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Brief paper

Constructive method for finite-dimensional observer-based control of 1-D parabolic PDEs[☆]

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ABSTRACT

The objective of the present paper is finite-dimensional observer-based control of the 1-D linear heat equation with constructive and feasible design conditions. We propose a method which is applicable to boundary or non-local sensing together with non-local actuation, or to Dirichlet actuation with non-local sensing. We use a modal decomposition approach. The dimension of the controller, N_0, is equal to the number of modes which decay slower than a given decay rate $\delta > 0$. The observer may have a larger dimension $N \geq N_0$. The observer and controller gains are found separately by solving $N_0 \times N_0$-dimensional Lyapunov inequalities. We suggest a direct Lyapunov approach to the full-order closed-loop system and provide linear matrix inequalities (LMIs) for finding N and the exponential decay rate of the closed-loop system. We prove that the LMIs are always feasible for large enough N. The proposed method is different from existing qualitative methods that do not give easily verifiable and efficient bounds on the observer-based controller dimension and the resulting closed-loop performance. Numerical examples demonstrate that our LMI conditions lead to non-conservative bounds on N and the resulting decay rate.

1. Introduction

Observer-based controllers for linear PDEs have been constructed by the modal decomposition approach (Curtain, 1982; Katz, Fridman, & Selivanov, 2020; Lasiecka & Triggiani, 2000; Orlov, Lou, & Christofides, 2004), the backstepping method (Krstic & Smyshlyaev, 2008) and by the spatial decomposition (sampling) method (Selivanov & Fridman, 2018), where the observer is found in the form of PDE. A PDE observer (that can be implemented via approximations Lasiecka & Triggiani, 2000) usually leads to separation of the controller and observer designs. Finite-dimensional observers and the resulting controllers, which are very attractive in applications, generally do not allow such separation. Therefore, design of the latter controllers is a very challenging control problem.

Finite-dimensional observer-based controllers for parabolic systems were designed by modal decomposition approach (Balas, 1988; Christofides, 2001; Curtain, 1982; Harkort & Deutscher, 2011). In particular, in the seminal work (Curtain, 1982) such controllers were suggested in case where the control input enters the PDE via one or several shape functions with special structure: these functions are linear combinations of a finite number of eigenfunctions. The latter assumption maintains the separation of state and error dynamics, but is highly restrictive in practice. For bounded control and observation operators, it was shown in Balas (1988) that the closed-loop system is stable provided the dimension of the controller is large enough. However, a constructive method for finding this dimension was not provided. A singular perturbation approach that reduces the controller design to a finite-dimensional slow system was suggested in Christofides (2001), without giving constructive and rigorous conditions for finding the dimension of the slow system that guarantees a desired closed-loop performance of the full-order system.

The first step towards constructive conditions for the finite-dimensional observer-based controller was done in Harkort and Deutscher (2011), where a quantitative bound on the controller and observer dimensions was suggested via modal decomposition in the case of bounded control and observation operators. However, as mentioned in Harkort and Deutscher (2011), the obtained bound may be difficult to compute and is highly conservative.

In the framework of modal decomposition methods for parabolic PDEs, a direct Lyapunov method for either state-feedback or observer design has been suggested in Coron and Trélat (2004),

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Karafyllis, Ahmed-Ali, and Giri (2019) and Prieur and Trélat (2018). Note that in the latter papers, due to separation of the finite-dimensional part from the remainder, there is no problem of the feasibility of the conditions. Recently, a delayed finite-dimensional boundary observer was introduced for the 1-D heat equation in Selivanov and Fridman (2019).

The objective of the present paper is finite-dimensional observer-based control of 1-D heat equation with easily verifiable and nonconservative conditions. We propose a method which is applicable to boundary or non-local sensing together with non-local actuation, or to Dirichlet actuation with non-local sensing. We use a modal decomposition approach. The dimension of our controller, N_0, is equal to the number of modes which decay slower than a given decay rate $\delta > 0$. The observer may have a larger dimension $N \geq N_0$. The observer and controller gains are found separately by solving $N_0 \times N_0$-dimensional Lyapunov inequalities.

Inspired by Coron and Trélat (2004), Karafyllis et al. (2019) and Prieur and Trélat (2018), we suggest a direct Lyapunov approach to the full-order closed-loop system and provide LMIs for finding N and the resulting exponential decay rate. Numerical examples show that the LMIs lead to non-conservative bounds on N and the decay rate.

Differently from state-feedback or PDE observer-based control, the main challenge under the finite-dimensional observer is to prove that the LMIs are always feasible for large enough N (see e.g. proof of Theorem 3.1). Note that the size of our LMIs grows with N, which may lead to unboundedness of the norm of the closed-loop system matrix F and the norm of $P > 0$, which satisfies the Lyapunov inequality $PF + F^TP + 2\delta P < 0$. Unboundedness of the norm of P can cause the LMIs to be infeasible for all N. We propose a decomposition of F into a triangular Hurwitz matrix and a perturbation, and present a novel asymptotic perturbation analysis to obtain bounds on the norm of P in terms of N.

The article is organized as follows. In Section 2, some mathematical preliminaries are given. Section 3 is devoted to main results. Numerical examples are given in Section 4, and conclusions are drawn in Section 5. Some preliminary results confined to bounded control and observation operators have been presented in Katz and Fridman (2020).

Notation. We denote by $L^2(0,1)$ the Hilbert space of Lebesgue measurable and square integrable functions $f : [0,1] \to \mathbb{R}$ with the corresponding inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)dx$ and induced norm $\|f\|_{L^2}^2 := \langle f, f\rangle$. $H^1(0,1)$ is the Sobolev space of functions $f : [0,1] \to \mathbb{R}$ with a square integrable weak derivative. The norm defined on $H^1(0,1)$ is $\|f\|_{H^1}^2 := \|f\|_{L^2}^2 + \left\|f'\right\|_{L^2}^2$. The standard Euclidean norm on $\mathbb{R}^n$ will be denoted by $\|\cdot\|$. $H^2(0,1)$ is the Sobolev space of functions $f : [0,1] \to \mathbb{R}$ with a square integrable weak derivative of the second order. For $A \in \mathbb{R}^{n\times n}$, the operator norm of A, induced by $\|\cdot\|$, is denoted by $\|\cdot\|_2$. For $P \in \mathbb{R}^{n\times n}$, the notation $P > 0$ means that P is symmetric and positive definite. The sub-diagonal elements of a symmetric matrix will be denoted by $*$. For $U \in \mathbb{R}^{n\times n}$, $U > 0$ and $x \in \mathbb{R}^n$ we denote $\|x\|_U^2 := x^TUx$.

2. Mathematical preliminaries

Consider the Sturm–Liouville operator:

$$\begin{aligned} &\mathcal{A}h = -\tfrac{d}{dx}\left(p(x)\tfrac{d}{dx}h(x)\right) + q(x)h(x), \\ &\mathcal{D}(\mathcal{A}) = \left\{h \in H^2(0,1) \middle|\ h'(0) = h(1) = 0\right\}. \end{aligned} \tag{2.1}$$

Here $p \in C^2([0,1])$ and $q \in C([0,1])$ are subject to the following bounds:

$$0 < p_* \leq p(x) \leq p^*,\ \ 0 \leq q(x) \leq q^*, \quad x \in [0,1]. \tag{2.2}$$

The assumption $q(x) \in [0, q^*]$ is made for simplicity only. Otherwise one can consider the shifted operator $\mathcal{A} + \mu I$ with appropriate $\mu \in \mathbb{R}$. The Sturm–Liouville operator (2.1) has a sequence of eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots$ satisfying

$$\pi^2(n-1)^2 p_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^*, \quad n \geq 1 \tag{2.3}$$

with corresponding eigenfunctions $\phi_n(x), n \geq 1$ (Orlov, 2017). The eigenfunctions form a complete orthonormal system in $L^2(0,1)$. Note that in the case $p(x) \equiv 1$ and $q(x) \equiv 0$, λ_n and ϕ_n can be computed explicitly:

$$\lambda_n = \pi^2\left(n - \frac{1}{2}\right)^2, \quad \phi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}x),\ n \geq 1. \tag{2.4}$$

Following the assumptions on $p(x)$ and $q(x)$, $\mathcal{A}$ is positive and $-\mathcal{A}$ is a sectorial operator, which generates an *analytic semigroup* on $L^2(0,1)$. Furthermore, $\mathcal{A}$ has a square root $\mathcal{A}^{\frac{1}{2}} : \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to L^2(0,1)$, where

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \left\{h \in L^2(0,1) \middle|\ \textstyle\sum_{n=1}^{\infty} \lambda_n |\langle h, \phi_n\rangle|^2 < \infty\right\} \tag{2.5}$$

(for further details, see Tucsnak & Weiss, 2009, Section 3.5). Recall that $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ is the completion of $\mathcal{D}(\mathcal{A})$ with respect to the norm

$$\begin{aligned} &\|g\|_{\frac{1}{2}} = \left(\int_0^1 p(x)\left|g'(x)\right|^2 + q(x)\left|g(x)\right|^2 dx\right)^{\frac{1}{2}} \\ &= \sqrt{\langle \mathcal{A}g, g\rangle} = \sqrt{\textstyle\sum_{n=1}^{\infty} \lambda_n |\langle g, \phi_n\rangle|^2},\ \ g \in \mathcal{D}(\mathcal{A}). \end{aligned} \tag{2.6}$$

