Observer Design For a Class of Parabolic Systems With Large Delays and Sampled Measurements

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Abstract—In this paper, we design a novel observer for a class of semilinear heat one dimmensional (1-D) equations under the delayed and sampled point measurements. The main novelty is that the delay is arbitrary. To handle any arbitrary delay, the observer is constituted of a chain of subobservers. Each subobserver handles a fraction of the considered delay. The resulting estimation error system is shown to be exponentially stable under a sufficient number of subobservers is used. The stability analysis is based on a specific Lyapunov–Krasovskii functional and the stability conditions are expressed in terms of linear matrix inequalities (LMIs).

Index Terms—Chain observers, delayed output, parabolic systems.

I. INTRODUCTION

This paper deals with the design of observers for a class of parabolic partial differential equations (PDEs) with delayed and sampled measurements. This problem is highly challenging since time delays affecting output measurements appears in many applications. We can cite the well-known networked control systems (NCSs) which are systems controlled and supervised by remote controllers and observers through a communication device. The main works existing in the literature are focused on finite-dimensional systems described by ordinary differential equations (ODEs), e.g., cite [1] and reference list therein. The main idea consisted in a redesign of an existing exponentially convergent state observer for the delay-free system so that exponential convergence is preserved in the presence of time-delay. The modification essentially consists in introducing of state predictors to compensate for time delay. This has been illustrated with several classes of observers based on drift-observability property [2] or on high-gain observers [1], [3], [4]. However, for nonlinear ODEs, the designed predictor is useful in compensating the delay effect only up to some upper limit. Then to

enlarge the maximum time-delay, a chain of predictors simultaneously operating are introduced [2], [22], [23].

In parallel with the above "finite-dimensional" research activity, the "infinite-dimensional" backstepping transformation for linear systems-based approach has first been developed, e.g., [5] and references therein. This approach consists in letting the output sensor delay be captured by a first-order hyperbolic PDE. Then, full-order observers are designed that estimate both the system (finite-dimensional) state and the sensor (infinite-dimensional) state. The extension of this approach to (triangular) nonlinear systems has been studied in [6], where a high-gain type observer has been developed. The arbitrary-size delay effect has been compensated for by developing a PDE version of the chain observer concept.

The problem of observer design for nonlinear PDEs with arbitrary delays measurements has yet to be solved. In this paper, the problem is addressed for a class of parabolic PDEs under point measurements as in [7]. In the latter paper the results were confined to small delays. To compensate the effect of the arbitrary-size delay, the concept of chain-observer is extended to fit this class of systems. Accordingly, the initial delay PDE system representation is reexpressed in the form of fictive delayed subsystems. The observer is composed of elementary observers connected in series. The interconnection is such that the first elementary observer is directly driven by the physical system output. Then, the elementary observer is driven by a virtual output generated by the previous observer. Each elementary observer can be viewed as a predictor which compensates for the effects of the fractional time-delay. As in [7], using an appropriate Lyapunov–Krasovskii functional, sufficient conditions are established in terms of linear matrix inequalities (LMIs) via Halanay's inequality [14].

Note that in the existing results [3], [6], the exponential convergence can be proved by induction via input-to-state stability of the subobservers and by employing the fact that the input is exponentially converging. This approach is not applicable here because of Halanay's inequality that has been extended to the case of uniformly bounded inputs only (see Lemma 1 of [13]) that may lead to a practical stability. Here we suggest a novel proof which employs a special construction of a Lyapunov–Krasovskii functional for the augmented system of subobservers. The sufficient conditions involve a sufficient number of elementary observers; the larger the delay the larger the number of observers. Extension to sampled data delayed measurements is presented.

It has to be noticed that a conference version of the present work will be presented at [16]. The main differences between the present work and [16] are in the novel proof of Theorem 1 and in the more detailed proof of Theorem 2. We also add new simulations to more highlight the behavior of our algorithm and show the effect of the number of observers depending on the delay value. The paper is organized as follows: first, the observation problem under study is formulated in Section II; then, the observer design with delayed measurements and analysis are dealt within Sections III; In Section IV, the extension to sampled-data case is presented. In Section V, we illustrate our results

by some simulations on a numerical example involving both delays and sampling measurements.

Notations and Preliminaries

Throughout the paper the superscript T stands for matrix transposition, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|.|$, $\mathbf{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbf{R}^{n \times n}$, means that P is symmetric and positive definite. In matrices, symmetric terms are denoted by $*$; $\lambda_{\min}(P)(resp.\lambda_{\max}(P))$ denotes the smallest (resp. largest) eigenvalue. The notation $(t_i)_{i \geq 0}$ refers to a strictly increasing sequence such that $t_0 = 0$ and $\lim\limits_{k \to \infty} t_i = \infty$. The sampling periods are bounded, i.e., $0 < t_{i+1} - t_i < h$ for some scalar $0 < h < \infty$ and all $i = 0, 1, \ldots, \infty$. We also define the variable $\tau(t) = t - t_i, t \in [t_i, t_{i+1})$. $L_2(0, l)$ is the Hilbert space of square integrable functions $z(x), x \in [0, l]$ with the corresponding norm $\|z(x)\|_{L_2} = \sqrt{\int_0^l z^2(x)dx}$. $\mathcal{H}^1(0, l)$ is the Sobolev space of absolutely continuous functions $z : (0, l) \to \mathbf{R}$ with the square integrable derivative $\frac{d}{dx}$. $\mathcal{H}^2(0, l)$ is the Sobolev space of absolutely continuous functions $\frac{dz}{dx} : (0, l) \to \mathbf{R}$ and with $\frac{d^2 w}{dx^2} \in L_2(0, l)$. Given a two-argument function $u(x, t)$, its partial derivatives are denoted $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$. Throughout the paper the following lemma will be used to prove exponential convergence of our observer.

