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On delay-dependent conditions of ISS for generalized Persidskii systems

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***Abstract*— In this paper, input-to-state stability and stabilization conditions in time-delay generalized Persidskii systems are studied. These conditions are formulated in terms of linear matrix inequalities, which may depend on delay values, and be local or global in state space. Numerical examples of opinion dynamics and the Lotka-Volterra model illustrate the efficiency of the proposed results.**

I. Introduction

Studying the stability of nonlinear dynamical systems is a difficult problem, particularly in the presence of external perturbations [1]–[3]. The framework of input-to-state stability (ISS) [4], [5] is among the most general methods for stability analysis of perturbed systems. It quantifies the boundedness and the convergence of the state or the output of a nonlinear dynamical system with essentially bounded inputs. These stability properties can be verified by finding the corresponding Lyapunov functions [5]. However, the main drawback of applying the ISS theory is the lack of methodology for assigning Lyapunov functions to a generic nonlinear system. The inclusion of time delays implies utilizing even more complex stability analysis approaches, *e.g.*, using Lyapunov–Krasovskii functionals or Lyapunov-Razumikhin functions [6]. Moreover, the appearance of lags usually requires a redesign of regulation or estimation algorithms since time delays may degrade the performance or even lead to instability of the system.

Most existing approaches for synthesizing Lyapunov functions in nonlinear dynamics involve various canonical forms, such as Lur'e systems [7], Lipschitz dynamics, homogeneous models [8], and Persidskii systems [9]. In this paper, we focus on the latter case that has been extensively studied in the context of neural networks [10] and power systems [11], for instance. Such a choice is motivated by the existence of several canonical forms of Lyapunov functions for a Persidskii system [9], [12], [13]. In addition, due to recent advancements in [14], [15], the ISS and input-to-output stability (IOS) conditions are constructively formulated using linear matrix inequalities (LMIs). Our main goal is to extend these results to the delay-dependent conditions of input-to-state stability and stabilization for generalized Persidskii systems with time delays [a] based on the Lyapunov–Krasovskii functional approach, whose form is proposed specifically for the considered class of models. Numerical experiments are presented to illustrate the method.

The organization of this article is as follows. In Section II, the preliminaries are presented. The system under consideration and the problem statement are described in Section III. Section IV presents the ISS conditions of the considered systems, followed by a stabilization feedback design in Section V. In Section VI, we give examples to demonstrate the efficiency of the proposed results.

Notation

- The sets of natural, real and nonnegative real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The symbol $|\cdot|$ corresponds to the Euclidean norm on $\mathbb{R}^n$ (and the induced matrix norm $|A|$ for a matrix $A \in \mathbb{R}^{m\times n}$).
- For $p,\ n \in \mathbb{N}$ with $p \le n$, the notation $\overline{p,n}$ is used to represent the set $\{p, p+1, \ldots, n\}$. For all $i, j \in \overline{p,n}$, let $(B_{i,j})_{i,j=p}^n$ denote the block matrix $\begin{bmatrix} B_{p,p} & \cdots & B_{p,n} \\ \vdots & \ddots & \vdots \\ B_{n,p} & \cdots & B_{n,n} \end{bmatrix}$.
- The set of $n\times n$ diagonal matrices with nonnegative elements on the diagonal is denoted by $\mathbb{D}_+^n$. Also, I_n, $\mathbb{O}_{p\times n}$ and $\mathbf{1}_n$ stand for the $n\times n$ identity matrix, the $p\times n$ zero matrix and the n-dimensional vector with all elements equal 1, respectively. The Kronecker product is denoted by $\otimes$. Let $\mathbb{1}_{\mathcal{A}} : \mathcal{X} \to \{0,1\}$ denote the indicator function of a subset $\mathcal{A}$ of a set $\mathcal{X}$.
- For a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$, we use $\frac{\partial F(x)}{\partial x}$ to denote the Jacobian matrix of F at a point $x \in \mathbb{R}^n$.
- For $[t_1,\ t_2] \subset \mathbb{R}$ we denote by $C^n_{[t_1,t_2]}$ the Banach space of continuous functions $\psi : [t_1,t_2] \to \mathbb{R}^n$ with the norm $\|\psi\|_{[t_1,t_2]} = \sup_{t_1\le r\le t_2} |\psi(r)|$. For a Lebesgue measurable function $d\colon [t_1,t_2) \to \mathbb{R}^m$, define the norm $\|d\|_{[t_1,t_2)} = \operatorname{ess\,sup}_{t\in[t_1,t_2)} |d(t)|$ for $[t_1,t_2) \subset \mathbb{R}_+$. We denote by $\mathcal{L}^m_{[t_1,t_2)}$ (or $\mathcal{L}^m_\infty$) the Banach space of functions d with $\|d\|_{[t_1,t_2)} < +\infty$ (or $\|d\|_\infty := \|d\|_{[0,\infty)} < +\infty$). Denote by $\mathbb{W}_{[t_1,t_2]}$ the Banach (or Sobolev) space of absolutely continuous functions $\phi : [t_1,t_2] \to \mathbb{R}^n$ with the norm $\|\phi\|_{\mathbb{W}_{[t_1,t_2]}} := \|\phi\|_{[t_1,t_2]} + \|\dot\phi\|_{[t_1,t_2]} < +\infty$, where $\dot\phi(\ell) = \frac{\partial\phi(\ell)}{\partial\ell}$, $\ell \in [t_1,t_2] \subset \mathbb{R}$.
- A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also unbounded. A continuous function $\beta : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot,r) \in \mathcal{K}$ and $\beta(r,\cdot)$ is decreasing to zero for any fixed $r > 0$.

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[a] In the preliminary version of this research [16], only the delay-independent stability conditions have been obtained, illustrated by an academic example.

II. Preliminaries

Consider a nonlinear retarded system [17], [18]:

$$\dot{x}(t) = f(x_t, d(t)),\ t \in \mathbb{R}_+, \tag{1}$$

where $x(t) \in \mathbb{R}^n$; $x_t \in \mathbb{W}_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$, $\tau > 0$ is a constant delay; $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}_\infty^m$; $f : \mathbb{W}_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous functional, $f(0,0) = 0$, and it ensures the existence and the uniqueness of solutions in forward time for the system (1). With the initial condition $x_0 \in \mathbb{W}_{[-\tau,0]}$ and the input $d \in \mathcal{L}_\infty^m$, such a unique solution is defined as $x(t, x_0, d)$, for which $x_t(s, x_0, d) = x(t+s, x_0, d)$, $s \in [-\tau, 0]$ denotes the corresponding state function.

