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Tutorial on Lyapunov-based methods for time-delay systems ☆

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ABSTRACT

Time-delay naturally appears in many control systems, and it is frequently a source of instability. However, for some systems, the presence of delay can have a stabilizing effect. Therefore, stability and control of time-delay systems is of theoretical and practical importance. Modern control systems usually employ digital technology for controller implementation, i.e. sampled-data control. A time-delay approach to sampled-data control, where the system is modeled as a continuous-time system with the delayed input/output became popular in the networked control systems, where the plant and the controller exchange data via communication network. In the present tutorial, introduction to Lyapunov-based methods for stability of time-delay systems is given together with some advanced results on the topic.

1. Introduction

Time-delay systems (TDSs) are also called systems with after effect or dead-time, hereditary systems, equations with deviating argument or differential-difference equations. They belong to the class of *functional differential equations* which are infinite-dimensional, as opposed to ordinary differential equations (ODEs). The simplest example of such a system is

$$\dot{x}(t) = -x(t-h), \quad x(t) \in \mathbb{R},$$

where $h>0$ is the time-delay.

Time-delays appear in many engineering systems – aircraft, chemical control systems, in laser models, in Internet, biology, medicine [31,41]. Delays are strongly involved in challenging areas of communication and information technologies: stability of networked control systems or high-speed communication networks [62].

Time-delay is, in many cases, a source of instability. However, for some systems, the presence of delay can have a stabilizing effect. In the well-known example

$$\ddot{y}(t) + y(t) - y(t-h) = 0,$$

the system is unstable for $h=0$, but is asymptotically stable for $h=1$. The approximation $\dot{y}(t) \simeq [y(t)-y(t-h)]h^{-1}$ explains the damping effect. The stability analysis and robust control of time-delay systems are, therefore, of theoretical and practical importance.

As in systems without delay, an efficient method for stability analysis of TDSs is the Lyapunov method. For TDSs, there exist two main Lyapunov methods: the *Krasovskii* method of Lyapunov *functionals* [43] and the *Razumikhin* method of Lyapunov *functions* [61]. The two Lyapunov methods for linear TDSs result in Linear Matrix Inequalities (LMIs) conditions. The LMI approach to analysis and design of TDSs provides constructive finite-dimensional conditions, in spite of significant model uncertainties [1].

Modern control systems usually employ digital technology for controller implementation, i.e. sampled-data control. Consider a sampled-data control system

$$\dot{x}(t) = Ax(t) + BKx(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, \ldots, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, A, B, K are constant matrices and $\lim_{k\to\infty} t_k = \infty$. This system can be represented as a continuous system with time-varying delay $\tau(t) = t - t_k$ [52,6]:

$$\dot{x}(t) = Ax(t) + BKx(t-\tau(t)), \quad t \in [t_k, t_{k+1}), \tag{2}$$

where the delay is piecewise-linear (sawtooth) with $\dot{\tau} = 1$ for $t \neq t_k$. Modeling of continuous-time systems with *digital control* in the form of *continuous-time systems with time-varying delay* and the extension of Krasovskii method to TDSs without any constraints on the delay derivative [20] and to discontinuous delays [18] have allowed the development of the *time-delay approach to sampled-data and to network-based control.*

Bernoulli, Euler and Concordet were (among) the first to study equations with delay (the 18-th century). Systematical study started at the 1940s by A. Myshkis and R. Bellman. Since 1960

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there have appeared more than 50 monographs on the subject (see e.g. [5,28,31,41,57] to name a few). The beginning of the 21st century can be characterized as the "time-delay boom" leading to numerous important results. The emphasis in this Introduction to TDSs is on the Lyapunov-based analysis and design of time-delay and sampled-data systems.

The paper is organized as follows. Two main Lyapunov approaches for general TDSs are presented in Section 2. For linear systems with discrete time-varying delays, delay-independent and delay-dependent conditions are provided in Section 3. The section starts from the simple stability conditions and shows the ideas and tools that essentially improve the results. Section 3.3 presents recent Lyapunov-based results for the stability of sampled-data systems. Section 4 considers general (complete) Lyapunov functional for LTI systems with discrete delays corresponding to necessary stability conditions, and discusses the relation between simple, augmented and general Lyapunov functionals. Stability conditions for systems with distributed (finite and infinite) delays are presented in Section 5. Section 6 discusses the stability of some nonlinear systems. Finally the input–output approach to stability of linear TDSs is provided in Section 7 showing the relation of the input–output stability with the exponential stability of the linear TDSs.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. The space of functions $\phi:[-h,0]\to\mathbb{R}^n$, which are absolutely continuous on $[-h,0]$, and have square integrable first order derivatives is denoted by $W[-h,0]$ with the norm $\|\phi\|_W=\max_{\theta\in[-h,0]}|\phi(\theta)|+[\int_{-h}^0|\dot\phi(s)|^2\,ds]^{1/2}$. For $x:\mathbb{R}\to\mathbb{R}^n$ we denote $x_t(\theta)\triangleq x(t+\theta),\ \theta\in[-h,0]$.

2. General TDS and the direct Lyapunov method

Consider the following TDS:

$$\dot x(t)=f(t,x_t),\quad t\ge t_0,\tag{3}$$

where $f:\mathbb{R}\times C[-h,0]\to\mathbb{R}^n$ is continuous in both arguments and is locally Lipschitz continuous in the second argument. We assume that $f(t,0)=0$, which guarantees that (3) possesses a trivial solution $x(t)\equiv 0$.

Definition 1. The trivial solution of (3) is

- uniformly (in t_0) stable if $\forall t_0\in\mathbb{R}$ and $\forall\epsilon>0$, there exists a $\delta=\delta(\epsilon)>0$ such that $\|x_{t_0}\|_C<\delta(\epsilon)$ implies $|x(t)|<\epsilon$ for all $t\ge t_0$;
- *uniformly asymptotically stable* if it is uniformly stable and there exists a $\delta_a>0$ such that for any $\eta>0$ there exists a $T(\delta_a,\eta)$ such that $\|x_{t_0}\|_C<\delta_a$ implies $|x(t)|<\eta$ for all $t\ge t_0+T(\delta_a,\eta)$ and $t_0\in\mathbb{R}$.
- *globally uniformly asymptotically stable* if δ_a can be an arbitrarily large, finite number.

The system is uniformly asymptotically stable if its trivial solution is uniformly asymptotically stable.

Note that the stability notions are not different from their counterparts for systems without delay [36]. In this tutorial we shall only be concerned with uniform asymptotic stability, that sometimes will be referred as asymptotic stability.

Prior to N.N. Krasovskii's papers on Lyapunov functionals and B.S. Razumikhin's papers on Lyapunov functions, L.E. El'sgol'tz (see [5] and references therein) considered the stability problem of the solution $x(t)\equiv 0$ of TDSs by proving that the function $\overline{V}(t)=V(x(t))$ is decreasing in t, where V is some Lyapunov function. This is possible only in some rare special cases. We shall show this on the example of the scalar autonomous Retarded Differential Equation (RDE)

$$\dot x(t)=f(x(t),x(t-h)),\quad f(0,0)=0,$$

where $f(x,y)$ is locally Lipschitz in its arguments. Let us assume that $V(x)=x^2$, which is a typical Lyapunov function for $n=1$. Then we have along the system

$$\frac{d}{dt}[V(x(t))]=2x(t)\dot x(t)=2x(t)f(x(t),x(t-h)).$$

For the feasibility of inequality $(d/dt)[V(x(t))]\le 0$, we need to require that

$$x(t)f(x(t),x(t-h))\le 0$$

for all sufficiently small $|x(t)|$ and $|x(t-h)|$. This essentially restricts the class of equations considered. For example,

$$\dot x(t)=-x(t)x^2(t-h)$$

is stable by the above arguments.

2.1. Lyapunov–Krasovskii approach

Let $V:\mathbb{R}\times C[-h,0]\to\mathbb{R}$ be a continuous functional, and let $x_\tau(t,\phi)$ be the solution of (3) at time $\tau\ge t$ with the initial condition $x_t=\phi$. We define the right upper derivative $\dot V(t,\phi)$ along (3) as follows:

$$\dot V(t,\phi)=\lim\sup_{\Delta t\to 0^+}\frac{1}{\Delta t}[V(t+\Delta t,x_{t+\Delta t}(t,\phi))-V(t,\phi)].$$

Intuitively, a non-positive $\dot V(t,x_t)$ indicates that x_t does not grow with t, meaning that the system under consideration is stable.

Theorem 1 (*Lyapunov–Krasovskii Theorem, Gu et al. [28]*). *Suppose $f:\mathbb{R}\times C[-h,0]\to\mathbb{R}^n$ maps $\mathbb{R}\times$ (bounded sets in $C[-h,0]$) into bounded sets of $\mathbb{R}^n$ and that $u,v,w:\mathbb{R}_+\to\mathbb{R}_+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s>0$, and $u(0)=v(0)=0$. The trivial solution of (3) is uniformly stable if there exists a continuous functional $V:\mathbb{R}\times C[-h,0]\to\mathbb{R}^+$, which is positive-definite, i.e.*

$$u(|\phi(0)|)\le V(t,\phi)\le v(\|\phi\|_C),\tag{4}$$

and such that its derivative along (3) is non-positive in the sense that

$$\dot V(t,\phi)\le -w(|\phi(0)|).\tag{5}$$

If $w(s)>0$ for $s>0$, then the trivial solution is uniformly asymptotically stable. If in addition $\lim_{s\to\infty}u(s)=\infty$, then it is globally uniformly asymptotically stable.

In some cases functionals $V(t,x_t,\dot x_t)$ that depend on the state-derivatives are useful (see [41, p. 337]). Denote by $W[-h,0]$ the Banach space of absolutely continuous functions $\phi:[-h,0]\to\mathbb{R}^n$ with $\dot\phi\in L_2(-h,0)$ (the space of square integrable functions) with the norm

$$\|\phi\|_W=\max_{s\in[-h,0]}|\phi(s)|+\left[\int_{-h}^0|\dot\phi(s)|^2\,ds\right]^{1/2}.$$

Theorem 1is then extended to continuous functionals

$$V:\mathbb{R}\times W[-h,0]\times L_2(-h,0)\to\mathbb{R}_+,$$

where inequalities (4) and (5) are modified as follows:

$$u(|x(t)|)\le V(t,x_t,\dot x_t)\le v(\|x_t\|_W)\tag{6}$$

and

$$\dot{V}(t,x_t,\dot{x}_t) \le -w(|x(t)|). \tag{7}$$

Note that the functionals $V(t,x_t,\dot{x}_t)$ can be applied to solutions of RDEs with the initial functions x_{t_0} from $W[-h,0]$. However, for RDE the stability results corresponding to continuous initial functions and to absolutely continuous initial functions are equivalent [5].

