

## SHORT COMMUNICATION

# On complete Lyapunov–Krasovskii functional techniques for uncertain systems with fast-varying delays

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### SUMMARY

Stability of linear systems with norm-bounded uncertainties and uncertain time-varying delays is considered. The delay is supposed to be bounded and fast varying (without any constraints on the delay derivative). *Sufficient* stability conditions are derived by direct Lyapunov method based on the *complete Lyapunov–Krasovskii functional* (LKF). A novel complete LKF construction is presented: the derivative condition for the nominal LKF (i.e. for the LKF, which corresponds to the system with the nominal values of the coefficients and of the delay) depends on the ‘present’ state only. The comprehensive technique for stability analysis of uncertain time-delay systems is extended to the case of complete LKF: the application of free weighting matrices (instead of descriptor model transformation) and of Jensen’s inequality (instead of the cross-terms bounding). Numerical examples illustrate the efficiency of the method, and complete the paper. Copyright © 2007 John Wiley & Sons, Ltd.

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### 1. INTRODUCTION

The stability and control of time-delay systems is a subject of recurring interest, and a lot of research has been devoted to the field in the last decade using both frequency- and time-domain methods. Most of the results devoted to the robust stability of systems with norm-bounded uncertainties and uncertain delays consider as assumption the *stability of the system free of delays*, and next, in the time domain, use appropriate Lyapunov–Razumikhin functions or Lyapunov–Krasovskii functionals (LKFs) combined with linear matrix inequalities (LMIs) to

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derive some bounds on the delay values  $\mu$  (finite), such that the uncertain system will be stable for all delays intervals of the form  $[0, \mu]$ . Without any loss of generality, we can define such a case as stability characterization of *uncertain ‘small’ delays* (see, for instance, the results of [1–5]).

Next there exist cases (high-speed networks, biological systems, see some examples in [6]) where such an assumption is not realistic, and the procedures and the methods mentioned above fail. In other words, the system free of uncertainty and free of delays is *not* necessarily asymptotically stable, but it may be stable for some *‘non-zero’ delays*. Such a case can be called *‘non-small’ delays*, and the analysis becomes largely more complicated as in the ‘small’ delays case. As expected, their stability analysis cannot be performed by using *simple* LKFs [7]. *Complete* LKF (which corresponds to necessary and sufficient conditions for the stability of the nominal system) should be applied to stability analysis in this case. Note that the discretized Lyapunov functional method [7, 8], which gives the sufficient conditions only, cannot always be applied efficiently.

There exist two main methods for robust stability of uncertain linear systems: *direct* Lyapunov method and *input–output* approach to stability [7]. The direct Lyapunov method *via* complete LKF has been developed in the case of known constant delays and norm-bounded uncertainties [9, 10], or in the case of uncertain ‘non-small’ delays and known coefficients [11, 12]. Robust stability of uncertain systems with ‘non-small’ delay has been analysed also *via* input–output approach to stability [7, 13]. However, only direct Lyapunov method can be applied to the problems, where the knowledge of the initial function is important (see e.g. [14] for application of LKF to the estimate on the domain of attraction of the nonlinear system, modelled as a linear uncertain system).

In [9, 11], the complete LKF was constructed for the uncertain system, which did not explicitly depend on the bounds of the uncertainties. As a result, the conditions were rather complicated, and induced some conservatism. In the case of constant delay and uncertain system matrices, a less conservative condition was obtained by inserting a cross-term into the derivative of LKF in [10].

Recently, a new construction of complete LKF for stability analysis of systems with non-small delays was suggested [12]: to a nominal LKF, which is appropriate to the nominal system (with nominal delays), the additional terms are added. These terms correspond to the perturbed system and they vanish when the delay uncertainties approach 0. Unlike the existing complete LKFs (see e.g. [9, 11, 15–17]), the derivative of the complete nominal LKF of [12] along the trajectories of the nominal system depended on the state and the *state derivative* which allowed a less conservative treatment of the *delay perturbation*.