For a continuous functional $V : \mathbb{R} \times \mathbb{W}_{[-\tau,0]} \times C^n_{[-\tau,0]} \to \mathbb{R}_+$, we define the following derivative along the solutions of (1) [19]:

$$D^+V(t,\phi,\ell) = \limsup_{h\to 0^+} \frac{V(t+h, x_h(\phi,\ell), \dot{x}_h(\phi,\ell)) - V\left(t,\phi,\dot{\phi}\right)}{h},$$

$$x_h(\phi,\ell)(s) = \begin{cases} \phi(s+h), & s \in [-\tau,-h] \\ \phi(0) + (h+s)\cdot f(\phi,\ell), & s \in [-h, 0] \end{cases}$$

for any $\phi \in \mathbb{W}_{[-\tau,0]}$ and $\ell \in \mathbb{R}^m$.

Definition 1. *[18], [20] The system* (1) *is called* input-to-state stable *(ISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that*

$$|x(t,x_0,d)| \le \beta(\|x_0\|_{\mathbb{W}_{[-\tau,0]}}, t) + \gamma(\|d\|_{[0,t)}), \quad \forall t \in \mathbb{R}_+$$

for all $x_0 \in \mathbb{W}_{[-\tau,0]}$ and $d \in \mathcal{L}_\infty^m$.

Definition 2. *[18], [20] The system* (1) *is said to possess the* asymptotic gain (AG) property, *if there exists $\gamma \in \mathcal{K}$ such that $\limsup_{t\to+\infty} |x(t,x_0,d)| \le \gamma(\|d\|_\infty)$ for all $x_0 \in \mathbb{W}_{[-\tau,0]}$ and $d \in \mathcal{L}_\infty^m$.*

Definition 3. *[18], [20] A continuous functional $V : \mathbb{R} \times \mathbb{W}_{[-\tau,0]} \times C^n_{[-\tau,0]} \to \mathbb{R}_+$ is called an* ISS Lyapunov-Krasovskii functional *(LKF) if there exist some α_1, $\alpha_2 \in \mathcal{K}_\infty$, α_3, $\chi \in \mathcal{K}$ such that*

$$\alpha_1(|\phi(0)|) \le V\left(t,\phi,\dot{\phi}\right) \le \alpha_2\left(\|\phi\|_{\mathbb{W}_{[-\tau,0]}}\right),$$

$$V\left(t,\phi,\dot{\phi}\right) \ge \chi(|d|) \quad \Rightarrow \quad D^+V(t,\phi,d) \le -\alpha_3\left(\|\phi\|_{\mathbb{W}_{[-\tau,0]}}\right)$$

for all $t \in \mathbb{R}_+$, $\phi \in \mathbb{W}_{[-\tau,0]}$ and $d \in \mathbb{R}^m$.

Theorem 1. *[18], [20] If the system* (1) *admits an ISS LKF, then it is ISS with $\gamma = \alpha_1^{-1} \circ \chi$.*

The existence of an LKF can also be necessary for ISS property under additional restrictions on continuity of f in (1) [21], [22], for instance, the function f is required to be Lipschitz on bounded sets in [23], [24].

Remark 1. *In many cases, it is technically proficient to use $\dot{x}_t$ as an argument of LKF (see,* e.g., *[22], [25]), and in such a situation, the stability of the system is analyzed in a Banach space $\mathbb{W}_{[-\tau,0]}$. Nevertheless, for example, in [16] $\dot{x}_t$ is excluded, then the system* (1) *and definitions 1, 2, 3 save their meaning after substitution of $C^n_{[-\tau,0]}$, $\|\phi\|_{[-\tau,0]}$ in place of $\mathbb{W}_{[-\tau,0]}$, $\|\phi\|_{\mathbb{W}_{[-\tau,0]}}$.*

III. Problem Statement

The main goal of this paper is to consider the input-to-state stability of a class of nonlinear systems in generalized Persidskii form with constant time delays

$$\begin{aligned} \dot{x}(t) \;=\;& A_0 x(t) + \sum_{j=1}^{M} A_j F_j(x(t)) \\ &+ B_0 x(t-\tau_0) + \sum_{j=1}^{M} B_j F_j(x(t-\tau_j)) + d(t),\ t \in \mathbb{R}_+ \end{aligned} \tag{2}$$

with $x(t) = [x_1(t) \dots\ x_n(t)]^\top \in \mathbb{R}^n$ is the current value of the state; $0 < \tau_s < +\infty$ are constant delays for $s \in \overline{0,M}$, $\tau = \max_{s\in\overline{0,M}} \tau_s$; $A_s, B_s \in \mathbb{R}^{n\times n}$ for $s \in \overline{0,M}$; the functions $F_j : \mathbb{R}^n \to \mathbb{R}^n$ have diagonal structure, $F_j(x) = [f_j^1(x_1) \dots f_j^n(x_n)]^\top$, for $j \in \overline{1,M}$, and ensure the existence of the solutions of the system (2) in forward time, at least locally; $d(t) \in \mathbb{R}^n$ is the external perturbation/input, $d \in \mathcal{L}_\infty^n$.

In this work, if the upper bound of an index is smaller than the lower one, then the corresponding term in a sum or a sequence must be omitted.

Sector conditions on F_j, $j \in \overline{1,M}$ are imposed for the forthcoming analysis:

Assumption 1. *For any $i \in \overline{1,n}$ and $j \in \overline{1,M}$,*

$$\nu f_j^i(\nu) > 0,\ \forall \nu \in \mathbb{R}\backslash\{0\}.$$

Under Assumption 1, with a reordering of nonlinearities and their decomposition, there exists an index $\varpi \in \overline{0,M}$ such that for all $i \in \overline{1,n}$, $a \in \overline{1,\varpi}$

$$\lim_{\nu\to\pm\infty} f_a^i(\nu) = \pm\infty,$$

and there exists $\mu \in \overline{\varpi,M}$ such that for all $i \in \overline{1,n}$, $b \in \overline{1,\mu}$

$$\lim_{\nu\to\pm\infty} \int_0^\nu f_b^i(s)ds = +\infty.$$

In this case, $\varpi = 0$ implies that all nonlinearities are bounded (at least for negative or positive argument).

Assumption 2. *For any $i \in \overline{1,n}$ and j, $j' \in \overline{1,M}$, $z \in \overline{j+1,M}$, there exist $\eta^i_{0,j}$, $\eta^i_{1,jj'}$, $\eta^i_{2,jj'}$, $\eta^i_{3,jj'z} \ge 0$ such that*

$$\begin{aligned} 2\int_0^{\mathrm{x}_i} f_j^i(s)ds \;\le\;& \eta^i_{0,j}\mathrm{x}_i^2 + \sum_{j'=1}^{M} f^i_{j'}(\mathrm{x}_i)\Bigg(\eta^i_{1,jj'} f^i_{j'}(\mathrm{x}_i) \\ &+ 2\eta^i_{2,jj'}\mathrm{x}_i + 2\sum_{z=j+1}^{M} \eta^i_{3,jj'z} f^i_z(\mathrm{x}_i)\Bigg) \end{aligned}$$

for all $\mathrm{x} \in \mathbb{R}^n$.