By (2.6) and Wirtinger's inequality we have:

$$p_* \left\|g'\right\|_{L^2}^2 \leq \|g\|_{\frac{1}{2}}^2 \leq \tfrac{p^*\pi^2 + 4q^*}{\pi^2} \left\|g'\right\|_{L^2}^2, \tag{2.7}$$

implying for $h \in \mathcal{D}(\mathcal{A})$

$$\frac{\pi^2}{p^*\pi^2 + 4q^*}\sum_{n=1}^{\infty} \lambda_n h_n^2 \leq \left\|h'\right\|_{L^2}^2 \leq \frac{1}{p_*}\sum_{n=1}^{\infty} \lambda_n h_n^2. \tag{2.8}$$

Note that due to equivalence of $\|\cdot\|_{\frac{1}{2}}$ and $\|\cdot\|_{H^1}$ subject to $g(1) = 0$ for any $g \in \mathcal{D}(\mathcal{A})$ we have

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \left\{h \in H^1(0,1) \middle|\ h(1) = 0\right\}. \tag{2.9}$$

Finally, density of $\mathcal{D}(\mathcal{A})$ in $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ yields that (2.8) holds for any $h \overset{L^2(0,1)}{=} \sum_{n=1}^{\infty} h_n\phi_n \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. Summarizing:

Lemma 2.1. *Let $h \in L^2(0,1)$ be given by $h \overset{L^2(0,1)}{=} \sum_{n=1}^{\infty} h_n\phi_n$. Then, $h \in H^1(0,1)$ satisfies $h(1) = 0$ iff $\sum_{n=1}^{\infty} \lambda_n h_n^2 < \infty$. Moreover, in this case (2.8) holds.*

3. Observer–controller design

3.1. Non-local measurement and actuation: L^2-stability

Consider the reaction–diffusion system

$$\begin{aligned} &z_t(x,t) = \partial_x\left(p(x)z_x(x,t)\right) + (q_c - q(x))z(x,t) + b(x)u(t), \\ &z_x(0,t) = 0, \quad z(1,t) = 0, \end{aligned} \tag{3.1}$$

where $t \geq 0$, $x \in [0,1]$, $z(x,t) \in \mathbb{R}$, $q_c \in \mathbb{R}$ is a constant reaction coefficient, $b \in L^2(0,1)$ and $u(t)$ is the control input. For large enough $q_c > 0$ the open-loop system (3.1) is unstable. We consider non-local measurement

$$y(t) = \int_0^1 c(x)z(x,t)dx, \tag{3.2}$$

where $c \in L^2(0,1)$. Given any $\delta > 0$, our objective is exponential stabilization of (3.1) with a decay rate δ.

We begin by presenting the solution to (3.1) as

$$z(x,t)=\sum_{n=1}^{\infty} z_n(t)\phi_n(x),\ \ z_n(t)=\langle z(\cdot,t),\phi_n\rangle\,. \tag{3.3}$$

By differentiating under the integral sign, integrating by parts and using (2.1), we have

$$\begin{aligned}\dot z_n(t) &= \int_0^1 z_t(x,t)\phi_n(x)dx = q_c z_n(t)+b_n u(t)\\ &\quad+\int_0^1\left[\partial_x\left(p(x)z_x(x,t)\right)-q(x)z(x,t)\right]\phi_n(x)dx\\ &=(-\lambda_n+q_c)z_n(t)+b_n u(t),\\ z_n(0)&=\langle z_0,\phi_n\rangle =: z_{0,n},\quad b_n=\langle b,\phi_n\rangle\,.\end{aligned} \tag{3.4}$$

Since $\lim_{n\to\infty}\lambda_n=+\infty$, there exists $N_0\in\mathbb{N}$ such that

$$-\lambda_n+q_c<-\delta,\quad n>N_0. \tag{3.5}$$

N_0 will define the dimension of the controller, whereas $N\geq N_0$ will be the dimension of the observer.

We construct a finite-dimensional observer of the form

$$\hat z(x,t):=\sum_{n=1}^{N}\hat z_n(t)\phi_n(x), \tag{3.6}$$

where $\hat z_n(t)$ satisfy the ODEs

$$\begin{aligned}\dot{\hat z}_n(t) &= (-\lambda_n+q_c)\hat z_n(t)+b_n u(t)\\ &\quad -l_n\left[\int_0^1 c(x)\left(\sum_{n=1}^N \hat z_n(t)\phi_n(x)\right)dx-y(t)\right],\\ \hat z_n(0)&=0,\quad 1\leq n\leq N.\end{aligned} \tag{3.7}$$

Here l_n are scalar observer gains.

Denote

$$\begin{aligned}&A_0=\mathrm{diag}\left\{-\lambda_1+q_c,\ldots,-\lambda_{N_0}+q_c\right\},\\ &L_0=\left[l_1,\ldots,l_{N_0}\right]^T,\ \ C_0=\left[c_1,\ldots,c_{N_0}\right],\\ &c_n=\langle c,\phi_n\rangle\,,\quad n\geq 1.\end{aligned} \tag{3.8}$$

Assume that

$$c_n\neq 0,\quad 1\leq n\leq N_0. \tag{3.9}$$

Then, the pair (A_0,C_0) is observable by the Hautus lemma. We choose $l_1,\ldots,l_{N_0}$ such that L_0 satisfies the following Lyapunov inequality:

$$P_o(A_0-L_0C_0)+(A_0-L_0C_0)^TP_o<-2\delta P_o, \tag{3.10}$$

where $P_o\in\mathbb{R}^{N_0\times N_0}$ satisfies $P_o>0$. Furthermore, we choose $l_n=0,\ n>N_0$. We assume

$$b_n\neq 0,\quad 1\leq n\leq N_0, \tag{3.11}$$

where $b_n=\langle b,\phi_n\rangle$, and denote

$$B_0:=\begin{bmatrix}b_1 & \ldots & b_{N_0}\end{bmatrix}^T. \tag{3.12}$$

By the Hautus lemma the pair (A_0,B_0) is controllable. Let $K_0\in\mathbb{R}^{1\times N_0}$ satisfy

$$P_c(A_0+B_0K_0)+(A_0+B_0K_0)^TP_c<-2\delta P_c, \tag{3.13}$$

where $P_c\in\mathbb{R}^{N_0\times N_0}$ satisfies $P_c>0$. We propose a N_0-dimensional controller of the form

$$u(t)=K_0\hat z^{N_0}(t),\ \ \hat z^{N_0}(t)=\left[\hat z_1(t),\ldots,\hat z_{N_0}(t)\right]^T, \tag{3.14}$$

which is based on the N-dimensional observer (3.7).

Remark 3.1. In the state-feedback case, the control law $u(t)=K_0z^{N_0}(t)$ with $z^{N_0}(t)=\left[z_1(t),\ldots,z_{N_0}(t)\right]^T$ stabilizes the finite-dimensional part of (3.4), where $1\leq n\leq N_0$. Since the resulting closed-loop is separated from z_n with $n>N+1$, $u(t)=K_0z^{N_0}(t)$ stabilizes (3.1). A finite-dimensional observer-based controller was proposed in Curtain (1982) under very restrictive assumptions on b ($b_n=0,\ n>N_0$), which ensure separation of z_n with $n>N+1$ from the finite-dimensional part. For a general $b\in L^2(0,1)$ the observer, estimation error and remainder are coupled (see (3.22)).

For well-posedness of the closed-loop system (3.1) and (3.7) with $u(t)=K_0\hat z^{N_0}(t)$, let $\mathcal{H}:=\mathcal{D}\left(\mathcal{A}^{\frac12}\right)\times\mathbb{R}^N$, with $\mathcal{D}(\mathcal{A}^{\frac12})$ defined in (2.5). This is a Hilbert space with the norm $\|\cdot\|^2_{\mathcal{H}}:=\|\cdot\|^2_{H^2}+\|\cdot\|^2$. Defining the state $\xi(t)$ as

$$\xi(t)=\begin{bmatrix}z(\cdot,t) & \hat z^N(t)\end{bmatrix}^T\in\mathcal{H},\quad \hat z^N(t)=\left[\hat z_1(t),\ldots,\hat z_N(t)\right]^T,$$

the closed-loop system can be presented as

$$\begin{aligned}&\frac{d}{dt}\xi(t)+\tilde{\mathcal{A}}\xi(t)=\begin{bmatrix}f_1(\xi)\\ f_2(\xi)\end{bmatrix},\quad \tilde{\mathcal{A}}=\begin{bmatrix}\mathcal{A} & 0\\ 0 & \mathcal{A}_2\end{bmatrix},\\ &\mathcal{A}_2:\mathbb{R}^N\to\mathbb{R}^N,\quad \mathcal{A}_2v=\begin{bmatrix}-A_0-B_0K_0+L_0C_0 & L_0C_1\\ -B_1K_0 & -A_1\end{bmatrix}v,\\ &f_1(\xi)=b(\cdot)K_0\hat z^{N_0}(t)+q_cz(\cdot,t),\\ &f_2(\xi)=\begin{bmatrix}L_0\langle c,z(\cdot,t)\rangle & 0\end{bmatrix}^T,\\ &A_1=\mathrm{diag}\left\{-\lambda_{N_0+1}+q_c,\ldots,-\lambda_N+q_c\right\},\\ &C_1=\left[c_{N_0+1},\ldots,c_N\right],\ \ B_1=\left[b_{N_0+1},\ldots,b_N\right]^T.\end{aligned} \tag{3.15}$$

$\tilde{\mathcal{A}}$ has a domain $\mathcal{D}(\tilde{\mathcal{A}})=\mathcal{D}(\mathcal{A})\times\mathbb{R}^N$, with $\mathcal{D}(\mathcal{A})$ defined in (2.1). Furthermore, since $-\mathcal{A}$ generates an *analytic semigroup* on $L^2(0,1)$ and $\mathcal{A}_2$ is a bounded operator on $\mathbb{R}^N$, $-\tilde{\mathcal{A}}$ generates an *analytic semigroup* on $\mathcal{H}$.

