integrating (3.19) we have

$$\int_{t-s}^t y(\tau)d\tau = x(t) - x(t-s) - \int_{t-s}^t \chi(\tau)Cx(\tau)dw(\tau),$$
$$t \in \mathbb{R}. \tag{3.22}$$

Via (3.18)

$$y_1(t) = k_0x(t) - y_0(t) - \xi(t),$$
$$\xi(t) = \int_0^\infty \int_{t-s}^t \chi(\tau)Cx(\tau)dw(\tau)dK(s)$$
$$= \int_{-\infty}^t \int_{t-\tau}^\infty dK(s)\chi(\tau)Cx(\tau)dw(\tau) \tag{3.23}$$
$$= \int_0^t \int_{t-\tau}^\infty dK(s)Cx(\tau)dw(\tau).$$

So,

$$-y_1'(t)Ry_1(t) = -k_0^2x'(t)Rx(t)$$
$$+2k_0x'(t)Ry_0'(t) - y_0'(t)Ry_0(t) \tag{3.24}$$
$$+2k_0x'(t)R\xi(t) - 2y_0'(t)R\xi(t) - \xi'(t)R\xi(t).$$

From (3.23) by Ito's integral properties (see e.g. [8,20,21]) we have for any $n \times n$ matrix F

$$\mathbb{E}\xi'(t)F\xi(t)$$
$$= \int_0^t \left(\int_{t-\tau}^\infty dK(s)\right)^2 x'(\tau)C'FCx(\tau)d\tau. \tag{3.25}$$

We add the following term to the Lyapunov functional:

$$V_3(x_t) = \int_0^t \int_{t-\tau}^\infty \left(\int_\theta^\infty dK(s)\right)^2 d\theta x'(\tau)C'FCx(\tau)d\tau,$$
$$F > 0. \tag{3.26}$$

By using k_2 defined in (3.15) we find

$$LV_3(x_t) = k_2x'(t)C'FCx(t)$$
$$- \int_0^t \left(\int_{t-\tau}^\infty dK(s)\right)^2 x'(\tau)C'FCx(\tau)d\tau.$$

Then for the Lyapunov functional

$$V(x_t) = x'(t)Px(t) + V_2(x_t) + V_3(x_t) \tag{3.27}$$

Table 3.1
Example 3.1: the maximum h that preserves the stability.

σ	0.1	0.3	0.6	1.0	No vars
$\Phi_{1aug} < 0$	3.822	2.259	1.006	0.240	$2.5(n^2 + n)$
$\Phi_{1sim} < 0$	0.890	0.748	0.536	0.206	$n^2 + n$
$\Phi_{1\dot{x}} < 0$	3.295	1.533	0.559	0.058	$2(n^2 + n)$

we obtain

$$\mathbb{E}LV(x_t) \leq \mathbb{E}\eta'(t)\Phi_{1\dot{x}}\eta(t),$$

where $\eta(t) = col\{x(t), y_0(t), \xi(t)\}$ and $\Phi_{1\dot{x}} < 0$. $\square$

Remark 3.3. In the case of system (2.4) with distributed delay and kernel $K_0(s) = 1$, $s \in [0, h]$, $K_0(s) = 0$ $s > h$ the Lyapunov functional has the form (3.27) with

$$V_2(x_t) = h \int_{t-h}^t (\tau - t + h)x'(\tau)Sx(\tau)d\tau$$
$$+ \frac{h^2}{4} \int_{t-h}^t (\tau - t + h)^2 y'(\tau)Ry(\tau)d\tau,$$

$$y(t) = Ax(t) + A_1 \int_{t-h}^t x(s)ds$$

and with a novel "stochastic" term

$$V_3(x_t) = \frac{1}{3} \int_{t-h}^t (\tau - t + h)^3 x'(\tau)C'FCx(\tau)d\tau.$$

For $C = 0$ this functional coincides with the one from [23]. In the case of system (2.3) with discrete delay the Lyapunov functional has the form (3.27) with

$$V_2(x_t) = \int_{t-h}^t x'(\tau)Sx(\tau)d\tau + h \int_{t-h}^t (\tau - t + h)y'(\tau)Ry(\tau)d\tau,$$

$$y(t) = Ax(t) + A_1x(t - h),$$

$$V_3(x_t) = \int_{t-h}^t (\tau - t + h)x'(\tau)C'FCx(\tau)d\tau.$$

A similar V_3-term was considered in [10].