Assumptions 1 and 2 are satisfied by many nonlinear functions: for polynomial ones, for example, it is sufficient to select $\eta^i_{2,jj'} \neq 0$. In the sequel, we denote the diagonal matrices:

$$\eta_{0,j} = \mathrm{diag}(\eta^1_{0,j}, \dots, \eta^n_{0,j}),\ \eta_{1,jj'} = \mathrm{diag}(\eta^1_{1,jj'}, \dots, \eta^n_{1,jj'}),$$

$$\eta_{2,jj'} = \mathrm{diag}(\eta^1_{2,jj'}, \dots, \eta^n_{2,jj'}),\ \eta_{3,jj'z} = \mathrm{diag}(\eta^1_{3,jj'z}, \dots, \eta^n_{3,jj'z}).$$

IV. ISS ANALYSIS

Our goal is to propose constructive conditions for verifying the ISS property of (2). This system is highly nonlinear, with multiple delays appearing in linear and nonlinear parts. The following theorem is the main result of this paper, which formulates delay-dependent conditions based on a special ISS-LKF extending the previous results of [16].

Theorem 2. *Let assumptions 1 and 2 be satisfied and for given constants* $0 < w_0$, $0 < p_s < \delta_s$ $(s \in \overline{0,M})$ *and* $\rho \in \mathbb{R}$ *there exist* $0 \le P = P^\top$, $\Phi = \Phi^\top \in \mathbb{R}^{n\times n}$*; sets* $\{R_s\}_{s=0}^M$, $\{S_s\}_{s=0}^M$, $\{\Xi_s\}_{s=0}^M \subset \mathbb{R}^{n\times n}$ *of symmetric nonnegative definite matrices;* $\{\Omega_j\}_{j=1}^M \subset \mathbb{R}^{n\times n}$*;* P_2, P_3, $P_4 \in \mathbb{R}^{n\times n}$*;* $\left\{\Lambda_j = \mathrm{diag}(\Lambda_j^1,\dots,\Lambda_j^n)\right\}_{j=1}^M$, $\left\{\Upsilon_{s,r}\right\}_{0\le s<r\le M} \subset \mathbb{D}_+^n$ *such that*

$$P + \rho\sum_{j=1}^{\mu}\Lambda_j > 0, \tag{3}$$

$$Q = Q^\top = (Q_{a,b})_{a,b=1}^6 \le 0, \tag{4}$$

$$\mathbb{1}_{\{0\}}(s)\cdot\Xi^0 + \mathbb{1}_{\overline{1,M}}(s)\cdot\Xi^s \ge \xi\Bigg[\mathbb{1}_{\{0\}}(s)\Bigg(P + \sum_{j=1}^M \Lambda_j\eta_{0,j}\Bigg) + \mathbb{1}_{\overline{1,M}}(s)\cdot\Lambda_s\sum_{j'=1}^M \eta_{1,sj'}\Bigg], \tag{5}$$

$$\mathbb{1}_{\{0\}}(s)\cdot\Upsilon_{0,j} + \mathbb{1}_{\overline{1,M}}(s)\cdot\Upsilon_{s,z} \ge \xi\Bigg[\mathbb{1}_{\{0\}}(s)\cdot\Lambda_j\sum_{j'=1}^M \eta_{2,jj'} + \mathbb{1}_{\overline{1,M}}(s)\cdot\Lambda_s\sum_{j'=1}^M \eta_{3,sj'z}\Bigg],$$

$$s \in \overline{0,M},\ j \in \overline{1,M},\ z \in \overline{s+1,M}$$

for some $\xi \in \left(0, \min\left(w_0, \min_{s\in\overline{0,M}}\left(\frac{\delta_s - p_s}{\delta_s\tau_s}\right)\right)\right]$*, where*

$$Q_{1,1} = A_0^\top P_2 + P_2^\top A_0 + S_0 + \Xi^0 - p_0R_0;\ Q_{1,2} = P - P_2^\top + A_0^\top P_3,$$
$$Q_{1,3} = P_2^\top B_0 + p_0R_0 + A_0^\top P_4,$$
$$Q_{1,4} = P_2^\top A + A_0^\top\Omega + \begin{bmatrix}\Upsilon_{0,1} & \dots & \Upsilon_{0,M}\end{bmatrix},$$
$$Q_{1,5} = P_2^\top B;\ Q_{1,6} = P_2^\top,$$
$$Q_{2,2} = -P_3 - P_3^\top + \delta_0\tau_0^2R_0 + \sum_{j=1}^M \delta_j\tau_j^2\frac{\partial F_j(x)^\top}{\partial x}R_j\frac{\partial F_j(x)}{\partial x},$$
$$Q_{2,3} = P_3^\top B_0 - P_4;\ Q_{2,4} = P_3^\top A - \Omega + \Lambda;\ Q_{2,5} = P_3^\top B;\ Q_{2,6} = P_3^\top,$$
$$Q_{3,3} = -e^{-w_0\tau_0}S_0 - p_0R_0 + 2P_4^\top B_0;\ Q_{3,4} = B_0^\top\Omega + P_4^\top A;$$
$$Q_{3,5} = P_4^\top B,\ Q_{3,6} = P_4^\top,\ Q_{4,4} = Q_{4,4}^\top = (\widehat{Q}_{a,b})_{a,b=1}^M,$$
$$\widehat{Q}_{j,j} = A_j^\top\Omega_j + \Omega_j^\top A_j + \Xi^j + S_j - p_jR_j,\ j \in \overline{1,M},$$
$$\widehat{Q}_{s,z} = A_s^\top\Omega_z + \Omega_s^\top A_z + \Upsilon_{s,z},\ s \in \overline{1,M-1}, z \in \overline{s+1,M},$$
$$Q_{4,5} = \Omega^\top B + J;\ Q_{4,6} = \Omega^\top,$$
$$Q_{5,5} = \mathrm{diag}(-e^{-w_0\tau_1}S_1,\dots,-e^{-w_0\tau_M}S_M) - J,$$
$$Q_{5,6} = \mathbb{O}_{nM\times n};\ Q_{6,6} = -\Phi,$$
$$A = \begin{bmatrix}A_1 & \dots & A_M\end{bmatrix};\ B = \begin{bmatrix}B_1 & \dots & B_M\end{bmatrix},$$
$$\Lambda = \begin{bmatrix}\Lambda_1 & \dots & \Lambda_M\end{bmatrix};\ \Omega = \begin{bmatrix}\Omega_1 & \dots & \Omega_M\end{bmatrix},$$
$$J = \mathrm{diag}(p_1R_1,\dots,p_MR_M).$$

