Brief paper

Converse Lyapunov–Krasovskii theorem for ISS of neutral systems in Sobolev spaces☆

Denis Efimov [a,b,*], Emilia Fridman [c]

[a] Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France
[b] ITMO University, 197101 Saint Petersburg, Russia
[c] School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel

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ABSTRACT

The conditions of existence of a Lyapunov–Krasovskii functional (LKF) for nonlinear input-to-state stable (ISS) neutral type systems are proposed. The system under consideration depends nonlinearly on the delayed state and the delayed state derivative, and satisfies the conditions for the existence and uniqueness of the solutions. The LKF and the system properties are defined in a Sobolev space of absolutely continuous functions with bounded derivatives.

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1. Introduction

Stability analysis for dynamical systems constitutes an important area of research in different domains of science and technology, and especially in the control theory (Khalil, 2015). For generic dynamical systems, the key tool to study stability is the Lyapunov function method, which for time-delay systems has two extensions based on Lyapunov–Razumikhin functions and Lyapunov–Krasovskii functionals (Fridman, 2014; Hale, 1977; Khalil, 2015; Kolmanovskii & Myshkis, 1999; Kolmanovsky & Nosov, 1986). These approaches have also their development for systems with inputs, where one of the most popular concepts is ISS (Dashkovskiy, Efimov, & Sontag, 2011). It is a well-known fact that existence of a Lyapunov function is necessary and sufficient for asymptotic stability and ISS in the case of ordinary differential equations (Khalil, 2015). For time-delay systems, the conditions of such an equivalence appeared rather recently (for instance, check (Huang, 1989; Kharitonov & Zhabko, 2003) for asymptotic

stability, and (Karafyllis & Jiang, 2011; Karafyllis & Pepe, 2015; Lin & Wang, 2018a, 2018b; Mironchenko & Wirth, 2018; Pepe & Karafyllis, 2013) for ISS case). For the class of nonlinear neutral type time-delay systems, the papers (Pepe & Karafyllis, 2013; Pepe, Karafyllis, & Jiang, 2017) develop the equivalent conditions of asymptotic stability and ISS in terms of existence of LKF. In these works, the systems in the Hale's form have been analyzed, and an implicit expression of LKF has been proposed. A two steps procedure from Karafyllis and Jiang (2011) is used in Pepe and Karafyllis (2013) and Pepe et al. (2017), which at the last iteration includes an infinite summation over a partition of unity, then the form of dependence of LKF on the state function is complicated and not intuitive. Studying uniform asymptotic stability these results have been extended in Efimov and Fridman (2019) to a different class of neutral time-delay systems (not in the Hale's form) with a coercive LKF given in an explicit form. Differently from Karafyllis and Jiang (2011), Pepe and Karafyllis (2013) and Pepe et al. (2017), where the stability has been analyzed in the space $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ (the Sobolev space of continuous functions with essentially bounded derivatives), while also frequently the space $\mathbb{W}^{1,2}_{[-\tau,0]}$ is used (Fridman, 2014) (the state derivative is square integrable), in Efimov and Fridman (2019) the mathematical treatment has been performed in $\mathbb{W}^{1,1}_{[-\tau,0]}$ (the Sobolev space of continuous functions with integrable derivatives). The technical advantage of such a change consists in the established Lipschitz continuity of the solutions of the neutral type systems with locally Lipschitz continuous right-hand side in $\mathbb{W}^{1,1}_{[-\tau,0]}$ (in $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ such a property has been proven for the systems in the Hale's form only).

* Corresponding author at: Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France.
E-mail addresses: efde@mail.ru (D. Efimov), emilia@tauex.tau.ac.il (E. Fridman).

The main contribution of the present work consists in the formulation of equivalent conditions of ISS for neutral type nonlinear time-delay systems with essentially bounded inputs in $\mathbb{W}^{1,1}_{[-\tau,0]}$. As in Mironchenko and Wirth (2018) and Pepe et al. (2017), two types of the conditions are given: in terms of existence of an LKF, and in terms of uniform global asymptotic stability of an auxiliary system. A conference version of the paper restricted to the case of asymptotic stability was presented in Efimov and Fridman (2019).

The outline of this paper is as follows. Some preliminary results are introduced in Section 2. The problem statement is given in Section 3. The main result is established in Section 4.

2. Preliminaries

Denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real and natural numbers, respectively, and $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$.

2.1. Definitions of norms and spaces

Denote by $\mathbb{C}^n_{[a,b]}$, $[a,b] \subset \mathbb{R}$ the Banach space of continuous functions $\phi : [a,b] \to \mathbb{R}^n$ with the uniform norm $\|\phi\|_{[a,b]} = \sup_{s \in [a,b]} |\phi(s)|$, where $|\cdot|$ is the standard Euclidean norm in $\mathbb{R}^n$.

For a Lebesgue measurable function of time $d : [a,b] \to \mathbb{R}^m$, $[a,b] \subset \mathbb{R}$, define the norm $\|d\|_{[a,b]} = \text{ess sup}_{s \in [a,b]} |d(s)|$, then $\|d\|_\infty = \text{ess sup}_{s \geq 0} |d(s)|$ and the space of d with $\|d\|_{[a,b]} < +\infty$ ($\|d\|_\infty < +\infty$) we further denote as $\mathcal{L}^m_{[a,b]}$ ($\mathcal{L}^m_\infty$).

Denote by $\mathbb{W}^{1,p}_{[a,b]}$, $p \in \mathbb{N}$ and $\mathbb{W}^{1,\infty}_{[a,b]}$ the Sobolev spaces of absolutely continuous functions $\phi : [a,b] \to \mathbb{R}^n$, $[a,b] \subset \mathbb{R}$, with bounded derivatives having the respective norms $\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}} = \|\phi\|_{[a,b]} + \left(\int_a^b |\dot\phi(s)|^p ds \right)^{\frac{1}{p}} < +\infty$ and $\|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}} = \|\phi\|_{[a,b]} + \|\dot\phi\|_{[a,b]} < +\infty$, where $\dot\phi(s) = \frac{\partial \phi(s)}{\partial s}$ (it is a Lebesgue measurable essentially bounded function for $\phi \in \mathbb{W}^{1,\infty}_{[a,b]}$, i.e. $\dot\phi \in \mathcal{L}^n_{[a,b]}$).[1]