Remark 3.4. By arguments of [16,23], the LMI of Proposition 3.2 is always feasible if $A + k_0A_1$ is Hurwitz and $C = 0$.

3.3. Examples

Example 3.1. Consider the well-studied example with discrete delay (see e.g. [25]): (2.3) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad C = \sigma \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \sigma \in \mathbb{R}.$$

In the deterministic case ($C = 0$) LMIs of Proposition 3.1(i) ($\Psi_1 > 0$, $\Phi_{1aug} < 0$) and of Proposition 3.2 ($\Phi_{1\dot{x}} < 0$) give the same maximal value of delay $h = 4.4721$ that preserves stability (the same value was obtained in [25]), whereas the simplest LMI of Proposition 3.1(ii) ($\Phi_{1sim} < 0$) leads to $h = 0.9999$. In Table 3.1 the results are presented obtained from LMIs $\Psi_1 > 0$, $\Phi_{1aug} < 0$ (for brevity we write this throughout the section as $\Phi_{1aug} < 0$), $\Phi_{1sim} < 0$ and $\Phi_{1\dot{x}} < 0$ for various values of σ. Numbers of scalar decision variables are presented too. Note that an additional LMI for verification of the stability of the integral equation (similar to LMI of Lemma 4 in [5]), which is avoided by our method, adds $0.5(n^2 + n)$ variables to $n^2 + n$ variables of the conditions for $\Phi_{1sim} < 0$. It is seen that less conservative results are obtained from $\Phi_{1aug} < 0$, but on the account of computational complexity. The simplest LMI $\Phi_{1sim} < 0$ leads to efficient results for large stochastic perturbation $\sigma = 1$ that are close to the ones via $\Phi_{1aug} < 0$ and that essentially improve the results via $\Phi_{1\dot{x}} < 0$.

Table 3.2
Example 3.2: the stability intervals for h.

σ	0	0.2	0.4
$\Phi_{1aug} < 0$	[0.2001,1.6339]	[0.2334,1.3990]	[0.4048,1.1364]
$\Phi_{1sim} < 0$	[0.2001,1.4142]	[0.2335,1.2271]	[0.4052,1.0144]
$\Phi_{1\dot{x}} < 0$	[0.2001,1.6339]	[0.2568,1.2560]	–

Example 3.2 ([15]). Consider the system with distributed delay

$$\dot{x}(t) = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix} x(t) + \int_0^h \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-s)ds$$

$$+Cx(t)\dot{w}(t), \quad C = \sigma \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ \sigma \in \mathbb{R}.$$

The results obtained via $\Phi_{1aug} < 0$, $\Phi_{1sim} < 0$ and LMI $\Phi_{1\dot{x}} < 0$ are presented in Table 3.2 for various values of σ. Note that for $\sigma = 0.4$ the LMI $\Phi_{1\dot{x}} < 0$ does not give any interval for h, but for $\sigma = 0.39$ the following small stability interval is obtained: $h \in [0.6564, 0.7929]$. As in the previous example, less conservative results are obtained from $\Phi_{1aug} < 0$, but the simplest LMI $\Phi_{1sim} < 0$ leads to rather efficient results for large stochastic perturbations.