Then the system (2) *is ISS.*

Proof. Our goal is to check the conditions in Definition 3 for a LKF taken as follows:

$$\begin{aligned} V(x_t,\dot{x}_t) &= x(t)^\top Px(t) + \int_{t-\tau_0}^t e^{-w_0(t-s)}x(s)^\top S_0x(s)ds \\ &+2\sum_{j=1}^M\sum_{i=1}^M \Lambda_j^i\int_0^{x_i(t)} f_j^i(s)ds \\ &+\sum_{j=1}^M\int_{t-\tau_j}^t e^{-w_0(t-s)}F_j(x(s))^\top S_jF_j(x(s))ds \\ &+\delta_0\tau_0\int_{-\tau_0}^0\int_{t+\theta}^t \dot{x}(s)^\top R_0\dot{x}(s)dsd\theta \\ &+\sum_{j=1}^M\delta_j\tau_j\int_{t-\tau_j}^t\int_s^t\left(\dot{x}(r)^\top\frac{\partial F_j(x(r))^\top}{\partial x}R_j\frac{\partial F_j(x(r))}{\partial x}\dot{x}(r)\right)drds, \end{aligned} \tag{6}$$

which verifies the required lower (due to (3) and Finsler's lemma [26]) and upper (since all matrices are nonnegative definite) bounds given in Definition 3. The time derivative of V for (2) admits the following representation by using the descriptor method from [6], [25]:

$$\begin{aligned} \dot{V}(t,x_t,\dot{x}_t) &= \dot{x}(t)^\top Px(t) + x(t)^\top P\dot{x}(t) \\ &- w_0\int_{t-\tau_0}^t e^{-w_0(t-s)}x(s)^\top S_0x(s)ds + x(t)^\top S_0x(t) \\ &- e^{-w_0\tau_0}x(t-\tau_0)^\top S_0x(t-\tau_0) + 2\dot{x}(t)^\top\sum_{j=1}^M\Lambda_jF_j(x(t)) \\ &- w_0\sum_{j=1}^M\int_{t-\tau_j}^t e^{-w_0(t-s)}F_j(x(s))^\top S_jF_j(x(s))ds \\ &+\sum_{j=1}^M F_j(x(t))^\top S_jF_j(x(t)) \\ &-\sum_{j=1}^M e^{-w_0\tau_j}F_j(x(t-\tau_j))^\top S_jF_j(x(t-\tau_j)) \\ &+\delta_0\tau_0^2\dot{x}(t)^\top R_0\dot{x}(t) - \delta_0\tau_0\int_{t-\tau_0}^t \dot{x}^\top(s)R_0\dot{x}(s)ds \\ &+\sum_{j=1}^M\delta_j\tau_j^2\dot{x}(t)^\top\frac{\partial F_j(x(t))^\top}{\partial x}R_j\frac{\partial F_j(x(t))}{\partial x}\dot{x}(t) \\ &-\sum_{j=1}^M\delta_j\tau_j\int_{t-\tau_j}^t\dot{x}(s)^\top\frac{\partial F_j(x(s))^\top}{\partial x}R_j\frac{\partial F_j(x(s))}{\partial x}\dot{x}(s)ds \\ &+2\Bigg[x(t)^\top P_2^\top + \dot{x}(t)^\top P_3^\top + \sum_{j=1}^M F_j(x(t))^\top\Omega_j^\top \\ &+ x(t-\tau_0)^\top P_4^\top\Bigg]\cdot\Bigg[A_0x(t) + \sum_{j=1}^M A_jF_j(x(t)) \\ &+ B_0x(t-\tau_0) + \sum_{j=1}^M B_jF_j(x(t-\tau_j)) + d(t) - \dot{x}(t)\Bigg] \end{aligned}$$

$$
= \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-\tau_0) \\ F_1(x(t)) \\ \vdots \\ F_M(x(t)) \\ F_1(x(t-\tau_1)) \\ \vdots \\ F_M(x(t-\tau_M)) \\ d(t) \end{bmatrix}^\top Q \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-\tau_0) \\ F_1(x(t)) \\ \vdots \\ F_M(x(t)) \\ F_1(x(t-\tau_1)) \\ \vdots \\ F_M(x(t-\tau_M)) \\ d(t) \end{bmatrix}
$$

$$
\begin{aligned}
&- x(t)^\top \Xi^0 x(t) - \sum_{j=1}^{M} F_j(x(t))^\top \Xi^j F_j(x(t)) - 2\sum_{j=1}^{M} x(t)^\top \Upsilon_{0,j} F_j(x(t)) \\
&- 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} F_s(x(t))^\top \Upsilon_{s,z} F_z(x(t)) \\
&- w_0 \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&- w_0 \sum_{j} \int_{t-\tau_j}^{t} e^{-w_0(t-s)} F_j(x(s))^\top S_j F_j(x(s)) ds \\
&- (\delta_0 - p_0)\tau_0 \int_{t-\tau_0}^{t} \dot{x}(s)^\top R_0 \dot{x}(s) ds \\
&- \sum_{j=1}^{M} (\delta_j - p_j)\tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial F_j(x(s))^\top}{\partial x} R_j \frac{\partial F_j(x(s))}{\partial x} \dot{x}(s) ds \\
&+ d(t)^\top \Phi d(t)
\end{aligned}
$$

$$
\begin{aligned}
\leq & - x(t)^\top \Xi^0 x(t) - \sum_{j=1}^{M} F_j(x(t))^\top \Xi^j F_j(x(t)) \\
& - 2\sum_{j=1}^{M} x(t)^\top \Upsilon_{0,j} F_j(x(t)) - 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} F_s(x(t))^\top \Upsilon_{s,z} F_z(x(t)) \\
& - w_0 \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
& - w_0 \sum_{j} \int_{t-\tau_j}^{t} e^{-w_0(t-s)} F_j(x(s))^\top S_j F_j(x(s)) ds \\
& - (\delta_0 - p_0)\tau_0 \int_{t-\tau_0}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds \\
& - \sum_{j=1}^{M} (\delta_j - p_j)\tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial F_j(x(s))^\top}{\partial x} R_j \frac{\partial F_j(x(s))}{\partial x} \dot{x}(s) ds \\
& + d(t)^\top \Phi d(t).
\end{aligned}
$$

Here the condition (4) and the Jensen's inequalities

$$
\begin{aligned}
&-p_0\tau_0 \int_{t-\tau_0}^{t} \dot{x}(s)^\top R_0 \dot{x}(s) ds \\
&\leq -[x(t) - x(t-\tau_0)]^\top \cdot p_0 R_0 \cdot [x(t) - x(t-\tau_0)],
\end{aligned}
$$

$$
\begin{aligned}
&-p_j\tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial F_j(x(s))^\top}{\partial x} R_j \frac{\partial F_j(x(s))}{\partial x} \dot{x}(s) ds \\
&\leq -\left[F_j(x(t)) - F_j(x(t-\tau_j))\right]^\top \cdot p_j R_j \cdot \left[F_j(x(t)) - F_j(x(t-\tau_j))\right]
\end{aligned}
$$

were utilized.