Lemma 1 (*Efimov & Fridman, 2019*). *For any* $\phi \in \mathbb{W}^{1,\infty}_{[a,b]}$ *and* $p \in \mathbb{N} \cup \{+\infty\}$ *the following inequalities are satisfied:*

$$\min\{1, (b-a)^{\frac{1}{p}-1}\} \|\phi\|_{\mathbb{W}^{1,1}_{[a,b]}} \leq \|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}}$$
$$\leq \max\{1, (b-a)^{\frac{1}{p}}\} \|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}}.$$

Proof. From the norm definition we deduce:

$$\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}} = \|\phi\|_{[a,b]} + \left(\int_a^b |\dot\phi(s)|^p ds \right)^{\frac{1}{p}}$$
$$\leq \|\phi\|_{[a,b]} + \left(\int_a^b \|\dot\phi\|^p_{[a,b]} ds \right)^{\frac{1}{p}}$$
$$\leq \max\{1, (b-a)^{\frac{1}{p}}\} \|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}},$$

and using integral Jensen's inequality (Khalil, 2015):

$$\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}} = \|\phi\|_{[a,b]} + \left(\int_a^b |\dot\phi(s)|^p ds \right)^{\frac{1}{p}}$$
$$\geq \|\phi\|_{[a,b]} + (b-a)^{\frac{1}{p}-1} \int_a^b |\dot\phi(s)| ds$$
$$\geq \min\{1, (b-a)^{\frac{1}{p}-1}\} \|\phi\|_{\mathbb{W}^{1,1}_{[a,b]}}$$

that was necessary to prove.

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[1] In Fridman, Dambrine, and Yeganefar (2008) and Kolmanovsky and Nosov (1986) the norm with $p = 2$ has been only used for the state space of time-delay systems.

2.2. Comparison functions and their properties

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded.

Lemma 2 (*Karafyllis & Jiang, 2011*). *For any* $\alpha \in \mathcal{K}$ *there exists a continuous function* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *admitting the following properties:* $\gamma(0) = 0$, $\gamma(s) > 0$ *for all* $s > 0$, *and*

$$\gamma(s) \leq \alpha(s), \quad |\gamma(s) - \gamma(s')| \leq |s - s'| \quad \forall s, s' \in \mathbb{R}_+.$$

In addition,

$$\lim_{s \to +\infty} \gamma(s) = +\infty$$

provided that $\alpha \in \mathcal{K}_\infty$.

A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is a strictly decreasing to zero for any fixed $r \in \mathbb{R}_+$.

Lemma 3 (*Sontag, 1998*). *For any* $\beta \in \mathcal{KL}$ *there exist* $\theta_1, \theta_2 \in \mathcal{K}_\infty$ *such that*

$$\beta(s, t) \leq \theta_1 \left(\theta_2(s) e^{-t} \right) \quad \forall s \geq 0, \ t \geq 0.$$

2.3. Neutral systems under consideration

Consider an autonomous functional differential equation of the neutral type with inputs (Kolmanovsky & Nosov, 1986):

$$\dot{x}(t) = f(x_t, \dot{x}_t, d(t)), \ t \geq 0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $x_t \in \mathbb{C}^n_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$, with $\dot{x}_t \in \mathcal{L}^n_{[-\tau,0]}$; $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}^m_\infty$. The function $f : \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and Lipschitz in the second variable with a constant smaller than 1, ensuring forward uniqueness and existence of the system solutions at least locally in time (Kolmanovsky & Nosov, 1986). We assume $f(0,0,0) = 0$. For the initial function $x_0 \in \mathbb{C}^n_{[-\tau,0]}$ and disturbance $d \in \mathcal{L}^m_\infty$ denote a unique solution of the system (1) by $x(t, x_0, d)$, which is an absolutely continuous function defined on some time interval $[-\tau, T)$ for $T > 0$, then $x_t(x_0, d) \in \mathbb{C}^n_{[-\tau,0]}$ represents the corresponding state function, and $x_t(s, x_0, d) = x(t+s, x_0, d)$ for all $-\tau \leq s \leq 0$.

Given a continuous functional $V : \mathbb{R}_+ \times \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ define:

$$D^+ V(t, \phi, \dot\phi, d) = \limsup_{h \to 0^+} \frac{1}{h} [V(t+h, x_h(\phi, d), \dot{x}_h(\phi, d))$$
$$- V(t, \phi, \dot\phi)],$$

where $x_h(\phi, d)$ is a solution of the system (1) for $\phi \in \mathbb{C}^n_{[-\tau,0]}$, $\dot\phi \in \mathcal{L}^n_{[-\tau,0]}$ and $d \in \mathbb{R}^m$ is a constant.

2.4. Uniform stability

Denote $\mathcal{D} = \{d \in \mathcal{L}^m_\infty : \|d\|_\infty \leq 1\}$ and $\mathbb{D} = \{d \in \mathbb{R}^m : |d| \leq 1\}$ in order to analyze the behavior of (1) with a bounded input d. Then in the following we summarize the well-known results (Fridman, 2014; Kolmanovskii & Myshkis, 1999) on stability of (1).

Definition 1 (*Fridman et al., 2008; Lin & Wang, 2018b; Pepe & Jiang, 2006*). *The system* (1) *is called uniformly globally asymptotically stable* (uGAS), *if for all* $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ *and* $d \in \mathcal{D}$ *there exists* $\beta \in \mathcal{KL}$ *such that*

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) \quad \forall t \geq 0.$$

Instead of $\mathbb{W}^{1,1}_{[-\tau,0]}$ any other space $\mathbb{W}^{1,p}_{[-\tau,0]}$ can be used in this definition for $p \in \mathbb{N} \cup \{+\infty\}$.

As it is usually assumed (Fridman et al., 2008; Pepe & Jiang, 2006), in this case

$$|x(t, x_0, d)| \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) \quad \forall t \ge 0.$$

Recall that in El'sgol'ts and Norkin (1973), under the assumption that f is Lipschitz in the second variable $\dot{\phi}$ with a constant smaller than 1, it is established that the latter estimate is equivalent to the stability in $\mathbb{W}^{1,\infty}_{[-\tau,0]}$.

Definition 2. A continuous functional $V : \mathbb{R}_+ \times \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ is called simple if $D^+V(t, \phi, \dot{\phi}, d)$ is independent on $\dot{\phi}$.

For instance, a locally Lipschitz functional $V : \mathbb{C}^n_{[-\tau,0]} \to \mathbb{R}_+$ is simple.