4. Delay-induced stability for stochastic vector second-order system

Consider stochastic vector second-order system

$$\ddot{x}_1(t) = A_1 x_1(t) + A_2 \dot{x}_1(t) + Cx_1(t)\dot{w}(t) + Bu(t),$$
$$x_1(t) \in \mathbb{R}^n, \quad A_1, A_2, C \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \tag{4.1}$$

where $w(t)$ is the scalar standard Wiener process. The system can be presented as

$$\dot{x}(t) = \begin{bmatrix} 0 & I_n \\ A_1 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ C \end{bmatrix} x_1(t)\dot{w}(t) \tag{4.2}$$

with $x(t) = col\{x_1(t), x_2(t) = \dot{x}_1(t)\} \in \mathbb{R}^{2n}$.

Assume that (4.2) is stabilizable, i.e. there exists $\bar{K} = [\bar{K}_1 \ \bar{K}_1] \in \mathbb{R}^{m \times 2n}$ such that $u(t) = \bar{K}_1 x_1(t) + \bar{K}_2 x_2(t)$ exponentially stabilizes (4.2) with $C = 0$. The derivative $\dot{x}_1(t) = x_2(t)$ can be approximated by the finite-difference

$$\dot{x}_1(t) \approx \frac{x_1(t) - x_1(t-h)}{h}, \quad h > 0,$$

leading to the static output-feedback with a stabilizing delay $h > 0$:

$$u(t) = K_0 x_1(t) + K_1 x_1(t-h), \tag{4.3}$$

where $x_1(t) = 0$ for $t < 0$ and

$$K_0 = \bar{K}_0 + \frac{1}{h}\bar{K}_1, \quad K_1 = -\frac{1}{h}\bar{K}_1. \tag{4.4}$$

The closed-loop system (4.2), (4.3) has the form

$$\dot{x}(t) = \begin{bmatrix} 0 & I_n \\ A_1 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B \end{bmatrix}(K_1 x_1(t)$$
$$+ K_2 x_1(t-h)) + \begin{bmatrix} 0 \\ C \end{bmatrix} x_1(t)\dot{w}(t). \tag{4.5}$$

Differently from the previous section, for $h = 0$ and $C = 0$ system (4.5) is not exponentially stable.

4.1. Stability via neutral type model transformation

Put

$$G(t) = \int_{t-h}^t (s - t + h)K_2 x_2(s)ds,$$
$$G_0(t) = \int_{t-h}^t K_2 x_2(s)ds, \quad G, G_0 \in \mathbb{R}^m. \tag{4.6}$$

Then via $x_2(s)ds = dx_1(s)$ we have $\dot{G}(t) = hK_2 x_2(t) - G_0(t) = hK_2 x_2(t) - K_2 x_1(t) + K_2 x_1(t-h)$ or

$$K_2 x_1(t-h) = \dot{G}(t) - hK_2 x_2(t) + K_2 x_1(t). \tag{4.7}$$

Note also that $\begin{bmatrix} 0 \\ C \end{bmatrix} x_1(t) = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} x(t)$. So, substituting (4.7) into (4.5) we obtain

$$\dot{z}(t) = Dx(t) + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} x(t)\dot{w}(t),$$
$$z(t) = \begin{bmatrix} I_{2n} & -\begin{bmatrix} 0 \\ B \end{bmatrix} \end{bmatrix} \begin{bmatrix} x(t) \\ G(t) \end{bmatrix} \in \mathbb{R}^{2n}, \tag{4.8}$$
$$D = \begin{bmatrix} 0 & I_n \\ A_1 + B(K_1 + K_2) & A_2 - hBK_2 \end{bmatrix}.$$

Proposition 4.1. Given matrices $A_1, A_2, C \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $K_1, K_2 \in \mathbb{R}^{m \times n}$.