For $\xi \in \left(0, \min\left(w_0, \min_{s\in\overline{0,M}}\left(\frac{\delta_s - p_s}{\delta_s \tau_s}\right)\right)\right]$, we have

$$
\begin{aligned}
&\xi V(x_t, \dot{x}_t) = \xi \left(x(t)^\top P x(t) + 2\sum_{j=1}^{M}\sum_{i=1}^{n} \Lambda_j^i \int_0^{x_i(t)} f_j^i(s) ds \right) \\
&+ \xi \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&+ \xi \sum_{j=1}^{M} \int_{t-\tau_j}^{t} e^{-w_0(t-s)} F_j(x(s))^\top S_j F_j(x(s)) ds \\
&+ \xi \delta_0 \tau_0 \int_{-\tau_0}^{0} \int_{t+\theta}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds d\theta \\
&+ \xi \sum_{j=1}^{M} \delta_j \tau_j \int_{t-\tau_j}^{t} \int_{s}^{t} \left(\dot{x}(r)^\top \frac{\partial F_j(x(r))^\top}{\partial x} R_j \frac{\partial F_j(x(r))}{\partial x} \dot{x}(r) \right) dr ds \\
&\leq x(t)^\top \Xi^0 x(t) + \sum_{j=1}^{M} F_j(x(t))^\top \Xi^j F_j(x(t)) \\
&\quad + 2\sum_{j=1}^{M} x(t)^\top \Upsilon_{0,j} F_j(x(t)) + 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} F_s(x(t))^\top \Upsilon_{s,z} F_z(x(t)) \\
&\quad + w_0 \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&\quad + w_0 \sum_{j=1}^{M} \int_{t-\tau_j}^{t} e^{-w_0(t-s)} F_j(x(s))^\top S_j F_j(x(s)) ds \\
&\quad + (\delta_0 - p_0)\tau_0 \int_{t-\tau_0}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds \\
&\quad + \sum_{j=1}^{M} (\delta_j - p_j)\tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial F_j(x(s))^\top}{\partial x} R_j \frac{\partial F_j(x(s))}{\partial x} \dot{x}(s) ds
\end{aligned}
$$

due to the conditions (5), Assumption 2 and the relations

$$
\begin{aligned}
&\xi \delta_0 \tau_0 \int_{-\tau_0}^{0} \int_{t+\theta}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds d\theta \\
&\leq (\delta_0 - p_0)\tau_0 \int_{t-\tau_0}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds, \\
&\xi \delta_j \tau_j \int_{t-\tau_j}^{t} \int_{s}^{t} \left(\dot{x}(r)^\top \frac{\partial F_j(x(r))^\top}{\partial x} R_j \frac{\partial F_j(x(r))}{\partial x} \dot{x}(r) \right) dr ds \\
&\leq (\delta_j - p_j)\tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial F_j(x(s))^\top}{\partial x} R_j \frac{\partial F_j(x(s))}{\partial x} \dot{x}(s) ds.
\end{aligned}
$$

Under the restriction $\alpha(V) \geq d^\top \Phi d$ with $\alpha(s) = \frac{1}{2}\xi s$, it follows that

$$
\dot{V}(t, x_t, \dot{x}_t) \quad \leq -\tfrac{1}{2}\xi V(x_t, \dot{x}_t).
$$

By Definition 3 and Theorem 1, we can substantiate that system (2) is ISS as desired. □

Note that the ISS LKF (6) used for the proof of Theorem 2 depends explicitly on the delays τ_s for $s \in \overline{0,M}$ due to the presence of the last two terms. The delays also appear and play an important role in the matrix inequality (4) of Theorem 2, which is nonlinear (or state-dependent) due to the term $\frac{\partial F_j(x)^\top}{\partial x} R_j \frac{\partial F_j(x)}{\partial x} \in \mathbb{D}_+^n$ ($j \in \overline{1,M}$) of $Q_{2,2}$. For practical verification of the matrix inequality (4), the following nonrestrictive conditions can be imposed on these terms:

Assumption 3. *There exist the sets $\mathcal{X}_j \subseteq \mathbb{R}^n$, $j \in \overline{1,M}$ with $\bigcap_{j\in\overline{1,M}} \mathcal{X}_j = \mathcal{X} \neq \varnothing$ and the matrices $\{\tilde{R}_j\}_{j=1}^M \subset \mathbb{D}_+^n$ such that*

$$\mathcal{X}_j \subseteq \left\{ y \in \mathbb{R}^n \left| \frac{\partial F_j(y)^\top}{\partial y} R_j \frac{\partial F_j(y)}{\partial y} \leq \tilde{R}_j \right. \right\}.$$

In the case $\mathcal{X} = \mathbb{R}^n$, the term $\frac{\partial F_j(x)^\top}{\partial x} R_j \frac{\partial F_j(x)}{\partial x}$ is bounded by some $\tilde{R}_j \in \mathbb{D}_+^n$ for all $x \in \mathbb{R}^n$ and $j \in \overline{1,M}$, which is the case of bounded nonlinearities, *e.g.*, $F_j(x) = \tanh(x)$.

Denote by $Q^\dagger$ the block matrix Q from Theorem 2 under the substitutions $\frac{\partial F_j(x)^\top}{\partial x} R_j \frac{\partial F_j(x)}{\partial x} \to \tilde{R}_j$ for $j \in \overline{1,M}$.

Corollary 1. *If Assumption 3 with $\mathcal{X} = \mathbb{R}^n$ and the conditions of Theorem 2 under the substitution $Q \leq 0 \to Q^\dagger \leq 0$ are satisfied, then the system (2) is ISS.*

Proof. Note that under Assumption 3

$$Q^\dagger \leq 0 \quad \Rightarrow \quad Q \leq 0,$$

then the conditions of Corollary 1 imply that all counterparts in Theorem 2 are verified, and the conclusion follows. □

If $\mathcal{X} \subset \mathbb{R}^n$, then similarly local ISS property can be established for the initial conditions inside $\mathcal{X}$ and properly bounded inputs d.

V. Stabilization

In this section, we design a feedback control to stabilize a system as (2) and study the ISS property of the resulting closed-loop system.

Consider a variation of (2):

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{j=1}^M A_j F_j(x(t)) + B_0 x(t-\tau_0) \\ &\quad + \sum_{j=1}^M B_j F_j(x(t-\tau_j)) + Gu(t) + d(t), \end{aligned} \tag{7}$$

where all variables are defined as in (2), $G \in \mathbb{R}^{n\times q}$ and $u(t) \in \mathbb{R}^q$ is the control, which can be chosen for stabilization in the following general form:

$$\begin{aligned} u(t) &= K_{A,0} x(t) + \sum_{j=1}^M K_{A,j} F_j(x(t)) \\ &\quad + K_{B,0} x(t-\tau_0) + \sum_{j=1}^M K_{B,j} F_j(x(t-\tau_j)), \end{aligned} \tag{8}$$

where $K_{A,s}, K_{B,s} \in \mathbb{R}^{q\times n}$, $s \in \overline{0,M}$. Such a form of the control keeps the closed-loop system in the class of generalized Persidskii models:

$$\begin{aligned} \dot{x}(t) &= \widetilde{A}_0 x(t) + \sum_{j=1}^M \widetilde{A}_j F_j(x(t)) + \widetilde{B}_0 x(t-\tau_0) \\ &\quad + \sum_{j=1}^M \widetilde{B}_j F_j(x(t-\tau_j)) + d(t), \end{aligned} \tag{9}$$

where $\widetilde{A}_s = A_s + GK_{A,s}$, $\widetilde{B}_s = B_s + GK_{B,s}$ for $s \in \overline{0,M}$.