To the best of our knowledge, the stability of the systems with both, norm-bounded uncertainties and uncertain non-small delays, has not been studied yet *via* complete LKF. In the present paper we extend the construction of [12] to the systems with uncertain coefficients. Different from the existing papers on robust stability *via* complete LKF [9–12], the derivative of the present nominal LKF along the trajectories of the nominal system depends on the present state only. We extend to the case of complete LKF the comprehensive technique for stability analysis of uncertain time-delay systems: free weighting matrices [18] (instead of model transformation) and application of Jensen’s inequality [7] (instead of the cross-terms bounding). The terms depending on  $\dot{x}(t)$  are inserted into the derivative of LKF by application of a certain free weighting matrices (which is equivalent to the descriptor model transformation).

The remaining paper is organized as follows: Section 2 is devoted to the problem statement. The robust stability analysis is presented in Section 3. Next, numerical examples illustrate the efficiency of the method in Section 4, and some concluding remarks end the paper.

*Notation*

Throughout the paper the superscript ‘T’ stands for matrix transposition,  $\mathcal{R}^n$  denotes the  $n$  dimensional Euclidean space with vector norm  $|\cdot|$ ,  $\mathcal{R}^{n \times m}$  is the set of all  $n \times m$  real matrices, and the notation  $P > 0$ , for  $P \in \mathcal{R}^{n \times n}$  means that  $P$  is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by  $*$ .

2. PROBLEM FORMULATION

We consider the following linear system with uncertain coefficients and an uncertain time-varying delay  $\tau_1(t)$ :

$$\dot{x}(t) = (A_0 + H\Delta E_0)x(t) + (A_1 + H\Delta E_1)x(t - \tau_1(t)) \tag{1}$$

where  $x(t) \in \mathcal{R}^n$  is the system state,  $A_0, A_1, H, E_0$ , and  $E_1$  are constant matrices of appropriate dimensions and  $\Delta(t)$  is a time-varying uncertain matrix that satisfies

$$\Delta^T(t)\Delta(t) \leq I \tag{2}$$

The uncertain delay  $\tau_1(t)$  is supposed to have the following form:

$$\tau_1(t) = h_1 + \eta_1(t) \tag{3}$$

where  $h_1 > 0$  is a nominal constant value, and  $\eta_1$  is a time-varying piecewise continuous perturbation satisfying  $|\eta_1| \leq \mu_1$ , where  $\mu_1$  is a known upper bound.

As suggested in [12] we consider the following form of LKF:

$$V = V_n + V_a \tag{4}$$

where  $V_n$  is a nominal complete LKF which corresponds to the necessary and sufficient conditions for stability of the nominal system:

$$\dot{x}(t) = A_0x(t) + A_1x(t - h_1) \tag{5}$$

and  $V_a$  consists of additional terms and depends on  $\mu_1, H$ , and  $E_j$  ( $j = 0, 1$ ) and  $V_a \rightarrow 0$  for  $\mu_1 \rightarrow 0, H \rightarrow 0, E_j \rightarrow 0$ . The latter will guarantee that if the conditions for the stability of the nominal system are feasible, then the stability conditions for the perturbed system will be feasible for small enough delay perturbations and norm-bounded uncertainties.

3. ROBUST STABILITY VIA COMPLETE NOMINAL LKF

3.1. Complete LKF for the nominal system

We assume that the nominal system (5) is asymptotically stable. Then [17] there exists the nominal complete LKF  $V_n(x_t)$  such that  $V_n(x_t) > \varepsilon|x(t)|^2$ ,  $\varepsilon > 0$  and along the trajectories of the nominal system (5):

$$\dot{V}_n = -x^T(t)W_0x(t) \tag{6}$$

It has the following form:

$$\begin{aligned}
 V_n(\phi) &= \phi^T(0)U(0)\phi(0) + 2\phi^T(0) \int_{-h_1}^0 U^T(h_1 + \theta)A_1\phi(\theta) d\theta \\
 &+ \int_{-h_1}^0 \int_{-h_1}^0 \phi^T(\theta_2)A_1^T U(\theta_2 - \theta_1)A_1\phi(\theta_1) d\theta_1 d\theta_2
 \end{aligned} \tag{7}$$

where

$$U(\theta) = \int_0^\infty K^T(t)W_0K(t + \theta) dt, \quad \theta \in R \tag{8}$$

Here  $K(t)$  is a fundamental matrix associated with the nominal system (5), i.e.  $K(t)$  is an  $n \times n$ -matrix function satisfying

$$\dot{K}(t) = A_0K(t) + A_1K(t - h_1), \quad t \geq 0 \tag{9}$$

with the initial condition  $K(0) = I$  and  $K(t) = 0$  for  $t < 0$ .