Definition 3 (*Fridman et al., 2008; Lin & Wang, 2018b; Pepe & Jiang, 2006*). A continuous functional $V : \mathbb{R}_+ \times \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ is called LKF for the system (1) if it is simple, there exist $p \in \mathbb{N} \cup \{+\infty\}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{K}$ such that V is Lipschitz continuous on bounded sets in $\mathbb{W}^{1,p}_{[-\tau,0]} \setminus \{0\}$, and for all $t \in \mathbb{R}_+$, $d \in \mathbb{D}$ and $\phi \in \mathbb{W}^{1,p}_{[-\tau,0]}$:

$$\alpha_1(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}) \le V(t, \phi, \dot{\phi}) \le \alpha_2(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}),$$
$$D^+V(t, \phi, \dot{\phi}, d) \le -\alpha(V(t, \phi, \dot{\phi})).$$

Note that existence of such a LKF implies that for all $t \in \mathbb{R}_+$, $d \in \mathbb{D}$ and $\phi \in \mathbb{W}^{1,p}_{[-\tau,0]}$:

$$\alpha_1(|\phi(0)|) \le V(t, \phi, \dot{\phi}) \le \alpha_2(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}),$$
$$D^+V(t, \phi, \dot{\phi}, d) \le -\hat{\alpha}(|\phi(0)|),$$

where $\hat{\alpha}(s) = \alpha(\alpha_1(s))$ is a function from class $\mathcal{K}$, which is the standard LKF formulation used to establish asymptotic stability (Fridman, 2014).

Theorem 1 (*Fridman, 2014; Fridman et al., 2008*). *If there exists a LKF for the system* (1)*, then it is uGAS.*

There exist also some converse results to Theorem 1, see, e.g., Pepe and Karafyllis (2013), which are obtained for $V : \mathbb{C}^n_{[-\tau,0]} \to \mathbb{R}_+$ and a special class of f in the Hale's form, and (Karafyllis & Jiang, 2011) for the background framework. Here we will use the following counterpart of Theorem 1 given in Efimov and Fridman (2019) (since in the sequel we have to develop some steps of the proof from Efimov and Fridman (2019), it is presented in the Appendix):

Theorem 2. *Let the system* (1) *be uGAS in the sense of Theorem 1 and $f : \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ be uniformly Lipschitz continuous on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]}$ (i.e., for any closed and bounded subset $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$ there exists $L_\Upsilon > 0$ such that*

$$|f(\phi, \dot{\phi}, d) - f(\varphi, \dot{\varphi}, d)| \le L_\Upsilon \|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $\phi, \varphi \in \Upsilon$ and $d \in \mathbb{D}$). Then there exists a LKF for the system (1) *with $p = 1$.*

The expression for a LKF proposed in the proof of Theorem 2, see (4) in the Appendix, has an explicit form (if only negative definiteness of $D^+V(x_0, \dot{x}_0, d)$ is required), and it is more simple than in Karafyllis and Jiang (2011), Pepe and Karafyllis (2013) and Pepe et al. (2017), where a two step procedure for construction of LKF has been proposed.

2.5. Robust stability

The ISS property is an extension of the conventional stability paradigm to the systems with external inputs (Fridman et al., 2008; Pepe & Jiang, 2006; Teel, 1998).

Definition 4 (*Fridman et al., 2008; Pepe & Jiang, 2006*). The system (1) is called practical ISS, if for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$ there exist $q \ge 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $t \ge 0$

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) + \gamma(\|d\|_{[0,t)}) + q.$$

If $q = 0$ then (1) is called ISS.

Definition 5. The system (1) is said to possess the practical asymptotic gain (AG) property, if for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$ there exist $q \ge 0$ and $\gamma \in \mathcal{K}$ such that

$$\lim_{t \to +\infty} \|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \gamma(\|d\|_\infty) + q.$$

If $q = 0$ then (1) admits AG property.

Definition 6. The system (1) is said to have the practical global stability (GS) property, if for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$ there exist $q \ge 0$ and $\sigma_1, \sigma_2 \in \mathcal{K}$ such that for all $t \ge 0$

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \max\{\sigma_1(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}), \sigma_2(\|d\|_\infty)\} + q.$$

If $q = 0$ then (1) admits GS property.

Again, instead of $\mathbb{W}^{1,1}_{[-\tau,0]}$ any other space $\mathbb{W}^{1,p}_{[-\tau,0]}$ can be used in these definitions for $p \in \mathbb{N} \cup \{+\infty\}$.

As it follows from the definitions above, a (practical) ISS system has (practical) AG and (practical) GS properties, and for a system in (1) described by an ordinary differential equation (i.e., with $\tau = 0$) the converse implication also holds (Dashkovskiy et al., 2011).

As it has been observed in Fridman et al. (2008) and Pepe and Jiang (2006), a sufficient characterization of ISS property can be introduced for (1):

Definition 7. A continuous functional $V : \mathbb{R}_+ \times \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ is called practical ISS LKF for the system (1) if it is simple and there exist $p \in \mathbb{N} \cup \{+\infty\}$, $r \ge 0$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha, \chi \in \mathcal{K}$ such that V is Lipschitz continuous on bounded sets in $\mathbb{W}^{1,p}_{[-\tau,0]} \setminus \{0\}$, and for all $t \in \mathbb{R}_+$, $\phi \in \mathbb{W}^{1,p}_{[-\tau,0]}$ and $d \in \mathbb{R}^m$:

$$\alpha_1(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}) \le V(t, \phi, \dot{\phi}) \le \alpha_2(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}),$$
$$V(t, \phi, \dot{\phi}) \ge \max\{r, \chi(|d|)\} \implies D^+V(t, \phi, \dot{\phi}, d) \le -\alpha(V(t, \phi, \dot{\phi})).$$

If $r = 0$ then V is an ISS LKF.

Theorem 3 (*Fridman et al., 2008*). *If there exists a (practical) ISS LKF for the system* (1)*, then it is (practical) ISS with the AG $\gamma = \alpha_1^{-1} \circ \chi$.*

There exist also some converse results to Theorem 3, see e.g. Pepe et al. (2017), which are also obtained for $V : \mathbb{C}^n_{[-\tau,0]} \to \mathbb{R}_+$ and neutral systems in Hale's form.

3. Problem statement

The goal of this work is to propose a converse of Theorem 3 for $x_t \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and for another class of f than in Pepe and Karafyllis (2013) and Pepe et al. (2017). In particular, as in Theorem 2 the following hypothesis is accepted in the sequel:

Assumption 1. The system (1) is ISS in the sense of Definition 4, and $f : \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous on

bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]} \times \mathbb{R}^m$: for any closed and bounded subsets $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$ and $D \subset \mathbb{R}^m$ there exists $L_{\Upsilon,D} > 0$ such that

$$|f(\phi, \dot\phi, d) - f(\varphi, \dot\varphi, \delta)| \le L_{\Upsilon,D}(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |d - \delta|)$$

for all $\phi, \varphi \in \Upsilon$ and $d, \delta \in D$.