2.2. Lyapunov–Razumikhin approach

To give a precise formulation of Razumikhin method, we consider a differentiable *function* $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}_+$ and define the derivative of V along the solution $x(t)$ of (3) as

$$\dot{V}(t,x(t)) = \frac{d}{dt}V(t,x(t)) = \frac{\partial V(t,x(t))}{\partial t} + \frac{\partial V(t,x(t))}{\partial x}f(t,x_t).$$

Theorem 2 (*Lyapunov–Razumikhin Theorem, Gu et al. [28]*). *Suppose $f:\mathbb{R}\times C[-h,0]\to\mathbb{R}^n$ maps $\mathbb{R}\times$ (bounded sets in $C[-h,0]$) into bounded sets of $\mathbb{R}^n$ and that $u,v,w:\mathbb{R}_+\to\mathbb{R}_+$ are continuous nondecreasing functions $u(s)$ and $v(s)$ are positive for $s>0$ and $u(0)=v(0)=0$, v is strictly increasing. The trivial solution of (3) is uniformly stable if there exists a differentiable function $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^+$, which is positive-definite, i.e.*

$$u(|x|) \le V(t,x) \le v(|x|), \tag{8}$$

and such that the derivative of V along the solution $x(t)$ of (3) satisfies

$$\dot{V}(t,x(t)) \le -w(|x(t)|) \quad \text{if } V(t+\theta,x(t+\theta)) \le V(t,x(t)) \quad \forall\theta\in[-h,0].$$

If, in addition, $w(s)>0$ for $s>0$, and there exists a continuous nondecreasing function $\rho(s)>s$ for $s>0$ such that condition (9) is strengthened to

$$\dot{V}(t,x(t)) \le -w(|x(t)|) \quad \text{if } V(t+\theta,x(t+\theta)) \le \rho(V(t,x(t))) \ \forall\theta\in[-h,0], \tag{9}$$

then the trivial solution of (3) is uniformly asymptotically stable. If, in addition, $\lim_{s\to\infty}u(s)=\infty$, then it is globally uniformly asymptotically stable.

3. Stability of linear systems with discrete delays

Consider a simple linear TDS

$$\dot{x}(t) = Ax(t) + A_1x(t-\tau(t)), \quad t\ge t_0, \tag{10}$$

where $x(t)\in\mathbb{R}^n$, $\tau(t)\in[0,h]$ is a bounded time-varying delay. Here A and A_1 are constant $n\times n$-matrices. In this section, delay-independent and delay-dependent conditions will be presented. For simplicity only we consider a linear system with a single discrete delay. The results can be easily extended to a finite number of discrete delays.

3.1. Delay-independent conditions

The choice of Lyapunov–Krasovskii functional (that we will also call Lyapunov functional) is crucial for deriving stability criteria. Special forms of the functional lead to *delay-independent* and *delay-dependent* conditions. In this section we consider the delay-independent (i.e. h-independent) conditions for systems with time-varying delays.

A simple Lyapunov functional for (10) has the form

$$V(t,x_t) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)Qx(s)\,ds, \tag{11}$$

where $P>0$ and $Q>0$ are $n\times n$ matrices. Note that V of (11) depends on t because of $\tau(t)$. Note that in the case of constant delay $\tau\equiv h$, the functional in (11) is time-independent (i.e. $V(x_t)$). We further assume that the delay τ is a differentiable function with $\dot{\tau}\le d<1$ (this is the case of *slowly varying delays*). It is clear that V satisfies the positivity condition $\beta|x(t)|^2\le V(t,x_t)$ (for some $\beta>0$). Then, differentiating V along (10), we find

$$\dot{V}(t,x_t) = 2x^T(t)P\dot{x}(t) + x^T(t)Qx(t) \\ -(1-\dot{\tau})x^T(t-\tau)Qx(t-\tau).$$

We further substitute for $\dot{x}(t)$ the right-hand side of (10) and arrive at

$$\dot{V}(t,x_t) \le [x^T(t)\ x^T(t-\tau)]W\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix} \le -\varepsilon|x(t)|^2$$

for some $\varepsilon>0$ if

$$W = \begin{bmatrix} A^TP+PA+Q & PA_1 \\ A_1^TP & -(1-d)Q\end{bmatrix} < 0. \tag{12}$$

The LMI (12) does not depend on h and it is, therefore, delay-independent (but delay-derivative dependent). The feasibility of LMI (12) is a sufficient condition for the delay-independent asymptotic stability of systems with slowly varying delays. The feasibility of (12) yields the following:

(i) A and $A\pm A_1$ are Hurwitz matrices,

(ii) $A^{-1}A_1/\sqrt{1-d}$ is a Schur matrix, meaning that all its eigenvalues are inside the unit circle [8].

From (i) it follows that the delay-independent conditions cannot be applied for stabilization of unstable plants by a feedback with delay. For such systems *delay-dependent* (h-dependent) conditions are needed.

We next derive stability conditions by applying Razumikhin's method and using the Lyapunov function $V(x(t))=x^T(t)Px(t)$ with $P>0$, that satisfies the positivity condition (8). Consider the derivative of V along (10). We will apply the Lyapunov–Razumikhin theorem with $\rho(s)=\overline{\rho}\cdot s$, where the constant $\overline{\rho}>1$. Whenever Razumikhin's condition

$$\overline{\rho}x^T(t)Px(t) - x(t-\tau(t))^TPx(t-\tau(t)) \ge 0$$

holds for some $\overline{\rho}=1+\varepsilon$ ($\varepsilon>0$), we can conclude that, for any $q>0$ there exists $\alpha>0$ such that

$$\dot{V}(x(t)) = 2x^T(t)P[Ax(t)+A_1x(t-\tau(t))] \le 2x^T(t)P[Ax(t)+A_1x(t-\tau(t))] \\ + q[\overline{\rho}x^T(t)Px(t) - x(t-\tau(t))^TPx(t-\tau(t))] \le -\alpha|x(t)|^2$$

if

$$\begin{bmatrix} A^TP+PA+qP & PA_1 \\ A_1^TP & -qP\end{bmatrix} < 0. \tag{13}$$

The latter matrix inequality does not depend on h. Moreover, it does not depend on the delay derivative bound. Therefore, the feasibility of (13) is sufficient for delay-independent uniform asymptotic stability for systems with *fast-varying* delays (without any constraints on the delay-derivatives).

The Krasovskii-based LMI (12) for $d=0$ is less restrictive than the Razumikhin-based condition (13): the feasibility of (13) implies the feasibility of (12) with the same P and with $Q=qP$. Another advantage of the LMI (12) is that it is linear in the decision variables P and Q, whereas (13) is bilinear in q and P. The latter makes computation more difficult. One can treat (13) as an LMI with the tuning parameter $q>0$. However, till now only *Razumikhin method provides delay-independent conditions for systems with fast-varying delays.*

Halanay's inequality [30, 1960] that extends the Razumikhin method to the exponential stability can also lead to delay-independent conditions for systems with fast-varying delays:

Let $V:[t_0-h,+\infty)\to\mathbb{R}_+$ be bounded on $[t_0-h,t_0]$ and locally absolutely continuous on $[t_0,\infty)$. Assume that for some positive constants $\delta_1 < \delta_0$ the following inequality holds:

$$Hal \triangleq \dot{V}(t) + 2\delta_0 V(t) - 2\delta_1 \sup_{-h \le \theta \le 0} V(t+\theta) \le 0, \quad t \ge t_0.$$

Then

$$V(t) \le e^{-2\delta(t-t_0)} \sup_{-h \le \theta \le 0} V(t_0+\theta), \quad t \ge t_0,$$

where $\delta > 0$ is a unique positive solution of the equation $\delta = \delta_0 - \delta_1 e^{2\delta h}$.

We recover now the delay-independent stability condition (13) by using Halanay's inequality. Choose $V(t) = x^T P x(t) (P > 0)$, where $x(t)$ satisfies (10), $2\delta_1 = q > 0$ and $2\delta_0 = q(1+\varepsilon)(\varepsilon > 0)$. Then

$$Hal \le \dot{V}(t) + 2\delta_0 V(t) - 2\delta_1 V(t-\tau(t)) = 2x^T(t)P[Ax(t) + A_1 x(t-\tau(t))] + q(1+\varepsilon)x^T(t)Px(t) - qx(t-\tau(t))^T Px(t-\tau(t)).$$

Therefore, the feasibility of (13) guarantees that for small enough ε the Halanay inequality $Hal \le 0$ holds meaning that the system (10) is exponentially stable.

3.2. Delay-dependent conditions

The first delay-dependent (both, Krasovskii and Razumikhin-based) conditions were derived by using the relation

$$x(t-\tau(t)) = x(t) - \int_{t-\tau(t)}^{t} \dot{x}(s)\, ds \tag{14}$$

via different model transformations and by bounding the cross terms [44,59,42]. The widely used 1st Model Transformation, where (14) is substituted into (10) with $\dot{x}(s)$ substituted by the right-hand side of (10), has the form

$$\dot{x}(t) = [A + A_1]x(t) - A_1 \int_{t-\tau(t)}^{t} [Ax(s) + A_1 x(s-\tau(s))]\, ds. \tag{15}$$

Note that this transformation is valid for $t - \tau(t) \ge t_0$. The latter system is not equivalent to the original one possessing some additional dynamics [29,38]. The stability of the transformed system (15) guarantees the stability of the original one, but not vice versa.

The first delay-dependent conditions treated only the *slowly* varying delays with $\dot{\tau} \le d < 1$, whereas the fast-varying delay (without any constraints on the delay derivative) was analyzed via Lyapunov–Razumikhin functions.