Let $\xi_i\in\mathcal{H},\quad i\in\{1,2\}$. It can be easily verified that

$$\begin{aligned}&\|f_1(\xi_1)-f_1(\xi_2)\|_{L^2}\leq\left(\|b\|_{L^2}\|K_0\|+|q_c|\right)\|\xi_1-\xi_2\|_{\mathcal{H}}\,,\\ &\|f_2(\xi_1)-f_2(\xi_2)\|\leq\|c\|_{L^2}\|L_0\|\,\|\xi_1-\xi_2\|_{\mathcal{H}}\,.\end{aligned}$$

These estimates, together with the fact that $f_1(0)=f_2(0)=0$ imply that $\mathrm{col}\{f_1,f_2\}:\mathcal{H}\to\mathcal{H}$ satisfies assumptions (F) and (3.22) of Section 6.3 in Pazy (1983) with $\alpha=0$ (which is one of the cases discussed therein. See the last paragraph of p. 195). Theorems 6.3.1 and 6.3.3 in Pazy (1983) imply that system (3.1), (3.7) with $u(t)=K_0\hat z^{N_0}(t)$ and initial condition $z_0=z(\cdot,0)\in L^2(0,1)$ has a unique *classical* solution

$$\xi\in C\left([0,\infty);\mathcal{H}\right)\cap C^1\left((0,\infty);\mathcal{H}\right) \tag{3.16}$$

such that

$$\xi(t)\in\mathcal{D}\left(\tilde{\mathcal{A}}\right)=\mathcal{D}\left(\mathcal{A}\right)\times\mathbb{R}^N\quad\forall t>0. \tag{3.17}$$

The latter result follows from (3.20) and (3.21) in Pazy (1983), Section 6.3.

Let

$$e_n(t)=z_n(t)-\hat z_n(t),\ \ 1\leq n\leq N \tag{3.18}$$

be the estimation error. By using (3.3) and (3.6), the last term on the right-hand side of (3.7) can be written as

$$\begin{aligned}&\int_0^1 c(x)\left[\sum_{n=1}^N\hat z_n(t)\phi_n(x)-\sum_{n=1}^\infty z_n(t)\phi_n(x)\right]dx\\ &=-\sum_{n=1}^N c_ne_n(t)-\zeta(t),\quad \zeta(t)=\sum_{n=N+1}^\infty c_nz_n(t).\end{aligned} \tag{3.19}$$

Then the error equation has the form

$$\dot e_n(t)=(-\lambda_n+q)e_n(t)-l_n\left(\sum_{n=1}^N c_ne_n(t)+\zeta(t)\right),\ \ 1\leq n\leq N. \tag{3.20}$$

Denote

$$
\begin{aligned}
&e^{N_0}(t)=\left[e_1(t),\dots,e_{N_0}(t)\right]^T,\\
&e^{N-N_0}(t)=\left[e_{N_0+1}(t),\dots,e_N(t)\right]^T,\\
&\hat z^{N-N_0}(t)=\left[\hat z_{N_0+1}(t),\dots,\hat z_N(t)\right]^T,\\
&X(t)=\mathrm{col}\left\{\hat z^{N_0}(t),e^{N_0}(t),\hat z^{N-N_0}(t),e^{N-N_0}(t)\right\},\\
&\mathcal{L}=\mathrm{col}\left\{L_0,-L_0,0_{2(N-N_0)\times 1}\right\},\ \tilde K=\left[K_0,\quad 0_{1\times(2N-N_0)}\right],\\
&F=\begin{bmatrix} A_0+B_0K_0 & L_0C_0 & 0 & L_0C_1\\ 0 & A_0-L_0C_0 & 0 & -L_0C_1\\ B_1K_0 & 0 & A_1 & 0\\ 0 & 0 & 0 & A_1\end{bmatrix}.
\end{aligned}
\tag{3.21}
$$

From (3.4), (3.7), (3.14), (3.19) and (3.20), by using A_1, B_1, C_1 defined in (3.15), we present the closed-loop system for $\hat z$ given by (3.6) and the estimation error

$$z(x,t)-\hat z(x,t)=\sum_{n=1}^{N}e_n(t)\phi_n(x)+\sum_{n=N+1}^{\infty}z_n(t)\phi_n(x)$$

as follows:

$$
\begin{aligned}
&\dot X(t)=FX(t)+\mathcal{L}\zeta(t),\quad t\geq 0,\\
&\dot z_n(t)=(-\lambda_n+q_c)z_n(t)+b_n\tilde K X(t),\quad n>N.
\end{aligned}
\tag{3.22}
$$

Note that we consider the closed-loop system in $\ell^2(\mathbb{N})$ by using the isometry between $L^2(0,1)$ and $\ell^2(\mathbb{N})$ given by $L^2(0,1)\ni h\mapsto\{\langle h,\phi_n\rangle\}_{n=1}^{\infty}\in\ell^2(\mathbb{N})$. The Cauchy–Schwarz inequality implies the following estimate

$$
\begin{aligned}
\zeta^2(t)&\leq\left(\textstyle\sum_{n=N+1}^{\infty}c_n^2\right)\left(\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)\right)\\
&\leq\|c\|_{L^2}^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t).
\end{aligned}
\tag{3.23}
$$

For L^2-stability analysis of the closed-loop system (3.22), we define the Lyapunov function

$$V(t)=\|X(t)\|_P^2+\sum_{n=N+1}^{\infty}z_n^2(t),\tag{3.24}$$

where $P\in\mathbb{R}^{2N\times 2N}$ satisfies $P>0$. Using Parseval's equality and $\hat z_n^2+e_n^2=(z_n-e_n)^2+e_n^2\geq 0.5z_n^2$, we have

$$
\begin{aligned}
&V(t)\geq\lambda_{min}(P)\textstyle\sum_{n=1}^{N}\left[\hat z_n^2(t)+e_n^2(t)\right]+\sum_{n=N_0+1}^{\infty}z_n^2(t)\\
&\geq\min\left(\tfrac{\lambda_{min}(P)}{2},1\right)\|z(\cdot,t)\|_{L^2}^2,\\
&V(t)\leq\lambda_{max}(P)\textstyle\sum_{n=1}^{N}\left[\hat z_n^2(t)+e_n^2(t)\right]+\sum_{n=N_0+1}^{\infty}z_n^2(t)\\
&\leq\max\left(\tfrac{\lambda_{max}(P)}{2},1\right)\|z(\cdot,t)\|_{L^2}^2,\quad t\geq 0.
\end{aligned}
\tag{3.25}
$$

Since $z(\cdot,t)$ is a classical solution, the series $\sum_{n=N+1}^{\infty}z_n^2(t)$ can be differentiated term-by-term. Differentiation of $V(t)$ along (3.22) gives

$$
\begin{aligned}
&\dot V+2\delta V=X^T(t)\left[PF+F^TP+2\delta P\right]X(t)\\
&+2X^T(t)P\mathcal{L}\zeta(t)+2\textstyle\sum_{n=N+1}^{\infty}(-\lambda_n+q_c+\delta)z_n^2(t)\\
&+2\textstyle\sum_{n=N+1}^{\infty}z_n(t)b_n\tilde K X(t).
\end{aligned}
\tag{3.26}
$$

Furthermore, the Cauchy–Schwarz inequality implies

$$
\begin{aligned}
&\textstyle\sum_{n=N+1}^{\infty}2z_n(t)b_n\tilde K X(t)\leq\frac{1}{\alpha}\sum_{n=N+1}^{\infty}z_n^2(t)\\
&+\alpha\left(\textstyle\sum_{n=N+1}^{\infty}b_n^2\right)\left\|\tilde K X(t)\right\|^2\leq\frac{1}{\alpha}\sum_{n=N+1}^{\infty}z_n^2(t)\\
&+\alpha\|b\|_{L^2}^2\left\|\tilde K X(t)\right\|^2,
\end{aligned}
\tag{3.27}
$$

where $\alpha>0$. Denote $\eta(t)=\mathrm{col}\{X(t),\zeta(t)\}$. By (3.26) with (3.27) and taking into account (3.23) we obtain for some $\beta>0$

$$
\begin{aligned}
&\dot V+2\delta V+\beta\left(\|c\|_{L^2}^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)-\zeta^2(t)\right)\\
&\leq\eta^T(t)\Psi\eta(t)+2\textstyle\sum_{n=N+1}^{\infty}W_nz_n^2(t)\leq 0
\end{aligned}
\tag{3.28}
$$

if $W_n=-\lambda_n+q_c+\delta+\frac{1}{2\alpha}+\frac{\beta\|c\|_{L^2}^2}{2}<0,\quad n>N$ and

$$\Psi=\begin{bmatrix}PF+F^TP+2\delta P+\alpha\|b\|_{L^2}^2\tilde K^T\tilde K & P\mathcal{L}\\ * & -\beta\end{bmatrix}<0.\tag{3.29}$$

