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Bounds on the response of a drilling pipe model

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The drill pipe model described by the wave equation with boundary conditions is reduced through the d'Alembert transformation to a difference equation model. Assuming that the boundary condition at the bottom is perturbed by bounded additive noise, an ultimate bound for the velocity at the bottom of the pipe is obtained. The proposal of a Lyapunov functional for the distributed model allows to provide an ultimate bound for a measure of the distributed variables describing the system in terms of linear matrix inequality conditions. The two approaches are compared through an illustrative example.

Keywords: distributed parameter systems; time-delay systems; drill pipe model; ultimate boundedness.

1. Introduction

Drilling systems with actuator at the bottom are efficient but have a high risk of collapse and subsequent loss of the tool and of the perforation itself. This is why, even in deep perforations, the application of the torque at ground level is preferred. In this case, the distributed parameter nature of the system cannot be neglected. It should also be mentioned that a full description covering all relevant phenomena such as bit bouncing, whirling or stick slip is not reasonable in practice and authors usually study such mechanisms individually and simplifications are common for stability analysis purposes. See Challamel (2000), Fliess *et al.* (1995) and Rouchon (1998).

In particular, although the dimensional parameters of the plant are known, the linearization of the behaviour of the torque at the bottom hole boundary introduces uncertainty. Moreover, it is reasonable to consider the presence of a bounded additive noise signal $w(t)$ at the bottom in order to account for external disturbances and modelling errors.

It is clear that under these circumstances, exponential or asymptotic stability cannot be achieved, and we will seek instead ultimate boundedness of the solution (Khalil, 1992).

The problem is first treated, after appropriate simplifications, as a difference equation model that describes the angular velocity at the bottom of the pipe, which is the main variable of interest from an engineering view point. Then, the problem is addressed in the framework of distributed parameter systems as a special case of a wave equation and linear matrix inequality (LMI) type conditions for ultimate boundedness are derived from an appropriate energy function and Lyapunov functional following the ideas introduced in Nicaise & Pignotti (2008) and Fridman & Orlov (2009).

The paper is organized as follows: The distributed parameter model of the drill pipe and the ultimate boundedness problem formulation are introduced in Section 2. In Section 3, a difference equation description is obtained after simplifications and the corresponding ultimate bounds are found. In Section 4, ultimate boundedness conditions are derived from the proposal of an energy function for the distributed parameter model. The contribution ends with a comparison of the two approaches in the context of an example. A conference version of the paper has been presented in Saldivar *et al.* (2009).

2. Problem formulation

2.1 *Drill pipe model*

A sketch of a simplified drillstring system is shown on Fig. 1.

The main process during well drilling for oil is the creation of borehole by a rock-cutting tool called a bit. The drillstring consists of the bottom hole assembly (BHA) and drillpipes screwed end-to-end to each other to form a long pipe. The BHA comprises the bit, stabilizers (at least two spaced apart) which prevent the drillstring from balancing, and a series of pipe sections that are relatively heavy known as drill collars. While the length of the BHA remains constant, the total length of the drill pipes increases as the borehole depth does. An important element of the process is the drilling mud or fluid which among

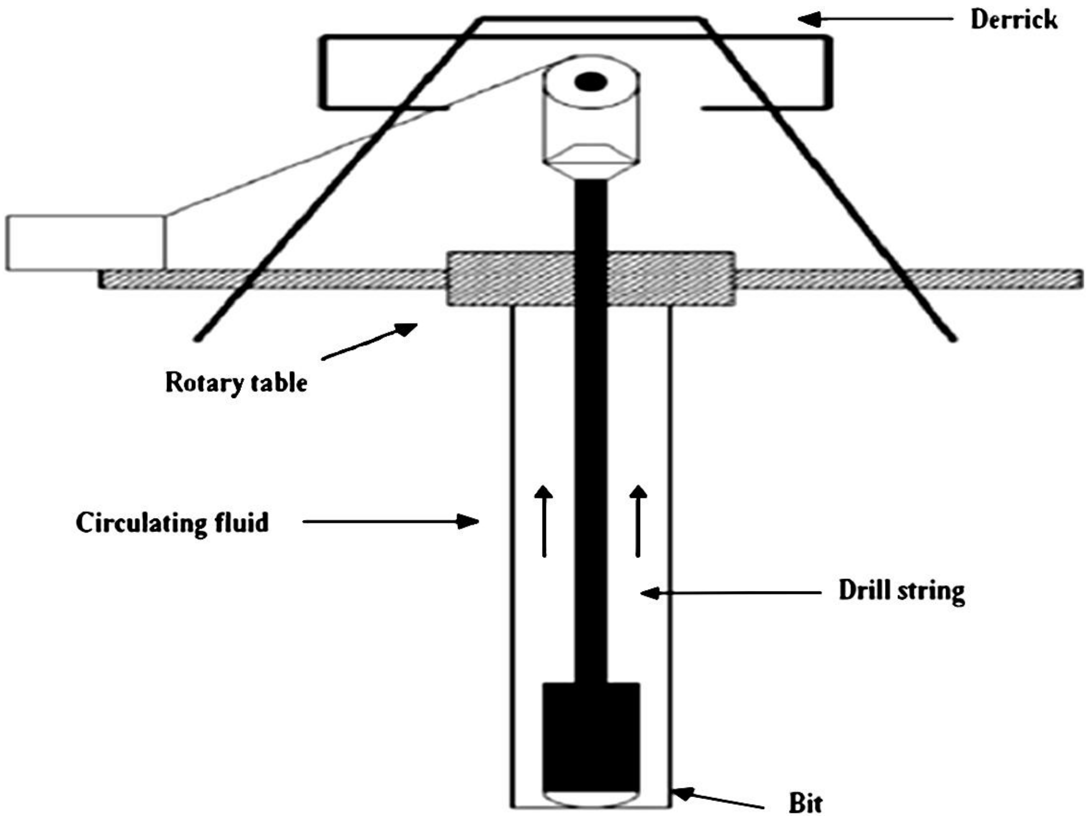

FIG. 1. The drilling system.

others has the function of cleaning, cooling and lubricating the bit. The drillstring is rotated from the surface by an electrical motor. The rotating mechanism can be of two types: a rotary table or a top drive.

