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H_∞ Control of Linear Singularly Perturbed Systems with Small State Delay

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An infinite horizon H_∞ state-feedback control problem for singularly perturbed linear systems with a small state delay is considered. An asymptotic solution of the hybrid system of Riccati-type algebraic, ordinary differential, and partial differential equations with deviating arguments, associated with this problem, is constructed. Based on this asymptotic solution, conditions for the existence of a solution of the original H_∞ problem, independent of the singular perturbation parameter, are derived. A simplified controller with parameter-independent gain matrices, solving the original problem for all sufficiently small values of this parameter, is obtained. An illustrative example is presented. © 2000 Academic Press

1. INTRODUCTION

For many years, controlled systems with disturbances (uncertainties) in dynamics have been extensively studied (see e.g. [20] and the list of references therein). One of the main problems in this topic which has been solved is constructing a feedback controller independent of the disturbance, which provides a required property of the closed-loop system for all realizations of the disturbance from a given set. Two classes of distur-

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bances are usually distinguished: (1) disturbances belonging to a known bounded set of Euclidean space; and (2) quadratically integrable disturbances. In this paper, we deal with the second class of disturbances. For controlled systems with quadratically integrable disturbance, the H_∞ problem is frequently considered (see e.g. [1, 4]).

The H_∞ control problem has been considered for systems without and with delay in the state variables (see e.g. [1, 3, 4, 9, 15]). For both types of systems, the solution of the H_∞ control problem can be reduced to a solution of a game-theoretic Riccati equation. In the case of undelayed systems, the Riccati equation is finite dimensional, while in the case of delayed systems it is infinite dimensional. The infinite dimensional Riccati equation can be reduced to a hybrid system of three matrix equations of Riccati type. Solving this system is a very complicated problem.

In various fields of science and engineering, systems with two-time-scale dynamics are often investigated. Mathematically, such systems are modelled by singularly perturbed equations (see e.g. [13, 29]). Control problems for singularly perturbed equations have been extensively investigated for many years (see [2, 18, 19, 21, 26, 28] and the references therein). However, most of these (and more recent) publications are devoted to problems with undelayed dynamics. Singularly perturbed control problems for systems with delays are less investigated. As far as is known to the authors, there are only few publications in this area [7, 8, 10–12, 24, 25].

In the present paper, we consider an infinite horizon H_∞ state-feedback control problem for singularly perturbed linear systems with a small state delay. The H_∞ control problem for singularly perturbed systems without delays has been studied in a number of papers [5, 17, 22, 23, 27, 30]. However, as far as is known to the authors, the H_∞ control problem for singularly perturbed systems with delays has not yet been considered in the open literature. The main results, obtained in this paper, are:

(a) an asymptotic solution of the hybrid system of Riccati equations, associated with the singularly perturbed H_∞ control problem with a small state delay;

(b) conditions for the existence of a solution of this H_∞ problem independent of the singular perturbation parameter $\varepsilon > 0$;

(c) the design of a simplified controller with ε-independent gain matrices, which solves the H_∞ problem for all sufficiently small $\varepsilon > 0$.

The approach proposed in this paper is valid for both standard and nonstandard forms of singularly perturbed delayed dynamics of the H_∞ control problem. These forms are an extension of the ones considered for singularly perturbed dynamics without delay in [16].

The paper is organized as follows. In Section 2, the H_∞ control problem for a singularly perturbed linear system with a small state delay is formulated. The hybrid system of Riccati equations, associated with this problem, is written out. In Section 3, the formal zero-order asymptotic solution of this system of equations is constructed. Reduced-order (slow) and boundary-layer (fast) H_∞ control problems, associated with the original one, are obtained, and their connection with the zero-order asymptotic solution is established. In Section 4, it is verified that the zero-order asymptotic solution is $O(\varepsilon)$-close to an exact solution. In Section 5, two controllers, solving the original H_∞ problem, are obtained. In Section 6, an example illustrating the results of the previous sections is presented. In Appendix, the auxiliary lemma, applied in the verification of the zero-order asymptotic solution, is proved.

The following main notations are applied in this paper: (1) E^n is the n-dimensional real Euclidean space; (2) $L_2[b,c;E^n]$ is the space of n-dimensional vector functions quadratically integrable on the interval (b,c); (3) $C[b,c;E^n]$ is the space of n-dimensional vector-functions continuous on the interval $[b,c]$; (4) $\|\cdot\|$ denotes the Euclidean norm either of the matrix or of the vector; (5) $\|\cdot\|_{L_2}$ denotes the norm in $L_2[b,c;E^n]$; (6) $\|\cdot\|_C$ denotes the norm in $C[b,c;E^n]$; (7) $\mathrm{col}\{x,y\}$, where $x\in E^n$, $y\in E^m$, denotes the column block-vector with upper block x and lower block y; (8) I_n is the n-dimensional identity matrix; (9) $\mathrm{Re}\,\lambda$ denotes the real part of a complex number λ; (10) $\dot{x}(t)\triangleq dx(t)/dt$; (11) $x_t\triangleq x(t+\theta)$, where $x\in E^n$, $t\geq 0$, $\theta\in[-b,0]$ $(b>0)$.

2. PROBLEM FORMULATION

Consider the system

$$\begin{aligned}\dot{x}(t) &= A_1x(t)+A_2y(t)+H_1x(t-\varepsilon h)+H_2y(t-\varepsilon h)\\ &\quad +B_1u(t)+F_1w(t),\qquad t>0,\end{aligned}\tag{2.1}$$

$$\begin{aligned}\varepsilon\dot{y}(t) &= A_3x(t)+A_4y(t)+H_3x(t-\varepsilon h)+H_4y(t-\varepsilon h)\\ &\quad +B_2u(t)+F_2w(t),\qquad t>0,\end{aligned}\tag{2.2}$$

$$v(t)=\mathrm{col}\{C_1x(t)+C_2y(t),u(t)\},\qquad t>0,\tag{2.3}$$

where $x\in E^n$ and $y\in E^m$ are state variables, $u\in E^r$ is a control, $w\in E^q$ is a disturbance, $v\in E^p$ is an observation, A_i,H_i,B_j,F_j,C_j $(i=1,\dots,4;$ $j=1,2)$ are constant matrices of the corresponding dimensions, $\varepsilon>0$ is a small parameter $(\varepsilon\ll 1)$ and $h>0$ is some constant.

Assuming that $w(t) \in L_2[0, +\infty; E^q]$, we consider the performance index

$$J(u,w) = \|v(t)\|_{L_2}^2 - \gamma^2 \|w(t)\|_{L_2}^2, \tag{2.4}$$

where $\gamma > 0$ is a given constant.

The H_∞ control problem for a performance level γ is to find a controller $u^*[x(\cdot), y(\cdot)]$ that internally stabilizes the system (2.1), (2.2) and ensures the inequality $J(u^*, w) \leq 0$ for all $w(t) \in L_2[0, +\infty; E^q]$ and for $x(t) = 0$, $y(t) = 0$, $t \leq 0$. Consider the matrices

$$A_\varepsilon = \begin{pmatrix} A_1 & A_2 \\ (1/\varepsilon)A_3 & (1/\varepsilon)A_4 \end{pmatrix}, \quad H_\varepsilon = \begin{pmatrix} H_1 & H_2 \\ (1/\varepsilon)H_3 & (1/\varepsilon)H_4 \end{pmatrix}, \tag{2.5a}$$

$$S_\varepsilon = \gamma^{-2} F_\varepsilon F'_\varepsilon - B_\varepsilon B'_\varepsilon, \qquad F_\varepsilon = \begin{pmatrix} F_1 \\ (1/\varepsilon)F_2 \end{pmatrix}, \qquad B_\varepsilon = \begin{pmatrix} B_1 \\ (1/\varepsilon)B_2 \end{pmatrix}, \tag{2.5b}$$

$$D = C'C, \qquad C = (C_1, C_2), \tag{2.5c}$$

where the prime denotes the transposition.

Consider the following hybrid system of matrix Riccati equations for $P, Q(\tau)$ and $R(\tau, \rho)$ in the domain $(\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$,

$$PA_\varepsilon + A'_\varepsilon P + PS_\varepsilon P + Q(0) + Q'(0) + D = 0, \tag{2.6}$$

$$dQ(\tau)/d\tau = [A'_\varepsilon + PS_\varepsilon]Q(\tau) + R(0, \tau), \quad Q(-\varepsilon h) = PH_\varepsilon, \tag{2.7}$$

$$\begin{aligned} (\partial/\partial\tau + \partial/\partial\rho)R(\tau, \rho) &= Q'(\tau)S_\varepsilon Q(\rho), \\ R(-\varepsilon h, \tau) = R'(\tau, -\varepsilon h) &= H'_\varepsilon Q(\tau). \end{aligned} \tag{2.8}$$

A solution of (2.6)–(2.8) is a triple of $(n+m) \times (n+m)$-matrices $\{P, Q(\tau), R(\tau, \rho)\}$, $(\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$, satisfying (2.6)–(2.8), where $Q(\tau)$ is continuously differentiable; $R(\tau, \rho)$ is continuous; and $\partial R(\tau, \rho)/\partial\tau$ and $\partial R(\tau, \rho)/\partial\rho$ are piecewise continuous, while $(\partial/\partial\tau + \partial/\partial\rho)R(\tau, \rho)$ is continuous.

Consider also the linear systems

$$\begin{aligned} \dot{z}(t) = {} & [A_\varepsilon - B_\varepsilon B'_\varepsilon P]z(t) + H_\varepsilon z(t - \varepsilon h) \\ & - B_\varepsilon B'_\varepsilon \int_{-\varepsilon h}^0 Q(\tau) z(t + \tau)\, d\tau, \qquad t > 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \dot{z}(t) = {} & [A_\varepsilon + S_\varepsilon P]z(t) + H_\varepsilon z(t - \varepsilon h) \\ & + S_\varepsilon \int_{-\varepsilon h}^0 Q(\tau) z(t + \tau)\, d\tau, \qquad t > 0. \end{aligned} \tag{2.10}$$

From [9] we obtain the following: if, for some $\varepsilon > 0$, the problem (2.6)–(2.8) has a solution $P(\varepsilon), Q(\tau, \varepsilon), R(\tau, \rho, \varepsilon)$ such that the systems (2.9) and (2.10) with $P = P(\varepsilon)$, $Q = Q(\tau, \varepsilon)$ are asymptotically stable, then, for this ε, the controller

$$u^*[x(\cdot), y(\cdot)](t) = -B'_\varepsilon\left[P(\varepsilon)z(t) + \int_{-\varepsilon h}^{0} Q(\tau, \varepsilon)z(t+\tau)\,d\tau\right],$$

$$z = \mathrm{col}\{x, y\}, \quad (2.11)$$

solves the H_∞ control problem (2.1)–(2.4).

The objectives of the present paper are:

1. To establish conditions (independent of ε) which ensure the existence of solution (2.11) of the H_∞ control problem (2.1)–(2.4) for all sufficiently small $\varepsilon > 0$.

2. To derive a controller much simpler than (2.11), which is constructed independently of ε and solves the H_∞ control problem (2.1)–(2.4) for all sufficiently small $\varepsilon > 0$.

The key point in reaching these objectives is the construction of a zero-order asymptotic solution to the problem (2.6)–(2.8).

3. ZERO-ORDER ASYMPTOTIC SOLUTION OF THE PROBLEM (2.6)–(2.8)

3.1. *Transformation of* (2.6)–(2.8) *and Formal Zero-Order Asymptotic Solution*

Let us transform the problem (2.6)–(2.8) to an explicit singular perturbation form. Following [10], we shall seek the solution of this problem in the form

$$P(\varepsilon) = \begin{pmatrix} P_1(\varepsilon) & \varepsilon P_2(\varepsilon) \\ \varepsilon P'_2(\varepsilon) & \varepsilon P_3(\varepsilon) \end{pmatrix}, \qquad Q(\tau, \varepsilon) = \begin{pmatrix} Q_1(\tau, \varepsilon) & Q_2(\tau, \varepsilon) \\ Q_3(\tau, \varepsilon) & Q_4(\tau, \varepsilon) \end{pmatrix}, \quad (3.1)$$

$$R(\tau, \rho, \varepsilon) = (1/\varepsilon)\begin{pmatrix} R_1(\tau, \rho, \varepsilon) & R_2(\tau, \rho, \varepsilon) \\ R'_2(\rho, \tau, \varepsilon) & R_3(\tau, \rho, \varepsilon) \end{pmatrix}, \quad (3.2)$$

where $P_i(\varepsilon)$ and $R_i(\tau, \rho, \varepsilon)$ $(i = 1, 2, 3)$ are matrices of the dimensions $n \times n$, $n \times m$, and $m \times m$ respectively; $Q_j(\tau, \varepsilon)$ $(j = 1, \ldots, 4)$ are matrices of the dimensions $n \times n$, $n \times m$, $m \times n$, and $m \times m$ respectively; $P_k(\varepsilon) = P'_k(\varepsilon)$, $R_k(\tau, \rho, \varepsilon) = R'_k(\rho, \tau, \varepsilon)$ $(k = 1, 3)$.