Lemma 1: (Halanay's type Inequalities [9])

Let $0 < \delta_1 < 2\delta$ and let $V : [t_0 - h, \infty) \to [0, \infty)$ be an absolutely continuous function which satisfies

$$\dot{V}(t) \leq -2\delta V(t) + \delta_1 \sup_{-h \leq s \leq 0} V(t + s). \tag{1}$$

Then

$$V(t) \leq e^{-2\alpha(t - t_0)} \sup_{-h \leq s \leq 0} V(s) \tag{2}$$

where α is the unique positive solution of the equation

$$\alpha = \delta - \frac{\delta_1 e^{2\alpha h}}{2}.$$

II. SYSTEM DESCRIPTION

We consider a semilinear diffusion equation

$$u_t(x, t) = u_{xx}(x, t) + f(u(x, t), x, t), t \geq t_0 \tag{3}$$

with the Dirichlet boundary conditions

$$u(0, t) = u(l, t) = 0.$$

As in [7], we assume that the points x_j divide the interval $[0, l]$ such that $0 = x_0 < \cdots < x_N = l$ and $x_{j+1} - x_j \leq \Delta$. The system output is given by

$$y(t) = u(\bar{x}_j, t - D), \bar{x}_j = \frac{x_{j+1} + x_j}{2}$$

$$j = 0, \ldots, N - 1, t \geq t_0 + D$$

$$y(t) = 0, t < t_0 + D$$

where N is the number of distributed sensors. The constant D represents an arbitrarily delay that may be large. It is also supposed that the function f is known, of class $\mathcal{C}^1$, and satisfying $m_f \leq f_u \leq M_f$, for some scalar constants m_f and M_f.

Remark 1: For simplicity we only consider the constant (equal to 1) diffusion coefficient. The presented results can be easily extended to a more general case

$$u_t(x, t) = \frac{\partial}{\partial x}[(a(x)u_x(x, t)] + f(u(x, t), x, t), t \geq t_0$$

with $a \in C^1$ and $a \geq a_0 > 0$ as considered in [7]. Note that the model of chemical reactor given in [18] is governed by (3) with globally Lipschitz nonlinearity.

III. OBSERVER DESIGN

We will present an observer, constituted by a chain of m subobservers, which ensures exponential convergence for an arbitrarily delay D. Each subobserver estimates the state $u(x, t + \frac{k}{m}D - D)$ by using the estimation provided by the previous one in the chain whereas the first subobserver uses the delayed measurement provided by sensors. The last subobserver in the chain provides the estimation of the $u(x, t)$. As we will see below, by using a suitable Lyapunov functional, we will derive sufficient conditions involving both delay D, and the number of subobservers in the chain m.

As in [2] we introduce the following notations for the delayed states:

$$u^0(x, t) = u(x, t - D)$$

$$u^k(x, t) = u\left(x, t + \frac{k}{m}D - D\right), k = 1 \ldots, m$$

Using these notations we easily check that

$$u^{k-1}(x, t) = u^k\left(x, t - \frac{D}{m}\right)$$

and

$$u^m(x, t) = u(x, t)$$

where m is the number of subobservers in the considered chain.

We propose the following observer structure:

for $k = 1$

$$\hat{u}_t^1(x, t) = \hat{u}_{xx}^1(x, t) + f\left(\hat{u}^1(x, t), x, t + \frac{1}{m}D - D\right)$$

$$- L\left(\hat{u}^1\left(\bar{x}_j, t - \frac{D}{m}\right) - y(t)\right)$$

$$\forall x \in [x_j, x_{j+1}), t \geq t_0 \tag{4}$$

for $k = 2, \ldots, m$

$$\hat{u}_t^k(x, t) = \hat{u}_{xx}^k(x, t) + f\left(\hat{u}^k(x, t), x, t + \frac{k}{m}D - D\right)$$

$$- L(\hat{u}^k\left(\bar{x}_j, t - \frac{D}{m}\right) - \hat{u}^{k-1}(\bar{x}_j, t))$$

$$\forall x \in [x_j, x_{j+1}), t \geq t_0. \tag{5}$$

with the initial conditions

$$\hat{u}^1(x, t) = \cdots = \hat{u}^m(x, t) = 0, t \leq t_0.$$

It is readily checked that the observation error $e^k(x, t) = \hat{u}^k(x, t) - u^k(x, t)$ undergoes the following equations:

for $k = 1$

$$e_t^1(x,t) = e_{xx}^1(x,t) + f\left(\hat{u}^1(x,t), x, t + \frac{1}{m}D - D\right) \quad (6)$$

$$- f\left(u^1(x,t), x, t + \frac{1}{m}D - D\right)$$

$$- Le^1\left(\bar{x}_j, t - \frac{D}{m}\right), t \geq t_0$$

$$\forall x \in [x_j, x_{j+1}) \quad (7)$$

for $k = 2, \ldots, m$

$$e_t^k(x,t) = e_{xx}^k(x,t) + f\left(\hat{u}^k(x,t), x, t + \frac{k}{m}D - D\right) \quad (8)$$

$$- f\left(u^k(x,t), x, t + \frac{k}{m}D - D\right)$$

$$- L\left(\hat{u}^k(\bar{x}_j, t - \frac{D}{m}) - \hat{u}^{k-1}(\bar{x}_j, t)\right), t \geq t_0$$

$$\forall x \in [x_j, x_{j+1}). \quad (9)$$

We will further derive stability conditions for the error system. For simplicity, we assume that $u(x,t) = u(x, t_0), t < t_0$ (that does not influence on the stability analysis).

