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NETWORKED CONTROL SYSTEMS IN THE PRESENCE OF SCHEDULING PROTOCOLS AND COMMUNICATION DELAYS*

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Abstract. This paper develops the time-delay approach to networked control systems in the presence of multiple sensor nodes, communication constraints, variable transmission delays, and sampling intervals. Due to communication constraints, only one sensor node is allowed to transmit its packet at a time. The scheduling of sensor information toward the controller is ruled by a weighted try-once-discard or by round-robin protocols. A unified hybrid system model under both protocols for the closed-loop system is presented; it contains time-varying delays in the continuous dynamics and in the reset conditions. A new Lyapunov–Krasovskii method, which is based on discontinuous in time Lyapunov functionals, is introduced for the stability analysis of the delayed hybrid systems. The resulting conditions can be applied to the system with polytopic type uncertainties. The efficiency of the time-delay approach is illustrated on the examples of uncertain cart-pendulum and of batch reactor.

Key words. networked control systems, time-delay approach, scheduling protocols, hybrid systems, Lyapunov–Krasovskii method

AMS subject classifications. 93D15, 93D05

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1. Introduction.

Networked control systems (NCSs) have received considerable attention in recent years due to many advantages, including reduced costs, ease of installation and maintenance, and increased flexibility [1]. In many NCSs, the transmissions are constrained by bandwidth limitations in the communication channels between spatially distributed sensors, controllers, and actuators. The communication constraints impose that only one node is allowed to transmit its packet at a time. Therefore, admissible protocols are needed to schedule which node is given access to the network per transmission.

Three main approaches have been used to model the sampled-data control and later to the NCSs: a discrete-time system [6, 12], an impulsive/hybrid system [21, 23], and a time-delay system [8, 10, 13, 14]. The hybrid system approach, which was inspired by [26], has been applied to nonlinear NCSs under try-once-discard (TOD) and round-robin (RR) protocols in [17, 23]. In the framework of discrete-time approach, network-based stabilization of linear time-invariant systems with TOD/RR protocols and communication delays has been considered in [6]. Variable sampling intervals or small communication delays (that are smaller than the sampling intervals) have been considered in the above works. Note that in the absence of scheduling protocols, all three approaches are applicable to nonsmall communication delays (see, e.g., [3, 22]).

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The time-delay approach that was recently suggested in [19] allowed, for the first time, to treat NCSs with $N = 2$ sensor nodes under the RR protocol in the presence of nonsmall communication delays. The closed-loop system was presented as a switched system with two and ordered time-varying delays. Some preliminary results on time-delay approach for NCSs with $N = 2$ sensor nodes and TOD scheduling were presented in [20] (see also [18] for the discrete-time case), where the closed-loop system was modeled as a hybrid delayed system. A time-dependent Lyapunov functional was introduced in [20] to derive stability conditions for the hybrid delayed system.

Note that the extension from $N = 2$ to a general $N \geq 2$ sensor nodes is far from being straightforward. It yields the following challenges:

1. The switched system model with multiple ordered delays for RR protocol may lead to complicated conditions.
2. The time-dependent Lyapunov functional of [20] is not applicable any more.

In the present paper, we consider linear (probably, uncertain) NCS with additive essentially bounded disturbances in the presence of multiple sensor nodes, scheduling protocols, variable transmission delays, and sampling intervals. The main contribution of this paper is as follows:

1. We give a unified hybrid system model under both TOD and RR protocols for the closed-loop system that contains time-varying delays in the continuous dynamics and in the reset conditions.
2. A novel Lyapunov–Krasovskii method is introduced for the stability analysis of the hybrid delayed systems, which is based on discontinuous in time Lyapunov functionals. The resulting conditions are computationally simpler under both protocols than the existing ones in [19].

Polytopic type uncertainties in the system model can be easily included in the analysis. The efficiency and advantages of the presented approach are illustrated by two examples. Some preliminary results were presented in [9] and [20].

Notation. Throughout the paper, the superscript T stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$, and $\lambda_{\min}(P)$ denotes the smallest eigenvalue of matrix P. The space of functions $\phi : [-\tau_M, 0] \to \mathbb{R}^n$, which are absolutely continuous on $[-\tau_M, 0]$, have a finite limit $\lim_{\theta\to 0^-} \phi(\theta)$, and have square integrable first-order derivatives, is denoted by $W[-\tau_M, 0]$ with the norm $\|\phi\|_W = \max_{\theta\in[-\tau_M,0]} |\phi(\theta)|+[\int_{-\tau_M}^0 |\dot\phi(s)|^2 ds]^{\frac{1}{2}}$. $\mathbb{Z}_+$ and $\mathbb{N}$ denote the set of nonnegative integers and positive integers, respectively. $\|w[t_0, t]\|_\infty$ stands for the essential supremum of the Euclidean norm $|w[t_0, t]|$, where $w : [t_0, t] \to \mathbb{R}^{n_w}$. MATI and MAD denote maximum allowable transmission interval and maximum allowable delay, respectively.

2. Problem formulation and the novel hybrid model.

2.1. The description of NCS.

Consider the system architecture in Figure 1 with plant

$$\dot x(t) = Ax(t) + Bu(t) + D\omega(t), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ denote the state and the control input, respectively; $\omega(t) \in \mathbb{R}^q$ is the essentially bounded disturbance. Assume that there exists a real number $\Delta > 0$ such that $\|\omega[0, t]\|_\infty \leq \Delta$ for all $t \geq 0$. The system matrices A, B, and D can be uncertain with polytopic type uncertainties.

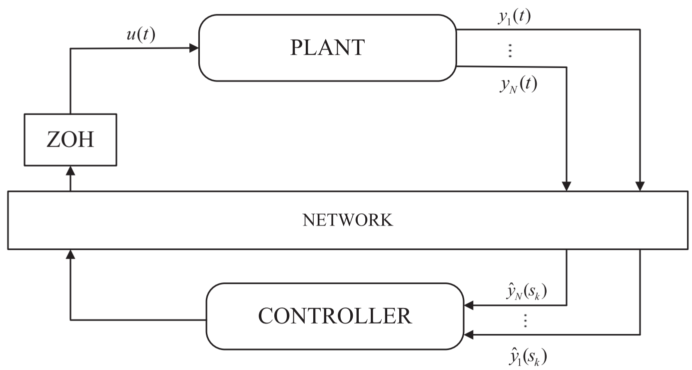

FIG. 1. *System architecture with N sensors.*

The system is equipped with N distributed sensors, a controller, and an actuator, which are connected via the network. The measurements are given by $y_i(t) = C_i x(t) \in \mathbb{R}^{n_i}$ $(i = 1, \dots, N,\ \sum_{i=1}^{N} n_i = n_y)$, which are sampled at s_k, satisfying

$$0 = s_0 < s_1 < \cdots < s_k < \cdots, \quad k \in \mathbb{Z}_+, \ \lim_{k\to\infty} s_k = \infty. \tag{2}$$

Then we denote $C = [\ C_1^T \quad \cdots \quad C_N^T\ ]^T$, $y(t) = [\ y_1^T(t) \quad \cdots \quad y_N^T(t)\ ]^T \in \mathbb{R}^{n_y}$.

At each sampling instant s_k, one of the outputs $y_i(s_k) \in \mathbb{R}^{n_i}$ is transmitted via the sensor network. We suppose that data loss does not occur and that the transmission of the information over the network experiences an uncertain, time-varying delay η_k. Then $t_k = s_k + \eta_k$ is the updating time instant of the zero-order hold (ZOH) device.

Assume that the maximum sampling interval and the maximum delay between the sampling instant s_k and its updating instant t_k are bounded by MATI and MAD, respectively. Following [19] and [22], we allow the transmission delays to be nonsmall provided that the transmission order of data packets is maintained for reception. Assume that the network-induced delay η_k and the time span between the updating and the most recent sampling instants are bounded,

$$t_{k+1} - t_k + \eta_k \le \tau_M, \ 0 \le \eta_m \le \eta_k \le \mathrm{MAD}, \ k \in \mathbb{Z}_+, \tag{3}$$

where τ_M denotes the maximum time span between the time

$$s_k = t_k - \eta_k \tag{4}$$

at which the state is sampled and the time t_{k+1} at which the next update arrives at the destination. Here, η_m and MAD are known bounds and $\tau_M = \mathrm{MATI} + \mathrm{MAD}$. Note that $\mathrm{MATI} = \tau_M - \mathrm{MAD} \le \tau_M - \eta_m$, $\eta_m > \frac{\tau_M}{2}$, i.e., $\eta_m > \tau_M - \eta_m$ leads to $\mathrm{MATI} \le \tau_M - \eta_m < \eta_m \le \eta_k$, which implies that the network delays are nonsmall. In the examples of section 6, we will show that our method is applicable for $\eta_m > \frac{\tau_M}{2}$.

2.2. A hybrid model via the time-delay approach. We will consider TOD and RR protocols that orchestrate the sensor data transmission. Denote by $\hat{y}(s_k) = [\ \hat{y}_1^T(s_k) \quad \cdots \quad \hat{y}_N^T(s_k)\ ]^T \in \mathbb{R}^{n_y}$ the output information submitted to the scheduling protocol. At each sampling instant s_k, one of the system nodes $i \in \{1, \dots, N\}$ is active, that is, only one of $\hat{y}_i(s_k)$ values is updated with the recent output $y_i(s_k)$. Let $i_k^* \in \{1, \ldots, N\}$ denote the active output node at the sampling instant s_k, which will be chosen due to scheduling protocols. Then

$$\hat{y}_i(s_k) = \begin{cases} y_i(s_k), & i = i_k^*, \\ \hat{y}_i(s_{k-1}), & i \neq i_k^*. \end{cases} \tag{5}$$

The choice of i_k^* will be either periodic (in RR protocol) or will depend on the transmission error (in TOD protocol), which is defined below. Consider the error between the system output $y(s_k)$ and the last available information $\hat{y}(s_{k-1})$:

$$\begin{aligned} e(t) = \operatorname{col}\{e_1(t), \ldots, e_N(t)\} \equiv \hat{y}(s_{k-1}) - y(s_k), \\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+,\ \hat{y}(s_{-1}) \stackrel{\Delta}{=} 0,\ e(t) \in \mathbb{R}^{n_y}. \end{aligned} \tag{6}$$

It is supposed that the controller and the actuator are event-driven (in the sense that the controller and the ZOH update their outputs as soon as they receive a new sample). The most recent output information on the controller side is denoted by $\hat{y}(s_k)$.

