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Brief paper

On input-to-state stability of systems with time-delay: A matrix inequalities approach☆

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ABSTRACT

Nonlinear matrix inequalities (NLMIs) approach, which is known to be efficient for stability and L_2-gain analysis, is extended to input-to-state stability (ISS). We first obtain sufficient conditions for ISS of systems with time-varying delays via Lyapunov–Krasovskii method. NLMIs are derived then for a class of systems with delayed state-feedback by using the $\mathcal{S}$-procedure. If NLMIs are feasible for all x, then the results are global. When NLMIs are feasible in a compact set containing the origin, bounds on the initial state and on the disturbance are given, which lead to bounded solutions. The numerical examples of sampled-data quantized stabilization illustrate the efficiency of the method.

1. Introduction

Nonlinear Matrix Inequalities or state-dependent Linear Matrix Inequalities (LMIs) (see Lu and Doyle (1997) and the references therein) constitute an efficient computational method for stabilization and H_∞ control of nonlinear systems. Recently, the NLMIs approach was extended to stabilization and H_∞ control of nonlinear systems with delay (Papachristodoulou, 2005), where the sufficient conditions were derived via corresponding Lyapunov–Krasovskii functionals.

To the best of our knowledge, NLMIs have not been applied yet to ISS property (Sontag, 1989). In the present paper we derive NLMIs, which give sufficient conditions for ISS, by applying the $\mathcal{S}$-procedure (Yakubovich, 1977) to Lyapunov-based sufficient conditions for ISS.

In order to derive NLMIs for ISS of systems with time-varying delays in the feedback, we first derive sufficient conditions for ISS for such systems via Lyapunov–Krasovskii functionals. We note that sufficient conditions for ISS systems with time-varying delays were derived via Razumikhin approach (Teel, 1998), which leads usually to more conservative results than Krasovskii method. For systems with constant delays, ISS sufficient conditions were recently derived in terms of Lyapunov–Krasovskii functionals in Pepe (2007) and Pepe and Jiang (2006).

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, R^n denotes the n-dimensional Euclidean space with norm $|\cdot|$, $R^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in R^{n\times n}$ means that P is symmetric and positive definite. In symmetric block matrices we use $*$ as an ellipsis for terms that are induced by symmetry. $\bar{\sigma}(P)$ and $\underline{\sigma}(P)$ denote the largest and the smallest eigenvalues of the symmetric matrix P. By $L_2([-r,0];R^n)$ is denoted the space of square integrable functions $\phi:[-r,0]\to R^n$.

The space of functions $\phi:[-r,0]\to R^n$, which are absolutely continuous on $[-r,0)$, have a finite $\lim_{\theta\to 0^-}\phi(\theta)$ and have square integrable first-order derivatives is denoted by W with the norm

$$\|\phi\|_W=\max_{\theta\in[-r,0]}|\phi(\theta)|+\left[\int_{-r}^{0}|\dot{\phi}(s)|^2\mathrm{d}s\right]^{\frac{1}{2}}.$$

We also denote $x_t(\theta)=x(t+\theta)$ $(\theta\in[-r,0])$.

A continuous function $\alpha:[0,a)\to[0,\infty)$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0)=0$. We will say that the function is of class $\mathcal{K}_\infty$ if $a=\infty$ and $\alpha(r)\to\infty$ when $r\to\infty$.

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A function $\beta : [0,\infty)^2 \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the mapping $\beta(s,t)$ is of class $\mathcal{K}$ and, for each fixed s, it is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$. The symbol $|\cdot|_\infty$ stands for essential supremum.

2. Input-to-state stability of systems with time-varying delays

We consider the following system

$$\dot{x}(t) = f(x(t), x(t-\tau(t)), w(t)), \tag{1}$$

where $x(t) \in R^n$ is the system state, $w(t) \in R^p$ is an exogenous signal, $f : R^n \times R^n \times R^p \to R^n$ is a continuously differentiable function, $f(0,0,0) = 0$ and $\tau(t)$ is the unknown piecewise-continuous time-delay that satisfies $0 \le \tau(t) \le r$. Given a measurable locally essentially bounded input w, Eq. (1) with initial condition $x_{t_0} = \phi \in W$ has a unique solution (Kolmanovskii & Myshkis, 1999, chapter 3, Section 2.4).