Recall that

$$u^k\left(x, t - \frac{D}{m}\right) = u^{k-1}(x,t)$$

Then, for $k = 1$

$$e_t^1(x,t) = e_{xx}^1(x,t) + \phi(x, t, e^1)e^1(x,t)$$

$$- Le^1\left(\bar{x}_j, t - \frac{D}{m}\right)$$

$$x \in [x_j, x_{j+1})$$

$$e^1(l,t) = e^1(0,t) = 0 \quad (10)$$

for $k = 2, \ldots, m$

$$e_t^k(x,t) = e_{xx}^k(x,t) + \phi(x, t, e^k)e^k(x,t)$$

$$- Le^k\left(\bar{x}_j, t - \frac{D}{m}\right) + Le^{k-1}(\bar{x}_j, t)$$

$$\forall x \in [x_j, x_{j+1})$$

$$e^k(l,t) = e^k(0,t) = 0 \quad (11)$$

where

$$\phi(x, t, e^k) = \int_0^1 f_u\left(\hat{u}^k + \theta e^k, x, t + \frac{k}{m}D - D\right) d\theta.$$

The initial condition is given by $e^k(x,t) = -u(x, t_0), t \leq t_0$.

Remark 2: The well-posedness for the system (3) and the error system (10), (11) can be proven similar to [7], [17]. For instance, consider

$$w(t) = e^1(\cdot, t) \quad (12)$$

of the error system (10). We apply the step method. For $t \in [t_0, t_0 + \frac{D}{m}]$, (10) can be rewritten as a differential equation in the Hilbert space $H = L_2(0, l)$

$$\dot{w}(t) = Aw(t) + F(t, w(t)), t \geq t_0 \quad (13)$$

where the operator A is defined by

$$A = \frac{\partial^2}{\partial x^2} \quad (14)$$

and has the dense domain

$$\mathcal{D}(A) = \{w \in \mathcal{H}^2(0, l) : w(0) = w(l) = 0\}. \quad (15)$$

The operator A generates an exponentially stable semigroup [19]. The nonlinear term $F : [0, \frac{D}{m}] \times L_2(0, l) \to L_2(0, l)$ is defined on functions $w(\cdot, t)$ as

$$F(t, w(\cdot, t)) = f\left(u^1(\cdot, t) + w(\cdot, t), \cdot, t + \frac{1}{m}D - D\right)$$

$$- f\left(u^1(\cdot, t), \cdot, t + \frac{1}{m}D - D\right) - Lw(\bar{x}_j, t_0)$$

where $u^1(\cdot, t) = u(\cdot, t + \frac{D}{m} - D) \equiv u(\cdot, t_0)$. The function F is continuous in t and globally Lipschitz in w

$$\|F(t, w) - F(t, \bar{w})\|_{L^2} \leq L\|w - \bar{w}\|_{L^2} \quad (16)$$

with some constant $L > 0$ for $w, \bar{w} \in L_2(0, l)$, $t \in [t_0, t_0 + \frac{D}{m}]$. We note that $F : [t_0, t_0 + \frac{D}{m}] \times L_2(0, 1) \to L_2(0, 1)$ is continuously differentiable. If $w(t_0) = -u(\cdot, t_0) \in \mathcal{D}(A)$, then there exists a unique classical solution of (13) $w \in \mathcal{C}^1([t_0, t_0 + \frac{D}{m}]; L_2(0, 1))$ with $w(t) \in \mathcal{D}(A)$ (see [20], Th. 6.1.5). By the same arguments there exists a unique classical solution of (3) starting from $u(\cdot, t_0) \in \mathcal{D}(A)$ for all $t \geq t_0$.

By using the same argument for (13) step by step on $[t_0 + \frac{D}{m}, t_0 + \frac{2D}{m}], \ldots$, we obtain the well-posedness of the error system for all $t \geq t_0$.

Theorem 1: Given D and m, consider the system (3) and the observer (4)–(5). Given positive constants $\Delta, \delta, L > M_f - \frac{\pi^2}{l^2}, R$ and δ_1 such that $2\delta > \delta_1$, let there exist positive scalars p_1, p_2, p_3, r, and g such that

$$\delta p_3 < p_2; \frac{\Delta}{\pi}LR^{-1}(p_3 + p_2) < \delta_1 p_3 \quad (17)$$

and

$$\Phi_{m_f} < 0; \Phi_{M_f} < 0 \quad (18)$$

where

$$\Phi_\phi = \begin{pmatrix} \Phi_{11} - \lambda & \Phi_{12} & \Phi_{13} \\ \Phi_{12} & \Phi_{22} & \Phi_{23} \\ \Phi_{13} & \Phi_{23} & \Phi_{33} \end{pmatrix} \quad (19)$$

with

$$\Phi_{11} = 2\delta p_1 + g - re^{-2\delta\frac{D}{m}} + 2p_2\left(\phi + \frac{\Delta}{2\pi}LR\right)$$

$$\Phi_{12} = -p_2 + p_1 + p_3\phi$$

$$\Phi_{13} = re^{-2\delta\frac{D}{m}} - p_2L$$

$$\Phi_{22} = \frac{\Delta LRp_3}{\pi} - 2p_3 + r\left(\frac{D}{m}\right)^2$$

$$\Phi_{23} = -Lp_3$$

$$\Phi_{33} = -(r+g)e^{-2\delta\frac{D}{m}}$$

$$\lambda = \frac{2\pi^2}{l^2}(p_2 - \delta p_3). \quad (20)$$

Then all the observation errors $\int_0^1 (e_x^k(x,t))^2 dx$ and $\int_0^1 (e_x^k(x,t))^2 dx$ ($k = 1, \ldots, m$) globally exponentially decay to zero as $t \to +\infty$. The above LMIs are always feasible for large enough m.