Static output feedback control. Assume that there exists a matrix $K = [\ K_1 \ \cdots \ K_N\ ]$, $K_i \in \mathbb{R}^{m \times n_i}$ such that $A + BKC$ is Hurwitz. Then, the static output feedback controller has a form

$$u(t) = K\hat{y}(s_k), \quad t \in [t_k, t_{k+1}).$$

So, due to (5), the controller can be presented as

$$u(t) = K_{i_k^*} y_{i_k^*}(t_k - \eta_k) + \sum_{i=1, i \neq i_k^*}^{N} K_i \hat{y}_i(t_{k-1} - \eta_{k-1}),\ t \in [t_k, t_{k+1}), \tag{7}$$

where i_k^* is the index of the active node at s_k and η_k is communication delay. We thus obtain the impulsive closed-loop model with the following continuous dynamics:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_1 x(t_k - \eta_k) + \sum_{i=1, i \neq i_k^*}^{N} B_i e_i(t) + D\omega(t), \\ &\dot{e}(t) = 0, \quad t \in [t_k, t_{k+1}), \end{aligned} \tag{8}$$

where $A_1 = BKC,\ B_i = BK_i,\ i = 1, \ldots, N$.

Taking into account (6), we obtain

$$\begin{aligned} e_i(t_{k+1}) &= \hat{y}_i(s_k) - y_i(s_{k+1}) = y_i(s_k) - y_i(s_{k+1}) \\ &= C_i x(s_k) - C_i x(s_{k+1}),\ i = i_k^*, \end{aligned}$$

and

$$\begin{aligned} e_i(t_{k+1}) &= \hat{y}_i(s_k) - y_i(s_{k+1}) = \hat{y}_i(s_{k-1}) - y_i(s_{k+1}) \\ &= \hat{y}_i(s_{k-1}) - y_i(s_k) + y_i(s_k) - y_i(s_{k+1}) \\ &= e_i(t_k) + C_i[x(s_k) - x(s_{k+1})],\ i \neq i_k^*,\ i \in \mathbb{N}. \end{aligned}$$

Thus, the delayed reset system is given by

$$
\begin{aligned}
&x(t_{k+1}) = x(t_{k+1}^-),\\
&e_i(t_{k+1}) = C_i[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})],\ i = i_k^*,\\
&e_i(t_{k+1}) = e_i(t_k) + C_i[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})],\ i \neq i_k^*,\ i \in \mathbb{N}.
\end{aligned} \tag{9}
$$

Therefore, (8)–(9) is a *novel hybrid model* of the NCS. Since $x(t_k-\eta_k) = x(t-\tau(t))$ for $t \in [t_k, t_{k+1})$ with $\tau(t) = t - t_k + \eta_k \in [\eta_m, \tau_M]$ (cf. (3)), the hybrid system model (8)–(9) contains the piecewise-continuous delay $\tau(t)$ in the continuous-time dynamics (8). Even for $\eta_k = 0$, we have the delayed state $x(t_k) = x(t-\tau(t))$ with $\tau(t) = t - t_k$.

Remark 2.1. In [17] (in the framework of the hybrid systems approach), a piecewise-continuous error $e(t) = \hat{y}(t_k) - y(t), t \in [s_k, s_{k+1}]$ is defined, which leads to the nondelayed continuous dynamics. The derivation of reset equations is based on the assumption of small communication delays, which is avoided in our approach. In our approach, $e(t)$ is different: it is given by (6) and is piecewise-constant. As a result, our hybrid model is different with the delayed continuous dynamics. Moreover, in the absence of scheduling protocols, the closed-loop system is given by nonhybrid system (8), where $e(t) \equiv 0$. The latter is consistent with the time-delay model considered, e.g., in [13, 14].

Note that in our model, the first updating time t_0 corresponds to the time instant when the first data is received by the actuator. Assume that initial conditions for (8)–(9) are given by

$$x_{t_0} \in W[-\tau_M, 0], \quad e(t_0) = -Cx(t_0 - \eta_0) = -Cx_0.$$

Dynamic output feedback control. Assuming that the controller is directly connected to the actuator, consider a dynamic output feedback controller of the form

$$
\begin{aligned}
\dot{x}_c(t) &= A_c x_c(t) + B_c \hat{y}(s_k),\\
u(t) &= C_c x_c(t) + D_c \hat{y}(s_k), \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+,
\end{aligned}
$$

where $x_c(t) \in \mathbb{R}^{n_c}$, $A_c, B_c, C_c,$ and D_c are the matrices with appropriate dimensions. Let $e_i(t)(i = 1, \ldots, N)$ be defined by (6). The closed-loop system can be presented in the form of (8)–(9), where x, e_i, and the matrices are changed by the ones with the bars as follows:

$$
\begin{aligned}
&\bar{x} = \begin{bmatrix} x \\ x_c \end{bmatrix},\ \bar{A} = \begin{bmatrix} A & BC_c \\ 0_{n_c\times n} & A_c \end{bmatrix}, \bar{B}_i = \begin{bmatrix} BD_c \\ B_c \end{bmatrix}, \bar{D} = \begin{bmatrix} D \\ 0_{n_c\times q} \end{bmatrix},\\
&\bar{A}_1 = \begin{bmatrix} BD_cC & 0_{n\times n_c} \\ B_cC & 0_{n_c\times n_c} \end{bmatrix}, \bar{C}_1 = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \bar{C}_i \in \mathbb{R}^{n_y\times(n+n_c)},\\
&\bar{C}_2 = \begin{bmatrix} 0_{n\times n_1} & C_2^T & 0 \\ 0_{n_c\times n_1} & 0 & 0 \end{bmatrix}^T, \ldots, \bar{C}_N = \begin{bmatrix} 0 & C_N^T \\ 0 & 0 \end{bmatrix}^T,\\
&\bar{e}_1(t) = [e_1^T(t)\ 0]^T, \bar{e}_2(t) = [0_{1\times n_1}\ e_2^T(t)\ \ 0]^T, \cdots \bar{e}_N(t) = [0\ \ e_N^T(t)]^T, \bar{e}_i(t) \in \mathbb{R}^{n_y}.
\end{aligned}
$$

2.3. Scheduling protocols.

TOD protocol. In the TOD protocol, the output node $i \in \{1, \ldots, N\}$ with the greatest (weighted) error will be granted the access to the network.

DEFINITION 2.2 (weighted TOD protocol). *Let $Q_i > 0 (i = 1, \ldots, N)$ be some weighting matrices. At the sampling instant s_k, the* weighted TOD protocol *is a* *protocol for which the active output node with the index* i_k^* *is defined as any index that satisfies*

$$|\sqrt{Q}_{i_k^*} e_{i_k^*}(t)|^2 \geq |\sqrt{Q}_i e_i(t)|^2,\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+,\ i = 1, \dots, N. \tag{10}$$

A possible choice of i_k^* is given by

$$i_k^* = \min \left\{ \arg \max_{i \in \{1,\dots,N\}} |\sqrt{Q}_i \left(\hat{y}_i(s_{k-1}) - y_i(s_k) \right)|^2 \right\}.$$

The conditions for computing the weighting matrices $Q_1, \dots, Q_N$ will be given in Theorem 3.2 below.

Remark 2.3. For the implementation of the TOD protocol in wireless networks, we refer to [4].

RR protocol. The active output node is chosen periodically:

$$\begin{aligned} &i_k^* = i_{k+N}^* \text{ for all } k \in \mathbb{Z}_+, \\ &i_j^* \neq i_l^* \text{ for } 0 \leq j < l \leq N-1. \end{aligned} \tag{11}$$

Remark 2.4. Note that another model for the closed-loop system under RR protocol was given in [19]. The model in [19] is a switched system with ordered delays $\tau_1(t) < \dots < \tau_N(t)$, where $\tau_i(t) = t - t_{k-i+1} + \eta_{k-i+1}, i = 1, \dots, N$. A Lyapunov–Krasovskii analysis of the latter model is based on the standard time-independent Lyapunov functional for interval delay.

Remark 2.5. The inclusion of packet dropouts under scheduling protocols is relatively easy if one assumes that there is an additional perfect (without packet dropouts) feedback channel to send a reception/dropout acknowledgement to the active sensor and if this acknowledgement is completed within one sampling period. Then, as in [17], packet dropouts can be modeled as prolongations of the transmission interval.

Definition 2.6. *The hybrid system* (8)–(9) *with essentially bounded disturbance* ω *is said to be partially input-to-state stable (ISS) with respect to* x *(or* x*-ISS) if there exist constants* $b > 0, \delta > 0,$ *and* $c > 0$ *such that the following holds for* $t \geq t_0$*:*

$$|x(t)|^2 \leq b e^{-\delta(t-t_0)} \left[\|x_{t_0}\|_W^2 + |e(t_0)|^2 \right] + c \|\omega[t_0, t]\|_\infty^2$$

for the solutions of the hybrid system initialized with $x_{t_0} = \phi \in W[-\tau_M, 0]$ *and* $e(t_0) \in \mathbb{R}^{n_y}$*. The hybrid system* (8)–(9) *is ISS if additionally the following bound is valid for* $t \geq t_0$*:*

$$|e(t)|^2 \leq b e^{-\delta(t-t_0)} \left[\|x_{t_0}\|_W^2 + |e(t_0)|^2 \right] + c \|\omega[t_0, t]\|_\infty^2.$$

Our objective is to derive linear matrix inequality (LMI) conditions for the partial ISS of the hybrid system (8)–(9) with respect to the variable of interest x. In [5], the notion of partial stability was also used. In section 3 below, ISS of (8)–(9) under TOD protocol with N sensor nodes will be studied. For $N = 2$, less restrictive conditions will be derived in section 4, and it will be shown that the same conditions guarantee x-ISS of (8)–(9) under the RR protocol. In section 5, the latter conditions will be extended to the RR protocol with $N \geq 2$.