(i) Let there exist matrices $P_1 \in \mathbb{R}^{2n \times 2n}$, $P_2 \in \mathbb{R}^{2n \times m}$, $P_3, R, S \in \mathbb{R}^{m \times m}$ that satisfy the following LMIs:

$$\Psi_2 = \begin{bmatrix} P_1 & P_2 - P_1 \begin{bmatrix} 0 \\ B \end{bmatrix} \\ * & \Psi_{22} \end{bmatrix} > 0,$$

$$\Psi_{22} = [0 \ B']P_1\begin{bmatrix} 0 \\ B \end{bmatrix} - [0 \ B']P_2 - P_2'\begin{bmatrix} 0 \\ B \end{bmatrix} \tag{4.9}$$
$$+P_3 + \frac{2}{h}S$$

and

$$\Phi_{2aug} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & -P_2 \\ * & -R & [0 \ B']P_2 - P_3 \\ * & * & -S \end{bmatrix} < 0,$$

$$\Phi_{11} = P_1 D + D'P_1 + hP_2[0 \ K_2] + h\begin{bmatrix} 0 \\ K_2' \end{bmatrix} P_2'$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & K_2'(h^2 S + \frac{1}{4}h^4 R)K_2 \end{bmatrix} + \begin{bmatrix} 0 & C' \\ 0 & 0 \end{bmatrix} P_1 \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}, \tag{4.10}$$

$$\Phi_{12} = D'\left(P_2 - P_1\begin{bmatrix} 0 \\ B \end{bmatrix}\right)$$
$$+ h\begin{bmatrix} 0 \\ K_2' \end{bmatrix}\left(P_3 - P_2'\begin{bmatrix} 0 \\ B \end{bmatrix}\right).$$

Then system (4.8) is asymptotically mean square stable.

(ii) If there exist $2n \times 2n$ matrix $P_1 > 0$ and $m \times m$ matrix R that satisfy the LMI

$$\Phi_{2sim} = \begin{bmatrix} \phi_{2s} & D'P_1\begin{bmatrix} 0 \\ B \end{bmatrix} \\ * & -R \end{bmatrix} < 0,$$

$$\phi_{2s} = P_1 D + D'P_1 \tag{4.11}$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{4}h^4 K_2' R K_2 \end{bmatrix} + \begin{bmatrix} 0 & C' \\ 0 & 0 \end{bmatrix} P_1 \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix},$$

then the system (4.8) is asymptotically mean square stable.

Proof. Consider V_1 given by (3.8), where $z(t)$ and $G(t)$ are defined in (4.8) and (4.6) respectively. Let L be the generator of Eq. (4.8). Differentiating V_1 along (4.8) we obtain

$$LV_1(x_t) = 2(x'(t) - G'(t)[0 \ B'])P_1 Dx(t)$$
$$+ 2(x'(t) - G'(t)[0 \ B'])P_2(hK_2 x_2(t) - G_0(t))$$
$$+ 2G'(t)P_3 (hK_2 x_2(t) - G_0(t)) \tag{4.12}$$
$$+ 2x'(t)D'P_2 G(t) + x'(t)\begin{bmatrix} 0 & C' \\ 0 & 0 \end{bmatrix} P_1 \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} x(t).$$

For the additional term of the Lyapunov functional

$$V_2(x_t) = h \int_{t-h}^t (s - t + h)x_2'(s)K_2'SK_2x_2(s)ds$$
$$+ \frac{1}{4}h^2 \int_{t-h}^t (s - t + h)^2 x_2'(s)K_2'RK_2x_2(s)ds,$$

via Jensen's inequalities (2.5)

$$G'(t)RG(t) \leq \frac{h^2}{2} \int_{t-h}^t (s - t + h)x_2'(s)K_2'RK_2x_2(s)ds,$$
$$G_0'(t)SG_0(t) \leq h \int_{t-h}^t x_2'(s)K_2'SK_2x_2(s)ds,$$

we have

$$
\begin{aligned}
LV_2(x_t) &= h^2 x_2'(t)K_2'SK_2x_2(t) \\
&- h \int_{t-h}^t x_2'(s)K_2'SK_2x_2(s)ds \\
&+ \frac{1}{4}h^4 x_2'(t)K_2'RK_2x_2(t) \\
&- \frac{1}{2}h^2 \int_{t-h}^t (s - t + h)x_2'(s)K_2'RK_2x_2(s)ds \\
&\leq x_2'(t)K_2' \left(h^2 S + \frac{1}{4}h^4 R \right) K_2x_2(t) \\
&- G'(t)RG(t) - G_0'(t)SG_0(t).
\end{aligned}
\tag{4.13}
$$