Definition 1. The system (1) is said to be uniformly globally ISS if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for any initial time t_0, any initial state $x_{t_0} = \phi \in W$ and any measurable, locally essentially bounded input w, the solution $x(t, t_0, \phi)$ exists for all $t \ge t_0$ and furthermore it satisfies

$$|x(t,t_0,\phi)| \le \max\left(\beta(\|\phi\|_W, t-t_0), \gamma\left(|w_{[t_0,t)}|_\infty\right)\right). \tag{2}$$

Given a continuous functional $V : R \times W \times L_2([-r,0], R^n) \to R^+$, define (see e.g., Kolmanovskii and Myshkis (1999))

$$\dot{V}(t,\phi,\dot{\phi}) = \lim\sup_{h\to 0^+} \frac{1}{h}[V(t+h, x_{t+h}(t,\phi), \dot{x}_{t+h}(t,\phi)) - V(t,\phi,\dot{\phi})],$$

where $x_t(t_0,\phi)$, for $t \ge t_0$, is a solution of the initial-value problem (1) with the initial condition $x_{t_0} = \phi \in W$.

Lemma 1. *Let there exists a locally Lipschitz with respect to the second and the third arguments functional $V : R \times W \times L_2([-r,0]; R^n) \to R^+$ such that the function $v(t) = V(t, x_t, \dot{x}_t)$ is absolutely continuous for measurable essentially bounded w. If additionally there exist functions α_1, α_2 of class $\mathcal{K}_\infty$, and functions α_3, θ of class $\mathcal{K}$ such that*

(i) $\alpha_1(|\phi(0)|) \le V(t,\phi,\dot{\phi}) \le \alpha_2(\|\phi\|_W)$,
(ii) $\dot{V}(t,\phi,\dot{\phi}) \le -\alpha_3(V(t,\phi,\dot{\phi}))$ *for* $V(t,\phi,\dot{\phi}) \ge \theta(|w|)$,

then (1) is uniformly globally ISS with $\gamma = \alpha_1^{-1} \circ \theta$.

Proof Follows the lines of Sontag (1989). Denote

$$\Lambda_\theta(t) = \left\{\phi \in W : V(t,\phi,\dot{\phi}) < \theta(|w_{[t_0,\infty)}|_\infty)\right\}.$$

(a) Let t_1 be the first time, when the solution $x_t = x_t(t_0,\phi)$ enters $\Lambda_\theta(t)$. For all $t \in [t_0, t_1)$, $v(t) = V(t, x_t, \dot{x}_t) \ge \theta(|w_{[t_0,\infty)}|_\infty)$ and thus $\dot{v}(t) \le -\alpha_3(v(t))$. Then (see Chapter 4 of Khalil (2002)), there exists a class $\mathcal{KL}$ function $\sigma(r,s)$ such that $v(t) \le \sigma(v(t_0), t-t_0)$. Hence,

$$\begin{aligned}|x(t,t_0,\phi)| &\le \alpha_1^{-1}(v(t)) \le \alpha_1^{-1}(\sigma(v(t_0), t-t_0)) \\ &\le \alpha_1^{-1}(\sigma(\alpha_2(\|\phi\|_W), t-t_0)) \triangleq \beta(\|\phi\|_W, t-t_0).\end{aligned}$$

Since $\beta(\|\phi\|_W, t-t_0) \to 0$ as $t \to \infty$, there is a finite time after which $\beta(\|\phi\|_W, t-t_0) < \theta(|w_{[t_0,\infty)}|_\infty)$ for all t. Therefore $t_1 < \infty$.

(b) We show now that $x_t \in \Lambda_\theta(t), \forall t \ge t_1$ which implies that $|x(t,t_0,\phi)| \le \alpha_1^{-1}(\theta(|w_{[t_0,\infty)}|_\infty))$. Suppose that it is not the case. Then, there exists an $\varepsilon > 0$ such that the set $\mathcal{T} = \{t \ge t_1 : v(t) > \theta(|w_{[t_0,\infty)}|_\infty) + \varepsilon\} \subset R$ is not empty. Let t^* denote its infimum. By (ii), $\dot{v}(t) < 0$ almost everywhere in a neighborhood of t^*, so v is strictly decreasing in a neighborhood of t^*, which is in contradiction with the fact that t^* is the infimum of $\mathcal{T}$.