3. ISS under TOD protocol: General N. Note that the differential equation for x given by (8) depends on $e_i(t) = e_i(t_k),\ t \in [t_k, t_{k+1})$ with $i \neq i_k^*$ only. Consider the following Lyapunov functional:

$$
\begin{aligned}
V_e(t) &= V(t, x_t, \dot{x}_t) + \sum_{i=1}^{N} e_i^T(t) Q_i e_i(t),\\
V(t, x_t, \dot{x}_t) &= \tilde{V}(t, x_t, \dot{x}_t) + V_G,\\
V_G &= \sum_{i=1}^{N} (\tau_M - \eta_m) \int_{s_k}^{t} e^{2\alpha(s-t)} |\sqrt{G_i} C_i \dot{x}(s)|^2 ds,\\
\tilde{V}(t, x_t, \dot{x}_t) &= x^T(t) P x(t) + \int_{t-\eta_m}^{t} e^{2\alpha(s-t)} x^T(s) S_0 x(s) ds\\
&\quad + \int_{t-\tau_M}^{t-\eta_m} e^{2\alpha(s-t)} x^T(s) S_1 x(s) ds\\
&\quad + \eta_m \int_{-\eta_m}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}^T(s) R_0 \dot{x}(s) ds d\theta\\
&\quad + (\tau_M - \eta_m) \int_{-\tau_M}^{-\eta_m} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta,\\
&P > 0, S_j > 0, R_j > 0, G_i > 0, Q_i > 0, \alpha > 0,\\
&j = 0, 1, i = 1, \dots, N,\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+,
\end{aligned}
\tag{12}
$$

where $x_t(\theta) \triangleq x(t+\theta),\ \theta \in [-\tau_M, 0]$ and where we define (for simplicity) $x(t) = x_0,\ \ t < 0$.

Here, the terms

$$e_i^T(t) Q_i e_i(t) \equiv e_i^T(t_k) Q_i e_i(t_k),\ t \in [t_k, t_{k+1})$$

are piecewise-constant, and $\tilde{V}(t, x_t, \dot{x}_t)$ presents the standard Lyapunov functional for systems with interval delays $\tau(t) \in [\eta_m, \tau_M]$. *The novel piecewise-continuous in time term V_G is inserted to cope with the delays in the reset conditions.* It is continuous on $[t_k, t_{k+1})$ and does not grow in the jumps (when $t = t_{k+1}$), since

$$
\begin{aligned}
&V_{G|t=t_{k+1}} - V_{G|t=t_{k+1}^-}\\
&= \sum_{i=1}^{N} (\tau_M - \eta_m) \int_{s_{k+1}}^{t_{k+1}} e^{2\alpha(s-t_{k+1})} |\sqrt{G_i} C_i \dot{x}(s)|^2 ds\\
&\quad - \sum_{i=1}^{N} (\tau_M - \eta_m) \int_{s_k}^{t_{k+1}^-} e^{2\alpha(s-t_{k+1})} |\sqrt{G_i} C_i \dot{x}(s)|^2 ds\\
&\leq -\sum_{i=1}^{N} (\tau_M - \eta_m) e^{-2\alpha\tau_M} \int_{s_k}^{s_{k+1}} |\sqrt{G_i} C_i \dot{x}(s)|^2 ds\\
&\leq -\sum_{i=1}^{N} e^{-2\alpha\tau_M} |\sqrt{G_i} C_i [x(s_k) - x(s_{k+1})]|^2,
\end{aligned}
\tag{13}
$$

where we applied Jensen's inequality (see, e.g., [16]). The function $V_e(t)$ is thus continuous and differentiable over $[t_k, t_{k+1})$. The following lemma gives sufficient conditions for the ISS of (8)–(10).

LEMMA 3.1. *Let there exist positive constants α, b, $0 < Q_i \in \mathbb{R}^{n_i \times n_i}$, $0 < U_i \in \mathbb{R}^{n_i \times n_i}$, $0 < G_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \dots, N$, and $V_e(t)$ of* (12) *such that along* (8) *the following inequality holds:*

$$\begin{aligned} &\dot{V}_e(t) + 2\alpha V_e(t) - \frac{1}{\tau_M - \eta_m} \sum_{i=1, i \neq i_k^*}^{N} |\sqrt{U_i} e_i(t)|^2 \\ &\qquad - 2\alpha \left| \sqrt{Q_{i_k^*}} e_{i_k^*}(t) \right|^2 - b|\omega(t)|^2 \leq 0, \ t \in [t_k, t_{k+1}). \end{aligned} \tag{14}$$

Assume additionally that

$$\Omega_i \triangleq \begin{bmatrix} -\frac{1-2\alpha(\tau_M - \eta_m)}{N-1} Q_i + U_i & Q_i \\ * & Q_i - G_i e^{-2\alpha\tau_M} \end{bmatrix} < 0, \ i = 1, \dots, N. \tag{15}$$

Then $V_e(t)$ does not grow in the jumps along (8)–(10):

$$\begin{aligned} \Theta &\triangleq V_e(t_{k+1}) - V_e(t_{k+1}^-) \\ &\quad + \sum_{i=1, i \neq i_k^*}^{N} |\sqrt{U_i} e_i(t_k)|^2 + 2\alpha(\tau_M - \eta_m) \left| \sqrt{Q_{i_k^*}} e_{i_k^*}(t_k) \right|^2 \leq 0. \end{aligned} \tag{16}$$

Moreover, the following bounds hold for the solutions of (8)–(10) *initialized by $x_{t_0} \in W[-\tau_M, 0], e(t_0) \in \mathbb{R}^{n_y}$:*

$$\begin{aligned} V(t, x_t, \dot{x}_t) &\leq e^{-2\alpha(t-t_0)} V_e(t_0) + \frac{b}{2\alpha} \Delta^2, \ t \geq t_0, \\ V_e(t_0) &= V(t_0, x_{t_0}, \dot{x}_{t_0}) + \sum_{i=1}^{N} |\sqrt{Q_i} e_i(t_0)|^2, \end{aligned} \tag{17}$$

and

$$\sum_{i=1}^{N} |\sqrt{Q_i} e_i(t)|^2 \leq \tilde{c} e^{-2\alpha(t-t_0)} V_e(t_0) + \frac{b}{2\alpha} \Delta^2, \tag{18}$$

where $\tilde{c} = e^{2\alpha(\tau_M - \eta_m)}$, implying ISS of (8)–(10).

Proof. Since $\int_{t_k}^{t} e^{-2\alpha(t-s)} ds \leq \tau_M - \eta_m$, $t \in [t_k, t_{k+1})$ and $|\omega(t)| \leq \Delta$, by the comparison principle, (14) implies

$$\begin{aligned} V_e(t) &\leq e^{-2\alpha(t-t_k)} V_e(t_k) + \sum_{i=1, i \neq i_k^*}^{N} \{ |\sqrt{U_i} e_i(t_k)|^2 \} \\ &\quad + 2\alpha(\tau_M - \eta_m) \left| \sqrt{Q_{i_k^*}} e_{i_k^*}(t_k) \right|^2 + b\Delta^2 \int_{t_k}^{t} e^{-2\alpha(t-s)} ds, \ t \in [t_k, t_{k+1}). \end{aligned} \tag{19}$$

Note that (15) yields $0 < 2\alpha(\tau_M - \eta_m) < 1$ and $U_i \leq \frac{1-2\alpha(\tau_M - \eta_m)}{N-1} Q_i \leq Q_i$, $i = 1, \dots, N$. Hence,

$$V(t, x_t, \dot{x}_t) \leq e^{-2\alpha(t-t_k)} V_e(t_k) + b\Delta^2 \int_{t_k}^{t} e^{-2\alpha(t-s)} ds, \ t \in [t_k, t_{k+1}). \tag{20}$$

Since $\tilde{V}_{|t=t_{k+1}} = \tilde{V}_{|t=t_{k+1}^-}$ and $e(t_{k+1}^-) = e(t_k)$, we obtain

$$\begin{aligned}\Theta = &\sum_{i=1}^{N}[|\sqrt{Q_i}e_i(t_{k+1})|^2 - |\sqrt{Q_i}e_i(t_k)|^2] + \sum_{i=1, i\neq i_k^*}^{N} |\sqrt{U_i}e_i(t_k)|^2 \\ &+ 2\alpha(\tau_M - \eta_m)\sqrt{Q_{i_k^*}}e_{i_k^*}(t_k)|^2 + V_{G|t=t_{k+1}} - V_{G|t=t_{k+1}^-}.\end{aligned}$$

Then, taking into account (13), we find

$$\begin{aligned}\Theta \le &|\sqrt{Q}_{i_k^*}e_{i_k^*}(t_{k+1})|^2 + \sum_{i=1,i\neq i_k^*}^{N} |\sqrt{Q_i}e_i(t_{k+1})|^2 - [1 - 2\alpha(\tau_M - \eta_m)] \left|\sqrt{Q_{i_k^*}}e_{i_k^*}(t_k)\right|^2 \\ &- \sum_{i=1,i\neq i_k^*}^{N} \{e_i^T(t_k)[Q_i - U_i]e_i(t_k)\} - \sum_{i=1}^{N} e^{-2\alpha\tau_M}|\sqrt{G_i}C_i[x(s_k) - x(s_{k+1})]|^2.\end{aligned}$$

Note that under the TOD protocol

$$-\left|\sqrt{Q_{i_k^*}}e_{i_k^*}(t_k)\right|^2 \le -\frac{1}{N-1}\sum_{i=1,i\neq i_k^*}^{N} |\sqrt{Q_i}e_i(t_k)|^2.$$

Denote $\zeta_i = \mathrm{col}\{e_i(t_k), C_i[x(s_k) - x(s_{k+1})]\}$. Then, employing (4) and (9), we arrive at

$$\Theta \le -\left|\sqrt{G_{i_k^*}e^{-2\alpha\tau_M} - Q_{i_k^*}}C_{i_k^*}[x(s_k) - x(s_{k+1})]\right|^2 + \sum_{i=1,i\neq i_k^*}^{N} \zeta_i^T\Omega_i\zeta_i \le 0,$$

which yields (16).