The drill pipe is considered here as a beam in torsion. A lumped inertia I_{B} is chosen to represent the assembly at the bottom hole and a damping $\beta \geqslant 0$, which includes the viscous and structural damping, is assumed along the structure. The speed of the surface ($\xi = 0$) is restricted to a constant value Ω. The other extremity ($\xi = L$), which symbolizes the bit, is subject to a torque T, which is a function of the bit speed. The mechanical system is described by the following equations:

$$
\begin{aligned}
GJ\frac{\partial^2 v}{\partial \xi^2}(\xi,t) - I\frac{\partial^2 v}{\partial t^2}(\xi,t) - \beta\frac{\partial v}{\partial t}(\xi,t) = 0, \quad \xi \in (0,L), t > 0,\\
v(0,t) = \Omega t,\\
GJ\frac{\partial v}{\partial \xi}(L,t) + I_{\mathrm{B}}\frac{\partial^2 v}{\partial t^2}(L,t) = -T\left(\frac{\partial v}{\partial t}(L,t)\right),
\end{aligned}
$$

where $v(\xi,t)$ is the angle of rotation, I is the inertia, G is the shear modulus and J is the geometrical moment of inertia.

The existence and uniqueness of the solution is assumed for all the initial conditions. The stationary solution of this system is

$$
v^0(\xi,t) = \Omega t - \left(\frac{T(\Omega)}{GJ} + \frac{\beta\Omega}{GJ}L\right)\xi + \frac{\beta\Omega}{2GJ}\xi^2.
$$

The change of variable $z(\xi,t) = v(\xi,t) - v^0(\xi,t)$ leads to an equivalent autonomous system for which the function $z^0(\xi,t) = 0$ is a solution. Non-linear phenomena at the bottom extremity such as stick slip and noise due to the bit interaction are modelled with the additive bounded disturbance $w(t)$ such that $|w(t)| \leqslant \overline{w} t \in (0,\infty)$. This additive noise is consistent with the model of the stick slip introduced in Navarro & Cortés (2007) in which an additive non-linear dry friction term is considered to approximate the rock-bit contact.

$$
\begin{aligned}
GJ\frac{\partial^2 z}{\partial \xi^2}(\xi,t) - I\frac{\partial^2 z}{\partial t^2}(\xi,t) - \beta\frac{\partial z}{\partial t}(\xi,t) = 0, \quad \xi \in (0,L),\\
z(0,t) = 0,\\
GJ\frac{\partial z}{\partial \xi}(L,t) + I_{\mathrm{B}}\frac{\partial^2 z}{\partial t^2}(L,t) = T(\Omega) - T\left(\Omega + \frac{\partial z}{\partial t}(L,t)\right) + w(t).
\end{aligned}
$$

The end $\xi = 0$ is now fixed and the wave propagation has not changed. The stability of the trivial solution $z^0(\xi,t)$ is equivalent to the stability of the equilibrium $v^0(\xi,t)$.

We consider a linearization of the torque T:

$$
T(\Omega) - T(\Omega + z_t(L,t)) = -T'(\Omega + \theta z_t(L,t))z_t(L,t), \quad \theta \in (0,1).
$$

For the sake of simplicity, we introduce the normalized rod length $\sigma = \xi/L$. The normalized drill pipe model is then:

$$
\begin{aligned}
\frac{GJ}{L^2}\frac{\partial^2 z}{\partial \sigma^2}(\sigma,t) - I\frac{\partial^2 z}{\partial t^2}(\sigma,t) - \beta\frac{\partial z}{\partial t}(\sigma,t) = 0, \quad \sigma \in (0,1),\\
z(0,t) = 0,\\
\frac{GJ}{L}\frac{\partial z}{\partial \sigma}(1,t) + I_{\mathrm{B}}\frac{\partial^2 z}{\partial t^2}(1,t) = -T'(\Omega + \theta z_t(L,t))z_t(1,t) + w(t), \quad \theta \in (0,1),
\end{aligned}
\tag{2.1}
$$

with initial conditions

$$
\begin{aligned}
&z(\sigma,0)=\zeta(\sigma), \quad z_\sigma(\sigma,0)=\dot{\zeta}(\sigma)\in L_2(0,1),\\
&z_t(\sigma,0)=\zeta_1(\sigma)\in L_2(0,1).
\end{aligned} \tag{2.2}
$$

2.2 *Useful inequalities*

When the disturbance term $w(t)$ is not identically zero, one cannot prove exponential stability of the solution. However, one can prove ultimate boundedness of the solutions for bounded $w(t) < \overline{w}$. We introduce the following technical lemma.