Substituting (2.5), (3.1), and (3.2) into the problem (2.6)–(2.8), we obtain the following system in the domain $(\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$ (in this system, for simplicity we omit the designation of the dependence of the unknown matrices on ε).

$$P_1 A_1 + P_2 A_3 + A_1' P_1 + A_3' P_2' + P_1 S_1 P_1 + P_2 S_2' P_1 + P_1 S_2 P_2' + P_2 S_3 P_2' + Q_1(0) + Q_1'(0) + D_1 = 0, \tag{3.3}$$

$$P_1 A_2 + P_2 A_4 + \varepsilon A_1' P_2 + A_3' P_3 + \varepsilon P_1 S_1 P_2 + \varepsilon P_2 S_2' P_2 + P_1 S_2 P_3 + P_2 S_3 P_3 + Q_2(0) + Q_3'(0) + D_2 = 0, \tag{3.4}$$

$$\varepsilon P_2' A_2 + P_3 A_4 + \varepsilon A_2' P_2 + A_4' P_3 + \varepsilon^2 P_2' S_1 P_2 + \varepsilon P_3 S_2' P_2 + \varepsilon P_2' S_2 P_3 + P_3 S_3 P_3 + Q_4(0) + Q_4'(0) + D_3 = 0, \tag{3.5}$$

$$\varepsilon dQ_1(\tau)/d\tau = \varepsilon(A_1' + P_1 S_1 + P_2 S_2')Q_1(\tau) + (A_3' + P_1 S_2 + P_2 S_3)Q_3(\tau) + R_1(0,\tau), \tag{3.6}$$

$$\varepsilon dQ_2(\tau)/d\tau = \varepsilon(A_1' + P_1 S_1 + P_2 S_2')Q_2(\tau) + (A_3' + P_1 S_2 + P_2 S_3)Q_4(\tau) + R_2(0,\tau), \tag{3.7}$$

$$\varepsilon dQ_3(\tau)/d\tau = \varepsilon(A_2' + \varepsilon P_2' S_1 + P_3 S_2')Q_1(\tau) + (A_4' + \varepsilon P_2' S_2 + P_3 S_3)Q_3(\tau) + R_2'(\tau,0), \tag{3.8}$$

$$\varepsilon dQ_4(\tau)/d\tau = \varepsilon(A_2' + \varepsilon P_2' S_1 + P_3 S_2')Q_2(\tau) + (A_4' + \varepsilon P_2' S_2 + P_3 S_3)Q_4(\tau) + R_3(0,\tau), \tag{3.9}$$

$$\varepsilon(\partial/\partial\tau + \partial/\partial\rho)R_1(\tau,\rho) = \varepsilon^2 Q_1'(\tau)S_1 Q_1(\rho) + \varepsilon Q_3'(\tau)S_2' Q_1(\rho) + \varepsilon Q_1'(\tau)S_2 Q_3(\rho) + Q_3'(\tau)S_3 Q_3(\rho), \tag{3.10}$$

$$\varepsilon(\partial/\partial\tau + \partial/\partial\rho)R_2(\tau,\rho) = \varepsilon^2 Q_1'(\tau)S_1 Q_2(\rho) + \varepsilon Q_3'(\tau)S_2' Q_2(\rho) + \varepsilon Q_1'(\tau)S_2 Q_4(\rho) + Q_3'(\tau)S_3 Q_4(\rho), \tag{3.11}$$

$$\varepsilon(\partial/\partial\tau + \partial/\partial\rho)R_3(\tau,\rho) = \varepsilon^2 Q_2'(\tau)S_1 Q_2(\rho) + \varepsilon Q_4'(\tau)S_2' Q_2(\rho) + \varepsilon Q_2'(\tau)S_2 Q_4(\rho) + Q_4'(\tau)S_3 Q_4(\rho), \tag{3.12}$$

$$\begin{aligned} Q_k(-\varepsilon h) &= P_1 H_k + P_2 H_{k+2}\ (k = 1,2), \\ Q_l(-\varepsilon h) &= \varepsilon P_2' H_{l-2} + P_3 H_l\ (l = 3,4), \end{aligned} \tag{3.13}$$

$$R_k(-\varepsilon h, \tau) = \varepsilon H_1' Q_k(\tau) + H_3' Q_{k+2}(\tau) \qquad (k = 1,2), \tag{3.14}$$

$$R_2(\tau, -\varepsilon h) = \varepsilon Q_1'(\tau) H_2 + Q_3'(\tau) H_4,$$
$$R_3(-\varepsilon h, \tau) = \varepsilon H_2' Q_2(\tau) + H_4' Q_4(\tau), \tag{3.15}$$

where $S_1 = \gamma^{-2} F_1 F_1' - B_1 B_1'$, $S_2 = \gamma^{-2} F_1 F_2' - B_1 B_2'$, $S_3 = \gamma^{-2} F_2 F_2' - B_2 B_2'$, $D_1 = C_1' C_1$, $D_2 = C_1' C_2$, $D_3 = C_2' C_2$.

The problem (3.3)–(3.15) has the explicit singular perturbation form. Now, let us construct the zero-order asymptotic solution of this problem. Similarly to [10], we shall seek the zero-order asymptotic solution of the problem (3.3)–(3.15) in the form

$$\bar{P}_{i0},\ Q_{j0}(\eta),\ R_{i0}(\eta, \chi), \qquad \eta = \tau/\varepsilon,\ \chi = \rho/\varepsilon\ (i = 1,2,3;\ j = 1,\dots,4). \tag{3.16}$$

Substituting (3.16) into (3.3)–(3.15) and equating coefficients of ε^0 in both parts of the resulting equations, we obtain the following system in the domain $(\eta, \chi) \in [-h, 0] \times [-h, 0]$.

$$\bar{P}_{10} A_1 + \bar{P}_{20} A_3 + A_1' \bar{P}_{10} + A_3' \bar{P}_{20}' + \bar{P}_{10} S_1 \bar{P}_{10} + \bar{P}_{20} S_2' \bar{P}_{10}$$
$$+ \bar{P}_{10} S_2 \bar{P}_{20}' + \bar{P}_{20} S_3 \bar{P}_{20}' + Q_{10}(0) + Q_{10}'(0) + D_1 = 0, \tag{3.17}$$

$$\bar{P}_{10} A_2 + \bar{P}_{20} A_4 + A_3' \bar{P}_{30} + \bar{P}_{10} S_2 \bar{P}_{30} + \bar{P}_{20} S_3 \bar{P}_{30}$$
$$+ Q_{20}(0) + Q_{30}'(0) + D_2 = 0, \tag{3.18}$$

$$\bar{P}_{30} A_4 + A_4' \bar{P}_{30} + \bar{P}_{30} S_3 \bar{P}_{30} + Q_{40}(0) + Q_{40}'(0) + D_3 = 0, \tag{3.19}$$

$$dQ_{10}(\eta)/d\eta = \left(A_3' + \bar{P}_{10} S_2 + \bar{P}_{20} S_3\right) Q_{30}(\eta) + R_{10}(0, \eta), \tag{3.20}$$

$$dQ_{20}(\eta)/d\eta = \left(A_3' + \bar{P}_{10} S_2 + \bar{P}_{20} S_3\right) Q_{40}(\eta) + R_{20}(0, \eta), \tag{3.21}$$

$$dQ_{30}(\eta)/d\eta = \left(A_4' + \bar{P}_{30} S_3\right) Q_{30}(\eta) + R_{20}'(\eta, 0), \tag{3.22}$$

$$dQ_{40}(\eta)/d\eta = \left(A_4' + \bar{P}_{30} S_3\right) Q_{40}(\eta) + R_{30}(0, \eta), \tag{3.23}$$

$$(\partial/\partial\eta + \partial/\partial\chi) R_{10}(\eta, \chi) = Q_{30}'(\eta) S_3 Q_{30}(\chi), \tag{3.24}$$

$$(\partial/\partial\eta + \partial/\partial\chi) R_{20}(\eta, \chi) = Q_{30}'(\eta) S_3 Q_{40}(\chi), \tag{3.25}$$

$$(\partial/\partial\eta + \partial/\partial\chi) R_{30}(\eta, \chi) = Q_{40}'(\eta) S_3 Q_{40}(\chi), \tag{3.26}$$

$$Q_{k0}(-h) = \bar{P}_{10} H_k + \bar{P}_{20} H_{k+2} \qquad (k = 1,2), \tag{3.27}$$

$$Q_{l0}(-h) = \bar{P}_{30} H_l \qquad (l = 3,4), \tag{3.28}$$

$$R_{k0}(-h, \eta) = H_3' Q_{k+2,0}(\eta) \qquad (k = 1,2), \tag{3.29}$$

$$R_{20}(\eta, -h) = Q'_{30}(\eta) H_4, \tag{3.30}$$

$$R_{30}(-h, \eta) = H'_4 Q_{40}(\eta). \tag{3.31}$$

Remark 3.1. The problem (3.17)–(3.31) can be divided into four simpler problems solved successively:

(i) The First Problem consists of (3.19), (3.23), (3.26), (3.28), with $l = 4$, and (3.31).

(ii) The Second Problem consists of (3.22), (3.25), (3.28), with $l = 3$, (3.29), with $k = 2$, and (3.30).

(iii) The Third Problem consists of (3.24) and (3.29) with $k = 1$.

(iv) The Fourth Problem consists of (3.17), (3.18), (3.20), (3.21), and (3.27).

3.2. *The First Problem and the Boundary-Layer H_∞ Control Problem*

We assume that:

A1. The First Problem has a solution $\overline{P}_{30}, Q_{40}(\eta), R_{30}(\eta, \chi), (\eta, \chi) \in [-h, 0] \times [-h, 0]$, such that $\overline{P}_{30} = \overline{P}'_{30}$, $R_{30}(\eta, \chi) = R'_{30}(\chi, \eta)$.

A2. All roots λ of the equation

$$\det\left[\lambda I_m - A_4 - S_3 \overline{P}_{30} - H_4 \exp(-\lambda h) - S_3 \int_{-h}^{0} Q_{40}(\eta) \exp(\lambda \eta)\, d\eta\right] = 0$$

lie inside the left-hand half-plane.

A3. All roots λ of the equation

$$\det\left[\lambda I_m - A_4 + B_2 B'_2 \overline{P}_{30} - H_4 \exp(-\lambda h) + B_2 B'_2 \int_{-h}^{0} Q_{40}(\eta) \exp(\lambda \eta)\, d\eta\right] = 0$$

lie inside the left-hand half-plane.

LEMMA 3.1. *Under the assumptions* A1–A3, *the matrix*

$$\begin{pmatrix} \overline{P}_{30} & Q_{40}(\chi) \\ Q'_{40}(\eta) & R_{30}(\eta, \chi) \end{pmatrix}$$

it is the kernel of linear bounded self-adjoint nonnegative operator mapping the space $E^m \times L_2[-h, 0; E^m]$ into itself.

Proof. The statement of the lemma is a direct consequence of results of [9] (see Lemma 1 and its proof). ∎

The First Problem is the hybrid system of matrix Riccati equations associated with the H_∞ control problem

$$d\tilde{y}(\sigma)/d\sigma = A_4\tilde{y}(\sigma) + H_4\tilde{y}(\sigma - h) + B_2\tilde{u}(\sigma) + F_2\tilde{w}(\sigma), \qquad \sigma > 0; \tag{3.32}$$

$$\tilde{y}(\sigma) = 0, \qquad \sigma \leq 0,$$

$$\begin{aligned}\tilde{J}(\tilde{u},\tilde{w}) &= \|\tilde{v}(\sigma)\|_{L_2}^2 - \gamma^2\|\tilde{w}(\sigma)\|_{L_2}^2,\\ \tilde{v}(\sigma) &= \mathrm{col}\{C_2\tilde{y}(\sigma), \tilde{u}(\sigma)\}, \ \sigma > 0,\end{aligned} \tag{3.33}$$

where $\tilde{y}$, $\tilde{u}$, $\tilde{w}$, and $\tilde{v}$ are state, control, disturbance, and observation respectively. In the following we shall call the problem (3.32), (3.33) the *boundary-layer* (fast) problem associated with the original H_∞ control problem (2.1)–(2.4).

LEMMA 3.2. *Under the assumptions* A1–A3, *the controller* $\tilde{u}^*[\tilde{y}(\cdot)](\sigma) = -B_2'[\overline{P}_{30}\tilde{y}(\sigma) + \int_{-h}^{0} Q_{40}(\eta)\tilde{y}(\sigma + \eta)\,d\eta]$ *solves the problem* (3.32), (3.33), *i.e.*, $\tilde{J}(\tilde{u}^*, \tilde{w}) \leq 0 \ \forall \tilde{w}(\sigma) \in L_2[0, +\infty; E^q]$.

Proof. The statement of the lemma directly follows from [9, Lemma 1]. ∎

3.3. *The Second and the Third Problems*

LEMMA 3.3. *The Second Problem and the Third Problem have the unique solutions* $\{Q_{30}(\eta), R_{20}(\eta, \chi)\}$ *and* $R_{10}(\eta, \chi)$, *respectively, for* $(\eta, \chi) \in [-h, 0] \times [-h, 0]$. *Moreover, the matrices* $R_{k0}(\eta, \chi)$ $(k = 1, 2)$ *have the form*

$$R_{k0}(\eta, \chi) = \Phi_k(\eta - \chi) + \int_{\max(\eta-\chi-h,\,-h)}^{\eta} Q_{30}'(s)S_3Q_{k+2,0}(s - \eta + \chi)\,ds \qquad (k = 1, 2), \tag{3.34}$$

where

$$\Phi_k(\sigma) = \begin{cases} H_3'Q_{k+2,0}(-\sigma - h), & -h \leq \sigma \leq 0,\\ Q_{30}'(\sigma - h)H_{k+2}, & 0 < \sigma \leq h,\end{cases} \tag{3.35}$$

and $Q_{30}(\eta)$ *is a unique solution to the linear integral-differential equation*

$$dQ_{30}(\eta)/d\eta = \left(A_4' + \overline{P}_{30}S_3\right)Q_{30}(\eta) + Q_{40}'(-\eta - h)H_3 + \int_{-h}^{\eta} Q_{40}'(s - \eta)S_3Q_{30}(s)\,ds \tag{3.36}$$

satisfying the initial condition (3.28) $(l = 3)$.