The inequalities (16) and (19) with $t = t_{k+1}^-$ imply

$$V_e(t_{k+1}) \le e^{-2\alpha(t_{k+1}-t_k)}V_e(t_k) + b\Delta^2\int_{t_k}^{t_{k+1}} e^{-2\alpha(t_{k+1}-s)}ds.$$

Then

$$\begin{aligned}V_e(t_{k+1}) &\le e^{-2\alpha(t_{k+1}-t_{k-1})}V_e(t_{k-1}) + b\Delta^2\int_{t_{k-1}}^{t_{k+1}} e^{-2\alpha(t_{k+1}-s)}ds \\ &\le e^{-2\alpha(t_{k+1}-t_0)}V_e(t_0) + b\Delta^2\int_{t_0}^{t_{k+1}} e^{-2\alpha(t_{k+1}-s)}ds.\end{aligned} \tag{21}$$

Replacing in (21) $k+1$ by k and using (20), we arrive at (17), which yields x-ISS of (8)–(10) because

$$\lambda_{\min}(P)|x(t)|^2 \le V(t, x_t, \dot{x}_t),\ V(t_0, x_{t_0}, \dot{x}_{t_0}) \le \delta\|x_{t_0}\|_W^2$$

for some scalar $\delta > 0$. Moreover, (21) with $k+1$ replaced by k implies (18), since for $t \in [t_k, t_{k+1})$

$$e^{-2\alpha(t_k-t_0)} = e^{-2\alpha(t-t_0)}e^{-2\alpha(t_k-t)} \le \tilde{c}e^{-2\alpha(t-t_0)}. \qquad \square$$

By using Lemma 3.1 and the standard arguments for the delay-dependent analysis, we derive LMI conditions for ISS of (8)–(10). (See Appendix A for the proof.)

THEOREM 3.2. *Given* $0 \le \eta_m < \tau_M$, $\alpha > 0$, *assume that there exist positive scalar* b, $n \times n$ *matrices* $P > 0$, $S_0 > 0$, $R_0 > 0$, $S_1 > 0$, $R_1 > 0$, S_{12}, *and* $n_i \times n_i$ *matrices* $Q_i > 0$, $U_i > 0$, $G_i > 0$, $i = 1, \dots, N$, *such that* (15) *and the following LMIs are feasible:*

$$\Phi = \begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \ge 0, \tag{22}$$

$$\begin{bmatrix} \Sigma_i - (F^i)^T \Phi F^i e^{-2\alpha\tau_M} & \Xi_i^T H \\ * & -H \end{bmatrix} < 0, \ i = 1, \dots, N, \tag{23}$$

where

$$\begin{aligned}
&H = \eta_m^2 R_0 + (\tau_M - \eta_m)^2 R_1 + (\tau_M - \eta_m) \sum_{l=1}^{N} C_l^T G_l C_l,\\
&\Sigma_i = (F_1^i)^T P \Xi_i + (\Xi^i)^T P F_1^i + \Upsilon_i - (F_2^i)^T R_0 F_2^i e^{-2\alpha\eta_m},\\
&F_1^i = [I_n \ 0_{n\times(3n+n_y-n_i+q)}],\\
&F_2^i = [I_n \ \ -I_n \ \ 0_{n\times(2n+n_y-n_i+q)}],\\
&F^i = \begin{bmatrix} 0_{n\times n} & I_n & -I_n & 0_{n\times n} & 0_{n\times(n_y-n_i+q)} \\ 0_{n\times n} & 0_{n\times n} & I_n & -I_n & 0_{n\times(n_y-n_i+q)} \end{bmatrix},\\
&\Xi_i = [A \ 0_{n\times n} \ A_1 \ 0_{n\times n} \ \tilde{\Xi}_i \ D],\\
&\tilde{\Xi}_i = [B_2 \ \cdots B_N], \ i = 1,\\
&\tilde{\Xi}_i = [B_1 \ \cdots B_{N-1}], \ i = N,\\
&\tilde{\Xi}_i = [B_1 \ \cdots \ B_{j_{|j\ne i}} \cdots B_N], \ i = 2, \dots N-1,\\
&\Upsilon_i = \mathrm{diag}\{S_0 + 2\alpha P, -(S_0 - S_1)e^{-2\alpha\eta_m}, 0_{n\times n}, -S_1 e^{-2\alpha\tau_M}, \ \phi_i, \ -bI_q\},\\
&\phi_i = \mathrm{diag}\{\psi_2, \dots, \psi_N\}, \ i = 1,\\
&\phi_i = \mathrm{diag}\{\psi_1, \dots, \psi_{N-1}\}, \ i = N,\\
&\phi_i = \mathrm{diag}\{\psi_1, \dots, \psi_{j_{|j\ne i}}, \dots, \psi_N\}, \ i = 2, \dots N-1,\\
&\psi_j = -\frac{1}{\tau_M - \eta_m} U_j + 2\alpha Q_j, \ j = 1, \dots, N.
\end{aligned} \tag{24}$$

Then solutions of the hybrid system (8)–(10) *satisfy the bound* (17), *where* $V(t, x_t, \dot{x}_t)$ *is given by* (12), *implying ISS of* (8)–(10). *If the above LMIs are feasible with* $\alpha = 0$, *then the bound* (17) *holds with a small enough* $\alpha_0 > 0$.

Remark 3.3. The LMIs of Theorem 3.2 are always feasible for small enough delay bound τ_M. For simplicity, we will explain this in the case of $\alpha = 0$. Choose $G_i = (N-1)Q_i$. Application of the Schur complement leads the LMI (15) with $\alpha = 0$ to a solution $U_i = \frac{Q_i}{(N-2)(N-1)}$, $N > 2$, $N \in \mathbb{N}$.

Consider next the LMIs (22) and (23). Since $A + A_1$ is Hurwitz, there always exists matrix $P > 0$ such that $P(A+A_1)+(A+A_1)^T P < 0$. The functional $\tilde{V}(t, x_t, \dot{x}_t)$ is a standard Lyapunov functional for delay-dependent analysis. Matrices $G_i > 0$ and $U_i > 0$, $i = 1, \dots, N$, appear only in H-term and on the main diagonal of the LMI (23), respectively. The latter LMI is independent of Q_i. Then, given $G_i > 0$ and $U_i > 0$, the LMIs (22) and (23) are feasible for small enough $\tau_M > 0$ [11]. Hence, the LMIs (15), (22), and (23) are feasible for small enough τ_M.

Remark 3.4. From (2)–(4), it follows that $t_{k+1}-t_k$ is lower bounded by 0. Under the additional assumption that there is a positive lower bound T on $t_{k+1}-t_k$, it is possible to relax the condition (16). Thus, for the stability analysis of (8)–(10) with $\omega(t)\equiv 0$, the inequality (16) can be replaced by a less restrictive one

$$V_e(t_{k+1})-\mu V_e(t_{k+1}^-)+\sum_{i=1,i\neq i_k^*}^{N}|\sqrt{U_i}e_i(t_k)|^2+2\alpha(\tau_M-\eta_m)\left|\sqrt{Q_{i_k^*}}e_{i_k^*}(t_k)\right|^2\leq 0 \tag{25}$$

with some $\mu>1$. Then, by the arguments of Lemma 3.1, the condition (25) holds if there exist $0<2\alpha(\tau_M-\eta_m)<1$ and $U_i<Q_i,\ i=1,\dots,N$, such that

$$\begin{bmatrix}-\frac{\mu-2\alpha(\tau_M-\eta_m)}{N-1}Q_i+U_i+(1-\mu)Q_i & Q_i\\ * & Q_i-\mu G_i e^{-2\alpha\tau_M}\end{bmatrix}<0,\ i=1,\dots,N. \tag{26}$$

Moreover, the following bound is achieved:

$$V(t,x_t,\dot{x}_t)\leq \mu^k e^{-2\alpha(t-t_0)}V_e(t_0)\leq e^{-(2\alpha-\ln\mu/T)(t-t_0)}V_e(t_0)$$

for $t\geq t_0$. The latter inequality holds due to $k\leq \frac{t-t_0}{T}$. Therefore, the exponential stability is guaranteed under the dwell time condition $T>\frac{\ln\mu}{2\alpha}$.

The LMI (26) is less restrictive than (15): given U_i, (26) allows smaller Q_i and, hence, smaller G_i, which may enlarge τ_M that solve (26), (22), and (23). Note that in the examples below (under the assumption of constant sampling and constant network-induced delays η, where $T=\tau_M-\eta$), the relaxed condition (25) does not improve the results (does not enlarge τ_M).