Note that monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and Schur's complement imply that $W_n<0$ for all $n>N$ iff

$$\begin{bmatrix}-\lambda_{N+1}+q_c+\delta+\frac{\beta\|c\|_{L^2}^2}{2} & \frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & -\alpha\end{bmatrix}<0.\tag{3.30}$$

Summarizing, we arrive at:

Theorem 3.1. *Consider (3.1) with $b\in L^2(0,1)$ satisfying (3.11), measurement (3.2) with $c\in L^2(0,1)$ satisfying (3.9), control law (3.14) and $z_0\in L^2(0,1)$. Let $\delta>0$ be a desired decay rate, $N_0\in\mathbb{N}$ satisfy (3.5) and $N\in\mathbb{N}$ satisfy $N_0\leq N$. Assume that L_0 and K_0 are obtained using (3.10) and (3.13), respectively. If there exist a positive definite matrix $P\in\mathbb{R}^{2N\times 2N}$ and scalars $\alpha,\beta>0$ which satisfy (3.29) and (3.30), then the solution $z(x,t)$ to (3.1) under the control law (3.14), (3.7) and the corresponding observer $\hat z(x,t)$ defined by (3.6) satisfy the following inequalities*

$$
\begin{aligned}
&\|z(\cdot,t)\|_{L^2}^2\leq Me^{-2\delta t}\|z_0\|_{L^2}^2,\\
&\left\|z(\cdot,t)-\hat z(\cdot,t)\right\|_{L^2}^2\leq Me^{-2\delta t}\|z_0\|_{L^2}^2,
\end{aligned}
\tag{3.31}
$$

with some constant $M>0$. Moreover, LMIs (3.29) and (3.30) are always feasible for large enough N.

Proof. To show (3.31), we note that (3.28) implies

$$V(t)\leq e^{-2\delta t}V(0),\quad t\geq 0.\tag{3.32}$$

By (3.24), for some $M_0>0$ we have

$$V(0)\leq M_0\|z_0\|_{L^2}^2\tag{3.33}$$

Finally, (3.32) and (3.25) imply (3.31).

For the proof of the feasibility of LMIs (3.29) and (3.30) we will first show that the solution to the Lyapunov equation

$$P(F+\delta I)+(F+\delta I)^TP=-I.\tag{3.34}$$

satisfies $\|P\|_2=O(1)$, uniformly in N. Note that this solution is given by

$$P=\int_0^{\infty}e^{(F+\delta I)^Tt}e^{(F+\delta I)t}dt.\tag{3.35}$$

So, it is sufficient to show that for some constants $\kappa_0>0$ and $M_0>0$, independent of N, the following inequality holds:

$$\left\|e^{(F+\delta I)t}\right\|_2\leq M_0e^{-\kappa_0 t},\quad t\geq 0.\tag{3.36}$$

To prove (3.36), we present $F+\delta I=\tilde F_1+\tilde F_2$, where

$$
\begin{aligned}
&\tilde F_1=\begin{bmatrix}A_0+B_0K_0 & L_0C_0 & 0 & 0\\ 0 & A_0-L_0C_0 & 0 & 0\\ 0 & 0 & A_1 & 0\\ 0 & 0 & 0 & A_1\end{bmatrix}+\delta I,\\
&\tilde F_2=F+\delta I-\tilde F_1.
\end{aligned}
$$

Since L_0 and K_0 satisfy (3.10) and (3.13), respectively, the block-diagonal matrix $\tilde F_1=\mathrm{diag}\{F_{10},F_{11}\}$ (F_{10} is a $2N_0\times 2N_0$ block) is Hurwitz. Thus, for some $\kappa>0$ and $M_1>1$, independent of N, we have:

$$
\begin{aligned}
&\left\|e^{F_{10}t}\right\|_2\leq M_1e^{-\kappa t},\quad t\geq 0,\\
&\left\|e^{\tilde F_1 t}\right\|_2\leq\max\{\|e^{F_{10}t}\|_2,e^{-\kappa t}\}\leq M_1e^{-\kappa t}.
\end{aligned}
\tag{3.37}
$$

By Parseval's equality,

$$
\begin{aligned}
&\|B_1K_0\|_2\leq\|B_1\|\,\|K_0\|\leq\|b\|_{L^2}\|K_0\|,\\
&\|L_0C_1\|_2\leq\|L_0\|\,\|C_1\|\leq\|c\|_{L^2}\|L_0\|.
\end{aligned}
\tag{3.38}
$$

Then, for some $M_2 > 0$, independent of N:

$$\begin{aligned}\left\|\tilde{F}_2\right\|_2 &\le M_2 \max(\|B_1K_0\|_2, \|L_0C_1\|_2)\\ &\le M_2\max(\|b\|_{L^2}\|K_0\|, \|c\|_{L^2}\|L_0\|).\end{aligned}\tag{3.39}$$

From (3.37) and (3.39) it can be easily verified that for all $t_1 \ge 0$ and $t_2 \ge 0$ there exists $M_3 > 0$, independent of N, such that

$$\begin{aligned}&\left\|\prod_{i=1}^2 e^{\tilde{F}_1 t_i}\tilde{F}_2\right\|_2\\ &\le M_3 e^{-\kappa(t_1+t_2)}\cdot\|b\|_{L^2}\cdot\|K_0\|\cdot\|c\|_{L^2}\cdot\|L_0\|.\end{aligned}\tag{3.40}$$

Moreover, it can be shown that the block-diagonal matrix $\tilde{F}_1$ and nilpotent matrix $\tilde{F}_2$ satisfy

$$\prod_{i=1}^3(\tilde{F}_1^{n_i}\tilde{F}_2) = 0 \quad n_i \in \{0,1,\dots\}$$

Then for any $t_i \ge 0$ $(i = 1, 2, 3)$ we have

$$\prod_{i=1}^3\left(e^{\tilde{F}_1 t_i}\tilde{F}_2\right) = 0.\tag{3.41}$$

For $t > 0$, we apply the following identity (see, e.g., Van Loan, 1977):

$$e^{(F+\delta I)t} = e^{\tilde{F}_1 t} + \int_0^t e^{\tilde{F}_1(t-t_1)}\tilde{F}_2 e^{(F+\delta I)t_1}dt_1.\tag{3.42}$$

By using (3.42) again with t, t_1 replaced by t_1, t_2, respectively, and substituting back into (3.42), we obtain

$$\begin{aligned}e^{(F+\delta I)t} =\ & e^{\tilde{F}_1 t} + \int_0^t e^{\tilde{F}_1(t-t_1)}\tilde{F}_2 e^{\tilde{F}_1 t_1}dt_1\\ &+ \int_0^t\int_0^{t_1} e^{\tilde{F}_1(t-t_1)}\tilde{F}_2 e^{\tilde{F}_1(t_1-t_2)}\tilde{F}_2 e^{(F+\delta I)t_2}dt_2dt_1.\end{aligned}$$

Finally, repeating this step again and using (3.41) in the resulting triple integral leads to

$$\begin{aligned}e^{(F+\delta I)t} =\ & e^{\tilde{F}_1 t} + \int_0^t e^{\tilde{F}_1(t-t_1)}\tilde{F}_2 e^{\tilde{F}_1 t_1}dt_1\\ &+ \int_0^t\int_0^{t_1} e^{\tilde{F}_1(t-t_1)}\tilde{F}_2 e^{\tilde{F}_1(t_1-t_2)}\tilde{F}_2 e^{\tilde{F}_1 t_2}dt_2dt_1.\end{aligned}\tag{3.43}$$

From (3.40) and (3.43) we find

$$\left\|e^{(F+\delta I)t}\right\|_2 \le M_4 e^{-\kappa t}\left(1+t+t^2\right),\tag{3.44}$$

where $M_4 > 0$ is independent of N. Hence, (3.36) holds and $\|P\|_2 = O(1)$, uniformly in N.