The matrix  $U(\theta)$  satisfies the following differential equation and boundary value condition:

$$\dot{U}(\theta) = U(\theta)A_0 + U(\theta - h)A_1, \quad \theta \geq 0$$

$$W_0 + U(0)A_0 + A_0^T U(0) + U^T(h)A_1 + A_1^T U(h) = 0 \tag{10}$$

Denote  $V(\theta) = U^T(-\theta + h) = U(\theta - h)$ ,  $\theta \geq 0$ . Then (10) can be represented in the form of the following boundary value problem for ordinary differential equations:

$$\dot{U}(\theta) = U(\theta)A_0 + V(\theta)A_1 \tag{11a}$$

$$\dot{V}(\theta) = -A_1^T U(\theta) - A_0^T V(\theta), \quad \theta \geq 0 \tag{11b}$$

$$-W_0 = U(0)A_0 + A_0^T U(0) + V(0)A_1 + A_1^T U(h) \tag{11c}$$

$$V(h) = U(0) \tag{11d}$$

For computation of  $U(\theta)$  one can apply Kronecker products of matrices (see e.g. [12, 19]). We remind that given  $n \times m$  matrix  $A$  with elements  $a_{ij}$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ , and  $p \times q$  matrix  $B$ , their Kronecker product  $A \otimes B$  is the  $np \times mq$  matrix with the block structure

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \dots & \vdots \\ a_{n1}B & \dots & a_{nm}B \end{bmatrix}$$

The stack of  $A$  is the vector formed by stacking the columns of  $A$  into  $nm \times 1$  vector

$$A^S \triangleq [a_{11} \dots a_{n1} \ a_{12} \dots a_{n2} \ \dots \ a_{m1} \dots a_{nm}]^T$$

The following holds  $(ABD)^S = (D^T \otimes A)B^S$ .

Representing (11) in the form

$$\begin{bmatrix} \dot{U}^S(\theta) \\ \dot{V}^S(\theta) \end{bmatrix} = \mathcal{A} \begin{bmatrix} U^S(\theta) \\ V^S(\theta) \end{bmatrix} \quad (12a)$$

$$\mathcal{A} = \begin{bmatrix} A_0^T \otimes I_n & A_1^T \otimes I_n \\ -I_n \otimes A_1^T & -I_n \otimes A_0^T \end{bmatrix} \quad (12b)$$

$$\begin{bmatrix} -W_0^S \\ 0_{n^2 \times 1} \end{bmatrix} = \mathcal{B} \begin{bmatrix} U^S(0) \\ V^S(0) \end{bmatrix} \quad (12c)$$

$$\mathcal{B} = \begin{bmatrix} (A_0^T \otimes I_n) + (I_n \otimes A_0^T) & A_1^T \otimes I_n \\ I_{n^2} & 0_{n^2 \times n^2} \end{bmatrix} + \begin{bmatrix} I_n \otimes A_1^T & 0_{n^2 \times n^2} \\ 0_{n^2 \times n^2} & -I_{n^2} \end{bmatrix} e^{\mathcal{A}h} \quad (12d)$$

and assuming that  $\mathcal{B}$  is non-singular we finally obtain

$$U^S(\theta) = [I \ 0] e^{\mathcal{A}\theta} \mathcal{B}^{-1} \begin{bmatrix} -W_0^S \\ 0_{n^2 \times 1} \end{bmatrix}, \quad \theta \in [0, h] \quad (13)$$

### 3.2. Robust stability analysis

Similar to [11, 12] we represent the perturbed system in the form:

$$\dot{x}(t) = (A_0 + H\Delta E_0)x(t) + (A_1 + H\Delta E_1)x(t - h_1) - (A_1 + H\Delta E_1) \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) \, ds \quad (14)$$

Differentiating  $V_n$  along the trajectories of (14), we find

$$\begin{aligned} \dot{V}_n(x_t) = & -x^T(t)W_0x(t) + 2 \left[ x^T(t)U^T(0) + \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta) \, d\theta \right] \left[ H\Delta \begin{bmatrix} E_0x(t) \\ E_1x(t-h_1) - E_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) \, ds \end{bmatrix} \right. \\ & \left. - A_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) \, ds \right] \end{aligned} \quad (15)$$