Via (4.12), (4.13) for the Lyapunov functional $V(x_t) = V_1(x_t) + V_2(x_t)$ we obtain $LV(x_t) \leq \eta'(t)\Phi_{2aug}\eta(t)$, where $\Phi_{2aug} < 0$ is defined in (4.10) and $\eta(t) = col\{x(t), G(t), G_0(t)\}$. Moreover, $V(x_t)$ is positive definite $V(x_t) \geq \begin{bmatrix} x(t) \\ G(t) \end{bmatrix}' \Psi_2 \begin{bmatrix} x(t) \\ G(t) \end{bmatrix}$, where $\Psi_2 > 0$ is defined in (4.9).

The proof of (ii) is similar to (ii) of Proposition 3.1. □

4.2. Stability via stochastic extension of simple Lyapunov functional depending on $\dot{x}_2$

Following [14], we use the following Taylor expansion with the integral form of the remainder:

$$
\begin{aligned}
x_1(t - h) &= x_1(t) - hx_2(t) + W(t), \\
W(t) &= \int_{t-h}^t (s - t + h)dx_2(s).
\end{aligned}
\tag{4.14}
$$

We further extend the results of [14] to stochastic case with $C \neq 0$. Denote

$$
\begin{aligned}
y_2(t) &= (A_1 + BK_1)x_1(t) + A_2x_2(t) \\
&+ BK_2x_1(t - h).
\end{aligned}
\tag{4.15}
$$

Then from (4.5) we have

$$dx_2(s) = y_2(s)ds + Cx_1(s)dw(s).$$

Therefore

$$
\begin{aligned}
W(t) &= r(t) + \xi(t), \\
r(t) &= \int_{t-h}^t (s - t + h)y_2(s)ds, \\
\xi(t) &= \int_{t-h}^t (s - t + h)Cx_1(s)dw(s).
\end{aligned}
\tag{4.16}
$$

Substituting (4.14), (4.16) into (4.5), we arrive at

$$
\begin{aligned}
\dot{x}(t) &= Dx(t) \\
&+ \begin{bmatrix} 0 \\ B \end{bmatrix} K_2[r(t) + \xi(t)] + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} x(t)\dot{w}(t), \\
D &= \begin{bmatrix} 0 & I_n \\ A_1 + B(K_1 + K_2) & A_2 - hBK_2 \end{bmatrix},
\end{aligned}
\tag{4.17}
$$

whereas y_2 defined by (4.15) has the form

$$y_2(t) = [0 \quad I_n]Dx(t) + BK_2[r(t) + \xi(t)]. \tag{4.18}$$

We choose the following Lyapunov functional:

$$V(x_t) = V_P(x(t)) + V_R(y_{2t}) + V_F(x_{1t}),$$

where

$$
\begin{aligned}
V_P(x(t)) &= x'(t)Px(t), \\
V_R(y_{2t}) &= \int_{t-h}^t (s - t + h)^2 y_2'(s)Ry_2(s)ds, \\
V_F(x_{1t}) &= \int_{t-h}^t (s - t + h)^3 x_1'(s)C'FCx_1(s)ds,
\end{aligned}
\tag{4.19}
$$

and matrices $P \in \mathbb{R}^{2n \times 2n}$, $R, F \in \mathbb{R}^{n \times n}$ are positive definite. Let L be the generator of (4.17). Then

$$
\begin{aligned}
LV_P(x(t)) &= 2x'(t)P \left(Dx(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} K_2(r(t) + \xi(t)) \right) \\
&+ x'(t) \begin{bmatrix} 0 & C' \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} x(t).
\end{aligned}
$$