We show next that (3.29) and (3.30) are feasible for large enough N with P which solves (3.34), $\alpha = N^{-1}$, $\beta = N$ and λ_{N+1} satisfying (2.3). By Schur complement, (3.29) and (3.30) with the chosen decision variables are feasible if and only if

$$\begin{aligned}&W_{N+1} \le -p_*N^2\pi^2 + q_c + \delta + \frac{N\left(1+\|c\|_{L^2}^2\right)}{2} < 0,\\ &\Xi = -I + \frac{\|b\|_{L^2}^2}{N}\tilde{K}^T\tilde{K} + \frac{1}{N}P\mathcal{L}\mathcal{L}^TP < 0.\end{aligned}$$

It is clear that $W_{N+1} < 0$ holds for large N. Since $\|P\|_2$, $\|\tilde{K}\|_2$, $\|\mathcal{L}\|_2$ are uniformly bounded in N, all of the eigenvalues of Ξ approach -1 uniformly in N. Hence, $\Xi < 0$ for N large enough. □

3.2. Non-local measurement and actuation: H^1-stability

Here we assume $b \in H^1(0,1)$ with $b(1) = 0$ in order to obtain a stronger H^1-stability result. Such stability is important e.g. in application of PDEs to multi-agent systems, where any b appropriate for PDE control may be adopted (Wei, Fridman, & Johansson, 2019). Then, by (2.8), we have

$$\sum_{n=1}^\infty \lambda_n b_n^2 \le \mu\left\|b'\right\|_{L^2}^2, \quad \mu = \left(p^*\pi^2 + 4q^*\right)\pi^{-2}.\tag{3.45}$$

Furthermore, we assume that $z_0 \in H^1(0,1)$ with $z_0(1) = 0$. We note that exponential H^1-convergence of the closed-loop system still holds under the assumption $z_0 \in L^2(0,1)$, due to the smoothing property of the heat equation (see Remark 3.2).

The observer and controller are defined as in Section 3.1. The closed-loop system is given by (3.22). Moreover, the estimate (3.23) continues to hold. For H^1-stability analysis, we modify $V(t)$, defined in (3.24), as follows

$$V(t) := \|X(t)\|_P^2 + \sum_{n=N+1}^\infty \lambda_n z_n^2(t).\tag{3.46}$$

Differentiating $V(t)$ along (3.22) gives

$$\begin{aligned}&\dot{V} + 2\delta V = X^T(t)\left[PF + F^TP + 2\delta P\right]X(t)\\ &+ 2X^T(t)P\mathcal{L}\zeta(t) + 2\textstyle\sum_{n=N+1}^\infty \lambda_n(-\lambda_n + q_c + \delta)z_n^2(t)\\ &+ \textstyle\sum_{n=N+1}^\infty 2z_n(t)\lambda_n b_n\tilde{K}X(t).\end{aligned}\tag{3.47}$$

Furthermore, (3.45) and the Cauchy–Schwarz inequality imply

$$\begin{aligned}&\textstyle\sum_{n=N+1}^\infty 2\lambda_n z_n(t)b_n\tilde{K}X(t) \le \frac{1}{\alpha}\sum_{n=N+1}^\infty \lambda_n z_n^2(t)\\ &+ \alpha\left(\textstyle\sum_{n=N+1}^\infty \lambda_n b_n^2\right)\left\|\tilde{K}X(t)\right\|^2 \le \frac{1}{\alpha}\sum_{n=N+1}^\infty \lambda_n z_n^2(t)\\ &+ \alpha\mu\left\|b'\right\|_{L^2}^2\left\|\tilde{K}X(t)\right\|^2.\end{aligned}\tag{3.48}$$

Denote $\eta(t) = \mathrm{col}\{X(t), \zeta(t)\}$. By combining (3.47) with (3.48) and taking into account (3.23) we obtain for some $\beta > 0$

$$\begin{aligned}&\dot{V} + 2\delta V + \beta\left(\|c\|_{L^2}^2\textstyle\sum_{n=N+1}^\infty z_n^2(t) - \zeta^2(t)\right)\\ &\le \eta^T(t)\Psi^1\eta(t) + 2\textstyle\sum_{n=N+1}^\infty \lambda_n W_n^{(1)}z_n^2(t) \le 0\end{aligned}\tag{3.49}$$

if

$$\begin{aligned}&W_n^{(1)} = -\lambda_n + q_c + \delta + \frac{1}{2\alpha} + \frac{\beta\|c\|_{L^2}^2}{2\lambda_n} < 0, \quad n > N,\\ &\Psi^1 = \begin{bmatrix} PF + F^TP + 2\delta P + \alpha\mu\left\|b'\right\|_{L^2}^2\tilde{K}^T\tilde{K} & P\mathcal{L}\\ * & -\beta\end{bmatrix} < 0.\end{aligned}\tag{3.50}$$

Monotonicity of $\{\lambda_n\}_{n=1}^\infty$ and Schur's complement imply that $W_n^{(1)} < 0$ for all $n > N$ iff

$$\begin{bmatrix} -\lambda_{N+1} + q_c + \delta + \frac{\beta\|c\|_{L^2}^2}{2\lambda_{N+1}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\alpha\end{bmatrix} < 0.\tag{3.51}$$

Summarizing, we arrive at:

Theorem 3.2. *Consider (3.1) with $b \in H^1(0,1)$, $b(1) = 0$ satisfying (3.11), measurement (3.2) with $c \in L^2(0,1)$ satisfying (3.9), control law (3.14) and $z_0 \in H^1(0,1)$, $z_0(1) = 0$. Let $\delta > 0$ be a desired decay rate, $N_0 \in \mathbb{N}$ satisfy (3.5) and $N \in \mathbb{N}$ satisfy $N_0 \le N$. Assume that L_0 and K_0 are obtained using (3.10) and (3.13), respectively. If there exist a positive definite matrix $P \in \mathbb{R}^{2N\times 2N}$ and scalars $\alpha, \beta > 0$ which satisfy (3.50) and (3.51), then the solution $z(x,t)$ to (3.1) under the control law (3.14), (3.7) and the corresponding observer $\hat{z}(x,t)$ defined by (3.6) satisfy the following inequalities*

$$\begin{aligned}&\|z(\cdot,t)\|_{H^1}^2 \le Me^{-2\delta t}\|z_0\|_{H^1}^2,\\ &\left\|z(\cdot,t) - \hat{z}(\cdot,t)\right\|_{H^1}^2 \le Me^{-2\delta t}\|z_0\|_{H^1}^2,\end{aligned}\tag{3.52}$$

with some constant $M > 0$. Moreover, LMIs (3.50) and (3.51) are always feasible for large enough N.

Proof. By Wirtinger's inequality (see Section 3.10 of Fridman, 2014), for each $t \ge 0$,

$$\|z_x(\cdot,t)\|_{L^2}^2 \le \|z(\cdot,t)\|_{H^1}^2 \le \left(1 + 4\pi^{-2}\right)\|z_x(\cdot,t).\|_{L^2}^2$$

Since $z(\cdot,t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$, by Lemma 2.1 we have

$$V(0) \le M_0\left\|z_0'\right\|_{L^2}^2 \le M_0\|z_0\|_{H^1}^2.\tag{3.53}$$

for some $M_0 > 0$. By positivity and monotonicity of λ_n, Wirtinger's inequality and Lemma 2.1 we obtain

$$V(t) \geq 0.5\sigma_{min}(P)\sum_{n=1}^{N} z_n^2(t) + \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t) \\ \geq M_1 \|z(\cdot,t)\|_{H^1}^2, \quad t \geq 0.$$

for some $M_1 > 0$. The rest of the proof of (3.52), as well as the feasibility of (3.50) and (3.51) for large enough N follow arguments of Theorem 3.1. □

Remark 3.2. In the case where $z_0 \in L^2(0,1)$, Theorem 3.2 still implies exponential H^1-convergence (although not exponential stability) of the closed-loop system. Indeed for $t_* > 0$ small enough, we have $z(\cdot,t_*) \in \mathcal{D}(\mathcal{A})$. Therefore, by applying Theorem 3.2 we obtain

$$\|z(\cdot,t)\|_{H^1}^2 \leq Me^{-2\delta(t-t_*)}\|z(\cdot,t_*)\|_{H^1}^2, \\ \left\|z(\cdot,t)-\hat{z}(\cdot,t)\right\|_{H^1}^2 \leq Me^{-2\delta(t-t_*)}\|z(\cdot,t_*)\|_{H^1}^2,$$

for all $t > t_*$, where $M > 0$ is some constant.

3.3. Boundary measurement and non-local actuation: H^1-convergence

Consider (3.1) with boundary measurement

$$y(t) = z(0,t). \tag{3.54}$$

As in Section 3.2, we assume $b \in H^1(0,1)$, $b(1) = 0$ and $z_0 \in H^1(0,1)$, $z_0(1) = 0$. This allows to use $V(t)$ of (3.46) in order to compensate $\zeta(t)$ defined by (3.56) below.

We present the solution to (3.1) as (3.3) with $z_n(t)$ satisfying (3.4). We construct a N-dimensional observer of the form (3.6), where $\hat{z}_n(t)$ satisfy

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + b_n u(t) \\ - l_n\left[\sum_{n=1}^{N}\hat{z}_n(t)\phi_n(0) - y(t)\right], \\ \hat{z}_n(0) = 0 \quad 1 \leq n \leq N. \tag{3.55}$$

Here l_n are scalar observer gains. Let L_0 defined in (3.8) satisfy (3.10) and choose $l_n = 0$, $n > N_0$. Define the controller (3.14) with $K_0 \in \mathbb{R}^{1\times N_0}$ subject to (3.13).