For the first time, systems with *fast varying delays* were analyzed by using Krasovskii method in [20], via the *descriptor model transformation* introduced in [7]:

$$\dot{x}(t) = y(t), \quad 0 = -y(t) + (A + A_1)x(t) - A_1 \int_{t-\tau(t)}^{t} y(s)\, ds. \tag{16}$$

The descriptor system (16) is equivalent to (10) in the sense of stability. In the descriptor approach, $\dot{x}(t)$ is not substituted by the right-hand side of the differential equation. Instead, it is considered as an additional state variable of the resulting descriptor system (16). Therefore, the novelty of the descriptor approach is not in $V = x^T(t)Px(t) + \cdots (P > 0)$, but in $\dot{V}$, where $(d/dt)[x^T(t)Px(t)]$ is found as

$$\frac{d}{dt}[x^T(t)Px(t)] = 2x^T(t)P\dot{x}(t) + 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T] \times \left[-\dot{x}(t) + (A + A_1)x(t) - A_1 \int_{t-\tau(t)}^{t} \dot{x}(s)\, ds\right], \tag{17}$$

and where $P_2 \in \mathbb{R}^{n \times n}$ and $P_3 \in \mathbb{R}^{n \times n}$ are "slack variables". This leads to $\dot{V} \le -\varepsilon(|x(t)|^2 + |\dot{x}(t)|^2)$, $\varepsilon > 0$.

The advantages of the descriptor method are

- less conservative conditions (even without delay) for *uncertain* systems,
- "unifying" LMIs for the discrete-time and for the continuous-time systems, having almost the same form and the same advantages [21],
- simple conditions in terms of LMIs can be derived for *neutral type systems* (these are systems with the delayed highest-order state derivative), where the LMIs imply the stability of the difference operator [8],
- efficient design is obtained for systems with state, input and output delays by choosing $P_3 = \varepsilon P_2$ with a tuning scalar parameter ε [70],
- simple delay-dependent conditions can be derived for diffusion partial differential equations [17].

Most of the recent Krasovskii-based results do not use model transformations and cross terms bounding. They are based on the application of *Jensen's inequality* [28]:

$$\int_{-h}^{0} \phi^T(s)R\phi(s)\, ds \ge \frac{1}{h} \int_{-h}^{0} \phi^T(s)\, dsR \int_{-h}^{0} \phi(s)\, ds, \quad \forall \phi \in L_2[-h,0], \ \forall R > 0. \tag{18}$$

3.2.1. Simple delay-dependent conditions

First Krasovskii-based LMI conditions for systems with fast-varying delays (without any restrictions on the delay-derivative) were derived in [20] via the descriptor method. We differentiate $x^T(t)Px(t)$ as in (17). To "compensate" $\int_{t-\tau(t)}^{t} \dot{x}(s)\, ds$ consider the double integral term [20]:

$$V_R(\dot{x}_t) = \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)\, ds\, d\theta, \quad R > 0. \tag{19}$$

The term V_R can be rewritten equivalently as

$$V_R(\dot{x}_t) = \int_{t-h}^{t} (h + s - t)\dot{x}^T(s)R\dot{x}(s)\, ds. \tag{20}$$

Differentiating $V_R(\dot{x}_t)$, we obtain

$$\begin{aligned} \frac{d}{dt}V_R(\dot{x}_t) &= -\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)\, ds + h\dot{x}^T(t)R\dot{x}(t) \\ &= -\int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)\, ds + h\dot{x}^T(t)R\dot{x}(t) \\ &\quad \underbrace{-\int_{t-h}^{t-\tau(t)} \dot{x}^T(s)R\dot{x}(s)\, ds}_{\text{will be ignored}}. \end{aligned} \tag{21}$$

We apply further Jensen's inequality

$$-\int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)\, ds \le -\frac{1}{h} \int_{t-\tau(t)}^{t} \dot{x}^T(s)\, dsR \int_{t-\tau(t)}^{t} \dot{x}(s)\, ds.$$

Then, for the Lyapunov functional

$$V(x(t), \dot{x}_t) = x^T(t)Px(t) + V_R(\dot{x}_t)$$

we find

$$\begin{aligned} \frac{d}{dt}V(x(t), \dot{x}_t) &\le 2x^T(t)P\dot{x}(t) + h\dot{x}^T(t)R\dot{x}(t) \\ &\quad -\frac{1}{h} \int_{t-\tau(t)}^{t} \dot{x}^T(s)\, ds\, R \int_{t-\tau(t)}^{t} \dot{x}(s)\, ds \\ &\quad + 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][(A + A_1)x(t) - A_1 \int_{t-\tau}^{t} \dot{x}(s)\, ds \\ &\quad - \dot{x}(t)] \le \eta^T(t)\Psi\eta(t) < -\varepsilon(|x(t)|^2 + |\dot{x}(t)|^2), \quad \varepsilon > 0, \end{aligned}$$

where $\eta(t) = col\{x(t), \dot{x}(t), (1/h)\int_{t-\tau}^{t} \dot{x}(s)\,ds\}$, if

$$\Psi_d = \begin{bmatrix} \Phi & P - P_2^T + (A+A_1)^T P_3 & -hP_2^T A_1 \\ * & -P_3 - P_3^T + hR & -hP_3^T A_1 \\ * & * & -hR \end{bmatrix} < 0,$$
$$\Phi = P_2^T (A+A_1) + (A+A_1)^T P_2. \quad (22)$$

As it was understood later [17,25,69] the equivalent delay-dependent conditions can be derived without the descriptor method, where $\dot{x}$ is substituted by the right-hand side of (10) and the Schur complement is applied further.

Note that $\Psi_d < 0$ yields that the eigenvalues of hA_1 are inside the unit circle. In the example $\dot{x}(t) = -x(t-\tau(t))$ with $A_1 = -1$, the simple delay-dependent conditions cannot guarantee the stability for $h \geq 1$, which is far from the analytical bound 1.5. This illustrates the conservatism of the simple conditions.

3.2.2. Improved delay-dependent conditions

The relation between $x(t-\tau(t))$ and $x(t-h)$ (and not only between $x(t-\tau(t))$ and $x(t)$) has been taken into account in [32]. The widely used by now Lyapunov–Krasovskii functional for delay-dependent stability has the form

$$V(t, x_t, \dot{x}_t) = x^T(t) P x(t) + \int_{t-h}^{t} x^T(s) S x(s)\,ds + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s)\,ds\,d\theta + \int_{t-\tau(t)}^{t} x^T(s) Q x(s)\,ds, \quad (23)$$

where $P > 0, R \geq 0, S \geq 0, Q \geq 0$. This functional depends on the state derivative. Moreover, this functional with $Q=0$ leads to delay-dependent conditions for systems with fast-varying delays, whereas for $R=S=0$ it leads to delay-independent conditions (for systems with slowly varying delays) and coincides with (11). The above V with $S=0$ was introduced in [20], whereas the S-dependent term was added in [32].

Differentiating V given by (23), we find

$$\frac{d}{dt}V \leq 2x^T(t) P \dot{x}(t) + h^2 \dot{x}^T(t) R \dot{x}(t) - h\int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s)\,ds + x^T(t)[S+Q]x(t) - x^T(t-h) S x(t-h) - (1-d) x^T(t-\tau(t)) Q x(t-\tau(t)) \quad (24)$$

and employ the representation

$$-h\int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s)\,ds = -h\int_{t-h}^{t-\tau(t)} \dot{x}^T(s) R \dot{x}(s)\,ds - h\int_{t-\tau(t)}^{t} \dot{x}^T(s) R \dot{x}(s)\,ds. \quad (25)$$

Applying Jensen's inequality (18) to *both* terms in (25) we arrive at

$$-h\int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s)\,ds \leq -\frac{h}{\tau(t)} e_1^T R e_1 - \frac{h}{h-\tau(t)} e_2^T R e_2, \quad (26)$$

where

$$e_1 = x(t) - x(t-\tau(t)), \quad e_2 = x(t-\tau(t)) - x(t-h).$$

Here, for $\tau = 0$ and $\tau = h$, we mean the following limits:

$$\lim_{\tau(t)\to 0} \frac{h}{\tau(t)} e_1^T R e_1 = h \lim_{\tau(t)\to 0} \tau(t) \dot{x}^T(t) R \dot{x}(t) = 0$$

and

$$\lim_{\tau(t)\to h} \frac{h}{h-\tau(t)} e_2^T R e_2 = 0.$$

Further, in [32] the right-hand side of (26) was upper-bounded by $-e_1^T R e_1 - e_2^T R e_2$ that was conservative. The convex analysis of [60] allowed to avoid the latter restrictive bounding. Similar to [65], we reformulate the result of [60] in a more convenient form for the Lyapunov-based analysis:

Lemma 1. *Let $R_1 \in \mathbb{R}^{n_1 \times n_1}, \ldots, R_N \in \mathbb{R}^{n_N \times n_N}$ be positive matrices. Then for all $e_1 \in \mathbb{R}^{n_1}, \ldots, e_N \in \mathbb{R}^{n_N}$, for all $\alpha_i > 0$ with $\sum_i \alpha_i = 1$ and for all $S_{ij} \in \mathbb{R}^{n_i \times n_j}$ $i = 1, \ldots, N$, $j = 1, \ldots, i-1$ such that*

$$\begin{bmatrix} R_i & S_{ij} \\ * & R_j \end{bmatrix} \geq 0 \quad (27)$$

the following inequality holds:

$$\sum_{i=1}^{N} \frac{1}{\alpha_i} e_i^T R_i e_i \geq \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}^T \begin{bmatrix} R_1 & S_{12} & \cdots & S_{1N} \\ * & R_2 & \cdots & S_{2N} \\ * & * & \ddots & \vdots \\ * & * & \cdots & R_N \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}. \quad (28)$$

By using Lemma 1 we arrive at the following statement:

Proposition 1. *Given $h \geq 0$, $d \in [0,1)$. If there exist $n \times n$ matrices $P > 0$, $S > 0, Q > 0$, $R > 0$ and S_{12} such that the following LMIs are feasible:*

$$\begin{bmatrix} A^T P + PA + S + Q - R & S_{12} & PA_1 + R - S_{12} & hA^T R \\ * & -S-R & R - S_{12}^T & 0 \\ * & * & \Phi_{33} & hA_1^T R \\ * & * & * & -R \end{bmatrix} < 0 \quad (29)$$

$$\begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \geq 0, \quad (30)$$

where $\Phi_{33} = -(1-d)Q - 2R + S_{12} + S_{12}^T$, then the system (10) is uniformly asymptotically stable for all delays $\tau(t) \in [0,h]$ such that $\dot{\tau}(t) \leq d$. Moreover, if the above conditions hold with $Q=0$ (or, equivalently, with $d=1$), then the system is uniformly asymptotically stable for all fast varying delays $\tau \in [0,h]$.