Proof: Consider (10) and the corresponding Lyapunov–Krasovskii functional (as in [7])

$$V^1(t) = p_1 \int_0^l \left(e^1(x,t)\right)^2 dx + p_3 \int_0^l \left(e_x^1(x,t)\right)^2 dx$$

$$+ g \int_0^l \left[\int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} \left(e^1(x,s)\right)^2 ds \right] dx$$

$$+ \frac{D}{m} r \int_0^l \left[\int_{-\frac{D}{m}}^0 \int_{t+\theta}^t e^{2\delta(s-t)} \left(e_s^1(x,s)\right)^2 ds d\theta \right] dx \quad (21)$$

Differentiating the above functional we find

$$\dot{V}^1(t) + 2\delta V^1(t) = 2p_1 \int_0^l e^1(x,t) e_t^1(x,t) dx$$

$$+ 2p_3 \int_0^l e_x^1(x,t) e_{xt}^1(x,t) dx$$

$$- \frac{D}{m} r \int_0^l \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} e_s^1(x,s)^2 ds dx$$

$$+ \int_0^l \left[\left(\frac{D}{m}\right)^2 r \left(e_t^1(x,t)^2 + g(e^1(x,t))^2\right) \right.$$

$$\left. - g e^{-2\delta \frac{D}{m}} \left(e^1\left(x, t-\frac{D}{m}\right)\right)^2 \right] dx$$

$$+ 2\delta p_1 \int_0^l (e^1(x,t))^2 dx$$

$$+ 2\delta p_3 \int_0^l (e_x^1(x,t))^2 dx \quad (22)$$

We use further Jensen's inequality

$$- \frac{D}{m} r \int_0^l \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} e_s^1(x,s)^2 ds dx \leq$$

$$- r; \int_0^l e^{-2\delta \frac{D}{m}} \left(\int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} e_s^1(x,s) ds \right)^2 dx \quad (23)$$

and employ th descriptor method [14], where the right-hand side of the following equation is added to $\dot{V}^1$:

$$0 = 2 \int_0^l \left[p_2 e^1(x,t) + p_3 e_t^1(x,t) \right] \left[-e_t^1(x,t) + e_{xx}^1(x,t) \right.$$

$$+ \Psi(x,t,e^1) e^1(x,t) - L e^1\left(x, t-\frac{D}{m}\right) \right] dx$$

$$+ 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left[p_2 e^1(x,t) + p_3 e_t^1(x,t) \right]$$

$$\times L \int_{\bar{x}_j}^x e_\xi^1\left(\xi, t-\frac{D}{m}\right) d\xi dx. \quad (24)$$

Here p_2 and p_3 are free parameters. ∎

From (22)–(24), by using Young and Wirtinger inequalities we arrive at

$$\dot{V}^1(t) + 2\delta V^1(t) \leq \int_0^l \eta^T \Phi_\phi \eta dx$$

$$+ \frac{\Delta}{\pi} L R^{-1} (p_3 + p_2) \int_0^l \left(e_x^1\left(x, t-\frac{D}{m}\right) \right)^2 dx$$

$$\quad (25)$$

where $\eta = col\{e^1(x,t), e_t^1(x,t), e^1(x, t-\frac{D}{m})\}$. Since Φ_ϕ is affine in ϕ, then under (18)

$$\int_0^l \eta^T \Phi_\phi \eta dx \leq 0. \quad (26)$$

From this we also deduce

$$\dot{V}^1(t) + 2\delta V^1(t) - \delta_1 V^1\left(t - \frac{D}{m}\right) \leq \int_0^l \eta^T \Phi_\phi \eta dx$$

$$+ \left(\frac{\Delta}{\pi} L R^{-1}(p_3 + p_2) - \delta_1 \right) \int_0^l \left(e_x^1\left(x, t-\frac{D}{m}\right) \right)^2 dx \quad (27)$$

Then we conclude under conditions of Theorem 1, that

$$\dot{V}^1(t) + 2\delta V^1(t) - \delta_1 V^1\left(t - \frac{D}{m}\right) \leq 0 \quad (28)$$

Consider further the observation error equations (11) with $k = 2, \ldots, m$. The only difference between the above system and the one of the case $k = 1$ is in the disturbing term $\int_0^{\bar{x}_j} e_x^{k-1}(x,t) dx$. So, under under the strict LMIs (18), the Lyapunov–Krasovskii functional

$$V^k(t) = p_1 \int_0^l \left(e^k(x,t)\right)^2 dx + p_3 \int_0^l \left(e_x^k(x,t)\right)^2 dx$$

$$+ g \int_0^l \left[\int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} \left(e^k(x,s)\right)^2 ds \right] dx$$

$$+ \frac{D}{m} r \int_0^l \left[\int_{-\frac{D}{m}}^0 \int_{t+\theta}^t e^{2\delta(s-t)} \left(e_s^k(x,s)\right)^2 ds d\theta \right] dx,$$

$$k = 2, \ldots, m$$

satisfies the following inequality:

$$\dot{V}^k(t) + (2\delta - \epsilon\gamma^2) V^k(t) - \delta_1 V^k\left(t - \frac{D}{m}\right) - \gamma^2 V^{k-1}(t) \leq 0 \quad (29)$$

along (11), where γ^2 is large enough and $\epsilon > 0$ is small enough subject to

$$2\delta - \epsilon\gamma^2 > \delta_1. \quad (30)$$

Similar to [21], consider next the following Lyapunov–Krasovskii functional for the augmented system (10), (11):

$$V(t) = \sum_{k=1}^m \epsilon^{k-1} V^k(t). \quad (31)$$

Then multiplying (29) by ϵ^{k-1} and summing with (28) we arrive at

$$\dot{V}(t) + (2\delta - \epsilon\gamma^2) V(t) - \delta_1 \sup_{-h \leq s \leq 0} V(t+s)$$

$$\leq \dot{V}(t) + (2\delta - \epsilon\gamma^2) V(t) - \delta_1 V(t - \frac{D}{m}) \leq 0.$$

that due to (30) and Halanay's inequality implies the exponential convergence of V.

Remark 3: The LMIs in Theorem 1 depend on the fraction D/m. If they are feasible for $H_{\max} = D/m$, then choosing $m \geq D/H_{\max}$ we have always a feasible LMI. Then for each delay D, we can find a sufficiently large m such that the LMIs of the Theorem 1 are verified. *More precisely, we aim to provide LMIs that give max of (D/m) that guarantee exponential convergence. The maximum ration D/m is derived numerically by using Matlab LMIs Solvers. Then by enlarging the value of m it is obvious that the LMIs remain feasible.*

Remark 4: It's obvious that the spatial discretization step Δ is directly related to the number of sensors and the exponential convergence is proven using the Halanay's Lemma. This Lemma gives a nice estimation of the exponential decay rate which depends on parameters δ, δ_1, and h. We can easily see that for fixed h and δ, the decay rate α increases if δ_1 decreases. Furthermore, It is not difficult to see that δ_1 decreases if spatial discretization step Δ decreases or if the number of sensors increases. Then, we conclude that by increasing the number of sensors p, we can enlarge the decay rate α and consequently improve the quality of exponential convergence of the observation error.

IV. SAMPLED MEASUREMENTS CASE

In this section, we present the extension of the above observer to sampled-measurements case. In this case the output is available only at sampling instants t_i

$$0 = t_0 < t_1 < \cdots < t_i < \cdots, \lim_{k \to \infty} t_i = \infty.$$

We assume that the sampling intervals may be variable, but upper-bounded by a known bound h

$$t_{i+1} - t_i \leq h \forall i = 0, 1, \ldots$$

The proposed observer has the following form:
for $k = 1$

$$\hat{u}_t^1(x, t) = \hat{u}_{xx}^1(x, t) + f\left(\hat{u}^1(x, t), x, t + \frac{1}{m}D - D\right)$$

$$- L\left(\hat{u}^1\left(\bar{x}_j, t_i - \frac{D}{m}\right) - y(t_i)\right)$$

$$\forall t \in [t_i, t_{i+1}), \forall x \in [x_j, x_{j+1}) \quad (32)$$

for $k = 2, \ldots, m$

$$\hat{u}_t^k(x, t) = \hat{u}_{xx}^k(x, t) + f\left(\hat{u}^k(x, t), x, t + \frac{k}{m}D - D\right)$$

$$- L(\hat{u}^k\left(\bar{x}_j, t - \frac{D}{m}\right) - \hat{u}^{k-1}(\bar{x}_j, t))$$

$$\forall t \in [t_i, t_{i+1}), \forall x \in [x_j, x_{j+1}). \quad (33)$$

Theorem 2: Given D, h and m, consider the system (3) and the observer (32), (33). Given positive constants scalars $\Delta, \delta, L > M_f - \frac{\pi^2}{l^2}, R$, and δ_1 such that $2\delta > \delta_1$, let there exist positive scalars p_1, p_2, p_3, r, W, and g such that

$$\delta p_3 < p_2; \frac{\Delta}{\pi}LR^{-1}(p_3 + p_2) < \delta_1 p_3 \quad (34)$$

and

$$\Phi_{m_f} < 0 \qquad \Phi_{M_f} < 0 \quad (35)$$

where

$$\Phi_\phi = \begin{pmatrix} \Phi_{11} - \lambda & \Phi_{12} & \Phi_{13} & p_2 L \\ \Phi_{12} & \Phi_{22} + Wh^2 e^{2\delta h} & \Phi_{23} & p_3 L \\ \Phi_{13} & \Phi_{23} & \Phi_{33} & 0 \\ p_2 L & p_3 L & 0 & -W\frac{\pi^2}{4} \end{pmatrix} \quad (36)$$

with $\Phi_{11}, \Phi_{12}, \Phi_{13}, \Phi_{22}, \Phi_{23}$, and Φ_{33} given by (20).

Then all the observation errors $\int_0^1 \left(e^k(x, t)\right)^2 dx$ and $\int_0^1 \left(e_x^k(x, t)\right)^2 dx$ $(k = 1, \ldots, m)$ globally exponentially decay to zero as $t \to +\infty$.