Proof. The lemma is an immediate consequence of the results of [10, Lemma 4.2]. ∎

3.4. *The Fourth Problem and the Reduced-Order H_∞ Control Problem*

Similarly to [10, pp. 498–499], one can rewrite the Fourth Problem in the equivalent form

$$\overline{P}_{10}\overline{A} + \overline{A}'\overline{P}_{10} + \overline{P}_{10}\overline{S}\,\overline{P}_{10} + \overline{D} = 0, \tag{3.37}$$

$$\overline{P}_{20} = -\left(\overline{P}_{10}N_1 + N_2 + \int_{-h}^{0} Q'_{30}(\eta)\,d\eta\right), \tag{3.38}$$

$$\begin{aligned} Q_{k0}(\eta) &= \overline{P}_{10}H_k + \overline{P}_{20}H_{k+2} \\ &\quad + \left(A'_3 + \overline{P}_{10}S_2 + \overline{P}_{20}S_3\right)\int_{-h}^{\eta} Q_{k+2,0}(s)\,ds \\ &\quad + \int_{-h}^{\eta} R_{k0}(0,s)\,ds \qquad (k = 1,2), \end{aligned} \tag{3.39}$$

where

$$\overline{A} = A_{H1} - N_1A_{H3} - S_2N'_2 + N_1S_3N'_2, \tag{3.40}$$

$$\overline{S} = \gamma^{-2}\overline{F}\,\overline{F}' - \overline{B}\,\overline{B}', \qquad \overline{F} = F_1 - N_1F_2, \qquad \overline{B} = B_1 - N_1B_2, \tag{3.41}$$

$$\overline{D} = D_1 - N_2A_{H3} - A'_{H3}N'_2 + N_2S_3N'_2, \tag{3.42}$$

$$N_1 = (A_{H2} + S_2G)M^{-1}, \qquad N_2 = (A'_{H3}G + D_2)M^{-1}, \tag{3.43}$$

$$\begin{aligned} &M = A_{H4} + S_3G, \qquad G = \overline{P}_{30} + \int_{-h}^{0} Q_{40}(\eta)\,d\eta, \\ &A_{Hi} = A_i + H_i \ (i = 1,\dots,4). \end{aligned} \tag{3.44}$$

From the assumption A2 we directly obtain that the matrix M is invertible.

In the following, we assume that:

A4. The equation (3.37) has a symmetric positive semidefinite solution $\overline{P}_{10}$.

A5. All eigenvalues of the matrix $(\overline{A} + \overline{S}\,\overline{P}_{10})$ lie inside the left-hand half-plane.

A6. All eigenvalues of the matrix $[\overline{A} + \Delta_A + (\overline{B}\,\overline{B}' + \Delta_B)\overline{P}_{10}]$ lie inside the left-hand half-plane, where $\Delta_A = \gamma^{-2}\overline{F}F'_2[N'_2 + G\overline{M}^{-1}(A_{H3} + B_2B'_2N'_2)]$, $\Delta_B = \gamma^{-2}\overline{F}F'_2G\overline{M}^{-1}B_2\overline{B}'$, $\overline{M} = A_{H4} - B_2B'_2G$.

From the assumption A3 we directly obtain that the matrix $\overline{M}$ is invertible.

Now, let us present an interpretation of Eq. (3.37) and the assumptions A4–A6. Setting $\varepsilon = 0$ in (2.1)–(2.4), one obtains the H_∞ control problem for the descriptor (algebraic–differential) system

$$E\dot{\bar{z}}(t) = A_H \bar{z}(t) + B\bar{u}(t) + F\bar{w}(t), t > 0, \qquad E\bar{z}(0) = 0, \quad (3.45)$$

$$\bar{J}(\bar{u}, \bar{w}) = \|\bar{v}(t)\|_{L_2}^2 - \gamma^2 \|\bar{w}(t)\|_{L_2}^2, \qquad \bar{v}(t) = \mathrm{col}\{C\bar{z}(t), \bar{u}(t)\}, t > 0, \quad (3.46)$$

where $\bar{z}$, $\bar{u}$, $\bar{w}$, and $\bar{v}$ are state, control, disturbance, and observation, respectively, and

$$E = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_H = \begin{pmatrix} A_{H1} & A_{H2} \\ A_{H3} & A_{H4} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \quad (3.47)$$

In the following, we shall call this problem the *reduced-order* (slow) one associated with the original H_∞ control problem (2.1)–(2.4).

Consider the generalized Riccati equation associated with the problem (3.45), (3.46)

$$K'A_H + A'_H K + K'SK + D = 0, \qquad EK = K'E, \qquad (3.48)$$

where $S = \gamma^{-2}FF' - BB'$.

LEMMA 3.4. *Under the assumptions* A1, A2, A4, *and* A5, *the matrix*

$$K_0 = \begin{pmatrix} \overline{P}_{10} & 0 \\ G_1 & G \end{pmatrix},$$

where $G_1 \triangleq \overline{P}'_{20} + \int_{-h}^{0} Q_{30}(\eta)\, d\eta = -(N'_1 \overline{P}_{10} + N'_2)$, *satisfies* (3.48), *and* EK_0 *is positive semidefinite.*

Proof. The lemma is proved by direct substitution of K_0 into (3.48), applying the block expansion of (3.48) (see [27]), and taking into account that G satisfies the Riccati equation $G'A_{H4} + A'_{H4}G + G'S_3G + D_3 = 0$ (see [10]). ▮

Note that (3.37) can be obtained from (3.48) by eliminating the lower left- and right-hand blocks of the matrix K of the dimensions $m \times n$ and $m \times m$ respectively.

LEMMA 3.5. *Under the assumptions* A1, A2, *and* A4, *the system*

$$E\dot{\bar{z}}(t) = (A_H + SK_0)\bar{z}(t) \tag{3.49}$$

is asymptotically stable iff the assumption A5 *is satisfied.*

Proof. Let $\bar{x}$ and $\bar{y}$ be the upper and lower blocks of the vector $\bar{z}$ of the dimensions n and m respectively. Since the matrix M is invertible (due to A2), one can rewrite (3.49) in the equivalent block form

$$\dot{\bar{x}}(t) = \left(\bar{A} + \bar{S}\bar{P}_{10}\right)\bar{x}(t), \quad \bar{y}(t) = -M^{-1}\left(A_{H3} + S_2'\bar{P}_{10} + S_3G_1\right)\bar{x}(t). \tag{3.50}$$

Now, the statement of the lemma directly follows from A5. ▌

LEMMA 3.6. *Under the assumption* A1, A2, A3, *and* A4, *the system*

$$E\dot{\bar{z}}(t) = (A_H - BB'K_0)\bar{z}(t) \tag{3.51}$$

*is asymptotically stable iff the assumption A*6 *is satisfied.*

Proof. The lemma is proved similarly to Lemma 3.5. ▌

LEMMA 3.7. *Under the assumptions* A1–A6, *the controller* $\bar{u}^*[\bar{z}(t)] = -B'K_0\bar{z}(t)$ *solves the* H_∞ *problem* (3.45), (3.46), *i.e.*, $\bar{J}(\bar{u}^*, \bar{w}) \le 0\ \forall \bar{w}(t) \in L_2[0, +\infty; E^q]$.

Proof. The lemma is a direct consequence of Lemmas 3.4–3.6, and it is proved similarly to [9, Lemma 1], applying the functional $V(\bar{z}) = \bar{z}'EK_0\bar{z}$. ▌

Thus, we have shown that Eq. (3.37) and the assumptions A4–A6 are associated with the reduced-order H_∞ control problem (3.45), (3.46) by conditions of the existence of its solution.

We have completed the construction of the zero-order asymptotic solution to the problem (3.3)–(3.15) and, hence, to (2.6)–(2.8). It is clear that the asymptotic approach to the problem (2.6)–(2.8) essentially simplifies a procedure of its solution. The original problem is reduced to three problems of lower dimensions solved successively. These problems are the First Problem, the problem (3.36), (3.28) ($l = 3$), and the equation (3.37). Note that these problems are independent of ε. The other components of the zero-order asymptotic solution are obtained from the explicit expressions (3.34), (3.38), (3.39).

In the next section, we shall verify the zero-order asymptotic solution to the problem (3.3)–(3.15) constructed in this section.

4. VERIFICATION OF THE ZERO-ORDER ASYMPTOTIC SOLUTION OF THE PROBLEM (3.3)–(3.15)

4.1. *Auxiliary Results*

In this subsection, we shall present some auxiliary results which will be applied in the verification of the zero-order asymptotic solution to the problem (3.3)–(3.15).

Consider the system

$$\dot{\varphi}(t) = \tilde{A}(\varepsilon)\varphi(t) + \tilde{H}(\varepsilon)\varphi(t-\varepsilon h) + \int_{-h}^{0} \tilde{G}(\eta,\varepsilon)\varphi(t+\varepsilon\eta)\,d\eta,$$
$$t > 0,\ \varphi \in E^{n+m}, \quad (4.1)$$

where

$$\tilde{A}(\varepsilon) = \begin{pmatrix} \tilde{A}_1(\varepsilon) & \tilde{A}_2(\varepsilon) \\ (1/\varepsilon)\tilde{A}_3(\varepsilon) & (1/\varepsilon)\tilde{A}_4(\varepsilon) \end{pmatrix},$$
$$\tilde{H}(\varepsilon) = \begin{pmatrix} \tilde{H}_1(\varepsilon) & \tilde{H}_2(\varepsilon) \\ (1/\varepsilon)\tilde{H}_3(\varepsilon) & (1/\varepsilon)\tilde{H}_4(\varepsilon) \end{pmatrix}, \quad (4.2)$$

$$\tilde{G}(\eta,\varepsilon) = \begin{pmatrix} \tilde{G}_1(\eta,\varepsilon) & \tilde{G}_2(\eta,\varepsilon) \\ (1/\varepsilon)\tilde{G}_3(\eta,\varepsilon) & (1/\varepsilon)\tilde{G}_4(\eta,\varepsilon) \end{pmatrix}, \quad (4.3)$$

the blocks $\tilde{A}_1(\varepsilon), \tilde{H}_1(\varepsilon), \tilde{G}_1(\eta,\varepsilon)$ are of dimension $n \times n$, and the blocks $\tilde{A}_4(\varepsilon), \tilde{H}_4(\varepsilon), \tilde{G}_4(\eta,\varepsilon)$ are of dimension $m \times m$.

We assume that:

A7. $\tilde{A}_i(\varepsilon)$, $\tilde{H}_i(\varepsilon)$, and $\tilde{G}_i(\eta,\varepsilon)$ $(i = 1,\dots,4)$ are differentiable functions of ε and (η,ε) for $\eta \in [-h, 0]$ and all sufficiently small $\varepsilon \geq 0$.

A8. The reduced-order subsystem associated with (4.1),

$$\dot{\bar{\varphi}}_1(t) = \Omega\bar{\varphi}_1(t), \qquad t > 0,\ \bar{\varphi}_1 \in E^n, \quad (4.4)$$

where

$$\Omega = \Omega_1 - \Omega_2\Omega_4^{-1}\Omega_3,$$
$$\Omega_i = \tilde{A}_i(0) + \tilde{H}_i(0) + \int_{-h}^{0} \tilde{G}_i(\eta,0)\,d\eta \qquad (i = 1,\dots,4) \quad (4.5)$$

is asymptotically stable.

A9. The boundary-layer subsystem associated with (4.1),

$$d\tilde{\varphi}_2(\sigma)/d\sigma = \tilde{A}_4(0)\tilde{\varphi}_2(\sigma) + \tilde{H}_4(0)\tilde{\varphi}_2(\sigma - h) + \int_{-h}^{0} \tilde{G}_4(\eta, 0)\tilde{\varphi}_2(\sigma + \eta)\, d\eta, \qquad \sigma > 0,\ \tilde{\varphi}_2 \in E^m, \quad (4.6)$$

is asymptotically stable.

Let $\Phi(t, \varepsilon)$ be the fundamental matrix of the system (4.1), i.e., it satisfies this system and the initial conditions

$$\Phi(0, \varepsilon) = I_{n+m}; \qquad \Phi(t, \varepsilon) = 0,\ t < 0. \quad (4.7)$$

LEMMA 4.1. *Let $\Phi_1(t, \varepsilon)$, $\Phi_2(t, \varepsilon)$, $\Phi_3(t, \varepsilon)$, and $\Phi_4(t, \varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand, and lower right-hand blocks of the matrix $\Phi(t, \varepsilon)$ of the dimensions $n \times n$, $n \times m$, $m \times n$, and $m \times m$ respectively. Under the assumptions A7–A9, for all $t \geq 0$ and sufficiently small $\varepsilon > 0$, the following inequalities are satisfied*:

$$\|\Phi_k(t, \varepsilon)\| \leq a \exp(-\alpha t)\ (k = 1, 3), \qquad \|\Phi_2(t, \varepsilon)\| \leq a\varepsilon \exp(-\alpha t),$$
$$\|\Phi_4(t, \varepsilon)\| \leq a \exp(-\alpha t)[\varepsilon + \exp(-\beta t/\varepsilon)],$$

where $a > 0$, $\alpha > 0$, and $\beta > 0$ are some constants independent of ε.

For a proof of the lemma, see Appendix.

Consider the particular case of the system (4.1) with the coefficients

$$\tilde{A}(\varepsilon) = A_\varepsilon + S_\varepsilon P_0(\varepsilon), \qquad \tilde{H}(\varepsilon) = H_\varepsilon,$$
$$\tilde{G}(\eta, \varepsilon) = \varepsilon[S_\varepsilon Q_0(\eta) + \Lambda_\varepsilon], \quad (4.8)$$

where

$$P_0(\varepsilon) = \begin{pmatrix} \bar{P}_{10} & \varepsilon \bar{P}_{20} \\ \varepsilon \bar{P}'_{20} & \varepsilon \bar{P}_{30} \end{pmatrix}, \qquad Q_0(\eta) = \begin{pmatrix} Q_{10}(\eta) & Q_{20}(\eta) \\ Q_{30}(\eta) & Q_{40}(\eta) \end{pmatrix}, \quad (4.9)$$

$$\Lambda_\varepsilon = \begin{pmatrix} S_2 \bar{P}'_{20} H_1 & S_2 \bar{P}'_{20} H_2 \\ (1/\varepsilon) S_3 \bar{P}'_{20} H_1 & (1/\varepsilon) S_3 \bar{P}'_{20} H_2 \end{pmatrix}. \quad (4.10)$$

Let $\Psi(t, \varepsilon)$ be the fundamental matrix of the system (4.1), (4.8). Let $\Psi_1(t, \varepsilon)$, $\Psi_2(t, \varepsilon)$, $\Psi_3(t, \varepsilon)$, and $\Psi_4(t, \varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand, and lower right-hand blocks of the matrix $\Psi(t,\varepsilon)$ of the dimensions $n \times n$, $n \times m$, $m \times n$, and $m \times m$ respectively.