By using (3.4) and the estimation error (3.18), the last term on the right-hand side of (3.55) can be presented as

$$\sum_{n=1}^{N}\phi_n(0)\hat{z}_n(t) - y(t) = -\sum_{n=1}^{N} c_n e_n(t) - \zeta(t) \\ c_n = \phi_n(0), \ \zeta(t) = z(0,t) - \sum_{n=1}^{N} c_n z_n(t). \tag{3.56}$$

Remark 3.3. In the case of boundary measurement, all components of C_0 are non-zero. Indeed, if for $n \geq 1$ we have $\phi_n(0) = \phi_n'(0) = 0$, then by uniqueness of solutions to the ODE $\mathcal{A}\phi_n = \phi_n$, we obtain $\phi_n \equiv 0$, which is a contradiction. Thus, assumption (3.9) is satisfied for all $N \in \mathbb{N}$. Note that by Orlov (2017), $c_n = O(1)$ for all $n \geq 1$.

For well-posedness of the closed-loop system (3.1), (3.55) with $u(t) = K_0\hat{z}^{N_0}(t)$, let $\mathcal{G} := \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times \mathbb{R}^N$, with $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ defined in (2.5). This is a Hilbert space endowed with the norm $\|\cdot\|_{\mathcal{G}}^2 = \|\cdot\|_{H^1}^2 + \|\cdot\|^2$. We present the system (3.1) and (3.55) as (3.15) with f_2 replaced by

$$\tilde{f}_2(\xi) = \begin{bmatrix} -L_0\int_0^1 z_x(x,t)dx & 0\end{bmatrix}^T.$$

Let $\xi_i = [w_i, y_i]^T \in \mathcal{G}$, $i \in \{1,2\}$. Then $\tilde{f}_2(0) = 0$ and, by using the Cauchy–Schwarz inequality, we have

$$\left\|\tilde{f}_2(\xi_1) - \tilde{f}_2(\xi_2)\right\| \leq \|L_0\|\,\|w_1 - w_2\|_{H^1} \leq \|L_0\|\,\|\xi_1 - \xi_2\|_{\mathcal{G}}.$$

These estimates imply $\text{col}\left\{f_1, \tilde{f}_2\right\} : \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times \mathbb{R}^N \to \mathcal{G}$ satisfy assumptions (F) and (3.22) in Pazy (1983), Section 6.3 with $\alpha = \frac{1}{2}$. By Theorems 6.3.1 and 6.3.3 in Pazy (1983), the system (3.1), (3.55) with $u(t) = K_0\hat{z}^{N_0}(t)$ and initial condition $z_0 \in H^1(0,1)$, $z_0(1) = 0$ has a unique *classical* solution $\xi(t)$ satisfying (3.16) and (3.17).

Since $z(\cdot,t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$, by using (2.1), the Cauchy–Schwarz inequality and Lemma 2.1, we obtain

$$\zeta^2(t) := \left[z(0,t) - \sum_{n=1}^{N}\phi_n(0)z_n(t)\right]^2 \\ = \left[\int_0^1\left(z_x(s,t) - \sum_{n=1}^{N}\phi_n'(s)z_n(t)\right)ds\right]^2 \\ \leq \left\|z_x(\cdot,t) - \sum_{n=1}^{N}\phi_n'(\cdot)z_n(t)\right\|_{L^2}^2 \leq \frac{1}{p_*}\sum_{n=N+1}^{\infty}\lambda_n z_n^2(t). \tag{3.57}$$

By using c_n and $\zeta(t)$ defined in (3.56) and the notations (3.8) and (3.21), we obtain the closed-loop system (3.22).

Taking into account the estimate (3.57), for exponential H^1-convergence, we consider the Lyapunov function (3.46). Let $\eta(t) = \text{col}\{X(t), \zeta(t)\}$. Differentiating (3.46) along (3.22) and using arguments similar to (3.47) and (3.48) we obtain for some $\beta > 0$

$$\dot{V} + 2\delta V + \beta\left(\frac{1}{p_*}\sum_{n=N+1}^{\infty}\lambda_n z_n^2(t) - \zeta^2(t)\right) \\ \leq \eta^T(t)\Psi^1\eta(t) + 2\sum_{n=N+1}^{\infty}\lambda_n W_n^{(2)} z_n^2(t) \leq 0 \tag{3.58}$$

if

$$W_n^{(2)} = -\lambda_n + q_c + \delta + \frac{1}{2\alpha} + \frac{\beta}{2p_*} < 0, \quad n > N, \\ \Psi^1 < 0, \tag{3.59}$$

where Ψ^1 is given by (3.50). Monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and Schur's complement imply that $W_n^{(2)} < 0$ for all $n > N$ iff

$$\begin{bmatrix} -\lambda_{N+1} + q_c + \delta + \frac{\beta}{2p_*} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\alpha \end{bmatrix} < 0. \tag{3.60}$$

Summarizing, we arrive at:

Theorem 3.3. *Consider (3.1) with $b \in H^1(0,1)$, $b(1) = 0$ satisfying (3.11), measurement (3.54), control law (3.14) and $z_0 \in H^1(0,1)$, $z_0(1) = 0$. Let $\delta > 0$ be a desired decay rate, $N_0 \in \mathbb{N}$ satisfy (3.5) and $N \in \mathbb{N}$ satisfy $N_0 \leq N$. Assume that L_0 and K_0 are obtained using (3.10) and (3.13), respectively. If there exist a positive definite matrix $P \in \mathbb{R}^{2N\times 2N}$ and scalars $\alpha, \beta > 0$ which satisfy (3.59) and (3.60), then the solution $z(x,t)$ to (3.1) under the control law (3.14), (3.7) and the corresponding observer $\hat{z}(x,t)$ defined by (3.6) satisfy (3.52) with some constant $M > 0$. Moreover, LMIs (3.59) and (3.60) are always feasible for large enough N.*

Proof. The feasibility of (3.59) and (3.60) for large enough N follow from arguments similar to proof of Theorem 3.1 with slight modifications. For completeness, we outline the differences. For the case of boundary measurement (3.43) and (3.40) continue to hold, while (3.38) is replaced by

$$\|B_1K_0\|_2 \leq \|B_1\|\,\|K_0\| \leq \|b\|_{L^2}\|K_0\|, \\ \|L_0C_1\|_2 \leq \|L_0\|\,\|C_1\| = \|L_0\|\cdot O(\sqrt{N}). \tag{3.61}$$

By using (3.43), (3.40) and (3.61) we obtain

$$\left\|\tilde{F}_2\right\|_2 \leq M_5\sqrt{N}, \quad \left\|e^{Ft}\right\|_2 \leq M_6 e^{-\kappa t}\sqrt{N}\left(1 + t + t^2\right), \tag{3.62}$$

where $M_5 > 0$ and $M_6 > 0$ are independent of N.

Consider the Lyapunov equation

$$P(F + \delta I) + (F + \delta I)^T P = -\frac{1}{N}I. \tag{3.63}$$

Then $\|P\|_2 = O(1)$, uniformly in N:

$$\begin{aligned}\|P\|_2 &\le \frac{1}{N}\int_0^\infty \left\|e^{F^T t}\right\|_2 \left\|e^{Ft}\right\|_2 dt\\ &\le M_7\int_0^\infty e^{-2\kappa t}\left(1+t+t^2\right)^2 dt < \infty,\end{aligned} \tag{3.64}$$

where $M_7 > 0$ is independent of N. Substitute $\alpha = N^{-1.5}$, $\beta = N^{1.5}$, λ_{N+1} satisfying (2.3) and (3.63) into (3.59) and (3.60). By taking Schur complement, (3.59) and (3.60) are feasible for large enough N. □

3.4. Boundary actuation and non-local measurement

In this section we consider the reaction–diffusion system

$$\begin{aligned}&z_t(x,t) = \partial_x\left(p(x)z_x(x,t)\right) + (q_c - q(x))z(x,t),\\ &z_x(0,t) = 0, \quad z(1,t) = u(t),\end{aligned} \tag{3.65}$$

where $t \ge 0$, $x \in [0,1]$, $z(x,t) \in \mathbb{R}$ and $q_c \in \mathbb{R}$ is a constant reaction coefficient, under Dirichlet actuation where $u(t)$ is the control. We consider measurement of the form (3.2) with $c \in H^1(0,1)$, $c(1) = 0$.

By presenting the solution to (3.65) as (3.3), performing calculations similar to (3.4) and using arguments from proof of Lemma 4.1 in Karafyllis and Krstic (2018) (see eq. (42) therein), we obtain the following ODEs for $z_n(t)$:

$$\begin{aligned}&\dot z_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t), \ t > 0,\\ &b_n = -p(1)\phi_n'(1).\end{aligned} \tag{3.66}$$

Using results of Orlov (2017) (see eq. (35)-(40) therein) one can obtain a constant $M_\phi > 0$, in terms of $p(x)$ and $q(x)$, such that

$$|b_n| \le p^*\left|\phi_n'(1)\right| \le M_\phi\sqrt{\lambda_n}, \quad \forall n \ge 1. \tag{3.67}$$

In the particular case $p(x) \equiv 1$ and $q(x) \equiv 0$, $\phi_n'(1)$ can be computed explicitly to obtain $\phi_n'(1) = (-1)^{n+1}\sqrt{2\lambda_n}$ with $M_\phi = \sqrt{2}$ in (3.67). Thus, unlike the previous sections, $|b_n|$ may be unbounded as $n \to \infty$.