Proof. Differentiating V given by (23), we find (24). Let the LMI (30) be feasible. Then by Jensen's inequality and Lemma 1

$$-h\int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s)\,ds \leq -\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} \frac{h}{\tau(t)} R & 0 \\ * & \frac{h}{h-\tau(t)} R \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \leq -\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (31)$$

Denote $\eta(t) = col\{x(t), x(t-h), x(t-\tau(t))\}$. Then employing (24), (31) and substituting for $\dot{x}(t)$ the right-hand side of (10), we arrive at

$$\frac{d}{dt}V \leq \eta^T(t) \Phi \eta(t) + \eta^T(t)[hRA \;\; 0 \;\; hRA_1] R^{-1} [hRA \;\; 0 \;\; hRA_1]^T \eta(t) \quad (32)$$

where

$$\Phi = \begin{bmatrix} A^T P + PA + S + Q - R & 0 & PA_1 + R \\ * & -S-R & R \\ * & * & -(1-d)Q - 2R \end{bmatrix}.$$

Applying the Schur complement to the last term in (32), we find that $(d/dt)V \leq -\varepsilon |x(t)|^2$ for some $\varepsilon > 0$ if (29) is feasible. □

3.2.3. Interval or non-small delay

The above conditions guarantee the stability for "small delay" $\tau(t) \in [0,h]$. Many applications motivate the stability analysis for interval (or non-small) delay $\tau(t) \in [h_0, h_1]$ with $h_0 > 0$ (see e.g.

[10,32,39]). Keeping in mind that (10) can be represented as

$$\dot{x}(t)=Ax(t)+A_1x(t-h_0)-A_1\int_{t-\tau(t)}^{t-h_0}\dot{x}(s)\,ds,$$

the stability of (10) can be analyzed via Lyapunov functionals of the form [10]

$$V(t,x_t,\dot{x}_t)=V_n(x_t,\dot{x}_t)+V_1(t,x_t,\dot{x}_t),$$

where V_n is a "nominal" functional for the "nominal" system with constant delay

$$\dot{x}(t)=Ax(t)+A_1x(t-h_0)$$

and where

$$V_1=\int_{t-h_1}^{t-h_0}x^T(s)S_1x(s)\,ds+\int_{t-\tau(t)}^{t-h_0}x^T(s)Q_1x(s)\,ds$$
$$+(h_1-h_0)\int_{-h_1}^{-h_0}\int_{t+\theta}^{t}\dot{x}^T(s)R_1\dot{x}(s)\,ds\,d\theta,$$
$$S_1>0,\quad Q_1>0,\quad R_1>0.$$

In the case where the nominal system is stable for all constant delays from $[0,h_0]$, V_n can be chosen in the form of (23), where $h=h_0$ and $Q=0$. Then the stability conditions in terms of LMIs can be derived by using arguments of Proposition 1.

3.3. Time-dependent Lyapunov functionals for sampled-data systems

Consider the sampled-data system given by (1), i.e.

$$\dot{x}(t)=Ax(t)+A_1x(t_k),\quad x(t)\in\mathbb{R}^n,\ k=0,1,\ldots,\tag{33}$$

where $A_1=BK$. It is assumed that the sampling intervals may be variable and uncertain (e.g. due to packet dropouts in networked control systems). However, it is assumed that $t_{k+1}-t_k\le h$, where $h>0$ is a known upper bound.

Till [12] the conventional time-independent Lyapunov functionals $V(x_t,\dot{x}_t)$ for systems with fast-varying delays were applied to sampled-data systems [18]. These functionals did not take advantage of the sawtooth evolution of the delays induced by sampled-and-hold. The latter drawback was removed in [12], where time-dependent Lyapunov functionals (inspired by [56]) were introduced for sampled-data system.

Lemma 2. *Let there exist positive numbers α, β and a functional $V:\mathbb{R}\times W[-h,0]\times L_2[-h,0]\to\mathbb{R}_+$ such that*

$$\alpha|\phi(0)|^2\le V(t,\phi,\dot{\phi})\le\beta\|\phi\|_W^2.$$

Consider the function $\overline{V}(t)=V(t,x_t,\dot{x}_t)$, which is continuous from the right for $x(t)$ satisfying (33), absolutely continuous for $t\ne t_k$ and which satisfies

$$\lim_{t\to t_k^-}\overline{V}(t)\ge\overline{V}(t_k).\tag{34}$$

If along (33) $(d/dt)\overline{V}(t)\le-\varepsilon|x(t)|^2,\ t\ne t_k$ for some scalar $\varepsilon>0$, then (33) is asymptotically stable.

Consider the following simple Lyapunov functional:

$$V_s(t,x(t),\dot{x}_t)=\overline{V}(t)=x^T(t)Px(t)+V_U(t,\dot{x}_t),\quad P>0,\tag{35}$$

where

$$V_U(t,\dot{x}_t)=(h-\tau(t))\int_{t-\tau(t)}^{t}\dot{x}^T(s)U\dot{x}(s)\,ds,$$
$$\tau(t)=t-t_k,\quad U>0.\tag{36}$$

The discontinuous term V_U does not increase along the jumps since $V_U\ge0$ and V_U vanishes after the jumps because $x(t)_{|t=t_k}=x(t-\tau(t))_{|t=t_k}$, i.e. (34) holds.

Since $(d/dt)x(t-\tau(t))=(1-\dot{\tau}(t))\dot{x}(t-\tau(t))=0$, we find

$$\frac{d}{dt}V_U(t,\dot{x}_t)=-\int_{t-\tau(t)}^{t}\dot{x}^T(s)U\dot{x}(s)\,ds+(h-\tau(t))\dot{x}^T(t)U\dot{x}(t)$$

and thus

$$\frac{d}{dt}\overline{V}(t)\le2\dot{x}^T(t)Px(t)-\int_{t-\tau(t)}^{t}\dot{x}^T(s)U\dot{x}(s)\,ds+(h-\tau(t))\dot{x}^T(t)U\dot{x}(t).\tag{37}$$

Denoting

$$v_1=\frac{1}{\tau(t)}\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds,$$

we understand by $v_{1|\tau(t)=0}$ the following: $\lim_{\tau(t)\to0}v_1=\dot{x}(t)$. We apply further Jensen's inequality

$$\int_{t-\tau(t)}^{t}\dot{x}^T(s)U\dot{x}(s)\,ds\ge\tau(t)v_1^TUv_1,$$

and the descriptor method, where

$$0=2[x^T(t)P_2^T+\dot{x}^T(t)P_3^T][(A+A_1)x(t)-\tau(t)A_1v_1-\dot{x}(t)],$$

with some $n\times n$ matrices P_2,P_3, is added to (37). Setting $\eta_1(t)=col\{x(t),\dot{x}(t),v_1\}$, we obtain that $(d/dt)\overline{V}(t)\le\eta_1^T(t)\Psi_s\eta_1(t)<-\varepsilon|x(t)|^2$ for some $\varepsilon>0$ if

$$\Psi_s=\begin{bmatrix}\Phi_{11} & P-P_2^T+(A+A_1)^TP_3 & -\tau(t)P_2^TA_1\\ * & -P_3-P_3^T+(h-\tau(t))U & -\tau(t)P_3^TA_1\\ * & * & -\tau(t)U\end{bmatrix}<0,$$

where $\Phi_{11}=P_2^T(A+A_1)+(A+A_1)^TP_2$. The latter matrix inequality for $\tau(t)\to0$ and $\tau(t)\to h$ leads to two LMIs:

$$\begin{bmatrix}\Phi_{11} & P-P_2^T+(A+A_1)^TP_3\\ * & -P_3-P_3^T+hU\end{bmatrix}<0,$$
$$\begin{bmatrix}\Phi_{11} & P-P_2^T+(A+A_1)^TP_3 & -hP_2^TA_1\\ * & -P_3-P_3^T & -hP_3^TA_1\\ * & * & -hU\end{bmatrix}<0.\tag{38}$$

We arrived at the following:

Proposition 2. *Let there exist $n\times n$ matrices $P>0$, $U>0$, P_2 and P_3 such that the LMIs (38) are feasible. Then (33) is asymptotically stable for all variable sampling instants $t_{k+1}-t_k\le h$.*

The conditions of Proposition 2 cannot be applied to (33) with A_1 from uncertainty polytope, since in the matrix Ψ_s the matrix A_1 is multiplied by $\tau(t)$. Moreover, additional terms in the Lyapunov functional may further improve the results. See [12,63] for various time-dependent construction of V. A different discontinuous in time Lyapunov functional was suggested in [46] which is based on *Wirtinger's inequality* [49]:

Let $z(t):(a,b)\to\mathbb{R}^n$ be absolutely continuous with $\dot{z}\in L_2[a,b]$ and $z(a)=0$. Then for any $n\times n$ matrix $W>0$ Wirtinger's inequality holds:

$$\int_a^b z^T(\xi)Wz(\xi)\,d\xi\le\frac{4(b-a)^2}{\pi^2}\int_a^b\dot{z}^T(\xi)W\dot{z}(\xi)\,d\xi.\tag{39}$$

Consider the following Lyapunov functional:

$$V(t,x_t,\dot{x}_t)=x^T(t)Px(t)+V_W(t,x_t,\dot{x}_t),\quad P>0,$$

where the Wirtinger-based term is given by

$$V_W(t,x_t,\dot{x}_t)=h^2\int_{t_k}^{t}\dot{x}^T(s)W\dot{x}(s)\,ds$$
$$-\frac{\pi^2}{4}\int_{t_k}^{t}[x(s)-x(t_k)]^TW[x(s)-x(t_k)]\,ds,$$
$$W>0,\ t_k\le t<t_{k+1},\ k=0,1,2,\ldots$$

Since $[x(s)-x(t_k)]_{|s=t_k}=0$, by Wirtinger's inequality (39) $V_W \geq 0$. Moreover, V_W vanishes at $t=t_k$, i.e. the condition (34) holds.