Here the boundedness of a subset is understood as boundedness of the least upper bound of the norm for difference of any two elements of the subset.

An example of such a system is given by dynamics with distributed delays in the state derivatives and general delays in the state:

$$f(\phi, \dot\phi, d) = F\left(\phi, \int_{-\tau}^{0} k(s)\dot\phi(s)ds, d\right)$$

with a Lipschitz continuous on bounded sets function $F : \mathbb{C}^n_{[-\tau,0]} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and an essentially bounded kernel $k : [-\tau, 0] \to \mathbb{R}$. An important example covered by Assumption 1 includes retarded system with a pointwise delay:

$$\dot{x}(t) = F(x(t), x(t - \tau), d(t)),$$

then $f(\phi, \dot\phi, d) = F(\phi(0), \phi(0) - \int_{-\tau}^{0} \dot\phi(s)ds, d)$ and it has been observed previously (Fridman, 2001, 2014) that a LKF in $\mathbb{W}^{1,2}_{[-\tau,0]}$ for this kind of dynamics is usually more efficient than in $\mathbb{C}^n_{[-\tau,0]}$ (especially for robustness analysis).

From the relations between the norms given in Lemma 1 we observe that boundedness of x_t in $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ implies immediately a similar property for all other norms. Therefore, a stability analysis performed in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$ seems to be less restrictive, which is the motivation for selection of that space in this problem formulation.

4. Main results

In this section, first, some preliminary results are established, which clarify the features and significance of the imposed assumptions, and second, a converse result is presented.

4.1. Lipschitz continuity of solutions

Under the introduced restrictions we have the following useful property for the solutions of system (1):

Proposition 1. *Let Assumption 1 be satisfied. Then in (1), for any $T > 0$, and closed and bounded subsets $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$ and $\Theta \subset \mathcal{L}^m_\infty$, there exists $M_{T,\Upsilon,\Theta} > 0$ such that*

$$\|x_t(\phi, d) - x_t(\varphi, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le M_{T,\Upsilon,\Theta}(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$
$$+ \|d - \delta\|_{[0,T)}) \quad \forall t \in [0, T]$$

for all $\phi, \varphi \in \Upsilon$ and $d, \delta \in \Theta$.

Proof. According to Assumption 1 there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (consider also $q \ge 0$)

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) + \gamma(\|d\|_\infty) + q$$

for all $t \ge 0$, for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$. For any Υ and Θ as above, $\phi, \varphi \in \Upsilon$ and $d, \delta \in \Theta$, there exists $\rho > 0$ such that $\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \rho$, $\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \rho$, $\|d\|_\infty \le \rho$ and $\|\delta\|_\infty \le \rho$. Hence, we have

$$\max\{\|x_t(\phi, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, \|x_t(\varphi, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\} \le \beta(\rho, 0)$$
$$+ \gamma(\rho) + q$$

and

$$|f(x_t(\phi, d), \dot{x}_t(\phi, d), d(t)) - f(x_t(\varphi, \delta), \dot{x}_t(\varphi, \delta), \delta(t))|$$
$$\le L_\rho(\|x_t(\phi, d) - x_t(\varphi, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |d(t) - \delta(t)|)$$

for almost all $t \ge 0$, where $L_\rho > 0$ represents the Lipschitz constant of f on the set $\{\phi \in \mathbb{W}^{1,1}_{[-\tau,0]} : \|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\rho, 0)+\gamma(\rho)+q\}$ and $\{d \in \mathcal{L}^m_\infty : \|d\|_\infty \le \rho\}$ (it exists due to Assumption 1). For any $\phi \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$ we have:

$$x(t, \phi, d) = x(0, \phi, d) + \int_0^t f(x_s(\phi, d), \dot{x}_s(\phi, d), d(s))ds$$

for all $t \ge 0$, and for $\phi, \varphi \in \Upsilon$ and $d, \delta \in \Theta$:

$$|x(t, \phi, d) - x(t, \varphi, \delta)| \le |x(0, \phi, d) - x(0, \varphi, \delta)|$$
$$+ \int_0^t |f(x_s(\phi, d), \dot{x}_s(\phi, d), d(s)) - f(x_s(\varphi, \delta), \dot{x}_s(\varphi, \delta), \delta(s))|ds$$
$$\le |x(0, \phi, d) - x(0, \varphi, \delta)|$$
$$+ L_\rho \int_0^t \|x_s(\phi, d) - x_s(\varphi, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |d(s) - \delta(s)|ds.$$

Therefore, for $t \ge 0$:

$$\|x_t(\phi, d) - x_t(\varphi, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} = \sup_{-\tau \le s \le 0} |x(t + s, \phi, d) - x(t + s, \varphi, \delta)|$$
$$+ \int_{-\tau}^0 |\dot{x}(t + \sigma, \phi, d) - \dot{x}(t + \sigma, \varphi, \delta)|d\sigma$$
$$\le \|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$
$$+ \int_0^t |f(x_\sigma(\phi, d), \dot{x}_\sigma(\phi, d), d(\sigma)) - f(x_\sigma(\varphi, \delta), \dot{x}_\sigma(\varphi, \delta), \delta(\sigma))|d\sigma$$
$$+ \int_{-\tau}^0 |\dot{x}(t + \sigma, \phi, d) - \dot{x}(t + \sigma, \varphi, \delta)|d\sigma$$
$$\le 2\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$
$$+ 2\int_0^t |f(x_\sigma(\phi, d), \dot{x}_\sigma(\phi, d), d(\sigma)) - f(x_\sigma(\varphi, \delta), \dot{x}_\sigma(\varphi, \delta), \delta(\sigma))|d\sigma$$
$$\le 2\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$
$$+ 2L_\rho \int_0^t \|x_\sigma(\phi, d) - x_\sigma(\varphi, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |d(\sigma) - \delta(\sigma)|d\sigma$$
$$\le 2\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + 2L_\rho t\|d - \delta\|_{[0,t)}$$
$$+ 2L_\rho \int_0^t \|x_\sigma(\phi, d) - x_\sigma(\varphi, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} d\sigma.$$

Next, using Gronwall's Lemma (Khalil, 2015) we obtain for all $t \in [0, T]$:

$$\|x_t(\phi, d) - x_t(\varphi, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le 2(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$
$$+ L_\rho T\|d - \delta\|_{[0,T)})e^{2L_\rho t}.$$

Consequently, take any $T > 0$ then

$$\|x_t(\phi, d) - x_t(\varphi, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le M_{T,\Upsilon,\Theta}(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$
$$+ \|d - \delta\|_{[0,T)}) \quad \forall t \in [0, T]$$

for $M_{T,\Upsilon,\Theta} = 2\max\{1, L_\rho T\}e^{2L_\rho T}$.