LEMMA 4.2. *Under the assumptions* A1, A2, A4, *and* A5, *the inequalities*

$$\|\Psi_k(t,\varepsilon)\| \le a\exp(-\alpha t) \; (k=1,3), \qquad \|\Psi_2(t,\varepsilon)\| \le a\varepsilon\exp(-\alpha t),$$
$$\|\Psi_4(t,\varepsilon)\| \le a\exp(-\alpha t)[\varepsilon + \exp(-\beta t/\varepsilon)],$$

are satisfied for all $t \ge 0$ *and sufficiently small* $\varepsilon > 0$; *where* $a > 0$, $\alpha > 0$, *and* $\beta > 0$ *are some constants independent of* ε.

Proof. Let us construct the reduced-order and the boundary-layer subsystems, associated with the system (4.1), (4.8), and show the asymptotic stability of these subsystems. From (4.2), (4.3), and (4.8) one has

$$\tilde{A}_1(\varepsilon) = A_1 + S_1\bar{P}_{10} + S_2\bar{P}'_{20}, \qquad \tilde{A}_2(\varepsilon) = A_2 + \varepsilon S_1\bar{P}_{20} + S_2\bar{P}_{30}, \tag{4.11}$$

$$\tilde{A}_3(\varepsilon) = A_3 + S'_2\bar{P}_{10} + S_3\bar{P}'_{20}, \qquad \tilde{A}_4(\varepsilon) = A_4 + \varepsilon S'_2\bar{P}_{20} + S_3\bar{P}_{30}, \tag{4.12}$$

$$\tilde{G}_k(\eta,\varepsilon) = \varepsilon S_1 Q_{k0}(\eta) + S_2 Q_{k+2,0}(\eta) + \varepsilon S_2\bar{P}'_{20}H_k \qquad (k=1,2), \tag{4.13}$$

$$\tilde{G}_l(\eta,\varepsilon) = \varepsilon S'_2 Q_{l-2,0}(\eta) + S_3 Q_{l0}(\eta) + \varepsilon S_3\bar{P}'_{20}H_{l-2} \qquad (l=3,4). \tag{4.14}$$

The block representation of the matrix H_ε is given in (2.5a).

Substituting (4.11)–(4.14) into (4.5), one obtains the matrix Ω of coefficients of the reduced-order subsystem (4.4) associated with the system (4.1), (4.8),

$$\Omega = A_{H1} + S_1\bar{P}_{10} + S_2G_1 - N_1\left(A_{H3} + S'_2\bar{P}_{10} + S_3G_1\right). \tag{4.15}$$

Under the assumption A2, the matrix M in the expression for N_1 is invertible. Substituting the expression for G_1 (see Lemma 3.4) into (4.15) yields, after some rearrangement, $\Omega = \bar{A} + \bar{S}\bar{P}_{10}$, which implies, along with the assumption A5, the asymptotic stability of the reduced-order subsystem, associated with the system (4.1), (4.8).

Replacing in (4.6) $\tilde{A}_4(0)$ with its expression from (4.12), $\tilde{H}_4(0)$ with H_4, and $\tilde{G}_4(\eta,0)$ with its expression from (4.14), we obtain the boundary-layer subsystem, associated with the system (4.1), (4.8):

$$d\tilde{\varphi}_2(\sigma)/d\sigma = \left(A_4 + S_3\overline{P}_{30}\right)\tilde{\varphi}_2(\sigma) + H_4\tilde{\varphi}_2(\sigma - h) + \int_{-h}^{0} S_3 Q_{40}(\eta)\tilde{\varphi}_2(\sigma+\eta)\,d\eta,$$

$$\sigma > 0,\ \tilde{\varphi}_2 \in E^m. \quad (4.16)$$

The assumption A2 directly implies the asymptotic stability of the boundary-layer subsystem (4.16). Now, the statement of the lemma is an immediate consequence of Lemma 4.1. ▮

4.2. *Estimation of the Remainder Term Corresponding to the Zero-Order Asymptotic Solution*

THEOREM 4.1. *Under the assumptions* A1, A2, A4, *and* A5, *the problem* (3.3)–(3.15) *has a solution* $P_i(\varepsilon), Q_j(\tau,\varepsilon), R_i(\tau,\rho,\varepsilon)$ $(i = 1,2,3;\ j = 1,\dots,4)$ *for all sufficiently small* $\varepsilon > 0$, *and this solution satisfies the inequalities*

$$\left\|P_i(\varepsilon) - \overline{P}_{i0}\right\| \le a\varepsilon, \qquad \left\|Q_j(\tau,\varepsilon) - Q_{j0}(\tau/\varepsilon)\right\| \le a\varepsilon,$$

$$\left\|R_i(\tau,\rho,\varepsilon) - R_{i0}(\tau/\varepsilon,\rho/\varepsilon)\right\| \le a\varepsilon,$$

where $(\tau,\rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$; $\overline{P}_{i0}$, $Q_{j0}(\eta)$ *and* $R_{i0}(\eta,\chi)$ *are defined in Section* 3; *and* $a > 0$ *is some constant independent of* ε.

Proof. Let us transform the variables in the problem (3.3)–(3.15) as

$$P_i(\varepsilon) = \overline{P}_{i0} + \theta_{Pi}(\varepsilon) \qquad (i = 1,2,3), \quad (4.17)$$

$$Q_k(\tau,\varepsilon) = Q_{k0}(\tau/\varepsilon) + \theta_{Qk}(\tau,\varepsilon),$$

$$Q_l(\tau,\varepsilon) = Q_{l0}(\tau/\varepsilon) + \varepsilon\overline{P}'_{20}H_{l-2} + \theta_{Ql}(\tau,\varepsilon), \quad (4.18)$$

$$(k = 1,2;\ l = 3,4),$$

$$R_1(\tau,\rho,\varepsilon) = R_{10}(\tau/\varepsilon,\rho/\varepsilon) + \varepsilon\left[H'_1 Q_{10}(\rho/\varepsilon) + H'_3\overline{P}'_{20}H_1\right] + \theta_{R1}(\tau,\rho,\varepsilon), \quad (4.19)$$

$$R_2(\tau,\rho,\varepsilon) = R_{20}(\tau/\varepsilon,\rho/\varepsilon) + \varepsilon\left\{H'_1\left[Q_{20}(\rho/\varepsilon) + \overline{P}_{20}H_4 - Q_{20}(-h)\right] + Q'_{10}(\tau/\varepsilon)H_2\right\} + \theta_{R2}(\tau,\rho,\varepsilon), \quad (4.20)$$

$$R_3(\tau,\rho,\varepsilon) = R_{30}(\tau/\varepsilon,\rho/\varepsilon) + \varepsilon\left[H'_2 Q_{20}(\rho/\varepsilon) + H'_4\overline{P}'_{20}H_2\right] + \theta_{R3}(\tau,\rho,\varepsilon). \quad (4.21)$$

The transformation (4.17)–(4.21) yields the following problem for the new variables $\theta_{Pi}(\varepsilon)$, $\theta_{Qj}(\tau, \varepsilon)$, and $\theta_{Ri}(\tau, \rho, \varepsilon)$ ($i = 1, 2, 3$; $j = 1, \dots, 4$) in the domain $(\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$.

$$\theta_P(\varepsilon)\tilde{A}(\varepsilon) + \tilde{A}'(\varepsilon)\theta_P(\varepsilon) + \theta_Q(0, \varepsilon) + \theta_Q'(0, \varepsilon) + D_P(\varepsilon) + \theta_P(\varepsilon)S_\varepsilon\theta_P(\varepsilon) = 0, \tag{4.22}$$

$$d\theta_Q(\tau, \varepsilon)/d\tau = \tilde{A}'(\varepsilon)\theta_Q(\tau, \varepsilon) + (1/\varepsilon)\theta_P(\varepsilon)\tilde{G}(\tau/\varepsilon, \varepsilon) + \theta_R(0, \tau, \varepsilon) + D_Q(\tau, \varepsilon) + \theta_P(\varepsilon)S_\varepsilon\theta_Q(\tau, \varepsilon), \tag{4.23}$$

$$(\partial/\partial\tau + \partial/\partial\rho)\theta_R(\tau, \rho, \varepsilon) = (1/\varepsilon)\theta_Q'(\tau, \varepsilon)\tilde{G}(\rho/\varepsilon, \varepsilon) + (1/\varepsilon)\tilde{G}'(\tau/\varepsilon, \varepsilon)\theta_Q(\rho, \varepsilon) + D_R(\tau, \rho, \varepsilon) + \theta_Q'(\tau, \varepsilon)S_\varepsilon\theta_Q(\rho, \varepsilon), \tag{4.24}$$

$$\begin{aligned} \theta_Q(-\varepsilon h, \varepsilon) &= \theta_P(\varepsilon)H_\varepsilon, \\ \theta_R(-\varepsilon h, \tau, \varepsilon) = \theta_R'(\tau, -\varepsilon h, \varepsilon) &= H_\varepsilon'\theta_Q(\tau, \varepsilon), \end{aligned} \tag{4.25}$$

where $\tilde{A}(\varepsilon)$ and $\tilde{G}(\eta, \varepsilon)$ are defined by (4.8). Also in (4.22)–(4.25),

$$\theta_P(\varepsilon) = \begin{pmatrix} \theta_{P1}(\varepsilon) & \varepsilon\theta_{P2}(\varepsilon) \\ \varepsilon\theta_{P2}'(\varepsilon) & \varepsilon\theta_{P3}(\varepsilon) \end{pmatrix},$$

$$\theta_Q(\tau, \varepsilon) = \begin{pmatrix} \theta_{Q1}(\tau, \varepsilon) & \theta_{Q2}(\tau, \varepsilon) \\ \theta_{Q3}(\tau, \varepsilon) & \theta_{Q4}(\tau, \varepsilon) \end{pmatrix},$$

$$\theta_R(\tau, \rho, \varepsilon) = (1/\varepsilon)\begin{pmatrix} \theta_{R1}(\tau, \rho, \varepsilon) & \theta_{R2}(\tau, \rho, \varepsilon) \\ \theta_{R2}'(\rho, \tau, \varepsilon) & \theta_{R3}(\tau, \rho, \varepsilon) \end{pmatrix},$$

$$D_P(\varepsilon) = \begin{pmatrix} 0 & D_{P2}(\varepsilon) \\ D_{P2}'(\varepsilon) & D_{P3}(\varepsilon) \end{pmatrix},$$

$$D_Q(\tau, \varepsilon) = \begin{pmatrix} D_{Q1}(\tau, \varepsilon) & D_{Q2}(\tau, \varepsilon) \\ D_{Q3}(\tau, \varepsilon) & D_{Q4}(\tau, \varepsilon) \end{pmatrix},$$

$$D_R(\tau, \rho, \varepsilon) = \begin{pmatrix} D_{R1}(\tau, \rho, \varepsilon) & D_{R2}(\tau, \rho, \varepsilon) \\ D_{R2}'(\rho, \tau, \varepsilon) & D_{R3}(\tau, \rho, \varepsilon) \end{pmatrix}.$$

The matrices $D_{Pk}(\varepsilon)$, $D_{Qj}(\tau, \varepsilon)$, $D_{Ri}(\tau, \rho, \varepsilon)$ are known functions of $\overline{P}_{i0}$, $Q_{j0}(\tau/\varepsilon)$, and $R_{i0}(\tau/\varepsilon, \rho/\varepsilon)$ ($k = 2, 3$; $i = 1, 2, 3$; $j = 1, \dots, 4$). These matrices are continuous in $(\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$, and for all sufficiently small $\varepsilon > 0$ they satisfy the inequalities

$$\|D_{Pk}(\varepsilon)\| \leq a\varepsilon, \; \|D_{Qj}(\tau, \varepsilon)\| \leq a, \; \|D_{Ri}(\tau, \rho, \varepsilon)\| \leq a/\varepsilon$$
$$(k = 2,3;\; j = 1,\dots,4;\; i = 1,2,3), \quad (4.26)$$

where $a > 0$ is some constant independent of ε.

Denote

$$\Gamma_P(\theta_P)(\varepsilon) = D_P(\varepsilon) + \theta_P(\varepsilon) S_\varepsilon \theta_P(\varepsilon), \quad (4.27)$$

$$\Gamma_Q(\theta_P, \theta_Q)(\tau, \varepsilon) = D_Q(\tau, \varepsilon) + \theta_P(\varepsilon) S_\varepsilon \theta_Q(\tau, \varepsilon), \quad (4.28)$$

$$\Gamma_R(\theta_Q)(\tau, \rho, \varepsilon) = D_R(\tau, \rho, \varepsilon) + \theta_Q'(\tau, \varepsilon) S_\varepsilon \theta_Q(\rho, \varepsilon), \quad (4.29)$$

$$\tilde{\Psi}(t, \tau, \varepsilon) = \Psi(t - \tau - \varepsilon h, \varepsilon) H_\varepsilon + (1/\varepsilon) \int_{-\tau}^{\varepsilon h} \Psi(t - \tau - \rho, \varepsilon) \tilde{G}(-\rho/\varepsilon, \varepsilon)\, d\rho. \quad (4.30)$$