Remark 2. *In the case that $K_{A,s}$, $K_{B,s}$, $s \in \overline{0,M}$ are given, we can directly formulate the results to analyze the input-to-state stability of the closed-loop system (9):*

If all conditions of Theorem 2 are satisfied under the substitutions $A_s \to \widetilde{A}_s$, $B_s \to \widetilde{B}_s$ for $s \in \overline{0,M}$, then the system (9) is ISS;

If Assumption 3 with $\mathcal{X} = \mathbb{R}^n$ and all conditions of Theorem 2 are satisfied under the substitutions $A_s \to \widetilde{A}_s$, $B_s \to \widetilde{B}_s$ for $s \in \overline{0,M}$, and $Q \leq 0 \to Q^\dagger \leq 0$ ($Q^\dagger$ as in Corollary 1), then the system (9) is ISS.

By introducing additional mild hypotheses, we now state a theorem for designing the feedback gains $K_{A,s}$, $K_{B,s}$, $s \in \overline{0,M}$ that guarantee the ISS property of the system (9):

Theorem 3. *Let assumptions 1 and 2 be satisfied and let $0 < w_0$, $0 < p_s < \delta_s$ ($s \in \overline{0,M}$) be given constants. If there exist matrices $0 < \widetilde{P}, \overline{P} \in \mathbb{D}_+^n$; $\left\{\overline{R}_s\right\}_{s=0}^M$, $\left\{\overline{\Lambda}_j = \mathrm{diag}(\overline{\Lambda}_j^1, \ldots, \overline{\Lambda}_j^n)\right\}_{j=1}^M$, $\left\{\overline{\Upsilon}_{s,r}\right\}_{0\leq s<r\leq M}$, $\left\{\overline{S}_k\right\}_{k=0}^M$, $\left\{\overline{\Xi}^k\right\}_{k=0}^M \subset \mathbb{D}_+^n$; $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$ and $\{U_k\}_{k=0}^M$, $\{L_k\}_{k=0}^M \subset \mathbb{R}^{q\times n}$ such that*

$$\overline{Q} = \overline{Q}^\top = \left(\overline{Q}_{a,b}\right)_{a,b=1}^6 \leq 0,$$

$$\begin{aligned} &\mathbb{1}_{\{0\}}(s) \cdot \overline{\Xi}^0 + \mathbb{1}_{\overline{1,M}}(s) \cdot \overline{\Xi}^s \geq \xi \Bigg[\mathbb{1}_{\{0\}}(s) \cdot \left(\widetilde{P} + \sum_{j=1}^M \overline{\Lambda}_j \eta_{0,j} \right) \\ &\qquad + \mathbb{1}_{\overline{1,M}}(s) \cdot \overline{\Lambda}_s \sum_{j'=1}^M \eta_{1,sj'} \Bigg], \\ &\mathbb{1}_{\{0\}}(s) \cdot \overline{\Upsilon}_{0,j} + \mathbb{1}_{\overline{1,M}}(s) \cdot \overline{\Upsilon}_{s,z} \geq \xi \Bigg[\mathbb{1}_{\{0\}}(s) \cdot \overline{\Lambda}_j \sum_{j'=1}^M \eta_{2,jj'} \\ &\qquad + \mathbb{1}_{\overline{1,M}}(s) \cdot \overline{\Lambda}_z \sum_{j'=1}^M \eta_{3,sj'z} \Bigg], \\ &s \in \overline{0,M},\ j \in \overline{1,M},\ z \in \overline{s+1,M} \end{aligned} \tag{10}$$

for some $\xi \in \left(0, \min\left(w_0, \left(\frac{\delta_0 - p_0}{\delta_0 \tau_0}\right)\right)\right]$, where

$$\overline{Q}_{1,1} = \overline{P} A_0^\top + U_0^\top G^\top + A_0 \overline{P} + GU_0 + \overline{S}_0 + \overline{\Xi}^0 - p_0 \overline{R}_0,$$

$$\overline{Q}_{1,2} = \widetilde{P} - \overline{P} + \overline{P} A_0^\top + U_0^\top G^\top;$$

$$Q_{1,3} = B_0 \overline{P} + GL_0 + p_0 \overline{R}_0 + \overline{P} A_0^\top + U_0^\top G^\top,$$

$$\overline{Q}_{1,4} = \begin{bmatrix} A_1 \overline{P} + GU_1 + \overline{P} A_0^\top + U_0^\top G^\top + \overline{\Upsilon}_{0,1} & \ldots \\ A_M \overline{P} + GU_M + \overline{P} A_0^\top + U_0^\top G^\top + \overline{\Upsilon}_{0,M} \end{bmatrix},$$

$$\overline{Q}_{1,5} = \begin{bmatrix} B_1\overline{P}+GL_1 & \dots & B_M\overline{P}+GL_M \end{bmatrix},$$
$$\overline{Q}_{1,6} = I_n;\ Q_{2,2} = -2\overline{P}+\delta_0\tau_0^2\overline{R}_0,$$
$$Q_{2,3} = B_0\overline{P}+GL_0-\overline{P},$$
$$\overline{Q}_{2,4} = \begin{bmatrix} A_1\overline{P}+GU_1-\overline{P}+\overline{\Lambda}_1 & \dots & A_M\overline{P}+GU_M-\overline{P}+\overline{\Lambda}_M \end{bmatrix},$$
$$\overline{Q}_{2,5} = \begin{bmatrix} B_1\overline{P}+GL_1 & \dots & B_M\overline{P}+GL_M \end{bmatrix};\ Q_{2,6} = I_n,$$
$$\overline{Q}_{3,3} = -e^{-w_0\tau_0}\overline{S}_0 - p_0\overline{R}_0 + 2B_0\overline{P} + 2GL_0,$$
$$\overline{Q}_{3,4} = \Big[\ \overline{P}B_0^\top + L_0^\top G^\top + A_1P + GU_1 \quad \dots$$
$$\overline{P}B_0^\top + L_0^\top G^\top + A_MP + GU_M\ \Big],$$
$$\overline{Q}_{3,5} = \begin{bmatrix} B_1P+GL_1 & \dots & B_MP+GL_M \end{bmatrix};\ \overline{Q}_{3,6} = \overline{P},$$
$$\overline{Q}_{4,4} = \overline{Q}_{4,4}^\top = (\hat{Q}'_{a,b})_{a,b=1}^M,$$
$$\hat{Q}'_{j,j} = \overline{P}A_j^\top + U_j^\top G^\top$$
$$+A_j\overline{P} + GU_j + \overline{\Xi}^j + \overline{S}_j,\ j\in\overline{1,M},$$
$$\hat{Q}'_{s,z} = \overline{P}A_s^\top + U_s^\top G^\top$$
$$+A_z\overline{P} + GU_z + \overline{\Upsilon}_{s,z},\ s\in\overline{1,M-1}, z\in\overline{s+1,M},$$
$$\overline{Q}_{4,5} = \begin{bmatrix} B_1\overline{P}+GL_1 & \cdots & B_M\overline{P}+GL_M \\ \vdots & \ddots & \vdots \\ B_1\overline{P}+GL_1 & \cdots & B_M+GL_M\overline{P} \end{bmatrix},$$
$$\overline{Q}_{4,6} = \mathbf{1}_M\otimes I_n,$$
$$\overline{Q}_{5,5} = \mathrm{diag}(-e^{-w_0\tau_1}\overline{S}_1,\dots,-e^{-w_0\tau_M}\overline{S}_M),$$
$$\overline{Q}_{5,6} = \mathbb{O}_{nM\times n};\ \overline{Q}_{6,6} = -\Phi.$$