LEMMA 2.1 (Fridman & Dambrine, 2009). Let $V\colon [0,\infty)\to R^+$ be an absolutely continuous function. If there exists $\delta>0$, $b>0$ such that the derivative of V satisfies almost everywhere the inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t}V(t)+2\delta V(t)-bw^2(t)\leqslant 0,
$$

then it follows that for all $|w(t)|\leqslant \overline{w}$,

$$
V(t)\leqslant \mathrm{e}^{-2\delta(t-t_0)}V(t_0)+(1-\mathrm{e}^{-2\delta(t-t_0)})\frac{b}{2\delta}\overline{w}^2.
$$

Proof. Multiplying by $\mathrm{e}^{2\delta(\theta-t)}$ the inequality $\frac{\mathrm{d}}{\mathrm{d}t}V+2\delta V\leqslant bw^2$ and integrating further from t_0 to t, we have

$$
\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{e}^{2\delta(\theta-t)}V(\theta))\mathrm{d}\theta\leqslant b\int_{t_0}^{t}\mathrm{e}^{2\delta(\theta-t)}w^2(\theta)\mathrm{d}\theta,
$$

and thus

$$
V(t)-\mathrm{e}^{-2\delta(t-t_0)}V(t_0)\leqslant \frac{b}{2\delta}(1-\mathrm{e}^{-2\delta(t-t_0)})\overline{w}^2.
$$

□

For later use, we recall the following.

LEMMA 2.2 (Wang, 1994). Let $z\in W^{1,2}([a,b],R)$ be a scalar function with $z(a)=0$. Then,

$$
\max_{\sigma\in[a,b]} z^2(\sigma)\leqslant (b-a)\int_a^b (z'(\sigma))^2\mathrm{d}\sigma. \tag{2.3}
$$

3. A difference equation approach

We assume in this section that T' is constant. Under the assumptions that the damping and the lumped inertia are negligible (i.e. $\beta=I_{\mathrm{B}}=0$) the model reduces to

$$
\frac{\partial^2 z}{\partial t^2}(\sigma,t)=a\frac{\partial^2 z}{\partial\sigma^2}(\sigma,t),\quad \sigma\in(0,1),\ \ t>0, \tag{3.1}
$$

$$z(0,t) = 0, \quad \frac{\partial z}{\partial \sigma}(1,t) = -k\frac{\partial z}{\partial t}(1,t) + r\omega(t), \tag{3.2}$$

where $a = \frac{GJ}{IL^2}$, $k = \frac{LT'}{GJ}$ and $r = \frac{L}{GJ} \in R$.

Note that neither the assumption of a constant torque derivative nor that of a negligible lumped inertia I_B are needed to apply the d'Alembert transformation. Nevertheless, such simplifications make it possible to reduce the distributed parameter model to a difference equation.

We use for this purpose the general solution of the 1D wave equation that may be written as

$$z(\sigma,t) = \phi(t+s\sigma) + \psi(t-s\sigma), \quad t > 0,$$

where ϕ, ψ are continuously differentiable real-valued functions of one variable and $s = \sqrt{\frac{1}{a}}$. We find

$$\begin{aligned} \frac{\partial z}{\partial t}(\sigma,t) &= \dot{\phi}(t+s\sigma) + \dot{\psi}(t-s\sigma), \\ \frac{\partial z}{\partial \sigma}(\sigma,t) &= s\dot{\phi}(t+s\sigma) - s\dot{\psi}(t-s\sigma). \end{aligned} \tag{3.3}$$

The initial conditions have the form

$$\begin{aligned} z_t(\sigma,0) &= \zeta_1(\sigma) = \dot{\phi}(s\sigma) + \dot{\psi}(-s\sigma), \\ z_\sigma(\sigma,0) &= \dot{\zeta}(\sigma) = s\dot{\phi}(s\sigma) - s\dot{\psi}(-s\sigma). \end{aligned} \tag{3.4}$$

Hence,

$$\begin{aligned} \dot{\phi}(s\sigma) &= 0.5[\zeta_1(\sigma) + \dot{\zeta}(\sigma)/s], \\ \dot{\psi}(-s\sigma) &= 0.5[\zeta_1(\sigma) - \dot{\zeta}(\sigma)/s]. \end{aligned} \tag{3.5}$$

The boundary conditions can be presented as

$$z(0,t) = \phi(t) + \psi(t) = 0, \quad t > 0, \tag{3.6}$$

$$\begin{aligned} \frac{\partial z}{\partial \sigma}(1,t) &= s\dot{\phi}(t+s) - s\dot{\psi}(t-s) \\ &= -k[\dot{\phi}(t+s) + \dot{\psi}(t-s)] + r\omega(t). \end{aligned} \tag{3.7}$$

It follows from (3.6) that

$$\phi(t) = -\psi(t), \quad t > 0, \tag{3.8}$$

and thus (3.7) takes the form

$$[s+k]\dot{\psi}(t+s) + [s-k]\dot{\psi}(t-s) = -r\omega(t). \tag{3.9}$$

This expression can be rewritten as

$$\dot{\psi}(t+s) = -c_0\dot{\psi}(t-s) - c_1\omega(t), \quad t > 0, \tag{3.10}$$

with $c_0 = \frac{(s-k)}{(s+k)}$ and $c_1 = \frac{r}{(s+k)}$.