4. ISS under TOD/RR protocol: $N = 2$.

For $N=2$, less restrictive conditions than those of Theorem 3.2 for the x-ISS of (8)–(9) will be derived via a Lyapunov functional, which is different from (12):

$$\begin{aligned}V_e(t)&=V(t,x_t,\dot{x}_t)+\frac{t_{k+1}-t}{\tau_M-\eta_m}\{e_i^T(t)Q_ie_i(t)\}_{|i\neq i_k^*},\\ &Q_1>0,\ Q_2>0,\ \alpha>0,\ t\in[t_k,t_{k+1}),\ k\in\mathbb{Z}_+,\end{aligned} \tag{27}$$

where $i_k^*\in\{1,2\}$ and $V(t,x_t,\dot{x}_t)$ is given by (12) with $G_i=Q_ie^{2\alpha\tau_M}$. The term $\frac{t_{k+1}-t}{\tau_M-\eta_m}\{e_i^T(t_k)Q_ie_i(t_k)\}$ is inspired by the similar construction of Lyapunov functionals for the sampled-data systems [7, 21, 25]. The following statement holds.

LEMMA 4.1. *Given $N=2$, if there exist positive constants α, b, and $V_e(t)$ of* (27) *such that along* (8)*, the following inequality holds*

$$\dot{V}_e(t)+2\alpha V_e(t)-b|\omega(t)|^2\leq 0,\quad t\in[t_k,t_{k+1}). \tag{28}$$

Then $V_e(t)$ does not grow in the jumps along (8)–(10) ((8), (9), (11)), *where*

$$\Theta\triangleq V_e(t_{k+1})-V_e(t_{k+1}^-)\leq 0. \tag{29}$$

The bound (17) *is valid for the solutions of* (8)–(10) ((8), (9), (11)) *with the initial condition $x_{t_0}\in W[-\tau_M,0]$, $e(t_0)\in\mathbb{R}^{n_y}$, implying the x-ISS of* (8)–(10) ((8), (9), (11)).

Proof. Since $|\omega(t)|\leq\Delta$, (28) implies

$$V_e(t)\leq e^{-2\alpha(t-t_k)}V_e(t_k)+b\Delta^2\int_{t_k}^{t}e^{-2\alpha(t-s)}ds,\ t\in[t_k,t_{k+1}). \tag{30}$$

Noting that

$$V_e(t_{k+1}) \leq \tilde{V}_{|t=t_{k+1}} + |\sqrt{Q}_i e_i(t_{k+1})|^2_{|i\neq i^*_{k+1}} + \sum_{i=1}^{2}(\tau_M - \eta_m)\int_{t_{k+1}-\eta_{k+1}}^{t_{k+1}} e^{2\alpha(s-t_{k+1})}|\sqrt{G_i}C_i\dot{x}(s)|^2 ds,$$

we obtain, employing (13),

$$\begin{aligned}\Theta &\leq e_i^T(t_{k+1})Q_i e_i(t_{k+1})_{|i\neq i^*_{k+1}} + V_{G|t=t_{k+1}} - V_{G|t=t^-_{k+1}} \\ &\leq e_i^T(t_{k+1})Q_i e_i(t_{k+1})_{|i\neq i^*_{k+1}} - \sum_{i=1}^{2}|\sqrt{Q_i}C_i[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})]|^2.\end{aligned}$$

We will prove that $\Theta \leq 0$ under the TOD and RR protocols, respectively. Under the TOD protocol, we have

$$e_i^T(t_{k+1})Q_i e_i(t_{k+1})_{|i\neq i^*_{k+1}} \leq e^T_{i^*_k}(t_{k+1})Q_{i^*_k}e_{i^*_k}(t_{k+1})$$

for $i^*_{k+1} = i^*_k$, whereas

$$e_i^T(t_{k+1})Q_i e_i(t_{k+1})_{|i\neq i^*_{k+1}} = e^T_{i^*_k}(t_{k+1})Q_{i^*_k}e_{i^*_k}(t_{k+1}) \tag{31}$$

for $i^*_{k+1} \neq i^*_k$. Then, taking into account (9), we obtain

$$\begin{aligned}\Theta &\leq |\sqrt{Q_i}C_i[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})]|^2_{|i=i^*_k} \\ &\quad - \sum_{i=1}^{2}|\sqrt{Q_i}C_i[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})]|^2 \leq 0.\end{aligned}$$

Under the RR protocol, we have $i^*_{k+1} \neq i^*_k$, meaning that (31) holds and that $\Theta \leq 0$. Then the result follows by the arguments of Lemma 3.1. ∎

Remark 4.2. Differently from Lemma 3.1, Lemma 4.1 guarantees (21), which does not give a bound on $e_{i^*_k}(t_k)$ since $V_e(t)$ for $t \in [t_k, t_{k+1})$ does not depend on $e_{i^*_k}(t_k)$. That is why Lemma 4.1 guarantees only x-ISS. However, as explained in Remark 5.4 below, under the RR protocol x-ISS implies the boundedness of e.

In the next section, we will extend the result of Lemma 4.1 under the RR protocol to the case of $N \geq 2$. Theorem 5.2 below (in the particular case of $N = 2$) will provide LMIs for the x-ISS of (8)–(10) ((8), (9), (11)).

5. ISS under the RR protocol: $N \geq 2$.

Under the RR protocol (11), the reset system (9) can be rewritten as

$$\begin{aligned}x(t_{k+1}) &= x(t^-_{k+1}), \\ e_{i^*_{k-j}}(t_{k+1}) &= C_{i^*_{k-j}}[x(s_{k-j}) - x(s_{k+1})], \\ &\quad j = 0,\dots,N-1 \text{ if } k \geq N-1,\end{aligned} \tag{32}$$

where the index $k-j$ corresponds to the last updated measurement in the node i^*_{k-j}.

Consider the following Lyapunov functional:

$$\begin{aligned}V_e(t) &= V(t, x_t, \dot{x}_t) + V_Q, \quad t \geq t_{N-1}, \\ V(t, x_t, \dot{x}_t) &= \tilde{V}(t, x_t, \dot{x}_t) + V_G,\end{aligned} \tag{33}$$

where $\tilde{V}(t, x_t, \dot{x}_t)$ is given by (12). The discontinuous in time terms V_Q and V_G are defined as follows:

$$
\begin{aligned}
V_Q &= \sum_{j=1}^{N-1} \frac{t_{k+1}-t}{j(\tau_M-\eta_m)} \left|\sqrt{Q_{i^*_{k-j}}} e_{i^*_{k-j}}(t)\right|^2, \ k \geq N-1, \ t \in [t_k, t_{k+1}),\\
V_G &= \begin{cases} \sum_{i=1}^{N}(\tau_M-\eta_m)\int_{s_k}^{t} e^{2\alpha(s-t)}|\sqrt{G_i}C_i\dot{x}(s)|^2 ds, \ k \geq N, \ t \in [t_k, t_{k+1}),\\ \sum_{i=1}^{N}(\tau_M-\eta_m)\int_{s_0}^{t} e^{2\alpha(s-t)}|\sqrt{G_i}C_i\dot{x}(s)|^2 ds, \ t \in [t_{N-1}, t_N), \end{cases}
\end{aligned} \tag{34}
$$

where for $i = 1, \ldots, N$

$$
G_i = (N-1)Q_i e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]} > 0. \tag{35}
$$

Here, V_e does not depend on $e_{i^*_k}(t_k)$. Note that given $i = 1, \ldots, N$, e_i-term appears $N-1$ times in V_Q for every N intervals $[t_{k+j}, t_{k+j+1})$, $j = 0, \ldots, N-1$ (except the interval with $i^*_{k+j} = i$). This motivates $N-1$ in (35) because V_G is supposed to compensate the V_Q term.

As in the previous sections, the term V_G is inserted to cope with the delays in the reset conditions. It is continuous on $[t_k, t_{k+1})$ and does not grow in the jumps (when $t = t_{k+1}$), since for $k > N-1$ (cf. (13))

$$
V_{G|t=t_{k+1}} - V_{G|t=t^-_{k+1}} \leq -\sum_{i=1}^{N}(\tau_M-\eta_m)\int_{s_k}^{s_{k+1}} e^{2\alpha(s-t_{k+1})}|\sqrt{G_i}C_i\dot{x}(s)|^2 ds \tag{36}
$$

and for $k = N-1$

$$
V_{G|t=t_N} - V_{G|t=t^-_N} \leq -\sum_{i=1}^{N}(\tau_M-\eta_m)\int_{s_0}^{s_N} e^{2\alpha(s-t_N)}|\sqrt{G_i}C_i\dot{x}(s)|^2 ds. \tag{37}
$$

The term V_Q grows in the jumps as follows:

$$
\begin{aligned}
V_{Q|t=t_{k+1}} - V_{Q|t=t^-_{k+1}} &= \sum_{j=1}^{N-1} \frac{t_{k+2}-t_{k+1}}{j(\tau_M-\eta_m)} \left|\sqrt{Q_{i^*_{k+1-j}}} e_{i^*_{k+1-j}}(t_{k+1})\right|^2\\
&\leq \sum_{j=0}^{N-2} \frac{1}{j+1} \left|\sqrt{Q_{i^*_{k-j}}} C_{i^*_{k-j}}[x(s_{k-j}) - x(s_{k+1})]\right|^2\\
&\leq \sum_{j=0}^{N-2} (\tau_M-\eta_m) \int_{s_{k-j}}^{s_{k+1}} \left|\sqrt{Q_{i^*_{k-j}}} C_{i^*_{k-j}} \dot{x}(s)\right|^2 ds,
\end{aligned}
$$

where we have used Jensen's inequality and the bound

$$
\begin{aligned}
s_{k+1} - s_{k-j} &= s_{k+1} - s_k + s_k - \cdots + s_{k-j+1} - s_{k-j}\\
&\leq (j+1)(\tau_M-\eta_m).
\end{aligned} \tag{38}
$$

Since $1 \leq e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]} e^{2\alpha(s_{k-j}-t_{k+1})}$ for $j = 0, \ldots, N-2$, we obtain

$$
\begin{aligned}
V_{Q|t=t_{k+1}} - V_{Q|t=t^-_{k+1}} \leq &\sum_{j=0}^{N-2} (\tau_M-\eta_m) e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]}\\
&\times \int_{s_{k-j}}^{s_{k+1}} e^{2\alpha(s-t_{k+1})} \left|\sqrt{Q_{i^*_{k-j}}} C_{i^*_{k-j}} \dot{x}(s)\right|^2 ds.
\end{aligned} \tag{39}
$$

The following lemma gives sufficient conditions for the x-ISS of (8), (11), (32). (See Appendix B for the proof.)