Then the closed-loop system (9) *is ISS with feedback gains*

$$K_{A,s} = U_s\overline{P}^{-1},\ K_{B,s} = L_s\overline{P}^{-1}, s\in\overline{0,M}.$$

Proof. Using the prescribed properties of $\widetilde{P}$, $\overline{\Lambda}_j\left(j\in\overline{1,M}\right)$, $\overline{R}_s\left(s\in\overline{0,M}\right)$, $\overline{S}_k\left(k\in\overline{0,M}\right)$ and $\overline{P}$, select the LKF V given by (6) in the proof of Theorem 2 with: $P = \overline{P}^{-1}\widetilde{P}\overline{P}^{-1}$, $\Lambda_j = \overline{P}^{-1}\overline{\Lambda}_j\overline{P}^{-1}$, $R_0 = \overline{P}^{-1}\overline{R}_0\overline{P}^{-1}$, $S_k = \overline{P}^{-1}\overline{S}_k\overline{P}^{-1}$, $R_j = 0$ ($j\in\overline{1,M}$), then V verifies the positive definiteness requirements of Definition 3. Furthermore, consider the conditions and the proof of Theorem 2 and denote by $\widetilde{Q}$ the block matrix Q (in Theorem 2) under the substitutions $R_j\to 0$ for $j\in\overline{1,M}$, $(P_2^{-1},P_3^{-1},P_4^{-1},\Omega_j^{-1})\to(\overline{P},\overline{P},\overline{P},\overline{P})$ for $j\in\overline{1,M}$, $A_s\to\widetilde{A}_s$, $B_s\to\widetilde{B}_s$ for $s\in\overline{0,M}$, define

$$\begin{aligned} \overline{Q} &= H^\top\widetilde{Q}H, \\ H &= \mathrm{diag}(P_2^{-1},P_3^{-1},P_2^{-1},\Omega_1^{-1},\dots,\Omega_M^{-1},\Omega_1^{-1},\dots,\Omega_M^{-1},I_n) \\ &= \mathrm{diag}(\underbrace{\overline{P},\dots,\overline{P}}_{2M+3},I_n) \end{aligned}$$

under the settings of

$$\begin{aligned} \Xi^k &= \overline{P}^{-1}\overline{\Xi}^k\overline{P}^{-1},\ k\in\overline{0,M} \\ \Upsilon_{s,z} &= \overline{P}^{-1}\overline{\Upsilon}_{s,z}\overline{P}^{-1}, s\in\overline{0,M-1},\ z\in\overline{s+1,M}, \end{aligned}$$

by which we can deduce that

$$\overline{Q}\le 0 \quad\Leftrightarrow\quad Q\le 0$$

and the conditions (10) are equivalent to (5). This completes the proof. □

To find the control gains as solutions of LMIs in Theorem 3, more restrictive conditions are imposed than in Theorem 2 (or in Remark 2): the matrices $\widetilde{P},\overline{\Lambda}_j,P_2,P_3,P_4$ are assumed to be diagonal and positive definite. In practice, Theorem 3 and Remark 2 can be applied iteratively: the former to find some guesses for $K_{A,s}$, $K_{B,s}$, $s\in\overline{0,M}$, while the latter to calculate more accurately the AGs from Definition 2 and to refine the restrictions on delays.

VI. Examples

A. Application to opinion dynamics

For modeling opinion dynamics among a network, the following equation can be used [27]–[29]:

$$\dot{x}(t) = -x(t) + \sum_{j=1}^M k_jA_j\overline{\tanh}(\alpha_jx(t)) \tag{11}$$
$$+\sum_{r=1}^L p_rB_r\overline{\tanh}(\beta_rx(t-\tau_r)) + Gu(t) + \varphi(t),$$

where $x(t)\in\mathbb{R}^n$ is the opinion variable of n agents, and $\mathrm{sign}(x_i(t))$ ($i\in\overline{1,n}$) describes the qualitative stance toward a binary choice (the bigger $|x_i(t)|$, more extreme is the opinion of the agent i); $(M+L)\ge 2$ ($M,L\ge 1$) is the number of social networks connecting the agents; $k_j,p_r>0$ denote the social interaction strength among agents in the network, $j\in\overline{1,M}, r\in\overline{1,L}$; $\tau_r>0$ is the time delay in the network $r\in\overline{1,L}$; $A_j,B_r\in\mathbb{R}^{n\times n}$ are the adjacency matrices, and $\alpha_j>0$ or $\beta_r>0$ characterizes the controversialness of the issue for j^{th} or r^{th} media; the function $\overline{\tanh}:\mathbb{R}^n\to\mathbb{R}^n$ and $\overline{\tanh}\left(\begin{bmatrix} g_1 & \dots & g_n\end{bmatrix}^\top\right) = \begin{bmatrix}\tanh(g_1) & \dots & \tanh(g_n)\end{bmatrix}^\top$ for $g_1,\dots,g_n\in\mathbb{R}$; $G\in\mathbb{R}^{n\times q}$; $u(t)\in\mathbb{R}^q$ is a controlling input for modifying the network connections among the agents (thus, it has to be of the form of (8), and any shape of control cannot be implemented); $\varphi(t)\in\mathbb{R}^n$ can be used to model the off-network influences on orientations of agents (*e.g.*, government communication). The detailed motivation for this model (for the case $M=1$ and time-varying matrix A_1) is given in [28], [29]. The system under feedback control takes the form of (2), and assumptions 1, 2 are satisfied.