From (3.5) and (3.9), we obtain the following initial condition:

$$\dot{\psi}(t) = -0.5[\zeta_1(t/s) + \dot{\zeta}(t/s)/s], \quad t \in (0, s),$$
$$\dot{\phi}(t) = 0.5[\zeta_1(t/s) - \dot{\zeta}(t/s)/s], \quad t \in (0, s). \tag{3.11}$$

It appears that (3.10) and (3.11) can be treated using difference equation techniques. Let $t = ls + \xi$, $\xi \in [-s, 0]$. Iterating l times (3.10), we obtain that for $l = 1, 2, \ldots$, the solution is described by

$$\dot{\psi}(2ls + \xi) = (-c_0)^l \dot{\psi}(\xi) - c_1 \sum_{i=0}^{l-1} (-c_0)^i w((2(l-i) - 1)s + \xi),$$
$$\dot{\psi}(2ls + s + \xi) = (-c_0)^l \dot{\psi}(s + \xi) - c_1 \sum_{i=0}^{l-1} (-c_0)^i w((2(l-i))s + \xi).$$

Taking into account $|c_0| < 1$ and setting $\lambda = -\ln(|c_0|)$, we arrive to $\left|(-c_0)^i\right| \leqslant \mathrm{e}^{-\lambda i}$. Hence, from $w(t) \leqslant \overline{w}$, $t > 0$, it follows that for $\xi \in [-s, 0]$

$$|\dot{\psi}(2ls + \xi)| \leqslant \mathrm{e}^{-\lambda l} |\dot{\psi}(\xi)| + |c_1| \overline{w} \sum_{i=0}^{l-1} \mathrm{e}^{-\lambda i},$$
$$|\dot{\psi}(2ls + s + \xi)| \leqslant \mathrm{e}^{-\lambda l} |\dot{\psi}(s + \xi)| + |c_1| \overline{w} \sum_{i=0}^{l-1} \mathrm{e}^{-\lambda i}.$$

Note that (3.11) implies $|\dot{\psi}(\xi)| \leqslant 0.5[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s]$ for $-s \leqslant \xi \leqslant s$. Moreover, from $\sum_{i=0}^{l-1} \mathrm{e}^{-\lambda i} \leqslant \frac{1}{1-\mathrm{e}^{-\lambda}}$, it follows that

$$|\dot{\psi}(2ls + \xi)| \leqslant 0.5\,\mathrm{e}^{-\lambda l}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{|c_1|}{1 - \mathrm{e}^{-\lambda}} \overline{w},$$
$$|\dot{\psi}(2ls + s + \xi)| \leqslant 0.5\,\mathrm{e}^{-\lambda l}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{|c_1|}{1 - \mathrm{e}^{-\lambda}} \overline{w}.$$

Since $\mathrm{e}^{-\lambda l} = \mathrm{e}^{-\frac{\lambda}{s}(t-\xi)} \leqslant \mathrm{e}^{-\frac{\lambda}{s}t}$ we obtain

$$|\dot{\psi}(2ls + \xi)| \leqslant 0.5\,\mathrm{e}^{-\frac{\lambda}{s}t}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{|c_1|}{1 - \mathrm{e}^{-\lambda}} \overline{w},$$
$$|\dot{\psi}(2ls + s + \xi)| \leqslant 0.5\,\mathrm{e}^{-\frac{\lambda}{s}t}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{|c_1|}{1 - \mathrm{e}^{-\lambda}} \overline{w}.$$

Then, for $t > 0$,

$$|\dot{\psi}(t)| \leqslant 0.5\,\mathrm{e}^{-\frac{\lambda}{s}t}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{|c_1|}{1 - \mathrm{e}^{-\lambda}} \overline{w}. \tag{3.12}$$

Furthermore, from (3.8), it follows that $|\dot{\phi}(t)|$ satisfies the same upper bound as $|\dot{\psi}(t)|$ for $t > -s$. Therefore, we obtain from (3.3) and from (3.4) that

$$|z_\sigma(1, t)| \leqslant s\,\mathrm{e}^{-\frac{\lambda}{s}t}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{2s|c_1|}{1 - \mathrm{e}^{-\lambda}} \overline{w}. \tag{3.13}$$

Finally, we find an ultimate bound for the variable of main interest, the angular velocity at the bottom $z_t(1,t)$. It follows from (3.3) that $\frac{\partial z}{\partial t}(1,t) = \dot{\phi}(t+s) + \dot{\psi}(t-s)$. In view of (3.8), $|\dot{\phi}(t)|$ and $|\dot{\psi}(t)|$ satisfy the same bound (3.12). It follows straightforwardly that

$$|z_t(1,t)| \leqslant \mathrm{e}^{-\frac{\lambda}{s}t}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{2|c_1|}{1-\mathrm{e}^{-\lambda}}\overline{w}. \tag{3.14}$$

We now summarize the above results:

THEOREM 3.1 Solutions of the boundary value problem (3.1), (3.2) with initial conditions (2.2) satisfy the inequalities (3.13) and (3.14).

4. A wave equation analysis

The difference equation description of the model provides an estimate of the behaviour at the bottom of the pipe. For a comprehensive estimate, we treat in this section the problem as a distributed parameter system. The lumped inertia I_{B} is considered to be negligible but, unlike in the previous section, the damping β is not. We assume in this section that T' is bounded: $0 < T_0 \leqslant T' \leqslant T_1$. We have the following equation:

$$z_{tt}(\sigma,t) = az_{\sigma\sigma}(\sigma,t) + \mathrm{d}z_t(\sigma,t), \quad t > t_0, \ \ 0 < \sigma < 1, \tag{4.1}$$

with the boundary conditions

$$\begin{aligned} &z(0,t) = 0, \\ &z_\sigma(1,t) = -kz_t(1,t) + rw(t), \quad t > 0, \end{aligned} \tag{4.2}$$

where $a = \frac{GJ}{IL^2}$, $d = \frac{-\beta}{I} \leqslant 0$, $r = \frac{L}{GJ}$ and $k = \frac{LT'}{GJ}$, satisfying $0 < k_0 \leqslant k \leqslant k_1$ with $k_i = \frac{LT_i}{GJ}$, $i = 0, 1$. The initial conditions are given by

$$\begin{aligned} &z(\sigma,0) = \zeta(\sigma), \quad z_\sigma(\sigma,0) = \dot{\zeta}(\sigma) \in L_2(0,1), \\ &z_t(\sigma,0) = \zeta_1(\sigma) \in L_2(0,1). \end{aligned} \tag{4.3}$$