Thus, under Assumption 1, for any bounded set of initial conditions and inputs, for a compact interval of time, for the solutions of the system (1) there is a kind of local Lipschitz property with respect to the initial conditions and inputs. It is worth to highlight that the result is substantiated in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$, and a similar conclusion in $\mathbb{W}^{1,\infty}_{[-\tau,0]}$ was obtained for the system (1) in the Hale's form (Pepe & Karafyllis, 2013; Pepe et al., 2017).

Remark 1. In addition, due to Lemma 1, for any closed and bounded subset $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$:

$$|f(\phi, \dot{\phi}, d) - f(\varphi, \dot{\varphi}, \delta)| \le L_{\Upsilon,D,p}(\|\phi - \varphi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} + |d - \delta|)$$

for all $\phi, \varphi \in \mathbb{W}^{1,p}_{[-\tau,0]}$ and $d, \delta \in D$ such that $\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} \le \min\{1, \tau^{\frac{1}{p}-1}\}\varrho$ and $\|\varphi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} \le \min\{1, \tau^{\frac{1}{p}-1}\}\varrho$, where $L_{\Upsilon,D,p} = \max\{1, \tau^{1-\frac{1}{p}}\}L_{\Upsilon,D}$ with $L_{\Upsilon,D}$ given in Assumption 1 and $\varrho > 0$ is the corresponding norm bound on Υ, i.e. the Lipschitz continuity of f in $\mathbb{W}^{1,p}_{[-\tau,0]}$ also follows.

4.2. A uniformly GAS system

Further consider an auxiliary system:

$$\dot{z}(t) = f(z_t, \dot{z}_t, \gamma^{-1}(\tfrac{1}{2}\|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t)), \quad t \ge 0, \tag{2}$$

where $z(t) \in \mathbb{R}^n$ and $z_t \in \mathbb{C}^n_{[-\tau,0]}$ is the state as before, $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$, γ is the AG function given for (1) in Definition 5 (which exists for (1) under Assumption 1, and we can always assume that $\gamma \in \mathcal{K}_\infty$, then its inverse is well defined), and $\delta \in \mathcal{D}$ is a uniformly bounded input. Obviously, the origin is an equilibrium of (2) since $f(0,0,0) = 0$ in (1). In addition, for any $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $\delta \in \mathcal{D}$ the solution $z_t(z_0, \delta)$ of (2) is defined on some interval of time $[0, T^{\max}_{z_0})$ and it coincides with the solution $x_t(z_0, d)$ of (1) for $d(t) = \gamma^{-1}(\tfrac{1}{2}\|z_t(z_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t)$ on this interval of time. Let us demonstrate a more strong relation between solutions of the systems (1) and (2) (as in Sontag and Wang (1996), see also Propositions 4.3 and 4.4 of Karafyllis and Jiang (2011)):

Proposition 2. *The following relations are true:*
(i) If the system (1) is ISS with the AG γ, then the system (2) is uGAS.
(ii) If the system (2) is uGAS with the decay $\beta \in \mathcal{KL}$, then the system (1) is ISS with the AG $\beta(2\gamma(\cdot), 0)$.

Proof. (i) If the system (1) is ISS, then according to Definition 4

$$\|x_t(x_{t_0}, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t - t_0) + \gamma(\|d\|_{[t_0,t)}),$$

for all $t \ge t_0$, for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and all $x_{t_0} \in \mathbb{W}^{1,1}_{[-\tau,0]}$, $d \in \mathcal{L}^m_\infty$, $t_0 \ge 0$. Since substituting $d(t) = \gamma^{-1}(\tfrac{1}{2}\|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t)$ in the system (2) we get (1), for the latter with $z_t = z_t(z_{t_0}, \delta)$ and any $z_{t_0} \in \mathbb{W}^{1,1}_{[-\tau,0]}$, $\delta \in \mathcal{D}$ we obtain:

$$\|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t - t_0)$$
$$+ \gamma(\sup_{s \in [t_0,t)} |\gamma^{-1}(\tfrac{1}{2}\|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(s)|)$$
$$= \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t - t_0) + \gamma(\sup_{s \in [t_0,t)} \gamma^{-1}(\tfrac{1}{2}\|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}))$$
$$= \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t - t_0) + \tfrac{1}{2}\sup_{s \in [t_0,t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $t \in [0, T^{\max}_{z_{t_0}})$. Hence,

$$\sup_{s \in [t_0,t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \sup_{s \in [t_0,t)} \{\beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, s - t_0)$$
$$+ \tfrac{1}{2}\sup_{\sigma \in [t_0,s)} \|z_\sigma\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\}$$
$$= \sup_{s \in [t_0,t)} \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, s - t_0) + \tfrac{1}{2}\sup_{s \in [t_0,t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}.$$

Take $t_0 = 0$, then from the last inequality

$$\tfrac{1}{2}\sup_{s \in [0,t)} \|z_s(z_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|z_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, 0)$$

for any $t \ge 0$, all $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and all $\delta \in \mathcal{D}$, which implies global uniform boundedness of the solutions of the system (2). Therefore, $T^{\max}_{z_{t_0}} = +\infty$ for all $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and all $\delta \in \mathcal{D}$. Next, select $t_0 = \frac{t}{2}$:

$$\sup_{s \in [\frac{t}{2},t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \sup_{s \in [\frac{t}{2},t)} \beta(\|z_{\frac{t}{2}}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, s - \tfrac{t}{2})$$
$$+ \tfrac{1}{2}\sup_{s \in [\frac{t}{2},t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},$$

which for $t \to +\infty$ (i.e., considering the asymptotic behavior of (2)) results in

$$\lim_{t \to +\infty} \sup_{s \in [\frac{t}{2},t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \tfrac{1}{2}\lim_{t \to +\infty} \sup_{s \in [\frac{t}{2},t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

that can be satisfied for $\lim_{t \to +\infty} \|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} = 0$ only. Consequently, the system (2) is uniformly converging to the origin, the uGAS property has been established.