For illustration, let $n=4$, $M=L=1$,

$$A_1 = \begin{bmatrix} 0&1&1&1\\1&0&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix},\ B_1 = \begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&1\\1&0&0&0\end{bmatrix},$$
$$G = \begin{bmatrix} 1&-1&0.2&1\\-1&0.5&1&-0.4\\1&-0.1&2&-1\\1&0.2&1&1\end{bmatrix},\ k_1=1.8,\ p_1=1.3,$$

$$\alpha_1 = 0.4,\ \beta_1 = 0.2,\ \tau_1 = 0.8,\ \varphi(t) = \begin{bmatrix} 1 \\ 0.3 \\ 0.7 \\ 0.1 \end{bmatrix},$$

$$u(t) = Z_1 \overline{\tanh}(\alpha_1 x(t)) + Z_2 \overline{\tanh}(\beta_1 x(t-\tau_1)),$$

$$Z_1 = \begin{bmatrix} 0.3321 & -0.0031 & 0.2004 & -1.1101 \\ 0.801 & 0.8805 & -0.2584 & -0.2578 \\ -0.9096 & -0.1803 & -0.4758 & 0.6053 \\ -0.5175 & -0.6344 & -0.0687 & 0.0637 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} 0.3432 & 0.7509 & -0.3315 & 0.2678 \\ -0.3231 & -0.8566 & 0.1478 & 0.2328 \\ -0.2667 & -1.1971 & 0.2777 & -0.2342 \\ -0.6731 & 0.3070 & 0.1869 & 0.3786 \end{bmatrix},$$

then the LMIs in Remark 2 are verified. The three sets of system trajectories $\left(x(t) \in \mathbb{R}^4\right)$ with different initial conditions are presented in Fig. 1, which illustrate that all agents converge to a common decision under the chosen control. Simulations of the system (11) with $u = 0$, $\varphi = 0$ demonstrate pluralism of opinions in the uncontrolled network.

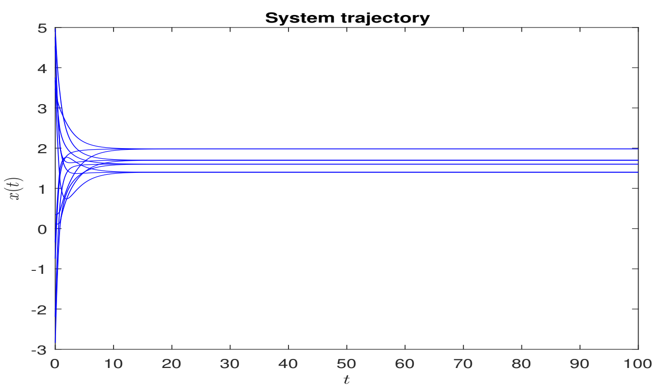

Fig. 1. The trajectories of the controlled system (11) versus the time t

B. Application to a modified Lotka-Volterra model

In this subsection, a modified Lotka-Volterra (LV) dynamics is considered. Different versions of this model have been widely investigated in infectious diseases, biology, finance, to mention a few [30]. The basic model does not reflect some important phenomena, such as time delays and stable coexistence. Thus many modified LV models have been proposed. Among them, the following one considers population dynamics with several delays [31]:

$$\dot{x}(t) = \mathrm{diag}\{x(t)\}\left[r_0 + r(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2)\right],\ t \in \mathbb{R}_+, \tag{12}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^\top \in \mathbb{R}_+^n$ contains the populations of n species; $r_0 \in \mathbb{R}^n$ models the birth and death rates; A_1, $A_2 \in \mathbb{R}^n$ represent the community matrices; τ_1, $\tau_2 > 0$ are delays corresponding to two different kinds of interactions between populations; the function $r : \mathbb{R}_+ \to \mathbb{R}^n$ is introduced to model the deviations of the rates from the nominal quantities $r_0 \in \mathbb{R}^n$.

Assuming the existence of a unique non-zero equilibrium point $x_e = \begin{bmatrix} x_e^1 & \ldots & x_e^n \end{bmatrix}^\top \in \mathbb{R}_+^n \setminus \{0\}$ for (12) with $r(t) = 0$, and defining

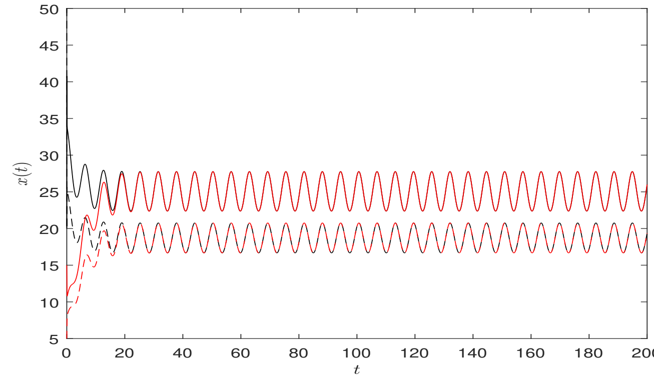

Fig. 2. The state trajectories versus the time t for LV model

$$\rho(t) = \begin{bmatrix} \rho_1(t) \\ \vdots \\ \rho_n(t) \end{bmatrix} = \begin{bmatrix} \ln(x_1(t)) - \ln(x_e^1) \\ \vdots \\ \ln(x_n(t)) - \ln(x_e^n) \end{bmatrix},$$

we have

$$\begin{aligned} \dot{\rho}(t) \ &= \ A_1 \,\mathrm{diag}(x_e) F_1(\rho(t-\tau_1)) \\ &\quad + A_2 \,\mathrm{diag}(x_e) F_1(\rho(t-\tau_2)) + r(t), \end{aligned} \tag{13}$$

where

$$F_1(\rho) = \begin{bmatrix} e^{\rho_1} \\ \vdots \\ e^{\rho_n} \end{bmatrix} - \mathbf{1}_n.$$

It is clear that F_1 satisfies Assumption 1 and Assumption 2 with $\eta_{0,j}^i = \eta_{2,jj'}^i = 1$. The requirements of Assumption 3 are not satisfied globally. However, as in [32], due to assumed existence of the global equilibrium x_e, it is possible to show that for $r_0 + r(t) \geq r_{\min}$ all trajectories converge to a neighborhood of the steady state, so that $x(t) > 0$ for all $t \in \mathbb{R}_+$, which results in well-posedness of (13). The analysis can be next performed without taking into account the unbounded deviations of the state.

The simulation results are given for

$$n = 2,\ \tau_1 = 0.001,\ \tau_2 = 0.02,\ A_1 = \begin{bmatrix} -0.6 & 0.4 \\ 0.5 & -0.6 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.3 & 0.8 \\ 0.6 & -0.9 \end{bmatrix},\ r_0 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix},\ r(t) = \begin{bmatrix} -0.2\sin(t) \\ 0.1\cos(t) \end{bmatrix}.$$

The LMIs of Corollary 1 are verified, and the state trajectories are shown in Fig. 2 for two sets of initial conditions.

VII. Conclusion

This paper has proposed input-to-state stability and stabilization conditions for generalized Persidskii systems with constant time delays. The formulated conditions were obtained in the form of linear matrix inequalities explicitly dependent on delays. Therefore they can be constructively verified. Two conditions were formulated, for a given control and the design of feedback gains. The simulations of opinion dynamics and a modified Lotka-Volterra model were presented to illustrate the proposed results. The future research directions include ISS analysis for the considered systems with unknown delays and study of other practical applications.

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