Now, we look for conditions such that the inequality $\frac{\mathrm{d}}{\mathrm{d}t}V + 2\delta V - bw^2 \leqslant 0$ holds. To this end, consider the Lyapunov functional

$$V(z_\sigma(\cdot,t), z_t(\cdot,t)) = p\int_0^1 az_\sigma^2(\sigma,t)\mathrm{d}\sigma + p\int_0^1 z_t^2(\sigma,t)\mathrm{d}\sigma + 2\chi\int_0^1 \sigma z_\sigma(\sigma,t)z_t(\sigma,t)\mathrm{d}\sigma$$

proposed in Nicaise & Pignotti (2008) with constants $p > 0$ and small enough χ. In Fridman & Orlov (2009), the following LMI

$$\begin{bmatrix} ap & \chi \\ \chi & p \end{bmatrix} > 0 \tag{4.4}$$

was introduced to guarantee that $V > 0$ for $\int_0^1 [z_\sigma^2(\sigma,t) + z_t^2(\sigma,t)]\mathrm{d}\sigma > 0$.

THEOREM 4.1 Given $\delta > 0$, let there exist $p > 0$ and $\chi > 0$ such that (4.4) and two LMIs

$$\begin{bmatrix} -2ak_i p + \chi & 0 & 0 & -a\chi k_i r + apr & ak_i\chi \\ * & \psi_2 & (2\delta + d)\chi & 0 & 0 \\ * & * & \psi_3 & 0 & 0 \\ * & * & * & -b + \chi a r^2 & 0 \\ * & * & * & * & -a\chi \end{bmatrix} < 0, \quad i = 0, 1, \tag{4.5}$$

where

$$\begin{aligned} \psi_2 &= -a\chi + 2\delta a p, \\ \psi_3 &= -\chi + 2pd + 2\delta p \end{aligned} \tag{4.6}$$

are feasible. Then solutions of the boundary value problem (4.1), (4.2) with initial conditions (4.3) satisfy the inequality

$$\max_{\sigma\in[0,1]} z^2(\sigma, t) \leqslant \int_0^1 [z_\sigma^2(\sigma, t) + z_t^2(\sigma, t)]\mathrm{d}\sigma \leqslant \frac{\alpha_2}{\alpha_1}\mathrm{e}^{-2\delta(t-t_0)} \int_0^1 [\zeta_1^2(\sigma) + \dot{\zeta}^2(\sigma)]\mathrm{d}\sigma + \frac{b}{\alpha_1 2\delta}\overline{w}^2, \tag{4.7}$$

where

$$\alpha_1 = \lambda_{\min}\begin{bmatrix} ap & 0 \\ 0 & p \end{bmatrix}, \quad \alpha_2 = \lambda_{\max}\begin{bmatrix} ap & \chi \\ \chi & p \end{bmatrix}. \tag{4.8}$$

Proof. As the LMI $\begin{bmatrix} ap & \chi\sigma \\ \chi\sigma & p \end{bmatrix} > 0$ is affine in $\sigma \in [0, 1]$, it follows from Schur complements and Rayleigh's Theorem that

$$\alpha_1 \int_0^1 [z_\sigma^2(\sigma, t) + z_t^2(\sigma, t)]\mathrm{d}\sigma \leqslant V(z_\sigma(\cdot,t), z_t(\cdot, t)) \leqslant \alpha_2 \int_0^1 [z_\sigma^2(\sigma, t) + z_t^2(\sigma, t)]\mathrm{d}\sigma, \tag{4.9}$$

with α_1 and α_2 satisfying (4.8).

Next, we find $\frac{\mathrm{d}}{\mathrm{d}t}V$. Following Fridman & Orlov (2009), we derive

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(2\int_0^1 \sigma z_t z_\sigma \,\mathrm{d}\sigma\right) &= 2\int_0^1 \sigma z_{tt} z_\sigma \,\mathrm{d}\sigma + 2\int_0^1 \sigma z_t z_{\sigma t}\,\mathrm{d}\sigma \\ &= 2a\int_0^1 \sigma z_{\sigma\sigma}(\sigma, t) z_\sigma \,\mathrm{d}\sigma + 2\int_0^1 \sigma z_t z_{\sigma t}\mathrm{d}\sigma + 2d\int_0^1 \sigma z_t(\sigma, t) z_\sigma \,\mathrm{d}\sigma. \end{aligned}$$

Integration by parts gives

$$2\int_0^1 \sigma z_t z_{\sigma t}\,\mathrm{d}\sigma = -2\int_0^1 \sigma z_{\sigma t} z_t \,\mathrm{d}\sigma - 2\int_0^1 z_t^2 d\sigma + 2z_t^2(1, t),$$

and

$$2\int_0^1 \sigma z_t z_{\sigma t}\,\mathrm{d}\sigma = -\int_0^1 z_t^2 d\sigma + z_t^2(1, t).$$

Similarly,

$$2\int_0^1 \sigma z_{\sigma\sigma}(\sigma,t)z_\sigma \,\mathrm{d}\sigma = -\int_0^1 z_\sigma^2 \,\mathrm{d}\sigma + z_\sigma^2(1,t).$$

Substitution of the boundary condition yields

$$2\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \sigma z_t z_\sigma \,\mathrm{d}\sigma\right) = -\int_0^1 (z_t^2 + a z_\sigma^2)\mathrm{d}\sigma + z_t^2(1,t) + a(-kz_t(1,t) + rw(t))^2$$
$$+2d\int_0^1 \sigma z_t(\sigma,t) z_\sigma \,\mathrm{d}\sigma.$$