Setting $v(t)=x(t_k)-x(t)$ and differentiating V_W, we have

$$\frac{d}{dt}V_W = h^2\dot{x}^T(t)W\dot{x}(t) - \frac{\pi^2}{4}v^T(t)Wv(t).$$

Then we arrive at the following stability condition (that recovers result of [49,53] derived via the small-gain theorem):

$$\begin{bmatrix} P(A+A_1)+(A+A_1)^TP & \frac{2h}{\pi}PA_1 & (A+A_1)^TW \\ * & -W & \frac{2h}{\pi}A_1^TR \\ * & * & -W \end{bmatrix} < 0, \quad P>0, \; W>0.$$

Example 1. Consider the scalar system

$$\dot{x}(t) = -x(t_k), \quad t_k \leq t < t_{k+1}, \; k=0,1,\ldots \tag{40}$$

We remind that the system $\dot{x}(t)=-x(t-\tau(t))$ with constant delay $\tau(t)$ is asymptotically stable for $\tau(t)<\pi/2$ and unstable for $\tau(t)>\pi/2$, whereas for the fast varying delay it is stable for $\tau(t)<1.5$ and there exists a destabilizing delay with an upper bound greater than 1.5 [41]. The latter means that all the existing methods, that are based on time-independent Lyapunov functionals, corresponding to stability analysis of systems with fast varying delays, cannot guarantee the stability for the samplings which may be greater than 1.5. Conditions of Proposition 1 guarantee asymptotic stability for all fast varying delays from the interval [0, 1.33].

By using discretization it can be easily found that the system remains asymptotically stable for all constant samplings less than 2 and becomes unstable for samplings greater than 2. By Wirtinger-based LMI, for all variable samplings up to 1.57 the system remains asymptotically stable. By Proposition 2 for all variable samplings up to 1.99 the system remains asymptotically stable.

The Wirtinger-based LMI is a single LMI with fewer decision variables than (38). More important, differently from the Lyapunov functionals of [12,63], the extension of the Wirtinger-based Lyapunov functional to a more general sampled-data system [46]

$$\dot{x}(t) = Ax(t)+BKx(t_k-\eta), \quad t\in[t_k,t_{k+1}) \tag{41}$$

with a constant delay $\eta>0$ leads to efficient stability conditions [46]. Note that taking into account (in an elegant and efficient manner) the special structure of delay in (41) with variable $\eta=\eta_k$ is still an open problem. Such kind of systems arises in networked control systems [23,45].

Discontinuous in time Lyapunov functionals appeared to be efficient for hybrid TDSs [47,48].

4. General Lyapunov functionals for LTI TDSs

A necessary condition for the application of the *simple* Lyapunov–Krasovskii functionals considered in the previous sections is the asymptotic stability of (10) with $\tau=0$. Consider e.g. the following system with a constant delay:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-h), \quad x(t)\in\mathbb{R}^2.$$

This system is unstable for $h=0$ and is asymptotically stable for the constant delay $h\in(0.1002, 1.7178)$ [28]. For analysis of such systems (particularly, for using delay for stabilization) the simple Lyapunov functionals considered in the previous sections are not suitable. One can use a general Lyapunov functional

$$V(x_t) = x(t)^TPx(t)+2x^T(t)\int_{-h}^{0}Q(\xi)x(t+\xi)\,d\xi + \int_{-h}^{0}\int_{-h}^{0}x^T(t+s)R(s,\xi)x(t+\xi)\,ds\,d\xi \tag{42}$$

(that corresponds to necessary and sufficient conditions for stability). However, this leads to a complicated system of PDEs with respect to P,Q,R (see e.g. [50]). LMI sufficient conditions via a general Lyapunov functional and *discretization* were found by Gu et al. [28]. See also recent results via augmented Lyapunov functional and improved integral inequality by Seuret and Gouaisbaut [64].

4.1. Necessary stability conditions and general Lyapunov functionals

Let the system with a constant delay $h>0$

$$\dot{x}(t) = Ax(t)+A_1x(t-h), \quad x(t)\in\mathbb{R}^n \tag{43}$$

be asymptotically (and thus exponentially) stable. Given an $n\times n$ matrix $W>0$, we look for V_W such that

$$\frac{d}{dt}V_W(x_t) = -x^T(t)Wx(t), \quad W>0, \tag{44}$$

and $V_W(0)=0$, where $x(t)=x(t,\phi)$ is a solution of (43) with $x_0=\phi\in C[-h,0]$. Note that

$$x(t) = X(t)\phi(0)+\int_{-h}^{0}X(t-\theta-h)A_1\phi(\theta)\,d\theta,$$

where $X(t)$ is the fundamental matrix of (43). The latter matrix satisfies (43) and also

$$\dot{X}(t) = X(t)A+X(t-h)A_1, \quad X(0)=I, \quad X(t)=0 \; (t<0).$$

Since (43) is exponentially stable, the fundamental matrix exponentially converges to zero in the sense that $|X(t)|\leq ce^{-\alpha t}$ for some $c\geq 1$ and $\alpha>0$.

Then

$$\int_0^\infty \frac{d}{dt}V_W(x_t)\,dt = -\int_0^\infty x^T(t)Wx(t)\,dt.$$

Since $V_W(x_\infty)=0$, we obtain $\int_0^\infty (d/dt)V_W(x_t)\,dt = -V_W(\phi)$ and

$$\begin{aligned} 0\leq V_W(\phi) &= \int_0^\infty x^T(t)Wx(t)\,dt \\ &= \phi^T(0)U_W(0)\phi(0) \\ &\quad +2\phi^T(0)\int_{-h}^{0}U_W(-h-\theta)A_1\phi(\theta)\,d\theta \\ &\quad +\int_{-h}^{0}\phi^T(\theta_2)A_1^T\int_{-h}^{0}U_W(\theta_2-\theta_1)A_1\phi(\theta_1)\,d\theta_1\,d\theta_2, \end{aligned}$$

where

$$U_W(\theta) = \int_0^\infty X^T(t)WX(t+\theta)\,dt < \infty. \tag{45}$$

Note that the latter integral converges due to the exponential convergence of $X(t)$ to zero.

Since for the autonomous system (43) $x(s+t,\phi)=x(s,x(t+\cdot,\phi))$, we have

$$\begin{aligned} V_W(x(t+\cdot,\phi)) &= \int_0^\infty x^T(s+t,\phi)Wx(s+t,\phi)\,ds \\ &= \int_t^\infty x^T(\theta,\phi)Wx(\theta,\phi)\,d\theta. \end{aligned}$$

Differentiating in t the latter equation we derive (44). Given $W>0$, it is easily seen that V_W is quadratically upper bounded: there exists $\beta>0$ such that $V_W(\phi)\leq\beta\|\phi\|_C^2$ for some $\beta>0$. However, as was shown in [33], this functional has a *cubic* lower bound.

A more general Lyapunov functional was introduced in [40]

$$\dot{V}(x_t) = -x^T(t)W_1x(t)-x^T(t-h)W_2x(t-h) - \int_{-h}^{0}x^T(t+s)W_3x(t+s)\,ds \tag{46}$$

with some $W_i>0$, $i=1,2,3$, leading to the following *complete* Lyapunov functional

$$V(\phi)=\phi^T(0)U(0)\phi(0)+2\phi^T(0)\int_{-h}^{0}U(-h-\theta)A_1\phi(\theta)\,d\theta$$
$$+\int_{-h}^{0}\phi^T(\theta_2)A_1^T\int_{-h}^{0}U(\theta_2-\theta_1)A_1\phi(\theta_1)\,d\theta_1\,d\theta_2$$
$$+\int_{-h}^{0}\phi^T(\theta)[W_2+(h+\theta)W_3]\phi(\theta)\,d\theta, \quad (47)$$

where $U(\theta)=U_{W_1+W_2+hW_3}(\theta)$.

It was proved in [40] that if the system (10) is asymptotically stable, then the complete Lyapunov functional has a *quadratic lower bound* $V(\phi)\geq\epsilon|\phi(0)|^2$ for some $\varepsilon>0$ and satisfies the derivative condition (46). Moreover, the Lyapunov matrix U_W can be found from the boundary value problem for a matrix linear ODE. Therefore, the complete Lyapunov functional can be found by fixing some $n\times n$ matrices $W_i>0, i=1,2,3$, and solving the resulting boundary value problems for the ODE and substituting the resulting U into (47).

In the case of multiple discrete delays, the complete Lyapunov functional has a form similar to (47). However, only in the case of commensurate delays the corresponding Lyapunov matrices can be found from the boundary value problems for ODEs.

Remark 1. The complete Lyapunov functional can be used for the robust stability analysis of linear uncertain systems provided the nominal LTI delayed system is asymptotically stable. See [40] for systems with uncertain matrices, [39] for systems with non-small slowly varying delays, and [11,16] for systems with non-small fast-varying delays. Note that for application of complete Lyapunov functionals one has to fix some matrices (like W_1 and W_2 above), which may lead to conservative results. An interesting application of complete Lyapunov functional to explicit necessary and sufficient stability conditions was suggested recently in [54]. See [37] for exhaustive treatment of complete Lyapunov functionals.

4.2. About the discretized Lyapunov functional method

As follows from the previous subsection, a general quadratic Lyapunov functional corresponding to necessary stability conditions for (43) with a *quadratic lower bound* has a form of

$$V(x_t)=x^T(t)Px(t)+2x^T(t)\int_{-h}^{0}Q(\xi)x(t+\xi)\,d\xi$$
$$+\int_{-h}^{0}\int_{-h}^{0}x^T(t+s)R(s,\xi)\,ds\,x(t+\xi)\,d\xi$$
$$+\int_{-h}^{0}x^T(t+\xi)S(\xi)x(t+\xi)\,d\xi, \quad (48)$$

where $0<P\in\mathbb{R}^n$ and where $n\times n$ matrix functions

$$Q(\xi),R(\xi,\eta)=R^T(\eta,\xi)\quad\text{and}\quad S(\xi)=S^T(\xi)$$

are absolutely continuous. For the sufficiency of (48), one has to formulate conditions for $V\geq\alpha_0|x(t)|^2$, $\alpha_0>0$ and $\dot V\leq-\alpha|x(t)|^2$, $\alpha>0$.