Proof: The observation error is described by the following equations:
for $k = 1$

$$e_t^1(x, t) = e_{xx}^1(x, t) + \phi(x, t, e^1)\left(\hat{u}^1(x, t) - u^1(x, t)\right)$$

$$- Le^1\left(\bar{x}_j, t_i - \frac{D}{m}\right)$$

$$\forall t \in [t_i, t_{i+1}), \qquad \forall x \in [x_j, x_{j+1})$$

$$e^1(l, t) = e^1(0, t) = 0 \quad (37)$$

for $k = 2, \ldots, m$

$$e_t^k(x, t) = e_{xx}^k(x, t) + \phi(x, t, e^k)\left(\hat{u}^k(x, t) - u^k(x, t)\right)$$

$$- Le^k\left(\bar{x}_j, t - \frac{D}{m}\right) + Le^{k-1}(\bar{x}_j, t)$$

$$\forall t \in [t_i, t_{i+1}), \qquad \forall x \in [x_j, x_{j+1})$$

$$e^k(l, t) = e^k(0, t) = 0 \quad (38)$$

As we can easily see, the unique difference with the observer without sampling measurements is for the first subobserver $(k = 1)$. In order to study the convergence of the case $k = 1$, we use the following modified Lyapunov–Krasvoskii inspired from [8], [15]:

$$V_W^1(t) = V^1(t) + W^1(t)$$

$$W^1(t) = Wh^2 e^{2\delta h} \int_0^l \int_{t_i - \frac{D}{m}}^t e^{2\delta(s-t)} \left(e_s^1(x, s)\right)^2 dsdx$$

$$- \frac{\pi^2}{4} W \int_0^l \int_{t_i - \frac{D}{m}}^{t - \frac{D}{m}} e^{2\delta(s-t)} \left[e^1(x, s) - e^1\left(x, t_i - \frac{D}{m}\right)\right]^2 dsdx. \quad (39)$$

By generalized Wirtinger's inequality [15], we deduce W^1 is nonnegative and does not grow in the jumps [15]. Moreover

$$\dot{W}^1(t) + 2\delta W^1(t) \leq h^2 e^{2\delta h} \int_0^l \left(e_t^1(x, s)\right)^2 dx$$

$$- \frac{\pi^2}{4} \int_0^l \left[e^1(x, t - \frac{D}{m}) - e^1(x, t_i - \frac{D}{m})\right]^2 dx.$$

Adding the latter inequality to (28) we conclude that under conditions of Theorem 2 $\forall t \in [t_i, t_{i+1})$ the following inequality holds:

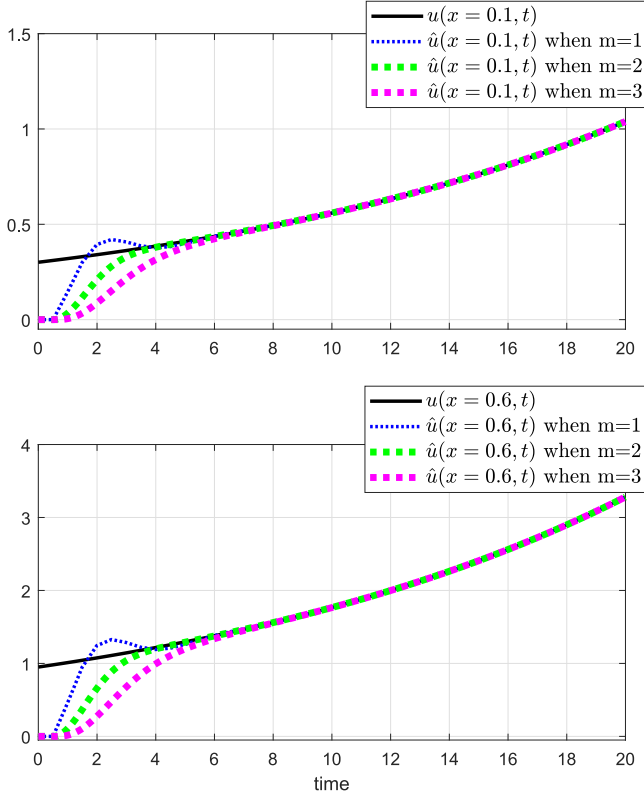$$\dot{V}_W^1(t) + 2\delta V_W^1(t) - \delta_1 V_W^1\left(t_i - \frac{D}{m}\right) \leq 0 \quad (40)$$

Fig. 1.   State $u(x,t)$ and its observations for $m = 1$ to $m = 3$ at $x = 0.1$ and $x = 0.6$ for a delay $D = 0.5$ s and sampling period $h = 0.5$ s.

For $k \geq 2$ the observation error is described by the following equation:

$$e_t^k(x,t) = e_{xx}^k(x,t) + \phi(x,t,e^k)e^k(x,t)$$
$$- Le^k\left(\bar{x}_j, t - \frac{D}{m}\right) + L\int_0^{\bar{x}_j} e_x^{k-1}(x,t)dx,$$
$$\forall x \in [x_j, x_{j+1})$$
$$e^k(l,t) = e^k(0,t) = 0, \tag{41}$$

As we can see the systems (41) and (11) are identical. Then, the exponential convergence of the observer (32), (33) can be proved by using arguments of Theorem 1 with V^1 changed by V_W^1. ∎

V. NUMERICAL ILLUSTRATION

Let us consider the following example:

$$u_t = u_{xx}(x,t) + 1.02\,\pi^2 u(x,t) \tag{42}$$

with $u(x,0) = sin(x)$ and let $y_j(t_k) = u(\bar{x}_j, t_k - D)$, $j = 1, \ldots, N-1$, where D is an arbitrarily delay and $\hat{u}(x,0) = 0$. It has to be noticed that the above system is unstable. It's well known that generally it is more difficult to estimate the states of unstable systems rather than stable systems because if the considered system is stable, then we can construct a state estimator without using the measurements. This is not the case for unstable systems where the delectability is required, and the correction term containing the measurements is necessary to ensure the convergence of the observation error.