Thus, differentiating V along (4.1), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V &= 2p\int_0^1 a z_\sigma(\sigma,t) z_{t\sigma}(\sigma,t)\mathrm{d}\sigma + 2p\int_0^1 z_t(\sigma,t)z_{tt}(\sigma,t)\mathrm{d}\xi + 2\chi\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \sigma z_t z_\sigma \,\mathrm{d}\sigma\right)\\ &= 2p\int_0^1 [a z_\sigma(\sigma,t) z_{t\sigma}(\sigma,t) + a z_t(\sigma,t) z_{\sigma\sigma}(\sigma,t)]\mathrm{d}\sigma\\ &\quad +2pd\int_0^1 z_t(\sigma,t) z_t(\sigma,t)\mathrm{d}\sigma + 2\chi\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \sigma z_t z_\sigma \,\mathrm{d}\sigma\right).\end{aligned}$$

Then, integrating by parts and substituting the boundary condition (4.2), we obtain

$$\begin{aligned}\int_0^1 z_t(\sigma,t) z_{\sigma\sigma}(\sigma,t)\mathrm{d}\sigma &= z_t(\sigma,t) z_\sigma(\sigma,t)|_0^1 - \int_0^1 z_{t\sigma}(\sigma,t) z_\sigma(\sigma,t)\mathrm{d}\sigma\\ &= z_t(1,t)(-kz_t(1,t) + rw(t)) - \int_0^1 z_{t\sigma}(\sigma,t) z_\sigma(\sigma,t)\mathrm{d}\sigma.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V &= -2apkz_t^2(1,t) + 2apz_t(1,t)rw(t) + 2pd\int_0^1 z_t(\sigma,t)z_t(\sigma,t)\mathrm{d}\sigma\\ &\quad +\chi\left[-\int_0^1 (z_t^2 + a z_\sigma^2)\mathrm{d}\sigma + z_t^2(1,\ t) + a z_\sigma^2(1,\ t) + 2d\int_0^1 \sigma z_t(\sigma,t) z_\sigma \,\mathrm{d}\sigma\right].\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V + 2\delta V - bw^2 &= -2apkz_t^2(1,t) + 2apz_t(1,t)rw(t) + 2pd\int_0^1 z_t(\sigma,t)z_t(\sigma,t)\mathrm{d}\sigma\\ &\quad +\chi\left[-\int_0^1 (z_t^2 + a z_\sigma^2)\mathrm{d}\sigma + z_t^2(1,t) + a(-kz_t(1,t) + rw(t))^2\right.\\ &\qquad \left. +2d\int_0^1 \sigma z_t(\sigma,t) z_\sigma \,\mathrm{d}\sigma\right]\\ &\quad +\int_0^1 2\delta[apz_\sigma^2(\sigma,t) + 2\chi\sigma z_\sigma(\sigma,t)z_t(\sigma,t) + pz_t^2(\sigma,t)]\mathrm{d}\sigma - bw^2.\end{aligned} \tag{4.10}$$

By setting $\vartheta^T(\sigma,t) = [z_t(1,t)\ z_\sigma(\sigma,t)\ z_t(\sigma,t)\ w(t)]$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}V + 2\delta V - bw^2 = \int_0^1 \vartheta^T(\sigma,t)\Psi\vartheta(\sigma,t)\mathrm{d}\sigma \ < 0$$

if

$$\Psi = \begin{bmatrix} -2akp + (1+ak^2)\chi & 0 & 0 & -a\chi kr + apr \\ * & \psi_2 & (2\delta + d)\chi\sigma & 0 \\ * & * & \psi_3 & 0 \\ * & * & * & -b + \chi ar^2 \end{bmatrix} < 0, \tag{4.11}$$

with notations given in (4.6). Applying Schur complements to $ak^2\chi$ in (4.11) and using the affinity of the resulting LMI in $\sigma \in [0,1]$ and $k \in [k_0, k_1]$, it is easy to see that (4.11) holds if (4.5) is feasible.

Then, if (4.5) is feasible, it follows from (4.9) and Lemma 2.1 that

$$\begin{aligned} \alpha_1 \int_0^1 [z_\sigma^2(\sigma,t) + z_t^2(\sigma,t)]\mathrm{d}\sigma &\leqslant V(z_\sigma(\cdot,t), z_t(\cdot,t)) \\ &\leqslant V(z_\sigma(\cdot,t_0), z_t(\cdot,t_0))\mathrm{e}^{-2\delta(t-t_0)} + \frac{b}{2\delta}(1 - \mathrm{e}^{-2\delta(t-t_0)})\overline{w}^2 \\ &\leqslant \alpha_2\,\mathrm{e}^{-2\delta(t-t_0)} \int_0^1 [\zeta_1^2(\sigma) + \dot{\zeta}^2(\sigma]\mathrm{d}\sigma + \frac{b}{2\delta}(1 - \mathrm{e}^{-2\delta(t-t_0)})\overline{w}^2. \end{aligned}$$

In addition, it follows from (2.3) that

$$\max_{\sigma\in[0,1]} z^2(\sigma,t) \leqslant \int_0^1 (z_\sigma(\sigma,t))^2\,\mathrm{d}\sigma \leqslant \int_0^1 [z_\sigma^2(\sigma,t) + z_t^2(\sigma,t)]\mathrm{d}\sigma.$$

□

REMARK 4.1 We note that inequality (4.7) means that (4.1), (4.2) is input-to-state stable. The conditions for exponential stability of the disturbance free system that follow from Theorem 4.1 coincide with the ones from Fridman & Orlov (2009).