Via Jensen's inequality and (4.16), (4.18) we obtain

$$
\begin{aligned}
LV_R(y_{2t}) &= h^2 y_2'(t)Ry_2(t) \\
&- 2 \int_{t-h}^t (s - t + h)y_2'(s)Ry_2(s)ds \\
&\leq h^2 \eta'(t) \left[[0 \ I_n]D \quad BK_2 \quad BK_2 \right]' R \\
&\times \left[[0 \ I_n]D \quad BK_2 \quad BK_2 \right] \eta(t) - \frac{4}{h^2} r'(t)Rr(t),
\end{aligned}
$$

where $\eta(t) = col\{x(t), r(t), \xi(t)\}$.

Via Ito's integral property

$$\mathbb{E}\xi'(t)F\xi(t) = \mathbb{E} \int_{t-h}^t (s - t + h)^2 x_1'(s)C'FCx_1(s)ds.$$

So, $\mathbb{E}LV_F(x_t) = h^3 \mathbb{E}x'(t)[C \ 0]'F[C \ 0]x(t) - 3\mathbb{E}\xi'(t)F\xi(t)$. Therefore, $\mathbb{E}LV(x_t) \leq \mathbb{E}\eta'(t)\Phi_{2\dot{x}}\eta(t)$, where

$$
\begin{aligned}
\Phi_{2\dot{x}} &= \begin{bmatrix} \phi_2 & P\begin{bmatrix} 0 \\ BK_2 \end{bmatrix} & P\begin{bmatrix} 0 \\ BK_2 \end{bmatrix} \\ * & -\frac{4}{h^2}R & 0 \\ * & * & -3F \end{bmatrix} \\
&+ h^2 \left[[0 \ I_n]D \quad BK_2 \quad BK_2 \right]' \\
&\times R \left[[0 \ I_n]D \quad BK_2 \quad BK_2 \right], \\
\phi_2 &= PD + D'P + \begin{bmatrix} 0 & C' \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \\
&+ h^3 \begin{bmatrix} C & 0 \end{bmatrix}' F \begin{bmatrix} C & 0 \end{bmatrix}.
\end{aligned}
\tag{4.20}
$$

Summarizing we arrive at the following

Proposition 4.2. *Given matrices $A_1, A_2, C \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $K_1, K_2 \in \mathbb{R}^{m \times n}$, let there exist positive definite matrices $P \in \mathbb{R}^{2n \times 2n}$, $R, F \in \mathbb{R}^{n \times n}$ that satisfy the LMI $\Phi_{2\dot{x}} < 0$, where $\Phi_{2\dot{x}}$ is defined by (4.20). Then (4.17) is asymptotically mean square stable.*

Remark 4.1. LMIs of Propositions 4.1 and 4.2 are feasible for small enough h provided D defined in these Propositions is Hurwitz [5]. The feasibility of $\Phi_{2\dot{x}} < 0$ yields that

$$-\frac{4}{h^2}R + h^2 K_2'B'RBK_2 < 0,$$

i.e. that all the eigenvalues of $\frac{h^2}{2}BK_2$ are inside of the unit circle. The feasibility of (4.11) implies the same conclusion. This can be proved by arguments of (iv) of Proposition 3.1. Indeed, denoting $\bar{P}' = -[0 \ I]D'P_1[0 \ I]'$ we find that (4.11) by Schur complements implies

$$-\bar{P} - \bar{P}' + \frac{h^4}{4}K_2'RK_2 + \bar{P}'BR^{-1}B'\bar{P} < 0.$$

that can be presented as

$$-R + \frac{h^4}{4}K_2'RK_2 + (\bar{P} - R)'R^{-1}(\bar{P} - R) < 0.$$

From the latter inequality it follows that

$$-R + \frac{h^4}{4}K_2'RK_2 < 0,$$

Table 4.1
Example 4.1: the stability intervals for h and different σ.