(ii) Conversely, let (2) be uGAS, then according to Definition 1:

$$\|z_t(z_{t_0}, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t - t_0), \quad \forall t \ge t_0,$$

for some $\beta \in \mathcal{KL}$ and all $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$, $\delta \in \mathcal{D}$, $t_0 \ge 0$. Consider the system (1), take any $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and any $d \in \mathcal{L}^m_\infty$, assume that $\|d\|_\infty \le \gamma^{-1}(\tfrac{1}{2}\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$ for $t \in [0, t_1)$ with some (possibly infinite) $t_1 \ge 0$, then there is $\delta \in \mathcal{D}$ such that $d(t) = \gamma^{-1}(\tfrac{1}{2}\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t)$ for all $t \in [0, t_1)$ and, hence, using the properties of (2) we obtain:

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t), \quad \forall t \in [0, t_1).$$

Next, assume that $\|d\|_\infty > \gamma^{-1}(\tfrac{1}{2}\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$ for $t \in [t_1, t_2)$ with some $t_2 > t_1$ (obviously, always $t_1 + t_2 > 0$ for any $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$), hence,

$$\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le 2\gamma(\|d\|_\infty), \quad \forall t \in [t_1, t_2).$$

Finally, let again $\|d\|_\infty \le \gamma^{-1}(\tfrac{1}{2}\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$ for $t \in [t_2, t_3)$ with some $t_3 > t_2$, similarly

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_{t_2}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t - t_2)$$
$$\le \beta(2\gamma(\|d\|_\infty), 0), \quad \forall t \in [t_2, t_3)$$

since in this case $\|x_{t_2}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} = 2\gamma(\|d\|_\infty)$ by construction. Next, all these steps can be repeated iteratively if necessary (they cover all possible scenarios), and it is clear that for all $t \ge 0$

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) + \beta(2\gamma(\|d\|_\infty), 0),$$

since by definition $\beta(s, 0) \ge s$ for all $s \ge 0$, which implies ISS property of (1). ∎

4.3. Converse design of LKF

For brevity of exposition the analysis is presented in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$ (we need the result of Proposition 1 formulated in this space) and for ISS case only.

Theorem 4. *Let Assumption 1 be satisfied, then there exists an ISS LKF for the system (1).*

Proof. Under the introduced hypothesis and Lemma 3 there are $\theta_1, \theta_2, \theta_3 \in \mathcal{K}_\infty$ such that

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq \theta_1 \left(\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) e^{-t} \right) + \theta_3(\|d\|_\infty)$$

for all $t \geq 0$, for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$. Recalling Lemma 2, there exists a continuous, positive definite and radially unbounded function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ admitting the following properties:

$$\psi(s) \leq \theta_3^{-1}(s), \quad |\psi(s) - \psi(s')| \leq |s - s'| \quad \forall s, s' \in \mathbb{R}_+.$$

Following the arguments of Proposition 2, the system (that is a variant of (2))

$$\dot{z}(t) = F(z_t, \dot{z}_t, \delta(t)), \quad t \geq 0, \tag{3}$$

where $z(t) \in \mathbb{R}^n$, $z_t \in \mathbb{C}^n_{[-\tau,0]}$, $\delta \in \mathcal{D}$ and $F(z_t, \dot{z}_t, \delta(t)) = f(z_t, \dot{z}_t, \psi(\frac{1}{2}\|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t))$, is uGAS. Indeed,

$$\theta_3\left(\sup_{s \in [t_0,t)} \psi(\frac{1}{2}\|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \right) \leq \theta_3\left(\sup_{s \in [t_0,t)} \theta_3^{-1}(\frac{1}{2}\|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \right)$$

$$= \frac{1}{2} \sup_{s \in [t_0,t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

as for (2) (and all other arguments of Proposition 2 stay unchanged). Now, take any closed and bounded subset $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$, according to Assumption 1 there exists $L_{\Upsilon, \mathbb{D}} > 0$ such that

$$|F(\phi, \dot{\phi}, \delta) - F(\varphi, \dot{\varphi}, \delta)| = |f(\phi, \dot{\phi}, \psi(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta)$$

$$-f(\varphi, \dot{\varphi}, \psi(\frac{1}{2}\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta)|$$

$$\leq L_{\Upsilon, \mathbb{D}}(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |\psi(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta$$

$$-\psi(\frac{1}{2}\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta|)$$

$$\leq L_{\Upsilon, \mathbb{D}}(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |\psi(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

$$-\psi(\frac{1}{2}\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})|)$$

$$\leq L_{\Upsilon, \mathbb{D}}(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + \frac{1}{2}|\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} - \|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}|)$$

$$\leq \frac{3}{2}L_{\Upsilon, \mathbb{D}}\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $\phi, \varphi \in \Upsilon$ and $\delta \in \mathbb{D}$, where the Lipschitz properties of ψ and $\|\cdot\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$ were used in the last two steps. Therefore, F is uniformly Lipschitz continuous on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]}$, then all conditions of Theorem 2 are satisfied and the system (3) admits a LKF V, which is Lipschitz continuous on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$, and for all $\delta \in \mathbb{D}$ and $\phi \in \mathbb{W}^{1,1}_{[-\tau,0]}$:

$$\alpha_1(\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \leq V(\phi, \dot{\phi}) \leq \alpha_2(\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}),$$

$$D^+V(\phi, \dot{\phi}, \delta) \leq -\alpha(V(\phi, \dot{\phi}))$$

for some $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$. Returning to the system (1), for any $\phi \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathbb{R}^m$, let $\underline{\psi}(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \geq |d|$, where $\underline{\psi}(s) = \frac{s}{1+s}\inf_{\sigma \geq s} \psi(\sigma)$ is a function from class $\mathcal{K}_\infty$, then $d = \underline{\psi}(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta' = \psi(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta$ for some $\delta', \delta \in \mathbb{D}$, hence, the following implication holds in (1):

$$V(\phi, \dot{\phi}) \geq \chi(|d|) \Rightarrow \|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \geq 2\underline{\psi}^{-1}(|d|)$$

$$\Rightarrow D^+V(\phi, \dot{\phi}, d) \leq -\alpha(V(\phi, \dot{\phi})),$$

where $\chi(s) = \alpha_2(2\underline{\psi}^{-1}(s))$, which implies that V is an ISS LKF for (1). $\blacksquare$

The obtained result shows that the existence of a coercive ISS LKF is necessary and sufficient for ISS of the system (1) in $\mathbb{W}^{1,1}_{[-\tau,0]}$ provided that the Lipschitz continuity property stated in Assumption 1 is verified. The proposed conditions establish a direct relation between a LKF for the auxiliary system (3) and the required ISS LKF of (1).