4.1 *Stick-slip oscillations and the non-growth of the energy*

In this subsection, we give a new look at the problem. We leave out the bounded additive noise signal $w(t)$ taken into account for external disturbances in previous analysis and we introduce a model for the torque on the bit T that allows us to perform a dissipativity analysis.

The drillstring interaction with the borehole gives rise to a wide variety of non-desired oscillations that are classified depending on the direction they appear. Three main types of vibrations can be distinguished: torsional (stick-slip oscillations), axial (bit bouncing phenomenon) and lateral (whirl motion due to the out of balance of the drillstring). Torsional drillstring vibrations appear due to downhole conditions, such as significant drag, tight hole and formation characteristics. It can cause the bit to stall in the formation while the rotary table continues to rotate. When the trapped torsional energy (similar to a wound-up spring) reaches a level that the bit can no longer resist, the bit suddenly comes loose, rotating and whipping at very high speeds. This stick-slip behaviour can generate a torsional wave that travels up the drillstring to the rotary top system. Because of the high inertia of the rotary table, it acts like a fixed end to the drillstring and reflects the torsional wave back down the drillstring to the bit. The bit may stall again, and the torsional wave cycle repeats as explained in Navarro & Suárez (2004). The whipping and high speed rotations of the bit in the slip phase can generate both severe axial and lateral vibrations at the bottom-hole assembly. The vibrations can originate problems such as drill pipe fatigue problems, drillstring components failures, wellbore instability. They contribute to drill pipe fatigue and are detrimental to bit life.

The following switched equation is introduced in Navarro & Cortés (2007) that allows to approximate the physical phenomenon at the bottom hole

$$T = c_{\mathrm{b}} z_t(1,t) + W_{\mathrm{ob}} R_{\mathrm{b}} \mu_{\mathrm{b}}(z_t(1,t)) \operatorname{sgn}(z_t(1,t)), \tag{4.12}$$

where the term $c_{\mathrm{b}} z_t(1,t)$ is a viscous damping torque at the bit that approximates the influence of the mud drilling and where the term $W_{\mathrm{ob}} R_{\mathrm{b}} \mu_{\mathrm{b}} \operatorname{sgn}(z_t(1,t))$ is a dry friction torque modelling the bit-rock contact. $R_{\mathrm{b}} > 0$ is the bit radius and $W_{\mathrm{ob}} > 0$ the weight on the bit. The bit dry friction coefficient $\mu_{\mathrm{b}}(z_t(1,t))$ is modelled as follows:

$$\mu_{\mathrm{b}}(z_t(1,t)) = \mu_{\mathrm{cb}} + (\mu_{\mathrm{sb}} - \mu_{\mathrm{cb}}) \mathrm{e}^{-\frac{\gamma_{\mathrm{b}}}{v_f}|z_t(1,t)|}, \tag{4.13}$$

where μ_{sb} and $\mu_{\mathrm{cb}} \in (0,1)$ are the static and Coulomb friction coefficients and $0 < \gamma_{\mathrm{b}} < 1$ is a constant defining the velocity decrease rate. The constant velocity $v_f > 0$ is introduced in order to have appropriate units.

The friction torque (4.12)–(4.13) leads to a decreasing torque on bit with increasing bit angular velocity for low velocities which acts as a negative damping (Stribeck effect) and is the cause of stick-slip self-excited vibrations. The exponential decaying behaviour of T coincides with experimental torque values.

The boundary conditions of the drilling system described by the wave equation (4.1) are then

$$\begin{aligned} &z_t(0,t) = 0, \\ &z_\sigma(1,t) = -k z_t(1,t) - q\mu_{\mathrm{b}}(z_t(1,t)) \operatorname{sgn}(z_t(1,t)) - h z_{tt}(1,t), \quad t > 0, \end{aligned} \tag{4.14}$$

where $k = \frac{c_{\mathrm{b}} L}{GJ}$, $q = \frac{W_{\mathrm{ob}} R_{\mathrm{b}} L}{GJ}$ and $h = \frac{I_{\mathrm{B}} L}{GJ}$.

Consider the energy function

$$E(t) = \int_0^1 a z_\sigma^2(\sigma,t)\mathrm{d}\sigma + \int_0^1 z_t^2(\sigma,t)\mathrm{d}\sigma + a h z_t^2(1,t). \tag{4.15}$$

Differentiating V along (4.1), yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(t) &= 2\int_0^1 a z_\sigma(\sigma,t) z_{t\sigma}(\sigma,t)\mathrm{d}\sigma \\ &\quad + 2\int_0^1 z_t(\sigma,t) z_{tt}(\sigma,t)\mathrm{d}\sigma + 2ahz_t(1,t)z_{tt}(1,t) \\ &= 2\int_0^1 [a z_\sigma(\sigma,t) z_{t\sigma}(\sigma,t) + a z_t(\sigma,t) z_{\sigma\sigma}(\sigma,t)]\mathrm{d}\sigma \\ &\quad + 2d\int_0^1 z_t(\sigma,t) z_t(\sigma,t)\mathrm{d}\sigma + 2ahz_t(1,t)z_{tt}(1,t). \end{aligned}$$