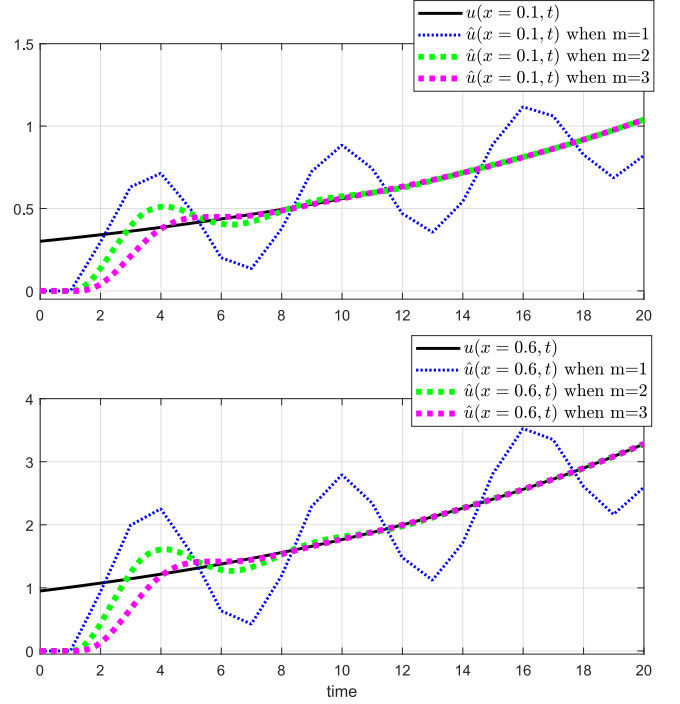We choose $L = 1$, $\Delta = \frac{1}{50}$, $\delta = 0.21$, and $\delta_1 = 0.1, l = 1, R = 1$.

Fig. 2.   State $u(x,t)$ and its observations for $m = 1$ to $m = 3$ at $x = 0.1$ and $x = 0.6$ for a delay $D = 1$ s and sampling period $h = 1$ s.
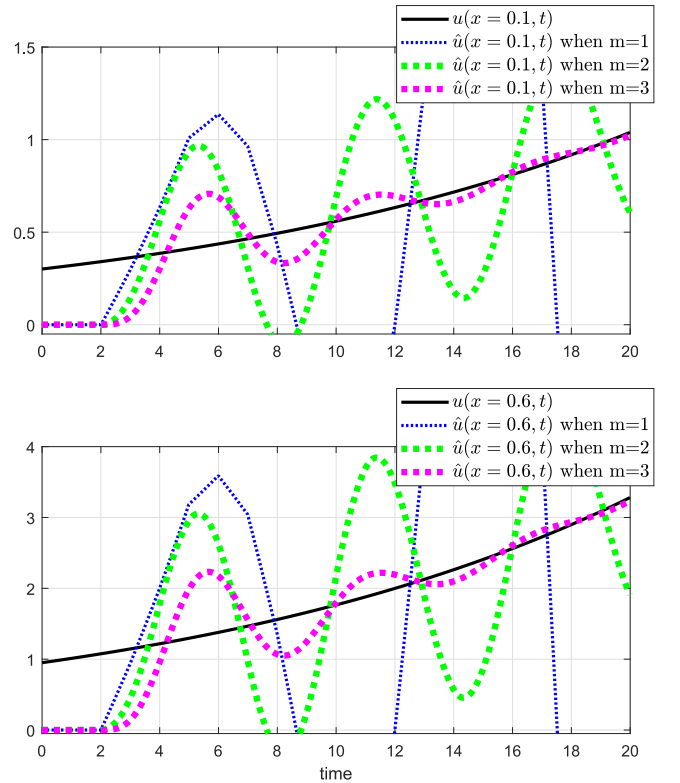

Fig. 3.   State $u(x,t)$ and its observations for $m = 1$ to $m = 3$ at $x = 0.1$ and $x = 0.6$ for a delay $D = 2$ s and sampling period $h = 1$ s.
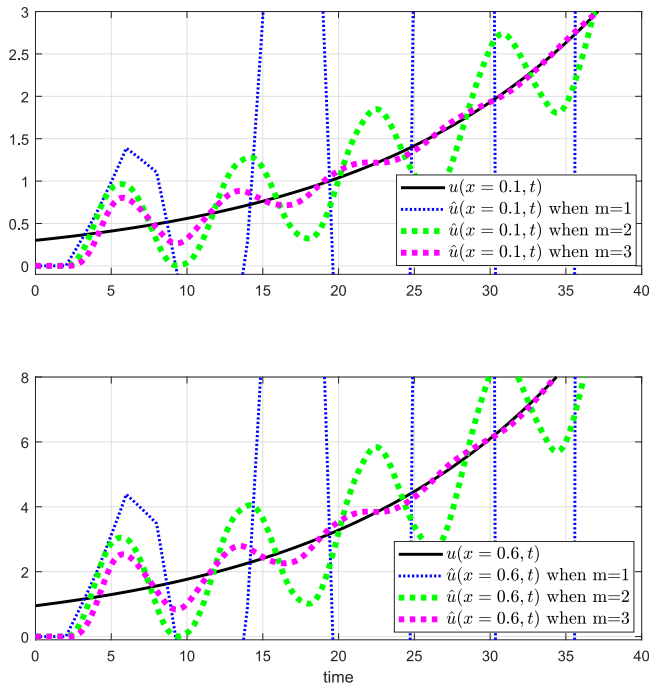
Fig. 4. State $u(x, t)$ and its observations for $m = 1$ to $m = 3$ at $x = 0.1$ and $x = 0.6$ for a delay $D = 2$ s and sampling period $h = 2$ s.