(c) By the causality argument, (a) and (b) imply (2) with the gain $\gamma = \alpha_1^{-1} \circ \theta$.

3. ISS analysis of systems with delayed feedback: NLMI approach

3.1. Global input-to-state stability

Consider a class of systems, affine in control $u(t) \in R^m$ and disturbance $w(t) \in R^p$:

$$\dot{x}(t) = A(x(t))x(t) + B_2(x(t))u(t) + B_1(x(t))w(t), \tag{3}$$

where $x(t) \in R^n$, A, B_2 and B_1 are continuously differentiable matrix functions. Given a state-feedback

$$u(t) = K(x(t-\tau(t))), \qquad K(x) = k(x)x, \tag{4}$$

where $k : R^n \to R^{m\times n}$ is a continuously differentiable function and where $\tau(t) \in [0,r]$ is the unknown piecewise-continuous delay, that often appears in the feedback. Following Liberzon (2006) and Teel (1998), we are looking for conditions that guarantee the ISS of the closed-loop system (3) and (4) with a gain γ for all $\tau(t) \in [0,r]$. We note that in this section we focus on the analysis results. For linear systems, the design procedure will be given in Section 4.

We assume that $K_x = \frac{\partial K}{\partial x}$ (the Jacobian matrix of K) satisfies the following inequality

$$|K_x|^2 \le N + a_2(|x|), \quad x \in R^n, \tag{5}$$

with some constant $N \ge 0$ and some class $\mathcal{K}$ function a_2.

We apply the relation

$$K(x(t-\tau(t))) = K(x(t)) - \int_{t-\tau(t)}^{t} K_x(x(s))\dot{x}(s)\mathrm{d}s,$$

and represent the closed-loop system (3) and (4) in the form:

$$\begin{aligned}\dot{x}(t) &= A(x(t))x(t) + B_2(x(t))K(x(t)) + B_1(x(t))w(t) \\ &\quad - B_2(x(t))\int_{t-\tau(t)}^{t} K_x(x(s))\dot{x}(s)\mathrm{d}s.\end{aligned} \tag{6}$$

Note that (6) is equivalent to (3) and (4).

Consider the following Lyapunov–Krasovskii functional

$$V(x_t, \dot{x}_t) = x^{\mathrm{T}}(t)Px(t) + \int_{t-r}^{t} D(s, x_t, \dot{x}_t)\mathrm{d}s,$$

$$D(s, x_t, \dot{x}_t) \triangleq \int_s^t \dot{x}^{\mathrm{T}}(\xi)K_x^{\mathrm{T}}(x(\xi))RK_x(x(\xi))\dot{x}(\xi)\mathrm{d}\xi, \quad P > 0,\ R > 0. \tag{7}$$

We verify that

$$\begin{aligned}&\int_{t-r}^{t} D(s,x_t,\dot{x}_t)\mathrm{d}s \\ &\quad = \int_{t-r}^{t}(\xi - t + r)\dot{x}^{\mathrm{T}}(\xi)K_x^{\mathrm{T}}(x(\xi))RK_x(x(\xi))\dot{x}(\xi)\mathrm{d}\xi \\ &\quad \le r\int_{t-r}^{t}\dot{x}^{\mathrm{T}}(\xi)K_x^{\mathrm{T}}(x(\xi))RK_x(x(\xi))\dot{x}(\xi)\mathrm{d}\xi.\end{aligned} \tag{8}$$

Applying further (5), we find

$$\begin{aligned}&\int_{t-r}^{t} D(s,x_t,\dot{x}_t)\mathrm{d}s \le r\bar{\sigma}(R)\int_{t-r}^{t}|K_x(x(s))|^2|\dot{x}(s)|^2\mathrm{d}s \\ &\le r\bar{\sigma}(R)(N + \max_{\theta\in[-r,0]} a_2(|x(t+\theta)|))\int_{t-r}^{t}|\dot{x}(s)|^2\mathrm{d}s \\ &\quad \le c_1\int_{t-r}^{t}|\dot{x}(s)|^2\mathrm{d}s + c_2 a_2^2(\max_{\theta\in[-r,0]}|x(t+\theta)|) + c_2\left(\int_{t-r}^{t}|\dot{x}(s)|^2\mathrm{d}s\right)^2,\end{aligned}$$