LMI sufficient conditions via general Lyapunov functional of (48) and *discretization* were found in [26], where $Q(\xi),R(\xi,\eta)=R^T(\eta,\xi)$ and $S(\xi)=S^T(\xi)\in\mathbb{R}^{n\times n}$ were continuous and piecewise-linear matrix-functions. The resulting LMI stability conditions appeared to be very efficient, leading in some examples to results close to analytical ones. For the discretized Lyapunov functional method see Section 5.7 of [28].

Till [9] no design problems were solved by this method due to bilinear terms in the resulting matrix inequalities. The latter terms arise from the substitution of $\dot x(t)$ by the right-hand side of the differential equation in $\dot V$. The descriptor discretized method suggested in [9] avoids this substitution. The descriptor discretized method was applied to state-feedback design of H_∞ controllers for neutral type systems with discrete and distributed delays [22] and to dynamic output-feedback H_∞ control of retarded systems with state, input and output delays [71]. For differential-algebraic systems with delay, the corresponding general Lyapunov–Krasovskii functionals were studied in [27].

4.3. Simple, augmented and general Lyapunov functionals

Consider a modified complete Lyapunov functional as suggested in [13]

$$\dot V(x_t)=-x^T(t)W_1x(t)-x^T(t-h)Sx(t-h)-\int_{-h}^{0}\dot x^T(t+s)R_0\dot x(t+s)\,ds \quad (49)$$

with some $W_1>0,S>0,R_0>0$, where $x(t+s)$ in the integral term of (46) is replaced by $\dot x(t+s)$ and $W_3=R_0$. This functional is defined for solutions of (43) with absolutely continuous initial functions $\phi\in W[-h,0]$. By changing the order of integrals we have

$$\int_0^\infty\int_{-h}^{0}\dot x^T(t+s)R_0\dot x(t+s)\,ds\,dt$$
$$=h\int_0^\infty\dot x^T(s)R_0\dot x(s)\,ds+\int_{-h}^{0}(s+h)\dot x^T(s)R_0\dot x(s)\,ds.$$

The form of the functional

$$V_{R_0}(\phi)=\int_0^\infty\dot x^T(s,\phi)R_0\dot x(s,\phi)\,ds \quad (50)$$

can be found by following the arguments of [11].

Denote

$$U_1(\theta)\triangleq\int_0^\infty\dot X^T(t)R_0\dot X(t+\theta)\,dt$$
$$=\int_0^\infty[A^TX^T(t)+A_1^TX^T(t-h)]R_0[X(t+\theta)A+X(t+\theta-h)A_1]\,dt,$$
$$\theta\in\mathbb{R}.$$

Let U_{R_0} be defined by (45) with $W=R_0$. It can be shown that

$$V_{R_0}(\phi)=\int_0^\infty\dot x^T(s,\phi)R_0\dot x(s,\phi)\,ds$$
$$=\phi^T(0)U_1(0)\phi(0)+2\phi^T(0)\int_{-h}^{0}U_1(-h-\theta)A_1\phi(\theta)\,d\theta$$
$$+\int_{-h}^{0}\phi^T(\theta_2)A_1^T\int_{-h}^{0}U_1(\theta_2-\theta_1)A_1\phi(\theta_1)\,d\theta_1\,d\theta_2+\tilde V,$$

where

$$\tilde V=\int_{-h}^{0}\phi^T(\theta_2)A_1^TR_0\Big[A_1\phi(\theta_2)+2[Ae^{A(\theta_2+h)}\phi(0)$$
$$+\int_{-h}^{\theta_2}Ae^{A(\theta_2-\theta_1)}A_1\phi(\theta_1)\,d\theta_1]\Big]d\theta_2.$$

Thus the functional defined by (49) has a form

$$V(\phi)=\phi^T(0)U(0)\phi(0)+2\phi^T(0)\int_{-h}^{0}U(-h-\theta)A_1\phi(\theta)\,d\theta$$
$$+\int_{-h}^{0}\phi^T(\theta_2)A_1^T\int_{-h}^{0}U(\theta_2-\theta_1)A_1\phi(\theta_1)\,d\theta_1\,d\theta_2$$
$$+\int_{-h}^{0}\phi^T(\theta)S\phi(\theta)d\theta+\int_{-h}^{0}\int_{\theta}^{0}\dot\phi^T(s)R_0\dot\phi(s)\,ds\,d\theta+h\overline{V}, \quad (51)$$

where $U(\theta)=U_{W_1+S}(\theta)+hU_1(\theta)$. The following can be proved [13]:

Proposition 3. *Let the system (43) be asymptotically stable. For all $n\times n$ matrices $W_1>0,S>0$ and $R_0>0$, and for small enough $\varepsilon>0$, the Lyapunov functional (51) satisfies (49) and $V(\phi)\geq\epsilon|\phi(0)|^2$.*

A general quadratic Lyapunov functional corresponding to (51) has a form of

$$V(x_t) = x^T(t)Px(t) + 2x^T(t)\int_{-h}^{0} Q(\xi)x(t+\xi)\,d\xi + \int_{-h}^{0}\int_{-h}^{0} x^T(t+s)R(s,\xi)\,ds\,x(t+\xi)\,d\xi + \int_{t-h}^{t} x^T(\xi)Sx(\xi)\,d\xi + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_0\dot{x}(s)\,ds\,d\theta, \quad (52)$$

where $P>0, S>0, R_0>0$. Matrix-functions $Q(\xi)\in\mathbb{R}^{n\times n}$ and $R(\xi,\eta)=R^T(\eta,\xi)\in\mathbb{R}^{n\times n}$ are absolutely continuous. For the sufficiency of (52), one has to formulate conditions for $V\ge\beta|x(t)|^2$, $\beta>0$ and $\dot{V}\le-\alpha|x(t)|^2$, $\alpha>0$.

Choosing in (52) $R=Q=0$ and replacing R_0 by hR we arrive at the simple Lyapunov functional (23), where $Q=0$.

Consider now (52) with constant $R\equiv Z$ and Q, and replace R_0 by hR. Then we arrive at the augmented Lyapunov functional of the form

$$V(x_t,\dot{x}_t) = \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(s)\,ds \end{bmatrix}^T \begin{bmatrix} P & Q \\ * & Z \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(s)\,ds \end{bmatrix} + \int_{t-h}^{t} x^T S x\,ds + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds\,d\theta,$$

$$\begin{bmatrix} P & Q \\ * & Z \end{bmatrix} > 0, \quad S>0, \quad R>0. \quad (53)$$

Note that the term $Q\neq0$ in (53) allows us to derive non-convex in h conditions that do not imply the stability of the original system with $h=0$. A remarkable result was obtained by Seuret and Gouaisbaut [64] for systems with constant discrete and distributed delays: by deriving an extended integral inequality, which includes Jensen's inequality as a particular case, and applying the augmented Lyapunov functional (53) the authors arrived at LMIs that may guarantee the stability of systems which are unstable with the zero delay (i.e. in the case of "stabilizing delay").

5. Lyapunov functionals for systems with distributed delays

Consider a linear system with the distributed delay

$$\dot{x}(t) = Ax(t) + A_d\int_{-h_d}^{0} x(t+s)\,ds, \quad x(t)\in\mathbb{R}^n,\ h_d>0, \quad (54)$$

where A and A_d are constant $n\times n$ matrices, $h_d<\infty$. We study stability in two cases: (1) A and $A+h_dA_d$ are Hurwitz, (2) A or $A+h_dA_d$ are Hurwitz.

In the 1st case the delayed term can be treated as a disturbance by using the following Lyapunov functional:

$$V_0(x_t) = x^T(t)Px(t) + V_{R_d}(x_t),$$

$$V_{R_d}(x_t) = h_d\int_{-h_d}^{0}\int_{t+\theta}^{t} x^T(\tau)R_dx(\tau)\,d\tau\,d\theta, \quad P>0, R_d>0.$$

In the 2nd case, keeping in mind that (54) can be represented in the following form:

$$\dot{x}(t) = (A+h_dA_d)x(t) + A_d\int_{t-h_d}^{t} [x(s)-x(t)]\,ds, \quad (55)$$

we have to "compensate" the perturbation given by the integral term in (55). This can be done by adding to V_0 a triple integral term as suggested in [2,68]

$$V(x_t,\dot{x}_t) = V_0(x_t) + V_{Z_d}(\dot{x}_t),$$

$$V_{Z_d}(\dot{x}_t) = \int_{-h_d}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^T(s)Z_d\dot{x}(s)\,ds\,d\lambda\,d\theta, \quad Z_d>0.$$

Then after differentiation we have

$$\frac{d}{dt}V(x_t,\dot{x}_t) = 2x^T(t)P\dot{x}(t) + h_d^2x^T(t)R_dx(t) + \frac{h_d^2}{2}\dot{x}^T(t)Z_d\dot{x}(t) - h_d\int_{t-h_d}^{t} x^T(s)R_dx(s)\,ds - \int_{-h_d}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_d\dot{x}(s)\,ds\,d\theta$$

Application of Jensen's inequality to $\int_{t-h_d}^{t} x^T(s)R_dx(s)\,ds$ and the extended Jensen's inequality to the double integral term [68]

$$-\int_{-h_d}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_d\dot{x}(s)\,ds\,d\theta \le -\frac{2}{h_d^2}\left(\int_{-h_d}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)\,ds\,d\theta\right)Z_d\left(\int_{-h_d}^{0}\int_{t+\theta}^{t} \dot{x}(s)\,ds\,d\theta\right) = -\frac{2}{h_d^2}\left(h_dx^T(t) - \int_{t-h_d}^{t} x^T(s)\,ds\right)Z_d\left(h_dx(t) - \int_{t-h_d}^{t} x(s)\,ds\right)$$

leads to the LMI stability condition for the distributed delay system (54):

$$\begin{bmatrix} PA+A^TP+h_d^2R_d-2Z_d & PA_d+\frac{2}{h_d}Z_d & A^TZ_d \\ * & -R_d-\frac{2}{h_d^2}Z_d & A_d^TZ_d \\ * & * & -\frac{2}{h_d^2}Z_d \end{bmatrix} < 0. \quad (56)$$

Example 2. Consider the system (54) with

$$A = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad (57)$$

where A is not Hurwitz. Here $A+h_dA_d$ is Hurwitz for $h_d>0.2$. By the LMI condition (56) the system (57) is asymptotically stable for any $h_d\in[0.2001, 1.6339]$.