Applying Lemma 4.2, one can directly show that the matrix $\tilde{\Psi}(t, \tau, \varepsilon)$ satisfies the inequalities for all $t \geq 0$, $\tau \in [-\varepsilon h, 0]$, and sufficiently small $\varepsilon > 0$,

$$\begin{aligned} \|\tilde{\Psi}_k(t, \tau, \varepsilon)\| &\leq a \exp(-\alpha t), \\ \|\tilde{\Psi}_l(t, \tau, \varepsilon)\| &\leq a \exp(-\alpha t)[1 + (1/\varepsilon)\exp(-\beta t/\varepsilon)], \end{aligned} \quad (4.31)$$

where $(k = 1,2;\; l = 3,4)$ $\tilde{\Psi}_j(t, \tau, \varepsilon)$ $(j = 1,\dots,4)$ are the upper left-hand, upper right-hand, lower left-hand, and lower right-hand blocks of this matrix of the dimensions $n \times n$, $n \times m$, $m \times n$, and $m \times m$ respectively; $a > 0$, $\alpha > 0$, and $\beta > 0$ are some constants independent of ε. Applying results of [10, pp. 501–502], we can rewrite the problem (4.22)–(4.25) in the equivalent form

$$\begin{aligned} \theta_P(\varepsilon) = \int_0^{+\infty} \Bigg[& \Psi'(t, \varepsilon) \Gamma_P(\theta_P)(\varepsilon) \Psi(t, \varepsilon) \\ & + \int_{-\varepsilon h}^{0} \Psi'(t, \varepsilon) \Gamma_Q(\theta_P, \theta_Q)(\tau, \varepsilon) \Psi(t + \tau, \varepsilon)\, d\tau \\ & + \int_{-\varepsilon h}^{0} \Psi'(t + \tau, \varepsilon) \Gamma_Q'(\theta_P, \theta_Q)(\tau, \varepsilon) \Psi(t, \varepsilon)\, d\tau \\ & + \int_{-\varepsilon h}^{0} \int_{-\varepsilon h}^{0} \Psi'(t + \tau, \varepsilon) \Gamma_R(\theta_Q)(\tau, \rho, \varepsilon) \\ & \times \Psi(t + \rho, \varepsilon)\, d\tau\, d\rho \Bigg] dt, \end{aligned} \quad (4.32)$$

$$\begin{aligned}
\theta_Q(\tau,\varepsilon) = \int_0^{+\infty} \Big[& \Psi'(t,\varepsilon)\Gamma_P(\theta_P)(\varepsilon)\tilde{\Psi}(t,\tau,\varepsilon) \\
& + \int_{-\varepsilon h}^{0} \Psi'(t,\varepsilon)\Gamma_Q(\theta_P,\theta_Q)(\rho,\varepsilon)\tilde{\Psi}(t+\rho,\tau,\varepsilon)\,d\rho \\
& + \int_{-\varepsilon h}^{0} \Psi'(t+\rho,\varepsilon)\Gamma_Q'(\theta_P,\theta_Q)(\rho,\varepsilon)\tilde{\Psi}(t,\tau,\varepsilon)\,d\rho \\
& + \int_{-\varepsilon h}^{0}\int_{-\varepsilon h}^{0} \Psi'(t+\rho,\varepsilon)\Gamma_R(\theta_Q)(\rho,\rho_1,\varepsilon) \\
& \times \tilde{\Psi}(t+\rho_1,\tau,\varepsilon)\,d\rho\,d\rho_1 \Big]\,dt \\
+ \int_0^{\tau+\varepsilon h} \Big[& \Psi'(t,\varepsilon)\Gamma_Q(\theta_P,\theta_Q)(\tau-t,\varepsilon) \\
& + \int_{-\varepsilon h}^{0} \Psi'(t+\rho,\varepsilon)\Gamma_R(\theta_Q)(\rho,\tau-t,\varepsilon)\,d\rho \Big]\,dt, \qquad (4.33)
\end{aligned}$$

$$\begin{aligned}
& \theta_R(\tau,\rho,\varepsilon) \\
& = \int_0^{+\infty} \Big[\tilde{\Psi}'(t,\tau,\varepsilon)\Gamma_P(\theta_P)(\varepsilon)\tilde{\Psi}(t,\rho,\varepsilon) \\
& \quad + \int_{-\varepsilon h}^{0} \tilde{\Psi}'(t,\tau,\varepsilon)\Gamma_Q(\theta_P,\theta_Q)(\rho_1,\varepsilon)\tilde{\Psi}(t+\rho_1,\rho)\,d\rho_1 \\
& \quad + \int_{-\varepsilon h}^{0} \tilde{\Psi}'(t+\rho_1,\tau,\varepsilon)\Gamma_Q'(\theta_P,\theta_Q)(\rho_1,\varepsilon)\tilde{\Psi}(t,\rho,\varepsilon)\,d\rho_1 \\
& \quad + \int_{-\varepsilon h}^{0}\int_{-\varepsilon h}^{0} \tilde{\Psi}'(t+\rho_1,\tau,\varepsilon)\Gamma_R(\theta_Q)(\rho_1,\rho_2,\varepsilon) \\
& \quad \times \tilde{\Psi}(t+\rho_2,\rho,\varepsilon)\,d\rho_1\,d\rho_2 \Big]\,dt \\
& + \int_0^{\tau+\varepsilon h} \Big[\Gamma_Q'(\theta_P,\theta_Q)(\tau-t,\varepsilon)\tilde{\Psi}(t,\rho,\varepsilon) \\
& \quad + \int_{-\varepsilon h}^{0} \Gamma_R(\theta_Q)(\rho_1,\tau-t,\varepsilon)\tilde{\Psi}(t+\rho_1,\rho,\varepsilon)\,d\rho_1 \Big]\,dt \\
& + \int_0^{\rho+\varepsilon h} \Big[\tilde{\Psi}'(t,\tau,\varepsilon)\Gamma_Q(\theta_P,\theta_Q)(\rho-t,\varepsilon) \\
& \quad + \int_{-\varepsilon h}^{0} \tilde{\Psi}'(t+\rho_1,\tau,\varepsilon)\Gamma_R(\theta_Q)(\rho_1,\rho-t,\varepsilon)\,d\rho_1 \Big]\,dt \\
& + \int_0^{\min(\tau+\varepsilon h,\,\rho+\varepsilon h)} \Gamma_R(\theta_Q)(\tau-t,\rho-t,\varepsilon)\,dt. \qquad (4.34)
\end{aligned}$$

It is obvious that

$$0 \le \min(\tau + \varepsilon h, \rho + \varepsilon h) \le \varepsilon h \qquad (\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]. \tag{4.35}$$

Now, applying the procedure of successive approximations to the system (4.32)–(4.34) and taking into account Lemma 4.2, the equations (4.27)–(4.29) and the inequalities (4.26), (4.31), (4.35), one directly obtains the existence of the solution $\theta_P(\varepsilon)$, $\theta_Q(\tau, \varepsilon)$, $\theta_R(\tau, \rho, \varepsilon)$ of the problem (4.32)–(4.34) (and, consequently, of the problem (4.22)–(4.25)), satisfying the inequalities for all sufficiently small $\varepsilon > 0$ and $(\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$,

$$\| \theta_{Pi}(\varepsilon) \| \le a\varepsilon, \qquad \| \theta_{Qj}(\tau, \varepsilon) \| \le a\varepsilon, \qquad \| \theta_{Ri}(\tau, \rho, \varepsilon) \| \le a\varepsilon$$
$$(i = 1, 2, 3;\ j = 1, \dots, 4), \tag{4.36}$$

where $a > 0$ is some constant independent of ε.

The inequalities (4.36) along with the equations (4.17)–(4.21) immediately yield the statements of the theorem. ∎

COROLLARY 4.1. *Under the assumptions* A1, A2, A4, *and* A5, *the system* (2.10), *where* $P = P(\varepsilon)$ *and* $Q(\tau) = Q(\tau, \varepsilon)$ *are defined in Theorem* 4.1, *is asymptotically stable for all sufficiently small* $\varepsilon > 0$.

Proof. The corollary is an immediate consequence of Theorem 4.1 and Lemma 4.1. It is proved similarly to Lemma 4.2. ∎

5. H_∞ CONTROLLERS FOR PROBLEM (2.1)–(2.4)

Consider the controller $u_\varepsilon^*[x(\cdot), y(\cdot)](t)$ of the form (2.11), where $P(\varepsilon)$ and $Q(\tau, \varepsilon)$ are defined by (3.1), and $P_i(\varepsilon)$ and $Q_j(\tau, \varepsilon)$ ($i = 1, 2, 3$; $j = 1, \dots, 4$) are components of the solution to the problem (3.3)–(3.15) mentioned in Theorem 4.1. Consider also the following controller with ε-independent gain matrices:

$$u_0^*[x(\cdot), y(\cdot)](t) = -\left(B_1'\bar{P}_{10} + B_2'G_1\right)x(t) - B_2'\left[\bar{P}_{30}y(t) + \int_{-h}^{0} Q_{40}(\eta)y(t + \varepsilon\eta)\, d\eta\right]. \tag{5.1}$$

LEMMA 5.1. *Under the assumptions* A1–A6, *the controller* $u_0^*[x(\cdot), y(\cdot)](t)$ *internally stabilizes the system* (2.1), (2.2) *for all sufficiently small* $\varepsilon > 0$.

Proof. Substituting (5.1) into (2.1), (2.2) and setting $w(t) = 0$, we obtain the system

$$\dot{x}(t) = \hat{A}_1 x(t) + \hat{A}_2 y(t) + H_1 x(t - \varepsilon h) + H_2 y(t - \varepsilon h) + \int_{-h}^{0} \hat{G}_1(\eta) y(t + \varepsilon\eta)\, d\eta, \qquad t > 0, \tag{5.2}$$

$$\varepsilon \dot{y}(t) = \hat{A}_3 x(t) + \hat{A}_4 y(t) + H_3 x(t - \varepsilon h) + H_4 y(t - \varepsilon h) + \int_{-h}^{0} \hat{G}_2(\eta) y(t + \varepsilon\eta)\, d\eta, \qquad t > 0, \tag{5.3}$$

where

$$\hat{A}_1 = A_1 - B_1 B_1' \overline{P}_{10} - B_1 B_2' G_1, \qquad \hat{A}_2 = A_2 - B_1 B_2' \overline{P}_{30}, \qquad \hat{G}_1(\eta) = -B_1 B_2' Q_{40}(\eta), \tag{5.4}$$

$$\hat{A}_3 = A_3 - B_2 B_1' \overline{P}_{10} - B_2 B_2' G_1, \qquad \hat{A}_4 = A_4 - B_2 B_2' \overline{P}_{30}, \qquad \hat{G}_2(\eta) = -B_2 B_2' Q_{40}(\eta). \tag{5.5}$$

Thus, in order to prove the lemma, one has to prove the asymptotic stability of the system (5.2), (5.3) for all sufficiently small $\varepsilon > 0$. Let us show the asymptotic stability of the reduced-order and the boundary-layer subsystems, associated with the system (5.2), (5.3). Setting $\varepsilon = 0$ in (5.2), (5.3), we obtain a system, coinciding with the system (3.51). Hence, due to Lemma 3.6, the reduced-order subsystem, associated with (5.2), (5.3), is asymptotically stable. The asymptotic stability of the boundary-layer subsystem, associated with (5.2), (5.3), is an immediate consequence of the assumption A3. Hence, by Lemma 4.1, the system (5.2), (5.3) is asymptotically stable for all sufficiently small $\varepsilon > 0$. ∎

From Theorem 4.1 and Lemma 5.1 we obtain the following corollary:

COROLLARY 5.1. *Under the assumptions* A1–A6, *the controller* $u_\varepsilon^*[x(\cdot), y(\cdot)](t)$ *internally stabilizes the system* (2.1), (2.2) *for all sufficiently small* $\varepsilon > 0$.

THEOREM 5.1. *Under the assumptions* A1–A6, *the controller* $u_\varepsilon^*[x(\cdot), y(\cdot)](t)$ *solves the* H_∞ *control problem* (2.1)–(2.4) *for all sufficiently small* $\varepsilon > 0$.

Proof. For all sufficiently small $\varepsilon > 0$, the theorem follows from [9] and Corollaries 4.1 and 5.1. ▮

THEOREM 5.2. *Under the assumptions* A1–A6, *the controller* $u_0^*[x(\cdot), y(\cdot)](t)$ *solves the* H_∞ *control problem* (2.1)–(2.4) *for all sufficiently small* $\varepsilon > 0$.

Proof. Substituting $u_0^*[x(\cdot), y(\cdot)](t)$ into (2.1)–(2.4), one has

$$\dot{x}(t) = \hat{A}_1 x(t) + \hat{A}_2 y(t) + H_1 x(t - \varepsilon h) + H_2 y(t - \varepsilon h) + \frac{1}{\varepsilon}\int_{-\varepsilon h}^{0} \hat{G}_1(\tau/\varepsilon) y(t + \tau)\, d\tau + F_1 w(t), \tag{5.6}$$

$$\varepsilon \dot{y}(t) = \hat{A}_3 x(t) + \hat{A}_4 y(t) + H_3 x(t - \varepsilon h) + H_4 y(t - \varepsilon h) + \frac{1}{\varepsilon}\int_{-\varepsilon h}^{0} \hat{G}_2(\tau/\varepsilon) y(t + \tau)\, d\tau + F_2 w(t), \tag{5.7}$$

$$v_0^*(t) = \mathrm{col}\{Cz(t), u_0^*[x(\cdot), y(\cdot)](t)\}, \qquad t > 0, \tag{5.8}$$

$$\begin{aligned} J_0^*(w) &= \| v_0^*(t) \|_{L_2}^2 - \gamma^2 \| w(t) \|_{L_2}^2 \\ &= \int_0^{+\infty} \Big[x'(t)\big(D_1 + \hat{D}_{P1}\big)x(t) + 2x'(t)\big(D_2 + \hat{D}_{P2}\big)y(t) \\ &\qquad + y'(t)\big(D_3 + \hat{D}_{P3}\big)y(t) \\ &\qquad + 2x'(t)\int_{-\varepsilon h}^{0} \hat{D}_{Q1}(\tau, \varepsilon) y(t + \tau)\, d\tau \\ &\qquad + 2y'(t)\int_{-\varepsilon h}^{0} \hat{D}_{Q2}(\tau, \varepsilon) y(t + \tau)\, d\tau \\ &\qquad + \int_{-\varepsilon h}^{0}\int_{-\varepsilon h}^{0} y'(t + \tau)\hat{D}_{R1}(\tau, \rho, \varepsilon) y(t + \rho)\, d\tau\, d\rho \Big] dt \\ &\quad - \gamma^2 \int_0^{+\infty} w'(t) w(t)\, dt, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} \hat{D}_{P1} &= \big(\bar{P}_{10} B_1 + G_1' B_2\big)\big(B_1' \bar{P}_{10} + B_2' G_1\big), \\ \hat{D}_{P2} &= \big(\bar{P}_{10} B_1 + G_1' B_2\big) B_2' \bar{P}_{30}, \end{aligned} \tag{5.10}$$

$$\hat{D}_{P3} = \overline{P}_{30} B_2 B_2' \overline{P}_{30}, \qquad \hat{D}_{Q1}(\tau, \varepsilon) = (1/\varepsilon)\big(\overline{P}_{10} B_1 + G_1' B_2\big) B_2' Q_{40}(\tau/\varepsilon), \tag{5.11}$$

$$\begin{aligned} \hat{D}_{Q2}(\tau, \varepsilon) &= (1/\varepsilon) \overline{P}_{30} B_2 B_2' Q_{40}(\tau/\varepsilon), \\ \hat{D}_{R1}(\tau, \rho, \varepsilon) &= (1/\varepsilon^2) Q_{40}'(\tau/\varepsilon) B_2 B_2' Q_{40}(\rho/\varepsilon). \end{aligned} \tag{5.12}$$

Note that, according to Lemma 5.1, the system (5.6), (5.7) is internally asymptotically stable for all sufficiently small $\varepsilon > 0$.