In order to treat effectively the fast-varying delay, we insert into  $\dot{V}_n$  negative quadratic terms, depending on  $\dot{x}(t)$  by adding to  $\dot{V}_n(x_t)$  the right side of the expression (we follow here the free weighting matrices technique of [18])

$$0 = 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T]\xi \quad (16)$$

where

$$\xi = -\dot{x}(t) + (A_0 + H\Delta E_0)x(t) + (A_1 + H\Delta E_1)x(t - h_1) - (A_1 + H\Delta E_1) \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) \, ds$$

and where  $P_2$  and  $P_3$  are  $n \times n$ -matrices. This is equivalent to descriptor model transformation [4]. We note that in the case of constant delay  $\tau_1 \equiv h_1$ , the matrices  $P_2$  and  $P_3$  may be chosen to

be zero. However, these matrices improve the stability analysis of systems with uncertain coefficients (see Example 1).

We have

$$\begin{aligned} \dot{V}_n(x_t) = & -x^T(t)W_0x(t) + \delta + 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][-\dot{x}(t) + A_0x(t) + A_1x(t-h_1)] \\ & - 2\left[x^T(t)(U^T(0) + P_2^T) + \dot{x}^T(t)P_3^T\right. \\ & \left. + \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta) d\theta\right] A_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) ds \end{aligned} \quad (17)$$

where

$$\begin{aligned} \delta = & 2\left[x^T(t)(U^T(0) + P_2^T) + \dot{x}^T(t)P_3^T\right. \\ & \left. + \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta) d\theta\right] H\Delta \left[E_0x(t) + E_1x(t-h_1) - E_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) ds\right] \end{aligned} \quad (18)$$

By applying the standard bounding, for a scalar  $\rho > 0$  the following is obtained:

$$\begin{aligned} \delta \leq & \rho^{-1} \left[x^T(t)(U^T(0) + P_2^T) + \dot{x}^T(t)P_3^T + \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta) d\theta\right] H \\ & \times H^T \left[x^T(t)(U^T(0) + P_2^T) + \dot{x}^T(t)P_3 + \int_{-h_1}^0 U^T(h_1+\theta)A_1x(t+\theta) d\theta\right] \\ & + \rho \left[x^T(t)E_0^T + x^T(t-h_1)E_1^T - \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}^T(s)E_1^T ds\right] \left[E_0x(t) + E_1x(t-h_1)\right. \\ & \left. - E_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) ds\right] \end{aligned} \quad (19)$$

We choose

$$\begin{aligned} V(x_t) = & V_n(x_t) + V_{a1}(x_t) + V_{a2}(x_t) + V_{a3}(x_t) \\ V_{a1}(x_t) = & \int_{t-h_1}^t x^T(s)Sx(s) ds, \quad S > 0 \\ V_{a2}(x_t) = & \int_{-\mu_1}^{\mu_1} \int_{t+\theta-h_1}^t \dot{x}^T(s)R\dot{x}(s) ds d\theta, \quad R > 0 \\ V_{a3}(x_t) = & r \int_{-h_1}^0 \int_{t+\theta}^t x^T(s)A_1^T U(h_1+\theta)U^T(h_1+\theta)A_1x(s) ds d\theta, \quad r > 0 \end{aligned} \quad (20)$$

where  $V_n$  is defined by (7),  $S, R$  are  $n \times n$ -matrices and  $r, \rho$  are scalars.

Derivative of  $V$  along (14) satisfies the following:

$$\begin{aligned}
 \dot{V} \leq & -x^T(t)W_0x(t) + 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][-\dot{x}(t) + A_0x(t) + A_1x(t-h_1)] \\
 & - 2\left[x^T(t)(U^T(0) + P_2^T) + \dot{x}^T(t)P_3^T + \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta) d\theta\right] A_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) ds \\
 & + \rho^{-1}\left[x^T(t)(U^T(0) + P_2^T) + \dot{x}^T(t)P_3^T + \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta) d\theta\right] H \\
 & \times H^T\left[x^T(t)(U^T(0) + P_2^T) + \dot{P}_3x(t) + \int_{-h_1}^0 U^T(h_1+\theta)A_1x(t+\theta) d\theta\right] \\
 & + \rho\left[x^T(t)E_0^T + x^T(t-h_1)E_1^T - \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}^T(s)E_1^T ds\right]\left[E_0x(t) + E_1x(t-h_1)\right. \\
 & \left.- E_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) ds\right] + x^T(t)Sx(t) - x^T(t-h_1)Sx(t-h_1) \\
 & + 2\mu_1\dot{x}^T(t)R\dot{x}(t) - \int_{t-h_1-\mu_1}^{t-h_1+\mu_1} \dot{x}^T(\theta)R\dot{x}(\theta) d\theta + h_1rx^T(t)A_1^T \varrho A_1x(t) \\
 & - r \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta)U^T(h_1+\theta)A_1x(t+\theta) d\theta
 \end{aligned} \tag{21}$$