σ	0	0.5	No var $(m = n)$
$\Phi_{2aug} < 0$	[0.106, 1.413]	[0.259, 1.370]	$5.5n^2 + 2.5n$
$\Phi_{2sim} < 0$ [5]	[0.106, 1.406]	[0.259, 1.370]	$2.5n^2 + 1.5n$ $3n^2 + 2n$
$\Phi_{2\dot{x}} < 0$	[0.106, 1.406]	[0.344, 0.955]	$3n^2 + 2n$

Table 4.2
Example 4.2: the stability intervals for h and different σ.

σ	0	0.7	1.3
$\Phi_{2aug} < 0$	(0, 1.0962]	[0.132, 1.079]	[0.616, 0.942]
$\Phi_{2sim} < 0$	(0, 0.999]	[0.132, 0.999]	[0.616, 0.942]
$\Phi_{2\dot{x}} < 0$	(0, 0.999]	[0.166, 0.723]	–

i.e. that the eigenvalues of $\frac{h^2}{2}BK_2$ are inside of the unit circle. The latter guarantees the stability of the integral equation $z(t) = x_2(t) - \int_{t-h}^{t}(s - t + h)BK_2x_2(s)ds = 0$ (see Remark 3 of [5]). Thus, we arrived at the additional justification of the fact that the stability condition for the integral equation can be omitted.

Differently from the augmented Lyapunov functional of Proposition 3.1, the augmented functional of Proposition 4.1 (for the delay-induced stability) does not essentially improve the stability analysis results in the examples even in the deterministic case. However, the augmented Lyapunov functional and the functional of Proposition 4.2 may be more efficient than the simplest one that corresponds to item (ii) of Proposition 4.2 in various control problems, where a lower bound of V in terms of $|x(t)|$ should be employed (e.g. for finding bounds on $|x(t)|$ or for finding domains of attraction for nonlinear systems [16]). These problems may be topics for future research.

4.3. Examples

Example 4.1 (*Delay-Induced Stability ([17], p. 176)*). The system

$$\ddot{x}_1(t) = (-2 + \sigma\dot{w}(t))x_1(t) + 0.1\dot{x}_1(t) + u, \quad u(t) = x_1(t - h),$$

is reduced to (4.1), (4.3) with $n = m = 1, A_1 = -2, A_2 = 0.1, B = 1, C = \sigma, K_1 = 0$ and $K_2 = 1$. The mean square asymptotic stability intervals via the augmented Lyapunov functional (LMIs $\Psi_2 > 0$, $\Phi_{2aug} < 0$ written for brevity as $\Phi_{2aug} < 0$) and simple Lyapunov functionals (LMI $\Phi_{2sim} < 0$ or LMI $\Phi_{2\dot{x}} < 0$) for $\sigma = 0$ and $\sigma = 0.5$ are presented in Table 4.1. Note that the number of scalar decision variables in these LMIs are: $2n^2 + n + 2nm + 1.5(m^2 + m)$ for $\Phi_{2aug} < 0, 2n^2 + n + 0.5(m^2 + m)$ for $\Phi_{2sim} < 0$ (whereas in [5] there are additional $0.5(n^2 + n)$ variables for the stability of the integral equation) and $3n^2 + 2n$ for $\Phi_{2\dot{x}} < 0$. It is seen that for $\sigma = 0.5$ the simplest LMI $\Phi_{2sim} < 0$ leads to the same result as $\Phi_{2aug} < 0$ essentially improving the result via $\Phi_{2\dot{x}} < 0$.