LEMMA 5.1. *If there exist positive constants α, b, and $V_e(t)$ of* (33) *such that along* (8) *the inequality* (28) *is satisfied for $k \geq N-1$, then the following bound holds along the solutions of* (8), (11), (32):

$$V_e(t_{k+1}) \leq e^{-2\alpha(t_{k+1}-t_{N-1})}V_e(t_{N-1}) + \Psi_{k+1} + b\Delta^2 \int_{t_{N-1}}^{t_{k+1}} e^{-2\alpha(t_{k+1}-s)}ds, \ k \geq N-1, \tag{40}$$

where

$$\begin{aligned}\Psi_{k+1} = &-(\tau_M - \eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]} \\ &\times \left[\sum_{j=0}^{N-3}(N-2-j) \int_{s_{k-j-1}}^{s_{k+1}} e^{2\alpha(s-t_{k+1})} \left|\sqrt{Q_{i^*_{k-j}}}C_{i^*_{k-j}}\dot{x}(s)\right|^2 ds \right. \\ &\left. + (N-1)\int_{s_k}^{s_{k+1}} e^{2\alpha(s-t_{k+1})}\left|\sqrt{Q_{i^*_{k+1}}}C_{i^*_{k+1}}\dot{x}(s)\right|^2 ds \right] \leq 0.\end{aligned} \tag{41}$$

Moreover, for all $t \geq t_{N-1}$

$$\begin{aligned} V(t, x_t, \dot{x}_t) &\leq e^{-2\alpha(t-t_{N-1})}V_e(t_{N-1}) + \frac{b}{2\alpha}\Delta^2, \\ V_e(t_{N-1}) &= V(t_{N-1}, x_{t_{N-1}}, \dot{x}_{t_{N-1}}) + \sum_{i=1}^{N} |\sqrt{Q_i}e_i(t_{N-1})|^2. \end{aligned} \tag{42}$$

The latter inequality guarantees the x-ISS of (8), (11), (32) *for $t \geq t_{N-1}$.*

By using Lemma 5.1, arguments of Theorem 3.2, and the fact that for $j = 1, \dots N-1$

$$\frac{d}{dt}\frac{t_{k+1}-t}{j(\tau_M-\eta_m)} = -\frac{1}{j(\tau_M-\eta_m)} \leq -\frac{1}{(N-1)(\tau_M-\eta_m)},$$

we arrive at the the following result.

THEOREM 5.2. *Given $0 \leq \eta_m < \tau_M$ and $\alpha > 0$, assume that there exist positive scalar b, $n \times n$ matrices $P > 0$, $S_0 > 0$, $R_0 > 0$, $S_1 > 0$, $R_1 > 0$, S_{12} and $n_i \times n_i$ matrices $Q_i > 0$ $(i = 1, \dots, N)$ such that* (22) *and* (23) *are feasible with $U_i = \frac{Q_i}{N-1}$, where G_i is given by* (35). *Then for $N > 2$ solutions of the hybrid system,* (8), (11), (32) *satisfy the bound* (42) *with $V(t, x_t, \dot{x}_t)$ given by* (33), *meaning x-ISS for $t \geq t_{N-1}$. For $N = 2$ solutions of the hybrid system,* (8)–(10) ((8), (9), (11)) *satisfy the bound* (17), *meaning x-ISS (for $t \geq t_0$). Moreover, if the above LMIs are feasible with $\alpha = 0$, then the solution bounds hold with a small enough $\alpha_0 > 0$.*

Remark 5.3. Let $N = 2$ and compare the number of scalar decision variables of the resulting LMIs under different protocols. The conditions of Theorem 5.2 are essentially simpler than those in [19] and lead to complementary results compared to the conditions in [19] (see the examples below). See Table 1 for the complexity of the LMI conditions under different protocols.

TABLE 1
The numerical complexity of stability conditions under different protocols (for y_1, $y_2 \in \mathbb{R}^{n/2}$).

Method	Decision variables	Number and order of LMIs
[19] (RR)	$8.5n^2 + 2.5n$	two of $6n \times 6n$, two of $3n \times 3n$
Theorem 3.2 (TOD)	$4.25n^2 + 4n$	two of $5.5n \times 5.5n$, two of $2n \times 2n$
Theorem 5.2 (TOD/RR)	$3.75n^2 + 3n$	two of $5.5n \times 5.5n$, one of $2n \times 2n$

Remark 5.4. For $N = 2$ and $\alpha = 0$, the LMIs of Theorem 3.2 are more restrictive than those of Theorem 5.2: (15) of Theorem 3.2 yields $(N-1)U_i < Q_i < G_i$, whereas in Theorem 5.2 we have $(N-1)^2U_i = (N-1)Q_i = G_i$ that leads to larger U_i for the same G_i. The latter helps for the feasibility of (23), where $U_i > 0$ appears on the main diagonal only (with minus). However, Theorem 3.2 achieves ISS with respect to the full state col$\{x, e\}$ and provides the solution bounds for $t \geq t_0$, while Theorem 5.2 guarantees only x-ISS.

Note that Theorem 5.2 under RR protocol guarantees the boundedness of e as well. From (38) and $s_0 = 0$, it follows that $s_{N-1} - s_0 \leq (N-1)(\tau_M - \eta_m)$. Due to (3), we have $t_N \leq s_{N-1} + \tau_M \leq N\tau_M$. Moreover, $e(t_N)$ in (32) depends on $x(0), \ldots, x(t_N - \eta_N)$. Therefore, relations (32) yield

$$|e_i(t)|^2 \leq c' sup_{\theta \in [-N\tau_M, 0]} |x(t+\theta)|^2, \ t \geq t_N$$

with some $c' > 0$, which together with (42) imply

$$|e_i(t)|^2 \leq c''[e^{-2\alpha(t-t_{N-1})}V_e(t_{N-1}) + \Delta^2]$$

for some $c'' > 0$ and all $t \geq t_N + N\tau_M$.

Remark 5.5. The LMIs of Theorems 3.2 and 5.2 are affine in the system matrices. Therefore, in the case of system matrices from an uncertain time-varying polytope

$$\Omega = \sum_{j=1}^{M} g_j(t)\Omega_j, \quad 0 \leq g_j(t) \leq 1,$$
$$\sum_{j=1}^{M} g_j(t) = 1, \quad \Omega_j = \begin{bmatrix} A^{(j)} & B^{(j)} & D^{(j)} \end{bmatrix},$$

where g_j, $j = 1, \ldots, M$, are uncertain time-varying parameters, one has to solve these LMIs simultaneously for all the M vertices Ω_j, applying the same decision matrices.

6. Examples.

6.1. Example 1: Uncertain inverted pendulum.

Consider an inverted pendulum mounted on a small car. We focus on the stability analysis in the absence of disturbance. Following [15], we assume that the friction coefficient between the air and the car, f_c, and the air and the bar, f_b, are not exactly known and time-varying: $f_c(t) \in [0.15, 0.25]$ and $f_b(t) \in [0.15, 0.25]$. The linearized model can be written as

TABLE 2
Example 1 (N = 2): *maximum value of* $\tau_M = \mathrm{MATI} + \mathrm{MAD}$.

$\tau_M \setminus \eta_m$	0	0.005	0.01	0.02	0.04	Decision variables
[19] (RR)	0.023	0.026	0.029	0.035	0.046	146
Theorem 3.2 (TOD)	0.014	0.018	0.022	0.029	0.044	84
Theorem 5.2 (TOD/RR)	0.025	0.028	0.031	0.036	0.047	72

(1), where the matrices $A = E^{-1}A_f$ and $B = E^{-1}B_0$ are determined from

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3/2 & -1/4 \\ 0 & 0 & -1/4 & 1/6 \end{bmatrix},$$

$$A_f = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(f_c + f_b) & f_b/2 \\ 0 & 5/2 & f_b/2 & -f_b/3 \end{bmatrix} \quad \text{and} \quad B_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Here, A belongs to uncertain polytope, defined by four vertices corresponding to $f_c/f_b = 0.15$ and $f_c/f_b = 0.25$. The pendulum can be stabilized by a state feedback $u(t) = Kx(t)$, where $x = [x_1, x_2, x_3, x_4]^T$ with the gain

$$K = [11.2062 \quad -128.8597 \quad 10.7823 \quad -22.2629]. \tag{43}$$

In this model, x_1 and x_2 represent cart position and velocity, whereas x_3, x_4 represent pendulum angle from vertical and its angular velocity, respectively. In practice, x_1, x_2 and x_3, x_4 (presenting spatially distributed components of the state of the pendulum-cart system) are not accessible simultaneously. Suppose that the state variables are not accessible simultaneously. Consider first $N = 2$ and

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The applied controller gain K has the following blocks:

$$K_1 = \begin{bmatrix} 11.2062 & -128.8597 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 10.7823 & -22.2629 \end{bmatrix}.$$

For the values of η_m given in Table 2, we apply Theorems 3.2 and 5.2 with $\alpha = 0$, $b = 0$ via Remark 5.5 and find the maximum values of $\tau_M = \mathrm{MATI} + \mathrm{MAD}$ that preserve the stability of the hybrid system (8)–(9) with $\omega(t) = 0$ with respect to x. From Table 2, it is observed that under the TOD or RR protocol the conditions of Theorem 5.2 possess fewer decision variables and stabilize the system for larger τ_M than the results in [19] under the RR protocol. Moreover, when $\eta_m > \frac{\tau_M}{2}(\eta_m = 0.02, 0.04)$, our method is still feasible. (Communication delays are larger than the sampling intervals.) The computational time for solving the LMIs (in seconds) under the TOD protocol is essentially less than that under the RR protocol in [19] (till 36% decrease).