where

$$\varrho = \int_{-h_1}^0 U(h_1+\theta)U^T(h_1+\theta) d\theta \tag{22}$$

Applying further Jensen inequality [7],

$$\begin{aligned}
 \mu_1 \int_{t-h_1-\mu_1}^{t-h_1+\mu_1} \dot{x}^T(\theta)R\dot{x}(\theta) d\theta & \geq \eta_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}^T(\theta)R\dot{x}(\theta) d\theta \geq \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}^T(s) ds R \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) ds \\
 h_1 \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta)U^T(h_1+\theta)A_1x(t+\theta) d\theta \\
 & \geq \int_{-h_1}^0 x^T(t+\theta)A_1^T U(h_1+\theta) d\theta \int_{-h_1}^0 U^T(h_1+\theta)A_1x(t+\theta) d\theta
 \end{aligned}$$

we find that

$$\begin{aligned} \dot{V} \leq & \zeta^T \Phi \zeta + \rho^{-1} \left[ x^T(t)(U^T(0) + P_2^T) + \dot{x}^T(t)P_3^T + \int_{-h_1}^0 x^T(t + \theta)A_1^T U(h_1 + \theta) d\theta \right] H \\ & \times H^T \left[ (U(0) + P_2)x(t) + \dot{P}_3 x(t) + \int_{-h_1}^0 U^T(h_1 + \theta)A_1 x(t + \theta) d\theta \right] \\ & + \rho \left[ x^T(t)E_0^T + x^T(t - h_1)E_1^T - \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}^T(s)E_1^T ds \right] \left[ E_0 x(t) + E_1 x(t - h_1) \right. \\ & \left. - E_1 \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}(s) ds \right] \end{aligned} \quad (23)$$

where

$$\zeta^T = \left[ x^T(t) \quad \dot{x}^T(t) \quad \frac{1}{\mu_1} \int_{t-h_1-\eta_1}^{t-h_1} \dot{x}^T(s) ds \quad \int_{-h_1}^0 x^T(t + \theta)A_1^T U(h_1 + \theta) d\theta \quad x^T(t - h_1) \right]$$

and

$$\Phi = \begin{bmatrix} -W_0 + P_2^T A_0 + A_0^T P_2 + S + h_1 r A_1^T \mathcal{Q} A_1 & A_0^T P_3 - P_2^T & -\mu_1 [U^T(0) + P_2^T] A_1 & 0 & P_2^T A_1 \\ * & -P_3^T - P_3 + 2\mu_1 R & -\mu_1 P_3^T A_1 & 0 & P_3^T A_1 \\ * & * & -\mu_1 R & -\mu_1 A_1 & 0 \\ * & * & * & -\frac{rI}{h_1} & 0 \\ * & * & * & * & -S \end{bmatrix} \quad (24)$$

Therefore, applying Schur complements to the last two terms of (23), we find that  $\dot{V} < 0$  (and thus (1) is asymptotically stable) if the following LMI is satisfied:

$$\left[ \begin{array}{c|cc} & (U^T(0) + P_2^T)H & \rho E_0^T \\ & P_3^T H & 0 \\ \Phi & 0 & -\mu_1 \rho E_1^T \\ & H & 0 \\ & 0 & \rho E_1^T \\ - & - & - \\ * & -\rho I & 0 \\ * & * & -\rho I \end{array} \right] < 0 \quad (25)$$

We have proved the following.