Example 4.2 (*Inverted Pendulum ([8], p. 209)*). Consider the controlled inverted pendulum with stochastic perturbations

$$\ddot{x}_1(t) = [1 + \sigma\dot{w}(t)]x_1(t) + u(t), \quad u(t) = -4x_1(t) + 2x_1(t - h).$$

It is reduced to Eq. (4.5) with $A_1 = 1, A_2 = 0, B = 1, C = \sigma, K_1 = -4, K_2 = 2$. The maximal (asymptotic mean square) stability intervals via different methods for various values of σ are presented in Table 4.2. Also in this example, for large values of stochastic perturbations, the simplest LMI $\Phi_{2sim} < 0$ leads to the best (in terms of conservatism and numerical complexity) results.

Table 4.3
Example 4.3: the obtained K_1 and the maximal σ.

Method	h	K_1	σ
$\Phi_{2aug} < 0$	0.025	[241.145 230.541]	3.224
$\Phi_{2aug} < 0$			3.234
$\Phi_{2sim} < 0$	0.018	[332.658 310.497]	3.234
$\Phi_{2\dot{x}} < 0$			2.228

Example 4.3 (*Inverted Pendulum on the Cart ([16], p. 313)*). Consider a model of the inverted pendulum on a cart

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{Ml} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ \frac{a}{M} \\ \frac{-a}{Ml} \end{bmatrix} u(t) + \sigma \begin{bmatrix} 0 \\ 0 \\ x \\ \theta \end{bmatrix} \dot{w}(t)$$

with $M = 3.9249$, $m = 0.2047$, $l = 0.2302$, $g = 9.81$, $a = 25.3$ and $\sigma \geq 0$. In this model, x and θ represent cart position coordinate and pendulum angle from vertical, respectively. The system can be stabilized by a state-feedback

$$u(t) = \bar{K}_1[x(t)\,\theta(t)]' + \bar{K}_2[\dot{x}(t)\,\dot{\theta}(t)]',$$

$$[\bar{K}_1\,\bar{K}_2] = [5.825\ 24.941\ 5.883\ 5.140].$$

Assume now that the measurement is given by $y(t) = col\{x(t), \theta(t)\}$, and we are looking for a static output-feedback

$$u(t) = K_1[x(t)\quad\theta(t)]' + K_2[x(t - h)\quad\theta(t - h)]'$$

that stabilizes by using delay h. The delays that ensure stability, the resulting gain K_1 found from (4.4) (for brevity K_2 is omitted) and maximal possible σ are shown in Table 4.3. It is seen that augmented Lyapunov functional allows to use a larger delay $h = 0.25$ that leads to a smaller gain.

5. Conclusions

In this paper two novel Lyapunov-based methods (via augmented Lyapunov functionals and via stochastic extension of Lyapunov functionals depending on $\dot{x}$) for two important classes of stochastic systems have been presented: for general retarded systems with the delay term in the form of Stieltjes integral (that are stable without delay) and for systems with delay-induced stability. The paper has introduced the first LMIs (even in the deterministic case) for systems with general delay term in the form of Stieltjes integral. These LMIs provide in the unified form conditions for both discrete and distributed delay.

The method via augmented Lyapunov functional that employs neutral type transformation is novel for both classes of systems (even in the deterministic case). The main novelty of the method via stochastic extension of Lyapunov functional depending on $\dot{x}$ is for the stochastic systems, where novel Lyapunov functionals are introduced. The paper simplifies the existing results based on neutral type transformation: the stability conditions for the integral equation are omitted (which is also new in the deterministic case).

Though results via the second method are less efficient for large stochastic perturbations in the numerical examples, this method and the method based on augmented Lyapunov functionals should be more efficient than the simplest method for finding bounds on $|x(t)|$. In this paper, the simplest conditions via Jensen's inequality were presented. The suggested methodology based on Stieltjes integral presentation may be useful for advanced results via other augmented Lyapunov functionals and less conservative integral inequalities (see e.g. [13,18,26]). These may be the topics for the future research.

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