By solving LMIs of Theorem 2, we deduce the following table which illustrates the maximum value of the ratio D/m for different values of h.

h	0.1	0.25	0.5	0.57
$\frac{D}{m}$	0.37	0.28	0.11	0.056

The simulations (Figs. 1–4) show clearly that the real values of the number (m) of required subobservers in the chain is less conservative than the one provided by LMIs.

For instance, in (Fig. 1), we can see that for $D = 0.5$ s and $h = 0.5$ s only $m = 1$ observer is required to ensure exponential convergence whereas the LMIs indicate that $m = 5$. This result confirms the fact that the LMIs are only sufficient conditions.

VI. CONCLUSION

In this paper, a novel observer is proposed for a class of parabolic systems with delayed and sampled measurements. The main advantage provided by this algorithm is that it can handle arbitrary delay with a sufficiently small maximum allowable sampling period. it has to be noticed that this method can be easily extended to several important classes of infinite-dimensional systems such that wave equation. Our main challenge in the future is the extension to adaptive observers case.

REFERENCES

[1] F. Cacace, A. Germani, and C. Manes, "Nonlinear systems with multiple time-varying measurement delays," *SIAM J. Control Optim.*, vol. 52, no. 3, pp. 1862–1885, 2014.

[2] A. Germani, C. Manes, and P. Pepe, "A new approach to state observation of nonlinear systems with delayed output," *IEEE Trans. Autom. Control*, vol. 47, no. 1, pp. 96–101, Jan. 2002.

[3] T. Ahmed-Ali, E. Cherrier, and F. Lamnabhi-Lagarrigue, "Cascade high predictors for a class of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 57, no. 1, pp. 221–226, Jan. 2012.

[4] G. Besançon, D. Georges, and Z. Benayache, "Asymptotic state prediction for contiunous-time systems with delayed input and application to control," in *Proc. Eur. Control Conf.*, 2007, pp. 2–5.

[5] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Cambridge, MA, USA: Birkhauser, 2009.

[6] T. Ahmed-Ali, F. Giri, and M. Krstic, and M. Kahelras, "PDE based observer design for nonlinear systems with large output delay," *Syst. Control Lett.*, vol. 113, pp. 1–8, 2018.

[7] E. Fridman and A. Blighovsky, "Robust sampled-data control of a class of semilinear parabolic systems," *Automatica*, vol. 48, pp. 826–836, 2012.

[8] K. Liu and E. Fridman, "Wirtinger inequality and Lyapunov-based sampled-data stabilization," *Automatica*, vol. 48, pp. 102–108, 2012.

[9] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*. New York, NY, USA: Academic, 1966.

[10] T. Ahmed-Ali, E. Fridman, F. Giri, L. Burlion, and F. Lamnabhi-Lagarrigue, "Using exponential time-varying gains for sampled-data stabilization and estimation," *Automatica*, vol. 67, pp. 244–251, 2016.

[11] M. Tucsnakand and G. Weiss, *Observation and Control for Operator Semigroups*. Cambridge, MA, USA: Birkhauser, 2009.

[12] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*. New York, NY, USA: Springer-Verlag, 1993.

[13] W. Kang and E. Fridman, "Boundary constrained control of delayed nonlinear schrodinger equation," *IEEE Trans. Autom. Control*, vol. 63, no. 11, pp. 3873–3880, Nov. 2018.

[14] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control. Systems and Control: Foundations and Applications*. Cambridge, MA, USA: Birkhauser, 2014.

[15] A. Selivanov and E. Fridman, "Observer-based input-to-state stabilization of networked control systems with large uncertain delays," *Automatica*, vol. 74, pp. 63–70, 2016.

[16] T. Ahmed-Ali, E. Fridman, F. Giri, M. Kahelras, F. Lamnabhi-Lagarrigue, and L. Burlion, "Observer design for a class of parabolic systems with arbitrarily delay measurements," in *Proc. IEEE Conf. Decis. Control*, Miami, USA, 2018, pp. 2199–2204.

[17] E. Fridman, "Observers and initial state recovering for a class of hyperbolic systems via Lyapunov method," *Automatica*, vol. 49, no. 7, pp. 2250–2260, 2013.

[18] P. D. Christofides, *Nonlinear and Robust Control of PDE Systems* (Systems & Control: Foundations & Applications). Boston, MA, USA: Birkhaüser, 2001.

[19] R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, vol. 21, New York, NY, USA: Springer, 1995.

[20] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44. New York, NY, USA: Springer 1983.

[21] E. Fridman, C. Bonnet, F. Mazenc, and W. Djema, "Stability of the cell dynamics in acute myeloid leukemia," *Syst. Control Lett.*, vol. 88, pp. 91–100, 2016.

[22] T. Ahmed-Ali, I. Karafyllis, and F. Lamnabhi-Lagarrigue, "Global exponential sampled-data observers for nonlinear systems with delayed measurements," *Syst. Control Lett.*, vol. 62, no. 7, pp. 539–549, 2013.

[23] I. Karafyllis and M. Krstic, *Predictor Feedback for Delay Systems: Implementations and Approximations* (Systems and Control: Foundations Applications). Boston, MA, USA: Birkhauser, 2017.