5. Conclusions

The problem of existence of an ISS LKF for nonlinear neutral type time-delay systems is solved considering the ISS property in $\mathbb{W}^{1,1}_{[-\tau,0]}$ space. It is shown that Lipschitz property of f defined in such a space can be transformed to the same property of the solutions of (1), and relations between ISS and uGAS properties of (1) and an auxiliary system (2) are established.

Appendix

Under the introduced hypothesis in Theorem 2, and Lemma 3, there are $\theta_1, \theta_2 \in \mathcal{K}_\infty$ such that

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq \theta_1 \left(\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) e^{-t} \right) \quad \forall t \geq 0$$

for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{D}$, then by recalling Lemma 2, there exists a continuous, positive definite and radially unbounded function $\varsigma : \mathbb{R}_+ \to \mathbb{R}_+$ admitting the following properties:

$$\varsigma(s) \leq \theta_1^{-1}(s), \quad |\varsigma(s) - \varsigma(s')| \leq |s - s'| \quad \forall s, s' \in \mathbb{R}_+.$$

Now, for any $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ select

$$V(x_0, \dot{x}_0) = \sup_{t \geq 0, d \in \mathcal{D}} \{\varsigma(\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \frac{\kappa_1 + t}{\kappa_2 + t}\} \tag{4}$$

with $\frac{\kappa_2}{1+\kappa_2} < \kappa_1 < \kappa_2 < +\infty$. Then

$$V(x_0, \dot{x}_0) \leq \sup_{t \geq 0, d \in \mathcal{D}} \{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) e^{-t} \frac{\kappa_1 + t}{\kappa_2 + t}\}$$

$$\leq \frac{\kappa_1}{\kappa_2} \theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

and

$$V(x_0, \dot{x}_0) \geq \sup_{t \geq 0, d \in \mathcal{D}} \{\varsigma(\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \frac{\kappa_1}{\kappa_2}\}$$

$$\geq \varsigma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \frac{\kappa_1}{\kappa_2}$$

since under introduced restrictions on κ_1 and κ_2 the functions $\frac{\kappa_1+t}{\kappa_2+t}$ and $e^{-t}\frac{\kappa_1+t}{\kappa_2+t}$ are strictly increasing and decreasing, respectively. Define $\underline{\varsigma}(s) = \frac{s}{1+s}\inf_{\sigma \geq s} \varsigma(\sigma)$, which is a function from class $\mathcal{K}_\infty$, then

$$\frac{\kappa_1}{\kappa_2}\underline{\varsigma}(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \leq V(x_0, \dot{x}_0) \leq \frac{\kappa_1}{\kappa_2}\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$, and the LKF $V(x_0, \dot{x}_0)$ is coercive admitting lower and upper bounds in terms of functions from the class $\mathcal{K}_\infty$ with respect to $\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$ as desired. From the decreasing of $e^{-t}\frac{\kappa_1+t}{\kappa_2+t}$ it also follows that there exists $T^{x_0} > 0$ such that

$$V(x_0, \dot{x}_0) = \sup_{0 \leq t \leq T^{x_0}, d \in \mathcal{D}} \{\varsigma(\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \frac{\kappa_1 + t}{\kappa_2 + t}\},$$

and since

$$\varsigma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \frac{\kappa_1}{\kappa_2} \leq V(x_0, \dot{x}_0)$$

$$\leq \sup_{0 \leq t \leq T^{x_0}, d \in \mathcal{D}} \{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) e^{-t}\},$$

then by the definition of T^{x_0}, it has an upper estimate:

$$T^{x_0} \leq \ln\left[\frac{\kappa_2}{\kappa_1}\frac{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})}{\varsigma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})}\right].$$

For any $0 < r < R < +\infty$ define a set

$$\Omega_{r,R} = \{x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]} : r \leq \|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq R\},$$

consequently, there exists a finite $T^{r,R} = \sup_{x_0 \in \Omega_{r,R}} T^{x_0}$, i.e. $T^{r,R} = \ln\left[\frac{\kappa_2}{\kappa_1}\frac{\theta_2(R)}{\varsigma(r)}\right]$.

Let us check the Lipschitz continuity of V on any bounded and closed subset in $\mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$ (here by $\{0\}$ we understand $x \in \mathbb{W}^{1,1}_{[-\tau,0]}$ with $\|x\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} = 0$). For any $x_0, x_1 \in \mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$ denote $T^{x_0,x_1} = \max\{T^{x_0}, T^{x_1}\}$ and consider:

$$|V(x_1, \dot{x}_1) - V(x_0, \dot{x}_0)| = |\sup_{0 \leq t \leq T^{x_1}, d \in \mathcal{D}}\{\varsigma(\|x_t(x_1, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + t}{\kappa_2 + t}\}$$

$$- \sup_{0 \leq t \leq T^{x_0}, d \in \mathcal{D}}\{\varsigma(\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + t}{\kappa_2 + t}\}|$$

$$\leq \sup_{0 \leq t \leq T^{x_0,x_1}, d \in \mathcal{D}}\left|[\varsigma(\|x_t(x_1, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) - \varsigma(\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})]\frac{\kappa_1 + t}{\kappa_2 + t}\right|$$

$$\leq \sup_{0 \leq t \leq T^{x_0,x_1}, d \in \mathcal{D}}\left|\|x_t(x_1, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} - \|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\right|$$

$$\leq \sup_{0 \leq t \leq T^{x_0,x_1}, d \in \mathcal{D}}\|x_t(x_1, d) - x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},$$

where in the last step and the step before the Lipschitz properties of the norm $\|\cdot\|_{\mathbb{W}^{1,\infty}_{[-\tau,0]}}$ and the function ς have been utilized, respectively. For any $x_0, x_1 \in \mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$ there exist $0 < r < R < +\infty$ such that $x_0, x_1 \in \Omega_{r,R}$, then $T^{x_0,x_1} \leq T^{r,R}$ and using Proposition 1 in Efimov and Fridman (2019) (see also Proposition 1) there exists $M_{T^{r,R},\Omega_{r,R}} > 0$ such that

$$\|x_t(x_1, d) - x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq M_{T^{r,R},\Omega_{r,R}}\|x_1 - x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $t \in [0, T^{r,R}]$ and all $d \in \mathcal{D}$, hence,

$$|V(x_1, \dot{x}_1) - V(x_0, \dot{x}_0)| \leq M_{T^{r,R},\Omega_{r,R}}\|x_1 - x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $x_0, x_1 \in \Omega_{r,R}$ and any $0 < r < R < +\infty$, which implies the required Lipschitz continuity of V on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$. The continuity at the origin follows from the upper estimate:

$$V(x_0, \dot{x}_0) \leq \frac{\kappa_1}{\kappa_2}\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

that is satisfied for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$.