Thus, in order to prove the theorem, we have to show that for all sufficiently small $\varepsilon > 0$

$$J_0^*(w) \le 0 \qquad \text{for all } w(t) \in L_2[0, +\infty; E^q] \text{ and for } x(t) = 0,\ y(t) = 0,\ t \le 0. \tag{5.13}$$

Consider the block matrices

$$\hat{A}_\varepsilon = \begin{pmatrix} \hat{A}_1 & \hat{A}_2 \\ (1/\varepsilon)\hat{A}_3 & (1/\varepsilon)\hat{A}_4 \end{pmatrix}, \qquad \hat{G}(\tau, \varepsilon) = \begin{pmatrix} 0 & (1/\varepsilon)\hat{G}_1(\tau/\varepsilon) \\ 0 & (1/\varepsilon^2)\hat{G}_2(\tau/\varepsilon) \end{pmatrix}, \tag{5.14}$$

$$\hat{D}_P = \begin{pmatrix} \hat{D}_{P1} & \hat{D}_{P2} \\ \hat{D}_{P2}' & \hat{D}_{P3} \end{pmatrix}, \qquad \hat{D}_Q(\tau, \varepsilon) = \begin{pmatrix} 0 & \hat{D}_{Q1}(\tau, \varepsilon) \\ 0 & \hat{D}_{Q2}(\tau, \varepsilon) \end{pmatrix},$$
$$\hat{D}_R(\tau, \rho, \varepsilon) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{D}_{R1}(\tau, \rho, \varepsilon) \end{pmatrix}, \tag{5.15}$$

and the hybrid system of matrix Riccati equations for $\hat{P}$, $\hat{Q}(\tau)$, and $\hat{R}(\tau, \rho)$ in the domain $(\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$,

$$\hat{P}\hat{A}_\varepsilon + \hat{A}_\varepsilon' \hat{P} + \hat{P}\hat{S}_\varepsilon \hat{P} + \hat{Q}(0) + \hat{Q}'(0) + D + \hat{D}_P = 0, \tag{5.16}$$

$$d\hat{Q}(\tau)/d\tau = \left[\hat{A}_\varepsilon' + \hat{P}\hat{S}_\varepsilon\right]\hat{Q}(\tau) + \hat{P}\hat{G}(\tau, \varepsilon) + \hat{R}(0, \tau) + \hat{D}_Q(\tau, \varepsilon), \tag{5.17}$$

$$\begin{aligned} (\partial/\partial\tau + \partial/\partial\rho)\hat{R}(\tau, \rho) = {} & \hat{G}'(\tau, \varepsilon)\hat{Q}(\rho) + \hat{Q}'(\tau)\hat{G}(\rho, \varepsilon) \\ & + \hat{Q}'(\tau)\hat{S}_\varepsilon \hat{Q}(\rho) + \hat{D}_R(\tau, \rho, \varepsilon), \end{aligned} \tag{5.18}$$

$$\hat{Q}(-\varepsilon h) = \hat{P} H_\varepsilon, \qquad \hat{R}(-\varepsilon h, \tau) = \hat{R}'(\tau, -\varepsilon h) = H_\varepsilon' \hat{Q}(\tau), \tag{5.19}$$

where $\hat{S}_\varepsilon = \gamma^{-2} F_\varepsilon F_\varepsilon'$.

Consider also the system

$$\dot{z}(t) = \left[\hat{A}_\varepsilon + \hat{S}_\varepsilon \hat{P}\right] z(t) + H_\varepsilon z(t - \varepsilon h) + \int_{-\varepsilon h}^{0} \left[\hat{S}_\varepsilon \hat{Q}(\tau) + \hat{G}(\tau/\varepsilon, \varepsilon)\right] z(t + \tau)\, d\tau, \qquad t > 0. \quad (5.20)$$

Similarly to [9], one can obtain the following: if for some $\varepsilon > 0$, such that (5.6), (5.7) is internally asymptotically stable, the problem (5.16)–(5.19) has a solution $\hat{P}(\varepsilon), \hat{Q}(\tau, \varepsilon), \hat{R}(\tau, \rho, \varepsilon)$, and the system (5.20) with $\hat{P} = \hat{P}(\varepsilon)$, $\hat{Q}(\tau) = \hat{Q}(\tau, \varepsilon)$ is asymptotically stable, then the inequality (5.13) is satisfied for this ε.

We shall seek the solution of the problem (5.16)–(5.19) in the form

$$\hat{P}(\varepsilon) = \begin{pmatrix} \hat{P}_1(\varepsilon) & \varepsilon \hat{P}_2(\varepsilon) \\ \varepsilon \hat{P}_2'(\varepsilon) & \varepsilon \hat{P}_3(\varepsilon) \end{pmatrix}, \qquad \hat{Q}(\tau, \varepsilon) = \begin{pmatrix} \hat{Q}_1(\tau, \varepsilon) & \hat{Q}_2(\tau, \varepsilon) \\ \hat{Q}_3(\tau, \varepsilon) & \hat{Q}_4(\tau, \varepsilon) \end{pmatrix}, \quad (5.21)$$

$$\hat{R}(\tau, \rho, \varepsilon) = (1/\varepsilon) \begin{pmatrix} \hat{R}_1(\tau, \rho, \varepsilon) & \hat{R}_2(\tau, \rho, \varepsilon) \\ \hat{R}_2'(\rho, \tau, \varepsilon) & \hat{R}_3(\tau, \rho, \varepsilon) \end{pmatrix}, \quad (5.22)$$

where the matrices $\hat{P}_1(\varepsilon)$, $\hat{Q}_1(\tau, \varepsilon)$, and $\hat{R}_1(\tau, \rho, \varepsilon)$ are of the dimension $n \times n$; the matrices $\hat{P}_3(\varepsilon)$, $\hat{Q}_4(\tau, \varepsilon)$, and $\hat{R}_3(\tau, \rho, \varepsilon)$ are of the dimension $m \times m$; $\hat{P}_k(\varepsilon) = \hat{P}_k'(\varepsilon)$, and $R_k(\tau, \rho, \varepsilon) = \hat{R}_k'(\rho, \tau, \varepsilon)$ $(k = 1, 3)$.

Similarly to Theorem 4.1, it can be verified that, for all sufficiently small $\varepsilon > 0$, the problem (5.16)–(5.19) has the solution in the form (5.21), (5.22) satisfying the inequalities

$$\left\| \hat{P}_i(\varepsilon) - \overline{P}_{i0} \right\| \le a\varepsilon, \qquad \left\| \hat{Q}_j(\tau, \varepsilon) - Q_{j0}(\tau/\varepsilon) \right\| \le a\varepsilon,$$
$$\left\| \hat{R}_i(\tau, \rho, \varepsilon) - R_{i0}(\tau/\varepsilon, \rho/\varepsilon) \right\| \le a\varepsilon,$$

where $(\tau, \rho) \in [-\varepsilon h, 0] \times [-\varepsilon h, 0]$ $(i = 1, 2, 3;\ j = 1, \dots, 4)$; $\overline{P}_{i0}, Q_{j0}(\eta)$ and $R_{i0}(\eta, \chi)$ are defined in Section 3; and $a > 0$ is some constant independent of ε.

Now, similarly to Corollary 4.1, we have that the system (5.20) with $\hat{P} = \hat{P}(\varepsilon)$, $\hat{Q}(\tau) = \hat{Q}(\tau, \varepsilon)$ is asymptotically stable for all sufficiently small $\varepsilon > 0$ which completes the proof of theorem.

6. EXAMPLE

Consider an example of the problem (2.1)–(2.4) with the following data: $n = m = r = q = 1$ and

$$A_1 = 3,\ A_2 = 1,\ A_3 = 1,\ A_4 = -2,\ H_1 = 2,\ H_2 = 1,\ H_3 = -1,\ H_4 = 1, \tag{6.1}$$

$$B_1 = 6,\ B_2 = 1,\ F_1 = 2,\ F_2 = 0.5,\ C_1 = 2,\ C_2 = 1,\ \gamma = 0.5. \tag{6.2}$$

In order to save the space, we do not rewrite the problem (3.3)–(3.15) with the data (6.1), (6.2). Applying the results of Section 3, we shall construct the asymptotic solution to this problem. Under Eqs. (6.1), (6.2), the First Problem (see Remark 3.1) becomes

$$-4\bar{P}_{30} + 2Q_{40}(0) + 1 = 0, \tag{6.3}$$

$$dQ_{40}(\eta)/d\eta = -2Q_{40}(\eta) + R_{30}(0,\eta),$$
$$\eta \in [-h,0],\ Q_{40}(-h) = \bar{P}_{30}, \tag{6.4}$$

$$(\partial/\partial\eta + \partial/\partial\chi)R_{30}(\eta,\chi) = 0,$$
$$(\eta,\chi) \in [-h,0]\times[-h,0],\ R_{30}(-h,\eta) = Q_{40}(\eta). \tag{6.5}$$

Solving (6.5), one directly has

$$R_{30}(\eta,\chi) = \begin{cases} Q_{40}(\chi-\eta-h), & \text{if } -h \le \eta \le \chi \le 0, \\ Q_{40}(\eta-\chi-h), & \text{if } -h \le \chi < \eta \le 0. \end{cases} \tag{6.6}$$

Substituting (6.6) into (6.4), we obtain the functional-differential equation for $Q_{40}(\eta)$,

$$dQ_{40}(\eta)/d\eta = -2Q_{40}(\eta) + Q_{40}(-\eta-h),$$
$$\eta \in [-h,0],\ Q_{40}(-h) = \bar{P}_{30}. \tag{6.7}$$

This equation has a unique solution

$$Q_{40}(\eta) = \bar{P}_{30}\,\omega(\eta),$$
$$\omega(\eta) = \left[f_1(h)\exp(\sqrt{3}\,\eta) + f_2(h)\exp(-\sqrt{3}\,\eta)\right]/f_0(h),\ \eta \in [-h,0], \tag{6.8}$$

where $f_1(h) = \sqrt{3} - 2 + \exp(\sqrt{3}\,h)$, $f_2(h) = \sqrt{3} + 2 - \exp(-\sqrt{3}\,h)$, and $f_0(h) = (2+\sqrt{3})\exp(\sqrt{3}\,h) - (2-\sqrt{3})\exp(-\sqrt{3}\,h)$. It is obvious that

$$f_k(h) > 0 \qquad \forall h \ge 0\ (k = 0,1,2). \tag{6.9}$$

Standard analysis of the function $\omega(\eta)$ yields

$$\max_{\eta\in[-h,0]} \omega(\eta) = \omega(-h) = 1 \qquad \forall h \geq 0. \tag{6.10}$$

Substituting (6.8) into (6.3) and solving the resulting equation with respect to $\bar{P}_{30}$, we have

$$\bar{P}_{30} = -f_0(h)/f(h), \tag{6.11}$$

where $f(h) = 4\sqrt{3} - (6 + 4\sqrt{3})\exp(\sqrt{3}h) + (6 - 4\sqrt{3})\exp(-\sqrt{3}h)$.

Some easy analysis shows that $f(h) < 0$ $\forall h \geq 0$ and, therefore,

$$\bar{P}_{30} > 0 \qquad \forall h \geq 0. \tag{6.12}$$

Moreover, it can be shown that

$$\max_{h\geq 0} \bar{P}_{30} = \bar{P}_{30}|_{h=0} = 0.5. \tag{6.13}$$

Using (6.8), (6.9), and (6.12), one directly has

$$Q_{40}(\eta) > 0, \qquad \eta \in [-h,0],\ h \geq 0. \tag{6.14}$$

Let us show that the assumptions A2 and A3 are satisfied. Begin with A2. Using the data (6.1), (6.2), we obtain the equation in A2, $\Lambda_1(\lambda) \triangleq \lambda + 2 - \exp(-\lambda h) = 0$. Further, we have for any $h \geq 0$,

$$\mathrm{Re}[\Lambda_1(\lambda)] = 2 + \mathrm{Re}\,\lambda - \mathrm{Re}[\exp(-\lambda h)] \geq 1 + \mathrm{Re}\,\lambda \geq 1 \qquad \forall \lambda\colon \mathrm{Re}\,\lambda \geq 0. \tag{6.15}$$

Hence, all roots λ of the equation $\Lambda_1(\lambda) = 0$ lie inside the left-hand half-plane for all $h \geq 0$. Now, let us proceed to A3. The equation in A3 is

$$\Lambda_2(\lambda) \triangleq \lambda + 2 + \bar{P}_{30} - \exp(-\lambda h) - \int_{-h}^{0} Q_{40}(\eta)\exp(\lambda\eta)\,d\eta = 0. \tag{6.16}$$

Taking into account (6.8), (6.10), (6.12), and (6.14), we have from (6.16)

$$\begin{aligned}\mathrm{Re}[\Lambda_2(\lambda)] &= 2 + \bar{P}_{30} + \mathrm{Re}\,\lambda - \mathrm{Re}[\exp(-\lambda h)] \\ &\quad - \int_{-h}^{0} Q_{40}(\eta)\mathrm{Re}[\exp(\lambda\eta)]\,d\eta \\ &\geq 1 + (1-h)\bar{P}_{30} + \mathrm{Re}\,\lambda \qquad \forall \lambda\colon \mathrm{Re}\,\lambda \geq 0. \end{aligned} \tag{6.17}$$

Consider the inequality with respect to $h \geq 0$,

$$1 + (1 - h)\bar{P}_{30} > 0. \tag{6.18}$$

The solution of this inequality is

$$0 \leq h < 4.4632. \tag{6.19}$$

Now, using (6.17)–(6.19), one has that all roots λ of the equation (6.16) lie inside the left-hand half-plane for all h satisfying (6.19). Hence, the assumption A3 is satisfied for all h satisfying (6.19).