Remark 2. Lyapunov functional constructions of this section can be easily extended to systems

$$\dot{x}(t) = Ax(t) + \int_{-h_d}^{0} A_d(s)x(t+s)\,ds, \quad x(t)\in\mathbb{R}^n,\ h_d>0$$

with variable $n\times n$ matrix kernels $A_d\in L_1(0,h_d)$. Thus, the following Lyapunov functional with a double integral term can be used [19]:

$$V(x_t) = x^T(t)Px(t) + h_d\int_{-h_d}^{0}\int_{t+s}^{t} x^T(\tau)A_d^T(s)R_dA_d(s)x(\tau)\,d\tau\,ds,$$

where $P>0, R_d>0$, leading to the following LMI:

$$\begin{bmatrix} PA+A^TP+h_d\int_{-h_d}^{0} A_d^T(s)R_dA_d(s)\,ds & P \\ * & -R_d \end{bmatrix} < 0.$$

In the latter LMI the decision variable R_d appears inside the integral. In order to verify the feasibility of this LMI by using MATLAB, one can assume that $R_d=r_dI$ (which is restrictive), where $r_d>0$ is a scalar. Another solution for the stability analysis in the case of variable kernels has been suggested in [67], where it is assumed that $A_d(s)=\sum_{i=1}^{m} A_{di}K_i(s)$ with constant matrices $A_{di}\in\mathbb{R}^{n\times n}$ and scalar kernel functions $K_i(s)$.

5.1. Systems with infinite delays

A linear system with infinite delay has a form

$$\dot{x}(t) = Ax(t) + A_d\int_{0}^{\infty} K(\theta)x(t-\theta)\,d\theta, \quad (58)$$

where $x(t)\in\mathbb{R}^n$, $A, A_d\in\mathbb{R}^{n\times n}$ are constant matrices. It is supposed that the scalar kernel function $K\in L_1[0,\infty)$ satisfies the inequality $\int_0^\infty |K(\theta)|\,d\theta<\infty$. A solution of (58) is uniquely determined for the uniformly continuous initial function $\phi \in C(-\infty, 0]$. This solution continuously depends on ϕ (see [41, Theorem 3.2.3]).

Assume that A or $A_0 = A + A_d \int_0^\infty K(\theta)\, d\theta$ are Hurwitz. The following Lyapunov functional can be applied to stability analysis of (58) [67]:

$$V(t) = V_P(t) + V_{R_d}(t) + V_{Z_d}(t), \quad V_P(t) = x^T(t)Px(t) \tag{59}$$

with

$$V_{R_d}(t) = \int_0^\infty \int_{t-\theta-\tau}^t |K(\theta)| x^T(s) R_d x(s)\, ds\, d\theta,$$

$$V_{Z_d}(t) = \int_0^\infty \int_0^{\theta+\tau} \int_{t-\lambda}^t |K(\theta)| \dot{x}^T(s) Z_d \dot{x}(s)\, ds\, d\lambda\, d\theta,$$

where P, R_d and Z_d are positive $n \times n$ matrices. Application of appropriately extended Jensen's inequalities leads to efficient LMI stability conditions [67].

A particular class of systems with infinite delays are systems with gamma-distributed delays $K(\theta) = \theta^{N-1} e^{-\theta/T} / T^N (N-1)!$, where $T > 0$ and $N = 1, 2, \ldots$ are parameters of distribution. Note that $\int_0^\infty K(\xi)\, d\xi = 1$. The corresponding average delay satisfies $\int_0^\infty \xi K(\xi)\, d\xi = NT$.

Gamma-distributed delays can be encountered in the problem of control over communication networks, in the population dynamics [4] and in the traffic flow dynamics [51]. For gamma-distributed delays by using augmented Lyapunov functionals, LMIs for the case of "stabilizing delays", where A and A_0 may be non-Hurwitz, have been derived in [67]. The latter stability problem is motivated e.g. by the traffic flow model on the ring [55], where $A = 0$ and where the zero eigenvalue of A_0 corresponds to the vehicles moving with the same velocity.

6. Stability of nonlinear systems

Till now the stability conditions for linear TDSs have been derived. This section discusses stability results for some classes of nonlinear systems.

Consider the following autonomous RDE:

$$\dot{x}(t) = Lx_t + g(x_t), \quad x(t) \in \mathbb{R}^n, \; t \geq 0, \tag{60}$$

where $L : C[-h, 0] \to \mathbb{R}^n$ is a linear bounded functional, $g : C[-h, 0] \to R^n$ is a locally Lipschitz continuous function that satisfies

$$|g(\psi)| \leq \beta(\|\psi\|_C)\|\psi\|_C \quad \forall \psi \in C[-h, 0],$$

where β is continuous and $\beta(0) = 0$.

The linear system

$$\dot{x}(t) = Lx_t, \quad t \geq 0 \tag{61}$$

is called the first approximation with respect to the original system (60). In fact, the linear system (61) can be considered as a linearization in the neighborhood of the trivial solution of the nonlinear system $\dot{x}(t) = f(x_t)$ with a smooth f such that $f(0) = 0$. As for non-delay systems, the stability of the nonlinear TDS with the asymptotically stable first approximation can be derived either by the (first) Lyapunov method with the quadratic Lyapunov function/functional or by using Gronwall's inequality. By using Gronwall's inequality the following can be proved [13, Proposition 3.17]:

Proposition 4. *If the linear system (61) is asymptotically stable, then the nonlinear system (60) is asymptotically stable.*

In the critical case, where some characteristic roots of (61) are on the imaginary axis, whereas all the others have negative real parts, either the direct Lyapunov method or the center manifold theory can be applied [31]. Thus, in the system

$$\dot{x}(t) = -ax^3(t) - a_1 x^3(t-h), \quad a > 0$$

the first approximation $\dot{x}(t) = 0$ has the zero eigenvalue. This is the critical case, where no conclusion can be done from the analysis of the first approximation. Application of the Lyapunov functional

$$V(x_t) = \frac{x^4(t)}{2a} + \int_{t-h}^t x^6(s)\, ds$$

leads to the following result [31]: the system is asymptotically delay-independently stable for $|a_1| < a$.

Lyapunov-based methods for asymptotic stability of linear systems considered in this paper can be usually extended to some quasilinear systems with e.g. Lipschitz nonlinearities. For example, consider the system

$$\dot{x}(t) = Ax(t) + A_1 x(t-\tau(t)) + g(t, x(t), x(t-\tau(t))), \quad x(t) \in \mathbb{R}^n, \; t \geq 0, \tag{62}$$

with a continuous $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which is locally Lipschitz continuous in the second and the third arguments and satisfies for all t the inequality

$$|g(t, x, y)|^2 \leq \begin{bmatrix} x^T & y^T \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix} \quad \forall x, y \in \mathbb{R}^n, \tag{63}$$

where $0 < M \in \mathbb{R}^{n \times n}$ and $M \leq \beta_0 I$, $\beta_0 \in \mathbb{R}_+$. Then, by using S-procedure together with the inequality (63) one can arrive to LMI condition for the global asymptotic stability of the quasilinear system (62).

Delay-dependent conditions have been extended to some classes of nonlinear systems (see e.g. [14]) Consider next a class of systems, affine in control $u(t) \in \mathbb{R}^m$

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t), \tag{64}$$

where $x(t) \in \mathbb{R}^n$, A and B are continuously differentiable matrix-functions. Given a state-feedback

$$u(t) = K(x(t-\tau(t))), \quad K(x) = k(x)x, \tag{65}$$

where $k : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ is a continuously differentiable function and where $\tau(t) \in [0, h]$ is the unknown piecewise-continuous delay that often appears in the feedback. We extend the relation $x(t-\tau(t)) = x(t) - \int_{t-\tau(t)}^t \dot{x}(s)\, ds$ to the nonlinear case as follows:

$$K(x(t-\tau(t))) = K(x(t)) - \int_{t-\tau(t)}^t K_x(x(s))\dot{x}(s)\, ds,$$

where $K_x = [(\partial/\partial x_1)K \ldots (\partial/\partial x_n)K]$, and represent the closed-loop system (64)–(65) in the form

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))K(x(t)) - B(x(t)) \int_{t-\tau(t)}^t K_x(x(s))\dot{x}(s)\, ds. \tag{66}$$

Note that (66) is equivalent to (64)–(65). The following Lyapunov functional

$$V(x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{t-r}^t D(s, x_t, \dot{x}_t)\, ds,$$

$$D(s, x_t, \dot{x}_t) \triangleq \int_s^t \dot{x}^T(\xi) K_x^T(x(\xi)) R K_x(x(\xi)) \dot{x}(\xi)\, d\xi, \quad P > 0, \; R > 0$$

leads to a state-dependent matrix inequality [14]. The feasibility of state-dependent LMIs may be studied by using a convex optimization approach (sum of squares) for nonlinear systems [58].

If a nonlinear system is locally (not globally) stable, it is of interest to find a domain of attraction. The direct Lyapunov method provides constructive tools for finding estimates on the domains of attraction of nonlinear TDSs [3].

7. The input–output approach to stability

Till now we have applied the direct Lyapunov approach to the stability analysis of (10). An alternative approach is the input–output approach that is based on the representation of the original system as a feedback interconnection of some auxiliary systems with additional inputs and outputs and application of the small-gain theorem. These two approaches sometimes lead to complementary results, improving each other and giving ideas for further improvements. The input–output approach was introduced for nonlinear time-varying finite-dimensional systems by Zames [72], it was extended to continuous-time linear systems with constant delays in [34,73] and with slowly varying delays in [28]. This approach was generalized to linear continuous-time with fast-varying delays and to discrete-time systems in [21,35,66].