Moreover, assumption (3.11) is satisfied for all $N \in \mathbb{N}$, by arguments similar to Remark 3.3. Using (2.3) and the integral convergence test, the following estimate is obtained:

$$\begin{aligned}\textstyle\sum_{n=N+1}^\infty \frac{b_n^2}{\lambda_n^2} &\le \frac{M_\phi^2}{p_*\pi^2}\textstyle\sum_{n=N}^\infty \frac{1}{n^2}\\ &\le \frac{M_\phi^2}{p_*\pi^2}\left(\frac{1}{N} + \int_N^\infty \frac{dx}{x^2}\right) \le \frac{M_\phi^2}{p_*\pi^2}\frac{2}{N}.\end{aligned} \tag{3.68}$$

For $p(x) \equiv 1$ and $q(x) \equiv 0$ using (2.4) we arrive at a less conservative bound $\frac{4}{\pi^2(2N-1)}$ in (3.68).

We construct a N-dimensional observer of the form (3.6), where $\hat z_n(t)$ satisfy (3.7) with scalar observer gains l_n. Moreover, let L_0 defined in (3.8) satisfy (3.10) and choose $l_n = 0$, $n > N_0$. We consider a controller (3.14) with $K_0 \in \mathbb{R}^{1\times N_0}$ subject to (3.13) and choose $k_n = 0$, $n > N_0$.

By arguments in Baudouin, Seuret, and Gouaisbaut (2019), the closed-loop system (3.65) and (3.7) has a unique solution $z_n(t)$, $1 \le n \le N$ and $z(\cdot,t)$ with

$$\begin{aligned}&z(\cdot,t) \in C\left([0,\infty); H^1(0,1)\right) \cap L^2\left((0,\infty); H^2(0,1)\right),\\ &z_t(\cdot,t) \in L^2\left((0,\infty); L^2(0,1)\right).\end{aligned}$$

We introduce the notations

$$\begin{aligned}&\rho_n(t) = \lambda_n^{-\frac12}\hat z_n(t), \ \nu_n(t) = \lambda_n^{-\frac12}e_n(t), \ N_0+1 \le n \le N,\\ &\rho^{N-N_0}(t) = \left[\rho_{N_0+1}(t), \dots, \rho_N(t)\right]^T,\\ &\nu^{N-N_0}(t) = \left[\nu_{N_0+1}(t), \dots, \nu_N(t)\right]^T,\\ &X(t) = \mathrm{col}\left\{\hat z^{N_0}(t), e^{N_0}(t), \rho^{N-N_0}(t), \nu^{N-N_0}(t)\right\}.\end{aligned} \tag{3.69}$$

Then, by arguments similar to (3.4), (3.7), (3.14) and (3.19), we obtain the closed-loop system (3.22) with C_1 and B_1 in F (defined in (3.21)) replaced by

$$\begin{aligned}&\tilde C_1 = \left[\lambda_{N_0+1}^{\frac12}c_{N_0+1}, \dots, \lambda_N^{\frac12}c_N\right],\\ &\tilde B_1 = \left[\lambda_{N_0+1}^{-\frac12}b_{N_0+1}, \dots, \lambda_N^{-\frac12}b_N\right]^T,\end{aligned} \tag{3.70}$$

respectively. Furthermore, by using the Cauchy–Schwarz inequality and Lemma 2.1 we obtain

$$\begin{aligned}\zeta^2(t) &= \left(\textstyle\sum_{n=N+1}^\infty c_n z_n(t)\right)^2\\ &\le \left(\textstyle\sum_{n=N+1}^\infty \lambda_n c_n^2\right)\left(\textstyle\sum_{n=N+1}^\infty \lambda_n^{-1}z_n^2(t)\right)\\ &\le \mu\left\|c'\right\|_{L^2}^2 \textstyle\sum_{n=N+1}^\infty \lambda_n^{-1}z_n^2(t),\end{aligned} \tag{3.71}$$

with μ defined in (3.45).

For convergence analysis of the closed-loop system, we introduce the Lyapunov function

$$V(t) := \|X(t)\|_P^2 + r\sum_{n=N+1}^\infty \lambda_n^{-1}z_n^2(t), \tag{3.72}$$

where $P \in \mathbb{R}^{2N\times 2N}$ satisfies $P > 0$ and $r \in \mathbb{R}$ is positive.

Remark 3.4. For $h = \sum_{n=1}^\infty h_n\phi_n \in L^2(0,1)$, one can define the $H^{-\frac12}$-norm of $h(x)$ as

$$\|h\|_{H^{-\frac12}}^2 := \sum_{n=1}^\infty \lambda_n^{-1}h_n^2.$$

It is easy to see that the norm induced by $V(t)$ is equivalent to $\|z(\cdot,t)\|_{H^{-\frac12}}^2$ (for additional information see e.g (Tucsnak & Weiss, 2009), p. 84). Therefore, the convergence analysis carried out in this section implies $H^{-\frac12}$-stability of the closed-loop system.

Differentiation of $V(t)$ along (3.22) gives

$$\begin{aligned}&\dot V + 2\delta V = X^T(t)\left[PF + F^TP + 2\delta P\right]X(t)\\ &+2X^T(t)P\mathcal{L}\zeta(t) + 2r\textstyle\sum_{n=N+1}^\infty\left(-1 + \frac{q_c+\delta}{\lambda_n}\right)z_n^2(t)\\ &+r\textstyle\sum_{n=N+1}^\infty 2z_n(t)\frac{b_n}{\lambda_n}\tilde K X(t).\end{aligned} \tag{3.73}$$

By (3.68) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}&r\sum_{n=N+1}^\infty 2z_n(t)\frac{b_n}{\lambda_n}\tilde K X(t) \le \frac{r}{\alpha}\sum_{n=N+1}^\infty z_n^2(t)\\ &+ r\alpha\left(\sum_{n=N+1}^\infty \frac{b_n^2}{\lambda_n^2}\right)\left\|\tilde K X(t)\right\|^2 \le \frac{r}{\alpha}\sum_{n=N+1}^\infty z_n^2(t)\\ &+ \frac{r\alpha M_\phi^2}{p_*\pi^2}\frac{2}{N}\left\|\tilde K X(t)\right\|^2,\end{aligned} \tag{3.74}$$

where $\alpha > 0$. Denote $\eta(t) = \mathrm{col}\{X(t), \zeta(t)\}$. By combining (3.73) with (3.74) and recalling (3.71) we obtain for μ in (3.50) and some $\beta > 0$

$$\begin{aligned}&\dot V + 2\delta V + \beta\left(\mu\left\|c'\right\|_{L^2}^2\sum_{n=N+1}^\infty \frac{1}{\lambda_n}z_n^2(t) - \zeta^2(t)\right)\\ &\le \eta^T(t)\Psi^2\eta(t) + 2r\sum_{n=N+1}^\infty W_n^{(3)}z_n^2(t) \le 0\end{aligned} \tag{3.75}$$

if

$$
\begin{aligned}
&W_n^{(3)}=-1+\frac{q_c+\delta}{\lambda_n}+\frac{1}{2\alpha}+\frac{\beta\mu\left\|c'\right\|_{L^2}^2}{2r\lambda_n}<0,\ \ n>N,\\
&\Psi^2=\begin{bmatrix} PF+F^TP+2\delta P+r\alpha\frac{2M_\phi^2}{Np_*\pi^2}\tilde{K}^T\tilde{K} & P\mathcal{L}\\ * & -\beta\end{bmatrix}<0.
\end{aligned}
\tag{3.76}
$$

Monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$, positivity of r and Schur's complement imply that $W_n^{(3)}<0$ for all $n>N$ iff

$$
\begin{bmatrix} -1+\frac{q_c+\delta}{\lambda_{N+1}}+\frac{\beta\mu\|c'\|_{L^2}^2}{2r\lambda_{N+1}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\alpha\end{bmatrix}<0. \tag{3.77}
$$

Summarizing, we obtain:

Theorem 3.4. *Consider (3.65) with control law (3.14), measurement (3.2) with $c\in H^1(0,1)$, $c(1)=0$ satisfying (3.9) and $z_0\in L^2(0,1)$. Let $\delta>0$ be a desired decay rate, $N_0\in\mathbb{N}$ satisfy (3.5) and $N\in\mathbb{N}$ satisfy $N_0\le N$. Assume that L_0 and K_0 are obtained using (3.10)* and *(3.13), respectively. If there exist a positive definite matrix $P\in\mathbb{R}^{2N}$ and scalars $\alpha,\beta,r>0$ which satisfy (3.76)* and *(3.77), then the following inequality holds*

$$
V(t)\le e^{-2\delta t}V(0),\quad t>0 \tag{3.78}
$$

for $V(t)$ defined in (3.72). Moreover, LMIs (3.76) and *(3.77) are always feasible for large enough N. For $p(x)\equiv 1$ and $q(x)\equiv 0$, the last term in the (1,1) entry of Ψ^2 should be changed to $\frac{4r\alpha}{(2N-1)\pi^2}\tilde{K}^T\tilde{K}$.*

Proof. Note that

$$
\begin{aligned}
&\left\|\tilde{B}_1K_0\right\|_2\le\|K_0\|\left(\sum_{n=N_0+1}^{N}\frac{b_n^2}{\lambda_n}\right)^{\frac12}\le\|K_0\|\cdot O(\sqrt{N}),\\
&\left\|L_0\tilde{C}_1\right\|_2\le\|L_0\|\left(\sum_{n=N_0+1}^{N}\lambda_n c_n^2\right)^{\frac12}\le\mu^{\frac12}\|L_0\|\left\|c'\right\|_{L^2},
\end{aligned}
$$

where the latter bound follows from Lemma 2.1. By substituting P which solves (3.63), $\alpha=N^{\frac14}$, $\beta=N^{\frac54}$, $r=N^{-\frac12}$ and λ_{N+1} satisfying (2.3) we find that the LMIs (3.76) and (3.77) are feasible for large enough N. □

4. Numerical examples

In all the examples, we choose $p(x)\equiv 1$, $q(x)\equiv 0$ and $q_c=10$. This choice corresponds to an unstable open-loop system. The gains L_0 and K_0 are found from (3.10) and (3.13), respectively. The LMIs are verified by using the standard Matlab LMI toolbox. The values of N start from the minimal ones that guarantee the LMIs feasibility.