Proceed to the Second and the Third Problems (see Remark 3.1). In order to obtain the solutions to these problems, one has (according to Lemma 3.3) to solve the equation (3.36) with the initial condition (3.28) ($l = 3$). Under the data (6.1), (6.2), the problem (3.36), (3.28) ($l = 3$) becomes

$$dQ_{30}(\eta)/d\eta = -2Q_{30}(\eta) - Q_{40}(-\eta - h),$$
$$\eta \in [-h, 0],\ Q_{30}(-h) = -\bar{P}_{30}. \tag{6.20}$$

Solving (6.20), we obtain

$$Q_{30}(\eta) = -\bar{P}_{30}\big[f_3(h)\exp(\sqrt{3}\,\eta) + f_4(h)\exp(-\sqrt{3}\,\eta)\big]/f_0(h),$$
$$\eta \in [-h, 0], \tag{6.21}$$

where

$$f_3(h) = f_2(h)\exp(\sqrt{3}\,h)/(2 + \sqrt{3}),$$
$$f_4(h) = f_1(h)\exp(-\sqrt{3}\,h)/(2 - \sqrt{3}).$$

Using (3.34), (3.35), we obtain for $k = 1, 2$

$$R_{k0}(\eta, \chi) = \begin{cases} -Q_{k+2,0}(\chi - \eta - h), & \text{if } -h \leq \eta \leq \chi \leq 0, \\ (-1)^k Q_{30}(\eta - \chi - h), & \text{if } -h \leq \chi < \eta \leq 0, \end{cases} \tag{6.22}$$

Now, let us proceed to the Fourth Problem (see Remark 3.1). In Section 3, this problem has been reduced to the equations (3.37)–(3.39). In order to solve these equations, we have to calculate the matrices defined in (3.40)–(3.44). Under the data (6.1), (6.2), these matrices become

$$N_1 = 2(G - 1),\ N_2 = -2,\ \bar{F} = 3 - G,\ \bar{B} = 2(4 - G), \tag{6.23}$$

$$\bar{A} = 1,\ \bar{S} = 4(2G - 7),\ \bar{D} = 4, \tag{6.24}$$

where G is defined in (3.44). It is clear that $G > 0\ \forall h \geq 0$.

Using (6.8), (6.10), and (6.12)–(6.14), one has for all h, satisfying (6.19), $G \leq (1+h)\overline{P}_{30} < 2.7316$. This inequality along with the expression for $\overline{S}$ (see (6.24)) yields

$$\overline{S} < 0 \qquad \forall h \in [0, 4.4632). \tag{6.25}$$

Substituting (6.24) into (3.37), we obtain after some rearrangement

$$2(2G - 7)\overline{P}_{10}^{2} + \overline{P}_{10} + 2 = 0. \tag{6.26}$$

The inequality (6.25) implies that (6.26) has the single positive solution for all h, satisfying (6.19)

$$\overline{P}_{10} = \left[1 + \sqrt{1 - 4(8G - 28)}\,\right] / [4(7 - 2G)]. \tag{6.27}$$

By direct calculation we have

$$\overline{A} + \overline{S}\,\overline{P}_{10} < 0 \qquad \forall h \in [0, 4.4632). \tag{6.28}$$

Hence, the assumption A5 is satisfied for all such h.

Now, let us verify that the assumption A6 holds. Calculating the matrices Δ_A and Δ_B, one has

$$\Delta_A = 4(G - 3)/(G + 1), \qquad \Delta_B = -4(G - 3)(G - 4)G/(G + 1). \tag{6.29}$$

Substitution $\overline{A} = 1$, $\overline{B} = 2(4 - G)$, and (6.29) into the matrix of A6 yields

$$\overline{A} + \Delta_A + \left(\overline{B}\,\overline{B}' + \Delta_B\right)\overline{P}_{10} = \left(5G - 11 + 16(G - 4)\overline{P}_{10}\right)/(G + 1). \tag{6.30}$$

Some easy analysis shows that the expression in the right-hand part of (6.30) is negative for all h, satisfying (6.19). Hence, the assumption A6 is satisfied for all h, satisfying (6.19).

Substituting the expressions for N_1 and N_2 (see (6.23)) into (3.38) yields

$$\overline{P}_{20} = 2(1 - G)\overline{P}_{10} - \int_{-h}^{0} Q_{30}(\eta)\, d\eta + 2. \tag{6.31}$$

Using (6.1), (6.2) and (6.22), (6.31), one obtains from (3.39)

$$Q_{10}(\eta) = 2G\overline{P}_{10} + \int_{-h}^{0} Q_{30}(s)\,ds - 2 + \left(1 - 2\overline{P}_{10}\right)\int_{-h}^{\eta} Q_{30}(s)\,ds - \int_{-h}^{\eta} Q_{30}(-s-h)\,ds, \tag{6.32}$$

$$Q_{20}(\eta) = (3 - 2G)\overline{P}_{10} - \int_{-h}^{0} Q_{30}(s)\,ds + 2 + \left(1 - 2\overline{P}_{10}\right)\int_{-h}^{\eta} Q_{40}(s)\,ds + \int_{-h}^{\eta} Q_{30}(-s-h)\,ds. \tag{6.33}$$

Thus, we have completed the construction of the zero-order asymptotic solution to the problem (3.3)–(3.15) for the data (6.1), (6.2). Having this asymptotic solution, one can construct the controller $u_0^*[x(\cdot), y(\cdot)]$, given by (5.1), which solves the H_∞ control problem (2.1)–(2.4) for all sufficiently small $\varepsilon > 0$. Giving any value of h, satisfying (6.19), one can obtain the numerical expression for $u_0^*[x(\cdot), y(\cdot)](t)$. Thus, for $h = 0.4$, we find

$$u_0^*[x(\cdot), y(\cdot)](t) = -5.1643x(t) - 0.3780y(t) - \int_{-0.4}^{0} \left[0.0893\exp\left(\sqrt{3}\,\eta\right) + 0.1667\exp\left(-\sqrt{3}\,\eta\right)\right] y(t + \varepsilon\eta)\,d\eta. \tag{6.34}$$

For $h = 0.6$, we find

$$u_0^*[x(\cdot), y(\cdot)](t) = -5.1643x(t) - 0.3491y(t) - \int_{-0.6}^{0} \left[0.0854\exp\left(\sqrt{3}\,\eta\right) + 0.1128\exp\left(-\sqrt{3}\,\eta\right)\right] y(t + \varepsilon\eta)\,d\eta. \tag{6.35}$$

APPENDIX: PROOF OF LEMMA 4.1

Proof of Lemma 4.1 is based on the decoupling transformation of a singularly perturbed system with a small delay (4.1). Such a transformation was introduced in [8] in the case when $\tilde{H}_k = 0$, $\tilde{G}_k = 0$ $(k = 1, 3)$. In Subsection A.1 we will generalize the results of [8] to the case of nonzero $\tilde{H}_k$, $\tilde{G}_k$ $(k = 1, 3)$.

A.1. *Slow–fast Decomposition of System* (4.1)

For each $\varepsilon > 0$ denote by $T(t)$: $C[-\varepsilon h, 0; E^n] \to C[-\varepsilon h, 0; E^n]$ and $S(t, \varepsilon)$: $C[-\varepsilon h, 0; E^m] \to C[-\varepsilon h, 0; E^m]$, $t \geq 0$, the semigroups of the solution operators, corresponding to the linear equations

$$\dot{x}(t) = 0, t \geq 0, \qquad x(\theta) = x_0(\theta), \qquad \theta \in [-\varepsilon h, 0], \qquad \text{(A.1)}$$

and

$$\begin{aligned} \varepsilon \dot{y}(t) &= A_b y_t, \ A_b y_t \\ &= \tilde{A}_4(0) y(t) + \tilde{H}_4(0) y(t - \varepsilon h) \\ &\quad + \int_{-h}^{0} \tilde{G}_4(\eta, 0) y(t + \varepsilon \eta)\, d\eta, \qquad t \geq 0; \\ y(\theta) &= y_0(\theta), \qquad \theta \in [-\varepsilon h, 0], \end{aligned} \qquad \text{(A.2)}$$

defined by

$$T(t) x_0(\theta) = x(t + \theta) \text{ and } S(t, \varepsilon) y_0(\theta) = y(t + \theta), \qquad \theta \in [-\varepsilon h, 0],$$

where $t \geq 0$ is considered as a parameter and $x_0(\theta)$ and $y_0(\theta)$ are continuous for $\theta \in [-\varepsilon h, 0]$.

Let $Y(t, \varepsilon)$, $t \in [-\varepsilon h, \infty)$ be the fundamental matrix of (A.2), $Y_0(0) = I_m$; $Y_0(\theta) = 0$, $\theta < 0$. Denote $S(t, \varepsilon) Y_0(\theta) = Y(t + \theta, \varepsilon)$ $(t \geq 0, \theta \in [-\varepsilon h, 0])$. Let $X(t) = I_n$, $t \geq 0$, $X(t) = 0$, $t < 0$, and $T(t) X_0(\theta) = X(t + \theta)$ $(t \geq 0, \theta \in [-\varepsilon h, 0])$, where $X_0(0) = I_n$, $X_0(\theta) = 0$, $\theta < 0$.

Introduce the new variable

$$z_t = x_t - x(t), \ z_t \in \mathscr{Q} = \{\phi(\theta) \in C[-\varepsilon h, 0; E^n]: \phi(0) = 0\} \qquad \forall t \geq 0.$$

Evidently $\mathscr{Q}$ is invariant with respect to $T(t)$. Under the assumption A7, we can represent the right-hand part of (4.1), where $\varphi(t + \varepsilon\eta) = \mathrm{col}\{x(t + \varepsilon\eta), y(t + \varepsilon\eta)\}$, in the form

$$\begin{pmatrix} 0 \\ (1/\varepsilon) A_b y_t \end{pmatrix} + \begin{pmatrix} A_{11}(\varepsilon) & A_{12}(\varepsilon) \\ (1/\varepsilon) A_{21}(\varepsilon) & A_{22}(\varepsilon) \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \qquad \text{(A.3)}$$

where $A_{ij}(\varepsilon)$ $(i = 1, 2,\ j = 1, 2)$ are linear operators on $C[-\varepsilon h, 0; E^n]$ and $C[-\varepsilon h, 0; E^m]$.

Applying the variation of constants formula [14] to (4.1) with the initial condition $\mathrm{col}\{x_0(\theta), y_0(\theta)\}$, we obtain the equivalent system of differential and integral equations with respect to $x(t)$, z_t, and y_t,

$$\dot{x}(t) = A_{11}(\varepsilon)(x(t) + z_t) + A_{12}(\varepsilon)y_t, \qquad x_t = x(t) + z_t,$$
$$z_t = T(t)z_0 + \int_0^t T(t-s)(X_0 - I_n)$$
$$\times\left[A_{11}(\varepsilon)(x(s) + z_s) + A_{12}(\varepsilon)y_s\right] ds, \tag{A.4}$$
$$y_t = S(t,\varepsilon)y_0 + (1/\varepsilon)\int_0^t S(t-s,\varepsilon)Y_0$$
$$\times\left[A_{21}(\varepsilon)x_s + \varepsilon A_{22}(\varepsilon)y_s\right] ds,$$

where $z_0 = z_t|_{t=0}$.

Note that (4.6) corresponds to (A.2) written in the fast time $\sigma = t/\varepsilon$. Then the asymptotic stability of (4.6) implies the following inequality for all $t \geq 0$ and sufficiently small $\varepsilon > 0$:

$$\|S(t,\varepsilon)y_0\|_C \leq a\exp(-\beta t/\varepsilon)\|y_0\|_C,$$
$$\sup_{\theta\in[-\varepsilon h,0]} \|S(t,\varepsilon)Y_0\| \leq a\exp(-\beta t/\varepsilon),$$
$$a > 0,\ \beta > 0. \tag{A.5}$$

Since $T(t)z_0 = 0$ and $T(t)(X_0 - I_n) = 0$ for $t \geq \varepsilon h$, and $z_0 \in \mathscr{Q}$, we have for all $t \geq 0$ and sufficiently small $\varepsilon > 0$

$$\|T(t)z_0\|_C \leq a\exp(-\beta t/\varepsilon)\|z_0\|_C,$$
$$\sup_{\theta\in[-\varepsilon h,0]} \|T(t)(X_0 - I_n)\| \leq a\exp(-\beta t/\varepsilon),$$
$$a > 0,\ \beta > 0. \tag{A.6}$$

By a standard argument for the existence of invariant manifolds (see e.g. [14, 6]), the system (A.4) has the center manifold for all sufficiently small $\varepsilon > 0$,

$$z_t = \mathscr{L}_1(\varepsilon)x(t), \qquad y_t = \mathscr{L}_2(\varepsilon)x(t), \tag{A.7}$$

where $\mathscr{L}_1(\varepsilon)$: $E^n \to \mathscr{Q}$, $\mathscr{L}_2(\varepsilon)$: $E^n \to C[-\varepsilon h, 0; E^m]$ are linear bounded operators. The flow on the center manifold is governed by the equation

$$\dot{\bar{x}}(t) = \left[A_{11}(\varepsilon)(I_n + \mathscr{L}_1(\varepsilon)) + A_{12}(\varepsilon)\mathscr{L}_2(\varepsilon)\right]\bar{x}(t). \tag{A.8}$$

For continuously differentiable functions $\phi \in C[-\varepsilon h, 0; E^n]$, $\psi \in C[-\varepsilon h, 0; E^m]$, denote

$$\mathscr{A}\phi = \begin{cases} \dot{\phi}, & \text{if } \theta \in [-\varepsilon h, 0), \\ 0, & \text{if } \theta = 0, \end{cases}$$

$$\mathscr{B}(\varepsilon)\psi = \begin{cases} \dot{\psi}, & \text{if } \theta \in [-\varepsilon h, 0), \\ (1/\varepsilon) A_b \psi, & \text{if } \theta = 0. \end{cases}$$

The latter operators are extensions of infinitesimal generators of the semigroups $T(t)$ and $S(t, \varepsilon)$ to the space of continuously differentiable functions [14]. Similarly to [6], the following proposition can be proved:

PROPOSITION A.1. *Under the assumptions* A7 *and* A9, *for all sufficiently small* $\varepsilon > 0$:

1. *the continuously differentiable in* $\theta \in [-\varepsilon h, 0]$ $(n \times n)$- *and* $(m \times n)$-*matrix functions* $\mathscr{L}_1(\varepsilon) = \mathscr{L}_1(\theta, \varepsilon)$ *and* $\mathscr{L}_2(\varepsilon) = \mathscr{L}_2(\theta, \varepsilon)$, *such that* $\mathscr{L}_1(0, \varepsilon) = 0$, *determine the center manifold* (A.7) *iff for every* $\theta \in [-\varepsilon h, 0]$ *they satisfy the equation*

$$\begin{pmatrix} \mathscr{L}_1(\varepsilon) \\ \mathscr{L}_2(\varepsilon) \end{pmatrix} [A_{11}(\varepsilon)(I_n + \mathscr{L}_1(\varepsilon)) + A_{12}(\varepsilon)\mathscr{L}_2(\varepsilon)]$$
$$= \begin{pmatrix} \mathscr{A}\mathscr{L}_1(\varepsilon) + (X_0 - I_n)[A_{11}(\varepsilon)(I_n + \mathscr{L}_1(\varepsilon)) + A_{12}(\varepsilon)\mathscr{L}_2(\varepsilon)] \\ \mathscr{B}(\varepsilon)\mathscr{L}_2(\varepsilon) + (1/\varepsilon)Y_0[A_{21}(\varepsilon)(I_n + \mathscr{L}_1(\varepsilon)) + \varepsilon A_{22}(\varepsilon)\mathscr{L}_2(\varepsilon)] \end{pmatrix} \tag{A.9}$$

2. *the matrix* Ω_4, *defined in* (4.5), *is nonsingular and the approximation*

$$\begin{pmatrix} \mathscr{L}_1(\varepsilon) \\ \mathscr{L}_2(\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathscr{L}_{20} \end{pmatrix} + O(\varepsilon), \qquad \mathscr{L}_{20} = -\Omega_4^{-1}\Omega_3, \tag{A.10}$$

where Ω_3 *is given in* (4.5), *holds for all* $\theta \in [-\varepsilon h, 0]$.