The input–output approach is applicable to stability and to L_2-gain analysis. Note that in the feedback interconnection of the systems, the initial conditions are supposed to be zero. Therefore, this approach cannot be directly applied to the bounds on the solutions of the original system, where the initial state may be non-zero (see e.g. [15] for related solution bounds depending on the initial conditions). For solution bounds the direct Lyapunov method seems to be preferable.

Consider first the delay-independent conditions, where the delayed state $x(t-\tau(t))$ is treated as a disturbance. This corresponds to the presentation of (10) as the following forward system

$$\dot{x}(t) = Ax(t) + A_1X^{-1}u(t),\\ y(t) = Xx(t) \tag{67}$$

with the feedback

$$u(t) = y(t-\tau(t)). \tag{68}$$

Here $X \in \mathbb{R}^{n\times n}$ is a non-singular scaling matrix. Indeed, substituting (68) and $y(t) = Xx(t)$ into the differential equation (67), we obtain (10). Assume that A is Hurwitz and $y(t) = 0$ for $t \le 0$.

In the simple delay-dependent conditions for $\tau(t) \in [0,h]$, the presentation

$$\dot{x}(t) = (A+A_1)x(t) - A_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds$$

is used, where $\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds$ is treated as a disturbance. This corresponds to the presentation of (10) as the following forward system

$$\dot{x}(t) = (A+A_1)x(t) + A_1X^{-1}u(t),\\ y(t) = X\dot{x}(t) = X[(A+A_1)x(t) + A_1X^{-1}u(t)] \tag{69}$$

with the feedback

$$u(t) = -\int_{t-\tau(t)}^{t} y(s)\,ds. \tag{70}$$

Here it is assumed that $A+A_1$ is Hurwitz and $y(t) = 0$ for $t \le 0$.

In both cases the forward system can be presented as $y = \mathbf{G}u$ and the feedback as $u = \Delta y$, where $\mathbf{G} : L_2[0,\infty) \to L_2[0,\infty)$ and $\Delta : L_2[0,\infty) \to L_2[0,\infty)$. The system $\mathbf{G} : L_2[0,\infty) \to L_2[0,\infty)$ is said to be *input–output* stable if it has a finite gain $\gamma_0(\mathbf{G})$ defined by

$$\gamma_0(\mathbf{G}) = \inf\{\gamma : \|\mathbf{G}u\|_{L_2} \le \gamma\|u\|_{L_2}\ \ \forall u \in L_2[0,\infty)\}.$$

The small gain theorem claims that the interconnected system feedback $(\mathbf{G}, \Delta)$ is well defined and input–output stable if $\gamma_0(\Delta)\gamma_0(\mathbf{G}) < 1$.

The following lemma provides upper bounds on the gains of the feedback systems (68) and (70):

Lemma 3 (*Lemma 3 [28,35]*). *For Δ given by (68) and for slowly varying delays with $\dot{\tau} \le d < 1$ the following holds:*

$$\gamma_0(\Delta) \le \frac{1}{\sqrt{1-d}}. \tag{71}$$

For Δ given by (70) and fast-varying delays $\tau \in [0,h]$ the following holds:

$$\gamma_0(\Delta) \le h. \tag{72}$$

Since $\gamma_0(\mathbf{G}) = \|G\|_\infty$, by the small-gain theorem, the feedback interconnection given by (67), (68) and (69), (70) is input–output stable if $\|G\|_\infty < 1/\gamma_0(\Delta)$. Deriving further LMI conditions for the last inequality, i.e. for

$$\dot{V} + y^Ty - \frac{1}{\gamma_0^2(\Delta)}u^Tu < 0 \quad \forall u \ne 0$$

by using $V(x) = x^TPx$, $0 < P \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ we can recover the delay-independent and simple delay-dependent conditions of Section 3.

7.1. Stability of systems with non-small delays

We consider (10), where we assume that the uncertain delay $\tau(t)$ has a form

$$\tau(t) = h + \eta(t), \qquad |\eta(t)| \le \mu \le h. \tag{73}$$

Here h is a known nominal delay value and μ is a known upper bound on the delay uncertainty. The delay is supposed to be either differentiable with $\dot{\tau} \le d$, where d is known, or piecewise-continuous (fast-varying). In the latter case we will say that d is unknown (though τ may be not differentiable).

Assume that the nominal system

$$\dot{x}(t) = Ax(t) + A_1x(t-h), \quad x(t) \in \mathbb{R}^n \tag{74}$$

is asymptotically stable.

We represent (10) in the form of the forward system

$$\dot{x}(t) = Ax(t) + A_1x(t-h) + A_1X^{-1}u(t),\\ y(t) = X\dot{x}(t), \tag{75}$$

with the feedback

$$u(t) = (\Delta y)(t) = -\int_{-h-\eta(t)}^{-h} y(t+s)\,ds, \tag{76}$$

where X is a scaling non-singular matrix.

Bounds on $\gamma_0(\Delta)$ for $\dot{\tau} \le d$ with $d > 1$ were found in [66]:

Lemma 4. *For the operator $u = \Delta y$ given by (76) with $y(s) = 0,\ s < 0$ the following holds:*

$$\gamma_0(\Delta) \le \mu\sqrt{\mathcal{F}(d)}, \quad \mathcal{F}(d) = \begin{cases} 1 & \text{if } -\infty \le d \le 1, \\ \dfrac{2d-1}{d} & \text{if } 1 < d < 2, \\ \dfrac{7d-8}{4d-4} & \text{if } d \ge 2, \\ \dfrac{7}{4} & \text{if } d \text{ is unknown}. \end{cases} \tag{77}$$

Note that $\mathcal{F}$ is an increasing continuous function satisfying for $d > 1$ the following inequalities:

$$1 = \mathcal{F}(1) < \mathcal{F}(d) < \lim_{d\to\infty}\mathcal{F}(d) = 1.75.$$

The value of 1 cannot be improved. Moreover, the value 1.75 for $\mathcal{F}(\infty)$ is not far from an optimal one, and it cannot be less than 1.5.

LMI conditions that guarantee the input–output stability of (10) can be derived by using some Lyapunov functional V_n that corresponds to the nominal system (74) and that satisfies along (75) the following inequality:

$$W \triangleq \dot{V}_n + y^Ty - \mu^{-2}\mathcal{F}^{-1}(d)u^Tu < 0 \quad \forall u \ne 0.$$

7.2. Relation between input–output and exponential stability

We will show below that input–output stability of LTV TDSs implies the exponential stability of these systems. This implication is based on Bohl–Perron principle that was generalized to TDSs (see [24]). Consider the following linear homogenous system:

$$\dot{x}(t)=\sum_{k=1}^{m}A_k(t)x(t-\tau_k(t))+\int_0^h A_d(t,\theta)x(t-\theta)\,d\theta,$$
$$x(s)=\phi(s),\quad s\in[-h,0],\ \phi\in C[-h,0] \tag{78}$$

with discrete and distributed delays. Here $\tau_0=0$, A_k and A_d are $n\times n$ matrices that are piecewise-continuous in their arguments, piecewise-continuous delays τ_k are bounded by h: $0\le\tau_k\le h$. Assume further that

$$\sup_{t\ge0}\left[\sum_{k=1}^{m}|A_k(t)|+\int_0^h|A_d(t,\theta)|\,d\theta\right]<\infty. \tag{79}$$

Consider next the corresponding non-homogeneous system with the zero initial condition:

$$\dot{x}(t)=\sum_{k=1}^{m}A_k(t)x(t-\tau_k(t))+\int_0^h A_d(t,\theta)x(t-\theta)\,d\theta+f(t),$$
$$x(s)=0,\quad s\in[-h,0], \tag{80}$$

where $f(t)\in L_p[0,\infty)$ $(1\le p\le\infty)$. For the existence of a solution $x\in L_p[0,\infty)$ to (80) with the zero initial condition see [24]. The following result is obtained (see [24] for the proof):

Theorem 3 (*Bohl–Perron principle*). *If for a* $p\ge1$ *and any* $f\in L_p[0,\infty)$, *the non-homogeneous system with the zero initial condition (80) has a solution* $x\in L_p[0,\infty)$, *and condition (79) holds, then the homogeneous system (78) is exponentially stable.*

We shall apply the Bohl–Perron principle with $p=2$. Consider the linear homogeneous system (10) with an uncertain delay $\tau(t)\in[0,h]$, where the following condition

$$\|G\|_\infty<1,\quad G(s)=sX(sI-A_0-A_1)^{-1}\mu A_1X^{-1} \tag{81}$$

guarantees the input–output stability of (10). Consider also the perturbed system

$$\dot{x}(t)=Ax(t)+A_1x(t-\tau(t))+\gamma^{-1}w(t),$$
$$z(t)=\varepsilon x(t),$$
$$x(s)=0,\quad s\le0,$$

where the positive scalars ε and γ^{-1} are small enough. The inequality (81) implies

$$\|G_\gamma\|_\infty<1,$$
$$G_\gamma(s)=\begin{bmatrix}sX\\ \varepsilon I_n\end{bmatrix}(sI-A_0-A_1)^{-1}\begin{bmatrix}\mu A_1X^{-1} & \frac{I_n}{\gamma}\end{bmatrix}$$

for some small enough ε and γ^{-1}. The latter means that for all $f=\gamma w\in L_2[0,\infty)$ the solution $x(t)$ of (10) has a bounded L_2-norm $\|x\|_{L_2}\le(1/\varepsilon)\|w\|_{L_2}$, i.e. $x\in L_2[0,\infty)$. Therefore, from the Bohl–Perron principle it follows that the condition (81) satisfied for some X and ρ implies the exponential stability of (10) with a non-zero initial condition $\phi\in C[-h,0]$. By the same arguments, other conditions for the input–output stability discussed in this section guarantee the exponential stability of the corresponding linear homogeneous system.

8. Conclusions

The methods presented in this tutorial for retarded type systems have been extended in the literature to neutral systems, to descriptor TDSs and to discrete-time TDSs. Lyapunov-based methods appeared to be efficient for the performance analysis of TDSs, as well as for control design in the presence of state, input or output delays. For detailed introduction to TDSs with applications to sampled-data and network-based control see [13].

There are a lot of open problems related to stability and control of TDSs. For example (to name a few), sufficient stability conditions taking into account a particular form of $\tau(t)$ or analytical stability bounds and necessary Lyapunov-based stability conditions for some classes of systems with time-varying delays.

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