Consider next $N = 4$, where $C_1, \ldots, C_4$ are the rows of I_4 and $K_1, \ldots, K_4$ are the entries of K given by (43). Here, the maximum values of τ_M that preserve the stability of (8)–(9) with $\omega(t) = 0$ with respect to x are given in Table 3. Also, here Theorem 5.2 leads to less conservative results than Theorem 3.2.

TABLE 3
Example 1 (N = 4): *maximum value of* $\tau_M = \text{MATI} + \text{MAD}$.

$\tau_M \setminus \eta_m$	0	0.01
Theorem 3.2 (TOD)	0.003	0.012
Theorem 5.2 (RR)	0.006	0.015

TABLE 4
Example 2: *maximum value of* $\tau_M = \text{MATI} + \text{MAD}$ *for different* η_m.

$\tau_M \setminus \eta_m$	0	0.004	0.01	0.02	0.03	0.04
[17](MAD = 0.004)	0.0108	0.0133	-	-	-	-
[6](MAD = 0.03)	0.069	0.069	0.069	0.069	0.069	-
Theorem 3.2 (TOD)	0.019	0.022	0.027	0.034	0.042	0.050
Theorem 5.2 (TOD/RR)	0.035	0.037	0.041	0.047	0.053	0.059
[19] (RR)	0.042	0.044	0.048	0.053	0.058	0.063

6.2. Example 2: Batch reactor. We illustrate the efficiency of the given conditions on the example of a batch reactor under the dynamic output feedback (see, e.g., [17]), where $N = 2$ and

$$A = \begin{bmatrix} 1.380 & -0.208 & 6.715 & -5.676 \\ -0.581 & -4.2902 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \quad C = \left[\begin{array}{c} C_1 \\ \hline C_2 \end{array}\right] = \left[\begin{array}{cccc} 1 & 0 & 1 & -1 \\ \hline 0 & 1 & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline -2 & 0 & 0 & -2 \\ 0 & 8 & 5 & 0 \end{array}\right].$$

For the values of η_m given in Table 4, we apply Theorems 3.2 and 5.2 with $\alpha = 0$, $b = 0$ and find the maximum values of $\tau_M = \text{MATI} + \text{MAD}$ that preserve the stability of the hybrid system (8)–(9) with $\omega(t) = 0$ with respect to x. From Table 4, it is seen that the results of our method essentially improve the results in [17], and are more conservative than those obtained via the discrete-time approach. Recently, in [2], the same result $\tau_M = 0.035$ as ours in Theorem 5.2 for $\eta_m = 0$, MAD = 0.01 has been achieved. In [2], the sum of squares method is developed in the framework of hybrid system approach. We note that the sum of squares method has not been applied yet to ISS. Moreover, our conditions are simple LMIs with a fewer decision variables. When $\eta_m > \frac{\tau_M}{2}(\eta_m = 0.03, 0.04)$, our method is still feasible. (Communication delays are larger than the sampling intervals.) The computational time under the TOD protocol is essentially less than that under RR protocol in [19] (till 32% decrease).

7. Conclusions. In this paper, a time-delay approach has been developed for the ISS of NCS with scheduling protocols, variable transmission delays, and variable sampling intervals. A unified hybrid system model with time-varying delays in the continuous dynamics and in the reset equations is introduced for the closed-loop system under both TOD and RR protocols, and a new Lyapunov–Krasovskii method is developed. The ISS conditions of the delayed hybrid system are derived in terms of LMIs. Differently from the existing (hybrid and discrete-time) methods on the stabilization of NCS with scheduling protocols, the time-delay approach allows nonsmall network-induced delay. Future work will involve consideration of more general NCS models, including quantization and scheduling protocols for the actuator nodes.

Appendix A. Proof of Theorem 3.2.

Proof. Consider $t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+$ and define $\xi_i(t) = \mathrm{col}\{x(t), x(t-\eta_m), x(t-\tau(t)), x(t-\tau_M), \bar{\xi}_i(t), \omega(t)\},\ i = 1, \dots, N$, where

$$\begin{aligned}
&\bar{\xi}_i(t) = \mathrm{col}\{e_2(t), \dots, e_N(t)\},\ i = 1,\\
&\bar{\xi}_i(t) = \mathrm{col}\{e_1(t), \dots, e_{N-1}(t)\},\ i = N,\\
&\bar{\xi}_i(t) = \mathrm{col}\{e_1(t), \dots, e_j(t)_{|j\neq i}, \dots, e_N(t)\},\ i = 2, \dots, N-1.
\end{aligned}$$

Let $i_k^* = i \in \mathbb{N}$. Differentiating $V_e(t)$ along (8) and applying Jensen's inequality, we have

$$\begin{aligned}
&\eta_m \int_{t-\eta_m}^{t} \dot{x}^T(s) R_0 \dot{x}(s) ds\\
&\quad \geq \int_{t-\eta_m}^{t} \dot{x}^T(s) ds R_0 \int_{t-\eta_m}^{t} \dot{x}(s) ds\\
&\quad = \xi_i^T(t) (F_2^i)^T R_0 F_2^i \xi_i(t),\\
&\qquad -(\tau_M - \eta_m) \int_{t-\tau_M}^{t-\eta_m} \dot{x}^T(s) R_1 \dot{x}(s) ds\\
&\quad = -(\tau_M - \eta_m) \int_{t-\tau(t)}^{t-\eta_m} \dot{x}^T(s) R_1 \dot{x}(s) ds - (\tau_M - \eta_m) \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s) R_1 \dot{x}(s) ds\\
&\quad \leq -\frac{\tau_M - \eta_m}{\tau(t) - \eta_m} \xi_i^T(t) \Big[[I_n\ 0_{n\times n}] F^i \Big]^T R_1 [I_n\ 0_{n\times n}] F^i \xi_i(t)\\
&\qquad - \frac{\tau_M - \eta_m}{\tau_M - \tau(t)} \xi_i^T(t) \Big[[0_{n\times n}\ I_n] F^i \Big]^T R_1 [0_{n\times n}\ I_n] F^i \xi_i(t)\\
&\quad \leq -\xi_i^T(t) (F^i)^T \Phi F^i \xi_i(t).
\end{aligned}$$

The latter inequality holds if (22) is feasible [24]. Then

$$\begin{aligned}
\dot{V}_e(t) + 2\alpha V_e(t) - \frac{1}{\tau_M - \eta_m} \sum_{l=1, l\neq i}^{N} |\sqrt{U_l} e_l(t)|^2 - 2\alpha |\sqrt{Q_i} e_i(t)|^2 - b|\omega(t)|^2&\\
\leq \xi_i^T(t) [\Sigma_i + \Xi_i^T H \Xi_i - (F^i)^T \Phi F^i e^{-2\alpha\tau_M}] \xi_i(t) \leq 0&
\end{aligned}$$

if $\Sigma_i + \Xi_i^T H \Xi_i - (F^i)^T \Phi F^i e^{-2\alpha\tau_M} < 0$, i.e., by Schur complement, if (23) is feasible. Thus, due to Lemma 3.1, inequalities (15), (22), and (23) imply (17). ∎

Appendix B. Proof of Lemma 5.1.

Proof. Since $|\omega(t)| \leq \Delta$, (28) implies

$$\begin{aligned}
V(t, x_t, \dot{x}_t) &\leq V_e(t)\\
&\leq e^{-2\alpha(t-t_k)} V_e(t_k) + b\Delta^2 \int_{t_k}^{t} e^{-2\alpha(t-s)} ds,\ t \in [t_k, t_{k+1}).
\end{aligned} \tag{44}$$

Note that $V_e(t_{k+1}) = \tilde{V}_{|t=t_{k+1}} + V_{Q_{|t=t_{k+1}}} + V_{G_{|t=t_{k+1}}}$. Taking into account (32) and the relations $\tilde{V}_{t=t_{k+1}} = \tilde{V}_{t=t_{k+1}^-}$, $e(t_{k+1}^-) = e(t_k)$, we obtain due to (35), (36), and (39) for $k > N-1$

$$\begin{aligned}\Theta_{k+1} &\triangleq V_e(t_{k+1}) - V_e(t_{k+1}^-)\\ &= [V_Q + V_G]_{t=t_{k+1}} - [V_Q + V_G]_{t=t_{k+1}^-}\\ &\leq (\tau_M - \eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]}\\ &\qquad \left[\sum_{j=0}^{N-2}\int_{s_{k-j}}^{s_{k+1}} e^{2\alpha(s-t_{k+1})}\left|\sqrt{Q^*_{i_{k-j}}}C^*_{i_{k-j}}\dot{x}(s)\right|^2 ds\right.\\ &\qquad \left.- \sum_{i=1}^{N}(N-1)\int_{s_k}^{s_{k+1}} e^{2\alpha(s-t_{k+1})}|\sqrt{Q_i}C_i\dot{x}(s)|^2 ds\right],\end{aligned} \tag{45}$$

whereas for $k = N-1$ due to (37) and (39)

$$\begin{aligned}\Theta_N &\leq \sum_{j=0}^{N-2}(\tau_M - \eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]}\\ &\quad\times\int_{s_{N-1-j}}^{s_N} e^{2\alpha(s-t_N)}\left|\sqrt{Q^*_{i_{N-1-j}}}C^*_{i_{N-1-j}}\dot{x}(s)\right|^2 ds\\ &\quad - \sum_{i=1}^{N}(\tau_M-\eta_m)\int_{s_0}^{s_N} e^{2\alpha(s-t_N)}|\sqrt{G_i}C_i\dot{x}(s)|^2 ds\\ &\leq -(\tau_M-\eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]}\int_{s_0}^{s_N} e^{2\alpha(s-t_N)}\\ &\quad\times\left[(N-2)\sum_{i=1}^{N}|\sqrt{Q_i}C_i\dot{x}(s)|^2 + |\sqrt{Q_l}C_l\dot{x}(s)|^2_{|l=i^*_N}\right] ds.\end{aligned}$$

We will prove (40) by induction. For $k = N-1$, we have

$$\begin{aligned}V_e(t_N) &\leq \Theta_N + V_e(t_N^-)\\ &\leq -(\tau_M-\eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]}\\ &\quad\times\int_{s_0}^{s_N} e^{2\alpha(s-t_N)}\left[(N-2)\sum_{i=1}^{N}|\sqrt{Q_i}C_i\dot{x}(s)|^2 + |\sqrt{Q_l}C_l\dot{x}(s)|^2_{|l=i^*_N}\right] ds\\ &\quad + e^{-2\alpha(t_N-t_{N-1})}V_e(t_{N-1}) + b\Delta^2\int_{t_{N-1}}^{t_N} e^{-2\alpha(t_N-s)}ds,\end{aligned}$$

which implies (40).