Integrating by parts and substituting the boundary condition, (4.14) gives

$$\begin{aligned}\int_0^1 z_t(\sigma,t)z_{\sigma\sigma}(\sigma,t)\mathrm{d}\sigma &= z_t(\sigma,t)z_\sigma(\sigma,t)|_0^1 - \int_0^1 z_{t\sigma}(\sigma,t)z_\sigma(\sigma,t)\mathrm{d}\sigma\\ &= z_t(1,t)(-kz_t(1,t) - q\mu_\mathrm{b}(z_t(1,t))\mathrm{sgn}(z_t(1,t)) - hz_{tt}(1,t))\\ &\quad - \int_0^1 z_{t\sigma}(\sigma,t)z_\sigma(\sigma,t)\mathrm{d}\sigma.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= 2\int_0^1 az_\sigma(\sigma,t)z_{t\sigma}(\sigma,t)\mathrm{d}\sigma + 2az_t(1,t)(-kz_t(1,t) - q\mu_\mathrm{b}(z_t(1,t))\mathrm{sgn}(z_t(1,t))\\ &\quad -hz_{tt}(1,t)) - 2a\int_0^1 z_{t\sigma}(\sigma,t)z_\sigma(\sigma,t)\mathrm{d}\sigma\\ &\quad +2d\int_0^1 z_t(\sigma,t)z_t(\sigma,t)\mathrm{d}\sigma + 2ahz_t(1,t)z_{tt}(1,t).\end{aligned}$$

Since $\mu_\mathrm{b}(z_t(1,t))\mathrm{sgn}(z_t(1,t))z_t(1,t) = \mu_\mathrm{b}(z_t(1,t))|z_t(1,t)|$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -2akz_t^2(1,t) - 2aq\mu_\mathrm{b}(z_t(1,t))|z_t(1,t)| + 2d\int_0^1 z_t^2(\sigma,t)\mathrm{d}\sigma.$$

Taking into account that $\mu_\mathrm{b}(z_t(1,t)) > 0$ and that $d \leqslant 0$, we find that $\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant -2akz_t^2(1,t) \leqslant 0$. The non-growth of the energy of the drilling system (which reflects the oscillatory behaviour of the system) is established.

PROPOSITION 4.1 For all solutions of (4.1) under the switched boundary condition (4.14), the energy given by (4.15) does not grow.

5. Numerical results

The numerical results presented below are for the parameter values given in Challamel (1999):

$$\begin{aligned}&G = 79.3\times 10^9\ \mathrm{N/m^2}, \quad I = 0.095\ \mathrm{kg\cdot m},\\ &T' = 3000\ \mathrm{N\cdot m}, \quad J = 1.19\times 10^{-5}\ \mathrm{m^4},\\ &L = 3145\ \mathrm{m},\end{aligned}$$

and the case where the damping is neglected ($\beta = 0$).

5.1 *A difference equation approach*

In this case, $c_0 = -0.8185$, $c_1 = 3.0011\times 10^{-4}$, $s = 0.9979$ and $\lambda = -\ln(|c_0|) = 0.2002$. Substituting these values into (3.13) and (3.14) yields

$$\begin{aligned}|z_\sigma(1,t)| &\leqslant 0.9979\,\mathrm{e}^{-0.2006t}[|\zeta_1(\xi/0.9979)| + 1.0021|\dot\zeta(\xi/0.9979)|] + 0.0033\overline{\varpi},\\ |z_t(1,t)| &\leqslant \mathrm{e}^{-0.2006t}[|\zeta_1(\xi/0.9979)| + 1.0021|\dot\zeta(\xi/0.9979)|] + 0.0033\overline{\varpi}.\end{aligned}$$

5.2 *A wave equation approach*

For the wave equation approach, the LMI conditions of Theorem 4.1 lead to the following pairs (δ, b).

Case	1	2	3	4	5
δ	0.08	0.06	0.04	0.01	0.0001
b	3.2521	1.0707	1.2145	1.5221	1.7951
α_1	5.0009	1.0934	1.2657	1.6273	1.9328
α_2	5.9854	1.3019	1.5074	1.9383	2.3023

For $\delta = 0.04$ and initial conditions $z_\sigma(\sigma, t_0) = \dot{\zeta}$, $z_t(\sigma, t_0) = \zeta_1(\sigma)$, the expression (4.7) in Theorem 4.1 provides the following ultimate boundedness condition:

$$\int_0^1 [z_\sigma^2(\sigma, t) + z_t^2(\sigma, t)]\mathrm{d}\sigma \leqslant 1.1909\, \mathrm{e}^{-0.08t} \int_0^1 [\zeta_1^2(\sigma) + \dot{\zeta}^2(\sigma)]\mathrm{d}\sigma + 11.9944\overline{\varpi}^2.$$

The wave equation approach also provides an ultimate bound when the damping β is not negligible. For $\beta = 0.1\ \mathrm{N} \cdot \mathrm{s}$, it is

$$\int_0^1 [z_\sigma^2(\sigma, t) + z_t^2(\sigma, t)]\mathrm{d}\sigma \leqslant 1.1854\, \mathrm{e}^{-0.08t} \int_0^1 [\zeta_1^2(\sigma) + \dot{\zeta}^2(\sigma)]\mathrm{d}\sigma + 18.8654\overline{\varpi}^2.$$

It appears that the two approaches complete each other: the difference equation approach leads to an ultimate bound for the main variable of interest, the angular velocity at the drill bottom $z_t(1, t)$, while the wave equation model provides an ultimate bound for the measure $\int_0^1 [z_\sigma^2(\sigma, t) + z_t^2(\sigma, t)]\mathrm{d}\sigma$ of the distributed behaviour nature of the system.

6. Concluding remarks

Ultimate bounds for a distributed drill pipe model are obtained through an analysis based on a difference equation model and on a wave equation description. Note that the estimate for the difference equation is obtained via direct computations while the estimate for the wave equation is achieved through direct Lyapunov method that usually involves some conservatism. It should be pointed out that the wave equation approach addresses a more general case where the damping is not neglected.

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