*Theorem 3.1*

Assume that the nominal system (5) is asymptotically stable. Given  $n \times n$ -matrix  $W_0 > 0$ . Let  $U(\theta), \theta \in [0, h_1]$  and  $\mathcal{Q}$  be defined by (8), (22). Then (1) is asymptotically stable if there exist  $n \times n$ -matrices  $P_2, P_3, R$ , and  $S$  and scalars  $r$  and  $\rho$  that satisfy LMI (25), where  $\Phi$  is given by (24).

*Remark 3.2*

For verification of the conditions of Theorem 3.1, one can compute  $U(\theta)$  by calculating the matrix  $\mathcal{B}$  given by (12d) and by using the relation (13).

In the case of known coefficients, where  $\Delta = 0$ , the stability condition is  $\Phi < 0$ . In the case of constant delays, we obtain the following corollary.

*Corollary 3.3*

Assume that the nominal system (5) is asymptotically stable and  $\eta_1 = 0$ . Given  $n \times n$ -matrix  $W_0 > 0$ . Let  $U(\theta), \theta \in [0, h_1]$  and  $\mathcal{Q}$  be defined by (8), (22). Then (1) is asymptotically stable if there exist an  $n \times n$ -matrix  $P_2, P_3, S$ , and scalars  $r$  and  $\rho$  that satisfy LMI:

$$\begin{bmatrix} -W_0 + P_2^T A_0 + A_0^T P_2 + S + h_1 r A_1^T \mathcal{Q} A_1 & A_0^T P_3 - P_2^T & 0 & P_2^T A_1 & (U^T(0) + P_2^T)H & \rho E_0^T \\ * & -P_3^T - P_3 & 0 & P_3^T A_1 & P_3^T H & 0 \\ * & * & -\frac{rI}{h_1} & 0 & H & 0 \\ * & * & * & -S & 0 & \rho E_1^T \\ * & * & * & * & -\rho I & 0 \\ * & * & * & * & * & -\rho I \end{bmatrix} < 0$$

4. EXAMPLES

In the sequel, we shall consider two examples from the literature, together with some comparisons between the results obtained by using our method with respect to the existing methodologies and techniques.

*Example 4.1 (Kharitonov and Niculescu [9])*

Consider (1) with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad H = I, \quad E_i = e_i I, \quad i = 0, 1, \quad e_i \in R \quad (26)$$

and with constant delay  $\tau_1 \equiv 1$ . It was found in [10] that the system is asymptotically stable for  $e_0 = 0.016$  and  $e_1 = 0.02$ . The latter bounds on the perturbations of the system matrices were less conservative than those given in [9]. The improvement was achieved due to the cross-terms which were inserted into the time derivative of the complete LKF. By applying a simpler nominal complete LKF of (7), (6), where  $W_0 = I$  and Corollary 3.2, we obtain a larger

stability region:  $e_0 = 0.11$  and  $e_1 = 0.11$ . For  $P_2 = P_3 = 0$  the results are more restrictive:  $e_0 = e_1 = 0.05$ .

Considering next the case of known coefficients ( $e_0 = e_1 = 0$ ) and the fast-varying delay  $\tau_1 = 1 + \eta_1(t)$  with  $|\eta_1| \leq \mu_1$ , we find that the system is asymptotically stable for  $\mu_1 = 0.14$ , which is less restrictive than [12].

Finally, in the case of uncertain coefficients with  $e_0 = e_1 = 0.05$ , we find that the system is asymptotically stable for  $\mu_1 = 0.1$ .

*Example 4.2 (Kharitonov and Niculescu [11])*

Consider (1) with

$$A_1 = \begin{bmatrix} 0 & 0 \\ -0.4 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \quad H = I, \quad E_i = e_i I, \quad i = 0, 1, \quad e_i \in R \quad (27)$$

which was analysed in [11]. The nominal non-delayed system (i.e. (27) with  $\tau_1 = 0$ ) is not asymptotically stable and thus the simple LKFs are not applicable. For the case of constant delay  $\tau_1 = 4 + \eta_1$  and  $e_1 = e_2 = 0$ , the following stability interval was found by the frequency domain analysis [11]:  $\tau_2 = 0, \quad -0.6209 < \eta_1(t) < 0.7963$ . We note that the discretized Lyapunov functional method of Gu *et al.* [7] applied to the nominal system with  $\tau_1 \equiv 4$  and  $e_0 = e_1 = 0$  does not converge for  $N \leq 8$  and verification of the corresponding LMI conditions takes a lot of computer time.