Assume that (40) holds for $k-1$ ($k \geq N-1$):

$$V_e(t_k) \leq e^{-2\alpha(t_k-t_{N-1})}V_e(t_{N-1}) + \Psi_k + b\Delta^2\int_{t_{N-1}}^{t_k} e^{-2\alpha(t_k-s)}ds.$$

Then, due to (44) for $t = t_{k+1}^-$ we obtain

$$V_e(t_{k+1}) \le \Theta_{k+1} + e^{-2\alpha(t_{k+1}-t_k)}\Psi_k + e^{-2\alpha(t_{k+1}-t_{N-1})}V_e(t_{N-1}) + b\Delta^2 \int_{t_{N-1}}^{t_{k+1}} e^{-2\alpha(t_{k+1}-s)}ds.$$

We have

$$\begin{aligned} e^{-2\alpha(t_{k+1}-t_k)}\Psi_k &= -(\tau_M-\eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]} \\ &\quad \times \Bigg[\sum_{l=0}^{N-3}(N-2-l)\int_{s_{k-l-2}}^{s_k} e^{2\alpha(s-t_{k+1})} \times \left|\sqrt{Q_{i^*_{k-1-l}}}C_{i^*_{k-1-l}}\dot{x}(s)\right|^2 ds \\ &\qquad + (N-1)\int_{s_{k-1}}^{s_k} e^{2\alpha(s-t_{k+1})}\left|\sqrt{Q_{i^*_k}}C_{i^*_k}\dot{x}(s)\right|^2 ds\Bigg] \\ &= -(\tau_M-\eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]} \\ &\quad \Bigg[\sum_{j=0}^{N-2}(N-1-j)\times\int_{s_{k-j-1}}^{s_k} e^{2\alpha(s-t_{k+1})}\left|\sqrt{Q_{i^*_{k-j}}}C_{i^*_{k-j}}\dot{x}(s)\right|^2 ds\Bigg]. \end{aligned}$$

Then, taking into account (45), we find

$$\begin{aligned} &\Theta_{k+1} + e^{-2\alpha(t_{k+1}-t_k)}\Psi_k \\ &\quad \le (\tau_M-\eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]} \\ &\quad\quad \times \Bigg[\sum_{j=0}^{N-2}\int_{s_{k-j-1}}^{s_{k+1}} e^{2\alpha(s-t_{k+1})}\left|\sqrt{Q^*_{i_{k-j}}}C^*_{i_{k-j}}\dot{x}(s)\right|^2 ds \\ &\quad\quad\quad - \sum_{i=1}^{N}(N-1)\int_{s_k}^{s_{k+1}} e^{2\alpha(s-t_{k+1})}|\sqrt{Q_i}C_i\dot{x}(s)|^2 ds \\ &\quad\quad\quad - \sum_{j=0}^{N-2}(N-1-j)\int_{s_{k-j-1}}^{s_k} e^{2\alpha(s-t_{k+1})}\times\left|\sqrt{Q_{i^*_{k-j}}}C_{i^*_{k-j}}\dot{x}(s)\right|^2 ds\Bigg] \\ &\quad \le -(\tau_M-\eta_m)e^{2\alpha[\tau_M+(N-2)(\tau_M-\eta_m)]} \\ &\quad\quad \times \Bigg[\sum_{j=0}^{N-2}(N-2-j)\int_{s_{k-j-1}}^{s_{k+1}} e^{2\alpha(s-t_{k+1})}\times\left|\sqrt{Q_{i^*_{k-j}}}C_{i^*_{k-j}}\dot{x}(s)\right|^2 ds \\ &\quad\quad\quad + (N-1)\int_{s_k}^{s_{k+1}} e^{2\alpha(s-t_{k+1})}\left|\sqrt{Q_{i^*_{k+1}}}C_{i^*_{k+1}}\dot{x}(s)\right|^2 ds\Bigg] \\ &\quad = \Psi_{k+1}, \end{aligned}$$

which implies (40). Hence, (40) and (44) yield (42). ∎

REFERENCES

[1] P. Antsaklis and J. Baillieul, *Special issue on technology of networked control systems*, Proc. IEEE, 95 (2007), pp. 5–8.

[2] N.W. Bauer, P.J.H. Maas, and W.P.M.H. Heemels, *Stability analysis of networked control systems: A sum of squares approach*, Automatica, 48 (2012), pp. 1514–1524.

[3] M.B.G. Cloosterman, L. Hetel, N. van de Wouw, W.P.M.H. Heemels, J. Daafouz, and H. Nijmeijer, *Controller synthesis for networked control systems*, Automatica, 46 (2010), pp. 1584–1594.
[4] D. Christmann, R. Gotzhein, S. Siegmund, and F. Wirth, *Realization of Try-Once-Discard in wireless multi-hop networks*, in Proceedings of the 18th World IFAC Congress, Milano, Italy, 2011.
[5] D. Dacic and D. Nesic, *Quadratic stabilization of linear networked control systems via simultaneous protocol and controller design*, Automatica, 43 (2007), pp. 1145–1155.
[6] M.C.F. Donkers, W.P.M.H. Heemels, N. van de Wouw, and L. Hetel, *Stability analysis of networked control systems using a switched linear systems approach*, IEEE Trans. Automat. Control, 56 (2011), pp. 2101–2115.
[7] E. Fridman, *A refined input delay approach to sampled-data control*, Automatica, 46 (2010), pp. 421–427.
[8] E. Fridman, *Introduction to Time-Delay Systems*, Birkhäuser, Basel, Switzerland, 2014.
[9] E. Fridman and K. Liu, *Networked control under Round-Robin protocol: Multiple sensors and non-small communication delays*, in Proceedings of the 53rd IEEE Conference on Decision and Control, Los Angeles, 2014.
[10] E. Fridman, A. Seuret, and J.P. Richard, *Robust sampled-data stabilization of linear systems: An input delay approach*, Automatica, 40 (2004), pp. 1441–1446.
[11] E. Fridman and U. Shaked, *Delay-dependent stability and H_∞ control: Constant and time-varying delays*, Internat. J. Control, 76 (2003), pp. 48–60.
[12] H. Fujioka, *A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices*, IEEE Trans. Automat. Control, 54 (2009), pp. 2440–2445.
[13] H. Gao, T. Chen, and T. Chai, *Passivity and passification for networked control systems*, SIAM J. Control Optim., 46 (2007), pp. 1299–1322.
[14] H. Gao, X. Meng, T. Chen, and J. Lam, *Stabilization of networked control systems via dynamic output-feedback controllers*, SIAM J. Control Optim., 48 (2010), pp. 3643–3658.
[15] J.C. Geromel, R.H. Korogui, and J. Bernussou, *H_2 and H_∞ robust output feedback control for continuous time polytopic systems*, IET Control Theory Appl., 1 (2007), pp. 1541–1549.
[16] K. Gu, V. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*, Birkhäuser, Basel, Switzerland, 2003.
[17] W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, and D. Nesic, *Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance*, IEEE Trans. Automat. Control, 55 (2010), pp. 1781–1796.
[18] K. Liu and E. Fridman, *Discrete-time network-based control under scheduling and actuator constraints*, Internat. J. Robust Nonlinear Control, 25 (2015), pp. 1816–1830.
[19] K. Liu, E. Fridman, and L. Hetel, *Stability and L_2-gain analysis of networked control systems under Round-Robin scheduling: A time-delay approach*, Systems Control Lett., 61 (2012), pp. 666–675.
[20] K. Liu, E. Fridman, and L. Hetel, *Network-based control via a novel analysis of hybrid systems with time-varying delays*, in Proceedings of the 51st IEEE Conference on Decision and Control, Maui, HI, 2012.
[21] P. Naghshtabrizi, J. Hespanha, and A.R. Teel, *Exponential stability of impulsive systems with application to uncertain sampled-data systems*, Systems Control Lett., 57 (2008), pp. 378–385.
[22] P. Naghshtabrizi, J. Hespanha, and A.R. Teel, *Stability of delay impulsive systems with application to networked control systems*, Trans. Inst. Measurement Control, 32 (2010), pp. 511–528.
[23] D. Nesic and A.R. Teel, *Input-output stability properties of networked control systems*, IEEE Trans. Automat. Control, 49 (2004), pp. 1650–1667.
[24] P. Park, J. Ko, and C. Jeong, *Reciprocally convex approach to stability of systems with time-varying delays*, Automatica, 47 (2011), pp. 235–238.
[25] A. Seuret, *A novel stability analysis of linear systems under asynchronous samplings*, Automatica, 48 (2012), pp. 177–182.
[26] G.C. Walsh, H. Ye, and L.G. Bushnell, *Stability analysis of networked control systems*, IEEE Trans. Control Syst. Technol., 10 (2002), pp. 438–446.