Changing the variables in (A.4)

$$\zeta_t = z_t - \mathscr{L}_1(\varepsilon)x(t) \ (\zeta_t \in \mathscr{Q}), \qquad \xi_t = y_t - \mathscr{L}_2(\varepsilon)x(t),$$

and using results of [14, Eq. (4.8)], we obtain the system

$$\dot{x}(t) = [A_{11}(\varepsilon)(I_n + \mathscr{L}_1(\varepsilon)) + A_{12}(\varepsilon)\mathscr{L}_2(\varepsilon)]x(t)$$
$$+ A_{11}(\varepsilon)\zeta_t + A_{12}(\varepsilon)\xi_t, \tag{A.11}$$

$$\zeta_t = T(t)\zeta_0 - \int_0^t T(t-s)\left[\mathscr{L}_1(\varepsilon) + (I_n - X_0)\right] \times \left[A_{11}(\varepsilon)\zeta_s + A_{12}(\varepsilon)\xi_s\right] ds, \tag{A.12}$$

$$\xi_t = S(t,\varepsilon)\xi_0 - (1/\varepsilon)\int_0^t S(t-s,\varepsilon)\left\{\varepsilon\mathscr{L}_2(\varepsilon)\left[A_{11}(\varepsilon)\zeta_s + A_{12}(\varepsilon)\xi_s\right] - Y_0\left[A_{21}(\varepsilon)\zeta_s + \varepsilon A_{22}(\varepsilon)\xi_s\right]\right\} ds, \tag{A.13}$$

where $\zeta_0 = \zeta_t|_{t=0}$, $\xi_0 = \xi_t|_{t=0}$.

From (A.5) and (A.6) one can derive the following exponential bounds on the solutions to (A.12) and (A.13) for all $t \geq 0$ and sufficiently small $\varepsilon > 0$:

$$\|\zeta_t\|_C + \|\xi_t\|_C \leq a\exp(-\beta t/\varepsilon)(\|\zeta_0\|_C + \|\xi_0\|_C), \qquad a > 0,\ \beta > 0. \tag{A.14}$$

Similarly to [6], one can show that the system (A.11)–(A.13) has the stable manifold for all sufficiently small $\varepsilon > 0$,

$$x(t) = \varepsilon\mathscr{M}_1(\varepsilon)\zeta_t + \varepsilon\mathscr{M}_2(\varepsilon)\xi_t, \tag{A.15}$$

where $\mathscr{M}_1(\varepsilon)$: $\mathscr{Q} \to E^n$, $\mathscr{M}_2(\varepsilon)$: $C[-\varepsilon h, 0; E^m] \to E^n$ are linear bounded (uniformly in ε) operators. Similarly to [8], we can show that after the following change of variables $\bar{x}(t) = x(t) - \varepsilon\mathscr{M}_1(\varepsilon)\zeta_t - \varepsilon\mathscr{M}_2(\varepsilon)\xi_t$ we obtain the decoupled system of (A.8) and (A.12), (A.13). Expressing $x(t)$, z_t, and y_t by $\bar{x}(t)$, ζ_t, and ξ_t, we obtain the following lemma.

LEMMA A.1. *Under the assumptions* A7 *and* A9, *for all sufficiently small* $\varepsilon > 0$ *there exists an invertible operator* $\mathscr{T}(\varepsilon)$: $E^n \times \mathscr{Q} \times C[-\varepsilon h, 0; E^m] \to E^n \times \mathscr{Q} \times C[-\varepsilon h, 0; E^m]$, *given by*

$$\operatorname{col}\{x(t), z_t, y_t\} = \mathscr{T}(\varepsilon)\operatorname{col}\{\bar{x}(t), \zeta_t, \xi_t\},$$

$$\mathscr{T}(\varepsilon) = \begin{pmatrix} I_n & \varepsilon\mathscr{M}_1(\varepsilon) & \varepsilon\mathscr{M}_2(\varepsilon) \\ \mathscr{L}_1(\varepsilon) & I_n + \varepsilon\mathscr{L}_1(\varepsilon)\mathscr{M}_1(\varepsilon) & \varepsilon\mathscr{L}_1(\varepsilon)\mathscr{M}_2(\varepsilon) \\ \mathscr{L}_2(\varepsilon) & \varepsilon\mathscr{L}_2(\varepsilon)\mathscr{M}_1(\varepsilon) & I_m + \varepsilon\mathscr{L}_2(\varepsilon)\mathscr{M}_2(\varepsilon) \end{pmatrix}, \tag{A.16}$$

and

$$\mathscr{T}^{-1}(\varepsilon) = \begin{pmatrix} I_n + \varepsilon\mathscr{M}_1(\varepsilon)\mathscr{L}_1(\varepsilon) + \varepsilon\mathscr{M}_2(\varepsilon)\mathscr{L}_2(\varepsilon) & -\varepsilon\mathscr{M}_1(\varepsilon) & -\varepsilon\mathscr{M}_2(\varepsilon) \\ -\mathscr{L}_1(\varepsilon) & I_n & 0 \\ -\mathscr{L}_2(\varepsilon) & 0 & I_m \end{pmatrix},$$

which transforms (A.4) *to the purely slow system* (A.8) *and the purely fast system of* (A.12) *and* (A.13). *Here,* I_n *and* I_m *denote the identity operators on the corresponding spaces.*

A.2. Proof of Lemma 4.1

From (A.10) it follows that (A.8) can be rewritten in the form

$$\dot{\bar{x}}(t) = [\Omega + O(\varepsilon)]\bar{x}(t),$$

where Ω is given by (4.5). Therefore, under the assumption A8, the solution of (A.8) satisfies the following inequality for all $t \geq 0$ and sufficiently small $\varepsilon > 0$,

$$\|\bar{x}(t)\| \leq a \exp(-\alpha t)\|\bar{x}(0)\|, \qquad a > 0, \alpha > 0. \tag{A.17}$$

Lemma A.1 and the inequalities (A.14) and (A.17) imply that the solution to (4.1) with the initial condition $\varphi_0 \in C[-\varepsilon h, 0; E^{n+m}]$ satisfies the following inequality for all $t \geq 0$ and sufficiently small $\varepsilon > 0$:

$$\|\varphi(t)\| \leq a \exp(-\alpha t), \qquad a > 0, \alpha > 0.$$

The latter inequality immediately implies the exponential bound for the fundamental matrix $\Phi(t, \varepsilon)$ for all $t \geq 0$ and sufficiently small $\varepsilon > 0$,

$$\|\Phi(t, \varepsilon)\| \leq a \exp(-\alpha t), \qquad a > 0, \alpha > 0. \tag{A.18}$$

Thus the inequalities for Φ_1 and Φ_3 of Lemma 4.1 are satisfied. Now, let us prove the inequalities for Φ_2 and Φ_4. Denoting

$$\Gamma(t, \varepsilon) = \begin{pmatrix} \Phi_2(t, \varepsilon) \\ \Phi_4(t, \varepsilon) \end{pmatrix}$$

and using (4.1), (4.7), we obtain the equation for $\Gamma(t, \varepsilon)$

$$d\Gamma(t, \varepsilon)/dt = \tilde{A}(\varepsilon)\Gamma(t, \varepsilon) + \tilde{H}(\varepsilon)\Gamma(t - \varepsilon h, \varepsilon) + \int_{-h}^{0} \tilde{G}(\eta, \varepsilon)\Gamma(t + \varepsilon\eta, \varepsilon)\, d\eta, \qquad t > 0, \tag{A.19}$$

and the initial conditions

$$\Gamma(0, \varepsilon) = \begin{pmatrix} 0 \\ I_m \end{pmatrix}; \qquad \Gamma(\theta, \varepsilon) = 0, \qquad \theta < 0. \tag{A.20}$$

Let the $(m \times m)$-matrix $\Theta(t, \varepsilon)$ satisfy the equation

$$\varepsilon d\Theta(t, \varepsilon)/dt = \tilde{A}_4(\varepsilon)\Theta(t, \varepsilon) + \tilde{H}_4(\varepsilon)\Theta(t - \varepsilon h, \varepsilon) + \int_{-h}^{0} \tilde{G}_4(\eta, \varepsilon)\Theta(t + \varepsilon\eta, \varepsilon)\, d\eta, \qquad t > 0, \quad \text{(A.21)}$$

and the initial conditions

$$\Theta(0, \varepsilon) = I_m; \qquad \Theta(\theta, \varepsilon) = 0, \theta < 0. \qquad \text{(A.22)}$$

Consider the equation

$$\det\left[\lambda I_m - \tilde{A}_4(\varepsilon) - \tilde{H}_4(\varepsilon)\exp(-\lambda h) - \int_{-h}^{0} \tilde{G}_4(\eta, \varepsilon)\exp(\lambda\eta)\, d\eta\right] = 0. \qquad \text{(A.23)}$$

Taking into account the assumptions A7 and A9, and applying results of [10, Proposition 4.3], one obtains that all roots λ of (A.23) lie inside the left-hand half-plane for all sufficiently small $\varepsilon \geq 0$. Moreover, similarly to this result of [10], it can be shown that

$$\operatorname{Re} \lambda \leq -2\beta, \qquad \beta > 0 \qquad \text{(A.24)}$$

for all sufficiently small $\varepsilon \geq 0$. Assumption A7 and (A.24) yield the following inequality for the solution to (A.21), (A.22) for all $t \geq 0$ and sufficiently small $\varepsilon > 0$:

$$\|\Theta(t, \varepsilon)\| \leq a \exp(-\beta t/\varepsilon), \qquad a > 0, \beta > 0. \qquad \text{(A.25)}$$

Changing the variable Γ in (A.19), (A.20) as

$$\Gamma(t, \varepsilon) = \tilde{\Gamma}(t, \varepsilon) + \Sigma(t, \varepsilon), \qquad \Sigma(t, \varepsilon) = \begin{pmatrix} 0 \\ \Theta(t, \varepsilon) \end{pmatrix}, \qquad \text{(A.26)}$$

we obtain the problem

$$d\tilde{\Gamma}(t, \varepsilon)/dt = \tilde{A}(\varepsilon)\tilde{\Gamma}(t, \varepsilon) + \tilde{H}(\varepsilon)\tilde{\Gamma}(t - \varepsilon h, \varepsilon) + \int_{-h}^{0} \tilde{G}(\eta, \varepsilon)\tilde{\Gamma}(t - \varepsilon\eta, \varepsilon)\, d\eta + \Delta_\Gamma(t, \varepsilon), \qquad \text{(A.27)}$$

$$\tilde{\Gamma}(\theta, \varepsilon) = 0, \qquad \theta \leq 0, \qquad \text{(A.28)}$$

where

$$\Delta_\Gamma(t, \varepsilon) = \begin{pmatrix} \tilde{A}_2(\varepsilon)\Theta(t, \varepsilon) + \tilde{H}_2(\varepsilon)\Theta(t - \varepsilon h, \varepsilon) + \int_{-h}^{0} \tilde{G}_2(\eta, \varepsilon)\Theta(t + \varepsilon\eta, \varepsilon)\, d\eta \\ 0 \end{pmatrix}.$$

From (A.25), we have for all $t \geq 0$ and for sufficiently small $\varepsilon > 0$,

$$\|\Delta_{\Gamma}(t, \varepsilon)\| \leq a \exp(-\beta t/\varepsilon), \qquad a > 0,\ \beta > 0. \tag{A.29}$$

Rewriting the problem (A.27), (A.28) in the equivalent integral form,

$$\tilde{\Gamma}(t, \varepsilon) = \int_0^t \Phi(t - s, \varepsilon)\Delta_{\Gamma}(s, \varepsilon)\, ds, \qquad t \geq 0,$$

and using the inequalities (A.18) and (A.29), one obtains for all $t \geq 0$ and sufficiently small $\varepsilon > 0$ that

$$\|\tilde{\Gamma}(t, \varepsilon)\| \leq a\varepsilon \exp(-\alpha t), \qquad a > 0,\ \alpha > 0. \tag{A.30}$$

The equation (A.26) and the inequalities (A.25), (A.30) yield the inequalities for $\Phi_2(t, \varepsilon)$ and $\Phi_4(t, \varepsilon)$ claimed in Lemma 4.1. Thus the lemma is proved. ∎

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