Finally, let us check the decreasing of the LKF V on the trajectories of the system (1) for $t > 0$:

$$V(x_t(x_0, d), \dot{x}_t(x_0, d))$$

$$= \sup_{\sigma \geq 0, \delta \in \mathcal{D}}\{\varsigma(\|x_\sigma(x_t(x_0, d), \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + \sigma}{\kappa_2 + \sigma}\}$$

$$= \sup_{\sigma \geq t, \delta \in \mathcal{D}}\{\varsigma(\|x_\sigma(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + \sigma - t}{\kappa_2 + \sigma - t}\}$$

$$< \sup_{\sigma \geq t, \delta \in \mathcal{D}}\{\varsigma(\|x_\sigma(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + \sigma}{\kappa_2 + \sigma}\}$$

$$\leq \sup_{\sigma \geq 0, \delta \in \mathcal{D}}\{\varsigma(\|x_\sigma(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + \sigma}{\kappa_2 + \sigma}\} = V(x_0, \dot{x}_0),$$

thus, V is strictly decreasing along the trajectories of (1) for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and any $d \in \mathcal{D}$. Moreover (this part is absent in Efimov and Fridman (2019)),

$$V(x_t(x_0, d), \dot{x}_t(x_0, d)) - V(x_0, \dot{x}_0)$$

$$= \sup_{\sigma \in [0, T^{x_t(x_0,d)}], \delta \in \mathcal{D}}\{\varsigma(\|x_\sigma(x_t(x_0, d), \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + \sigma}{\kappa_2 + \sigma}\}$$

$$- \sup_{\sigma \geq 0, \delta \in \mathcal{D}}\{\varsigma(\|x_\sigma(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + \sigma}{\kappa_2 + \sigma}\}$$

$$= \sup_{\sigma \in [0, T^{x_t(x_0,d)}], \delta \in \mathcal{D}}\{\varsigma(\|x_{t+\sigma}(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

$$\times\left(\frac{\kappa_1 + t + \sigma}{\kappa_2 + t + \sigma} + \frac{\kappa_1 + \sigma}{\kappa_2 + \sigma} - \frac{\kappa_1 + t + \sigma}{\kappa_2 + t + \sigma}\right)\}$$

$$- \sup_{\sigma \geq 0, \delta \in \mathcal{D}}\{\varsigma(\|x_\sigma(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1 + \sigma}{\kappa_2 + \sigma}\}$$

$$\leq \sup_{\sigma \in [0, T^{x_t(x_0,d)}], \delta \in \mathcal{D}}\{\varsigma(\|x_{t+\sigma}(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

$$\times\left(\frac{\kappa_1 + \sigma}{\kappa_2 + \sigma} - \frac{\kappa_1 + t + \sigma}{\kappa_2 + t + \sigma}\right)\},$$

and applying the Mean Value Theorem we obtain:

$$V(x_t(x_0, d), \dot{x}_t(x_0, d)) - V(x_0, \dot{x}_0)$$

$$\leq - \sup_{\sigma \in [0, T^{x_t(x_0,d)}], \delta \in \mathcal{D}}\{\varsigma(\|x_{t+\sigma}(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})t$$

$$\times\frac{\kappa_2 - \kappa_1}{(\kappa_2 + \sigma + \delta t)^2}\}$$

for some $\delta t \in (0, t)$. Finally,

$$D^+V(x_0, \dot{x}_0, d) = \limsup_{t \to 0^+}\frac{1}{t}[V(x_t(x_0, d), \dot{x}_t(x_0, d)) - V(x_0, \dot{x}_0)]$$

$$\leq -\limsup_{t \to 0^+}\sup_{\sigma \in [0, T^{x_t(x_0,d)}], \delta \in \mathcal{D}}\{\varsigma(\|x_{t+\sigma}(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_2 - \kappa_1}{(\kappa_2 + \sigma + \delta t)^2}\}$$

$$= - \sup_{\sigma \in [0, T^{x_0}], \delta \in \mathcal{D}}\{\varsigma(\|x_\sigma(x_0, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_2 - \kappa_1}{(\kappa_2 + \sigma)^2}\}$$

$$\leq -\frac{\kappa_2 - \kappa_1}{(\kappa_2 + T^{x_0})^2}\varsigma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

for any $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and any $d \in \mathbb{D}$, which is a negative definite function of V, but probably globally bounded. Now defining a new LKF $\tilde{V}(x_0, \dot{x}_0) = \int_0^{V(x_0, \dot{x}_0)} \rho(s)ds$ for a suitably defined function $\rho \in \mathcal{K}_\infty$ it is possible to obtain another LKF with a negative definite and properly unbounded derivative $D^+\tilde{V}(x_0, \dot{x}_0, d)$ for any $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and any $d \in \mathbb{D}$, as desired.

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Denis Efimov received Ph.D. degree in Automatic Control from the Saint-Petersburg State Electrical Engineering University (Russia) in 2001, and Dr.Sc. degree in Automatic control in 2006 from the Institute for Problems of Mechanical Engineering RAS (Saint-Petersburg, Russia). From 2006 to 2011 he was working in the S CNRS (Supelec, France), the Montefiore Institute (University of Liege, Belgium) and IMS CNRS lab (University of Bordeaux, France). In 2011, he joined Inria (Lille-Nord Europe center). Starting from 2018 he is the scientific head of Valse team. He has published more than 130 articles in international scientific journals. He is a member of several IFAC TCs and a Senior Member of IEEE. He is also serving as an Associate Editor for IEEE Transactions on Automatic Control, IFAC Journal on Nonlinear Analysis: Hybrid Systems and Asian Journal of Control.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden). Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published more than 150 articles in international scientific journals. She is the author of the monograph "Introduction to TimeDelay Systems: Analysis and Control" (Birkhauser, 2014). She serves/served as Associate Editor in Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control and Information. In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. Since 2018, she has been the incumbent for Chana and Heinrich Manderman Chair on System Control at Tel Aviv University. She is IEEE Fellow since 2019. She is currently a member of the IFAC Council.