Considering the case of constant delay  $\tau_1 \equiv 4$ , we apply Corollary 3.2. We find that the system is robustly stable for  $e_0 = e_1 = 10^{-3}$ .

In the case of uncertain coefficients and fast-varying delay, the conditions of Theorem 3.1 are feasible (and thus the system is asymptotically stable) for  $h_1 = 4, e_0 = e_1 = 10^{-4}$  and for  $\mu = 0.0025$ .

## 5. CONCLUSIONS

Stability of linear retarded type system with uncertain time-varying delays from given segments and norm-bounded uncertainties is analysed.

A new LKF construction, which was recently introduced for systems with uncertain delays, is extended to the case of norm-bounded uncertainties, where the nominal LKF (i.e. the LKF, which corresponds to the nominal system) is of the complete type. The derivative of the nominal complete LKF depends on the present state only. Similar to simple LKF, the additional terms of LKF insert the past state terms into the derivative of LKF. The state derivative terms are inserted into the derivative condition by application of free weighting matrices, which correspond to the descriptor method.

The new method treats, for the first time, systems with norm-bound uncertainties and with fast-varying delays *via* complete LKF method, improving and simplifying the existing methods based on complete LKF.

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## COMPLETE LYAPUNOV–KRASOVSKII FUNCTIONAL

### REFERENCES

1. Li X, de Souza C. Criteria for robust stability and stabilization of uncertain linear systems with state delay. *Automatica* 1997; **33**:1657–1662.
2. Moon YS, Park P, Kwon WH, Lee YS. Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control* 2001; **74**:1447–1455.
3. Niculescu SI. *Delay Effects on Stability: A Robust Control Approach*. Lecture Notes in Control and Information Sciences, vol. 269. Springer: London, 2001.
4. Fridman E. New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems and Control Letters* 2001; **43**:309–319.
5. Richard JP. Time-delay systems: an overview of some recent advances and open problems. *Automatica* 2003; **39**:1667–1694.
6. Kolmanovskii V, Myshkis A. *Applied Theory of Functional Differential Equations*. Kluwer: Dordrecht, 1999.
7. Gu K, Kharitonov V, Chen J. *Stability of Time-Delay Systems*. Birkhauser: Boston, 2003.
8. Han QL, Gu K. Stability of linear systems with time-varying delay: a generalized discretized Lyapunov functional approach. *Asian Journal of Control* 2001; **3**:170–180.
9. Kharitonov V, Zhabko A. Lyapunov–Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica* 2003; **39**:15–20.
10. Mondie S, Kharitonov V, Santos O. Complete Lyapunov–Krasovskii functional with a given cross term in the time derivative. *Proceedings of 44th Conference on Decision and Control*, Sevilla, Spain, 2005; 5060–5064.
11. Kharitonov V, Niculescu S. On the stability of linear systems with uncertain delay. *IEEE Transactions on Automatic Control* 2003; **48**:127–132.
12. Fridman E. Stability of systems with uncertain delays: a new ‘complete’ Lyapunov–Krasovskii functional. *IEEE Transactions on Automatic Control* 2006; **51**:885–890.
13. Kao CY, Lincoln B. Simple stability criteria for systems with time-varying delays. *Automatica* 2004; **40**:1429–1434.
14. Tarbouriech S, Gomes da Silva, J. Synthesis of controllers for continuous-time delay systems with saturating controls via LMI’s. *IEEE Transactions on Automatic Control* 2000; **45**:105–111.
15. Repin YM. Quadratic Lyapunov functionals for systems with delay. *Prikladnaya Matematika Mehanika* 1965; **29**:564–566 (in Russian).
16. Datko R. An algorithm for computing Lyapunov functionals for some differential-difference equations. In *Ordinary Differential Equations*, L. Weiss (ed.). Academic Press: New York, 1971; 387–398.
17. Infante EF, Castelan WB. A Lyapunov functional for a matrix difference-differential equation. *Journal of Differential Equations* 1978; **29**:439–451.
18. Wu M, He Y, She JH, Liu GP. Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica* 2004; **40**(8):1435–1439.
19. Louisell J. Numerics of the stability exponent and eigenvalue abscissas of a matrix delay system. *Stability and Control of Time-Delay Systems*. Lecture Notes in Control and Information Sciences, vol. 228. Springer: London, 1997; 140–157.