4.1. Non-local measurement and actuation

Consider system (3.1) with measurement (3.2), where

$$
\begin{aligned}
&c(x)=\sqrt{2}\cdot\chi_{[0.25,0.75]}(x),\\
&b(x)=\begin{cases}\sqrt{2}\,(4x-1), & x\in[0.25,0.5]\\ \sqrt{2}\,(-4x+3), & x\in[0.5,0.75]\\ 0, & x\notin[0.25,0.75].\end{cases}
\end{aligned}
\tag{4.1}
$$

Note that $b\in H^1(0,1)$, $b(1)=0$ and $c\in L^2(0,1)$. Let $N_0=1$ and $\delta=1$. The obtained observer and controller gains are $K_0=-57.6811$, $L_0=29.217$. The LMIs of Theorem 3.2 are feasible for $N=4$.

For the simulation of the solutions to the closed-loop system we chose $z_0(x)=x^2-1$ with $z_0(1)=0$. The simulation was carried out for the corresponding PDE (3.1) with $u(t)=K_0\hat{z}_1(t)$ (using the finite-difference FTCS scheme) and ODEs (3.7)

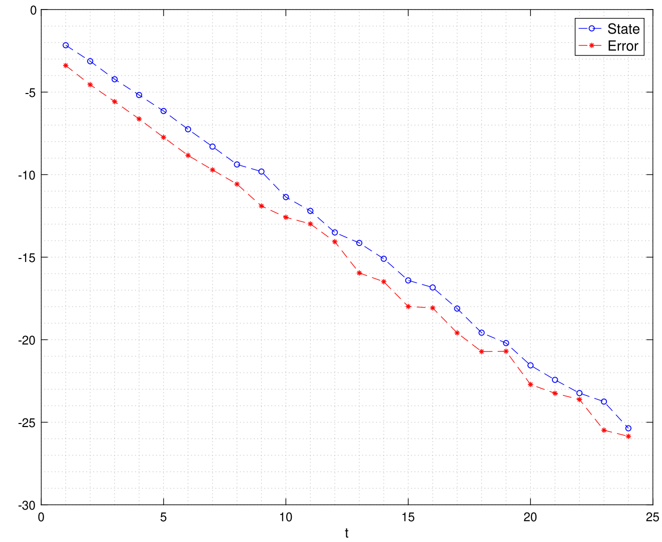

Fig. 1. Non-local measurement and actuation.

(using 4th order Runge–Kutta scheme). The norms $\|z_x(\cdot,t)\|_{L^2}$ and $\left\|z_x(\cdot,t)-\hat{z}_x(\cdot,t)\right\|_{L^2}$ for $t>0$ were estimated using (2.8) with $\|z_x\|_{L^2}^2=\sum_{n=1}^{40}\lambda_n z_n^2$, whereas z_n were found from simulation of ODEs (3.4) (note that these ODEs are not part of the closed-loop system). The $H^1(0,1)$ norms of the state and estimation error $e=z-\hat{z}$ are presented, on a logarithmic scale, in Fig. 1. The computed linear fits are given by $f_z(t)\approx -1.0031t-1.1824$, $f_e(t)\approx -0.9873t-2.0721$, which is consistent with a decay rate $\delta=1$ up to numerical errors. Numerical simulations showed that for $N=3$ the closed-loop system is unstable. Thus, our LMIs are not conservative.

4.2. Non-local actuation and boundary measurement

Consider (3.1) with the boundary measurement (3.54) and b given by (4.1). Let $N_0=1$, $\delta=1$. The obtained observer and controller gains are $K_0=-57.6811$, $L_0=14.2359$. The LMIs in Theorem 3.3 are feasible for $N=5$. For the simulation of the solution to the closed-loop system we chose initial condition $z_0(x)=x^2-1$ with $z_0(1)=0$. The $H^1([0,1])$ norms of the state and estimation error are presented, on a logarithmic scale, in Fig. 2. The computed linear fits are given by $f_z(t)\approx -1.0007t-1.6942$, $f_e(t)\approx -1.0049t-1.2269$ and correspond to theoretical $\delta=1$. Simulations show that for $N=4$ the closed-loop system is unstable, i.e. LMIs are not conservative.

4.3. Boundary actuation and non-local measurement

Consider system (3.65) and measurement (3.2) with $c\in H^1(0,1)$, $c(1)=0$ is equal to b appearing in (4.1). Let $N_0=1$ and $\delta=0.13$. The obtained gains are $K_0=-9.0629$, $L_0=22.6812$. The LMIs in Theorem 3.4 were feasible for $N=5$, $r=0.001$. The simulation was carried out with $z_0(x)=x^2-3$ for the ODEs (3.66) and the closed-loop system (3.22), with L_0C_1 and B_1K_0 in (3.21) replaced by (3.70). The evaluation of $V(t)$, defined in (3.72), was based on truncating the series presented therein after 40 coefficients. The values of $V(t)$ and the corresponding linear fit, on a logarithmic scale, are presented in Fig. 3. The computed linear fit $f_V(t)\approx -0.12981t-1.7385$ corresponds to $\delta=0.13$. In this case simulations for $N=6$ show instability, meaning that LMIs are not conservative.

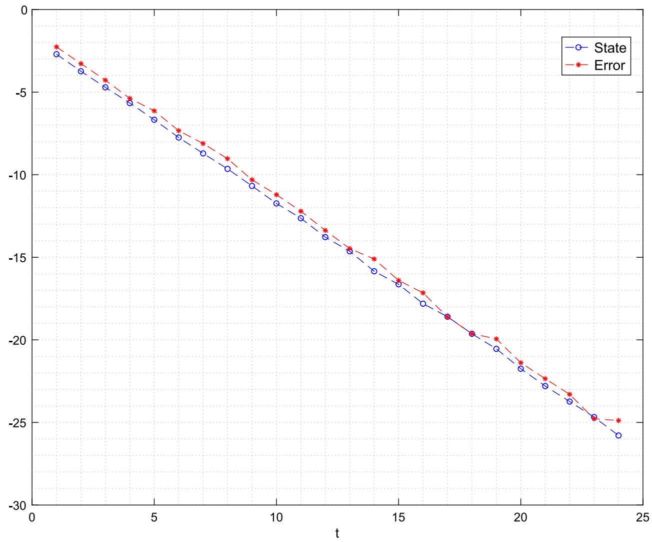

Fig. 2. Boundary measurement and non-local actuation.

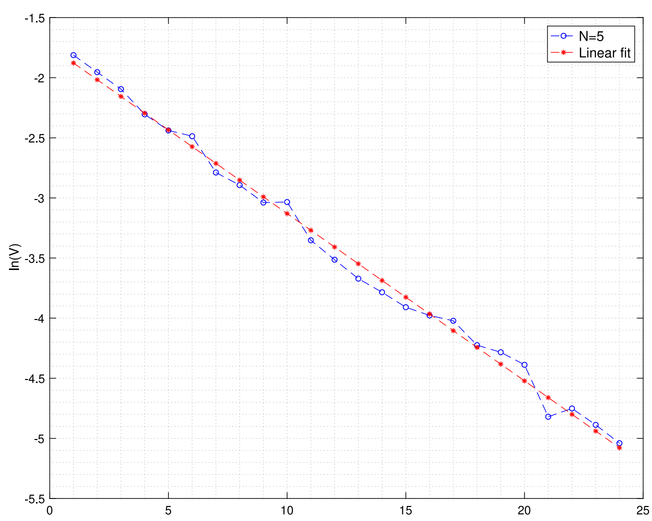

Fig. 3. Boundary actuation and non-local measurement.

5. Conclusions

The present paper has suggested the first LMI-based solution for the challenging finite-dimensional observer-based controller design in the case of 1-D linear heat equation. The method is demostrated to be applicable to the heat equation when at least one of the control or observation operators is bounded. This method is based on modal decomposition, and results in easily verifiable LMI conditions for finding the observer dimension N and the decay rate of the closed-loop system. The derived LMIs appear to be nonconservative in the examples. It is shown that the LMIs are always feasible for large enough N. The presented method gives tools for finite-dimensional observer-based control of other parabolic systems, and can be extended to design in the case of delayed and sampled-data inputs and outputs.

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