where $c_1 = r\bar{\sigma}(R)N$, $c_2 = r\bar{\sigma}(R)/2$. Since

$$|x(t)|^2\underline{\sigma}(P) \le x^{\mathrm{T}}(t)Px(t) \le |x(t)|^2\bar{\sigma}(P),$$

we conclude that (i) of Lemma 1 is satisfied with $\alpha_1(|x|) = |x|^2\underline{\sigma}(P)$ and with

$$\alpha_2(s) = \max\{c_1 s^2 + c_2 s^4, \bar{\sigma}(P)s^2 + c_2 a_2^2(s)\}.$$

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Moreover, the function $v(t) = V(x_t, \dot{x}_t)$ is absolutely continuous along the solutions of (3)–(4) with $x_{t_0} \in W$ and with measurable essentially bounded w.

To satisfy assumption (ii) of Lemma 1, we find conditions which guarantee that, for some constant $a_3 > 0$

$$\dot{V}(x_t, \dot{x}_t) + a_3 V(x_t, \dot{x}_t) < 0, \quad \text{for } V(x_t, \dot{x}_t) \geq |w(t)|^2.$$

By applying the $\mathcal{S}$-procedure (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Yakubovich, 1977), the latter condition is satisfied if, for some positive continuous scalar function $\lambda(x)$, the following holds:

$$U \triangleq \dot{V}(x_t, \dot{x}_t) + a_3 V(x_t, \dot{x}_t) + \lambda(x(t))(V(x_t, \dot{x}_t) - |w(t)|^2) < 0, \quad \forall x \neq 0. \tag{9}$$

Hence, differentiating V along the trajectories of (6) we find

$$\begin{aligned} U \leq\ & 2x^{\mathrm{T}}(t)P\Big[A(x(t))x(t) + B_2(x(t))K(x(t)) \\ & - B_2(x(t))\int_{t-\tau(t)}^{t} K_x(x(s))\dot{x}(s)\mathrm{d}s + B_1(x(t))w(t)\Big] \\ & + r\dot{x}^{\mathrm{T}}(t)K_x^{\mathrm{T}}(x(t))RK_x(x(t))\dot{x}(t) \\ & - \int_{t-r}^{t} \dot{x}^{\mathrm{T}}(s)K_x^{\mathrm{T}}(x(s))R(1 - r(a_3 + \lambda(x(t))))K_x(x(s))\dot{x}(s)\mathrm{d}s \\ & + x^{\mathrm{T}}(t)Px(t)[a_3 + \lambda(x(t))] - \lambda(x(t))w^{\mathrm{T}}(t)w(t). \end{aligned} \tag{10}$$

Applying further Jensen's inequality (Gu, Kharitonov, & Chen, 2003)

$$\begin{aligned} & \int_{t-r}^{t} \dot{x}^{\mathrm{T}}(s)K_x^{\mathrm{T}}(x(s))R(1 - r(a_3 + \lambda(x(t))))K_x(x(s))\dot{x}(s)\mathrm{d}s \\ & \geq \zeta_2(t)^{\mathrm{T}} rR(1 - r(a_3 + \lambda(x(t))))\zeta_2(t), \end{aligned} \tag{11}$$

where $\zeta_2(t) = \frac{1}{r}\int_{t-\tau(t)}^{t} K_x(x(s))\dot{x}(s)\mathrm{d}s$, we find that

$$U \leq \zeta^{\mathrm{T}}(t)\Psi\zeta(t) + r\dot{x}^{\mathrm{T}}(t)K_x^{\mathrm{T}}(x(t))RK_x(x(t))\dot{x}(t), \tag{12}$$

where

$$\zeta^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t), \quad \frac{1}{r}\int_{t-\tau(t)}^{t}[K_x(x(s))\dot{x}(s)]^{\mathrm{T}}\,\mathrm{d}s, \quad w^{\mathrm{T}}(t)\right],$$

$$\Psi = \begin{bmatrix} \Psi_{11} & -rPB_2 & PB_1 \\ * & -r(R - r(a_3 + \lambda)) & 0 \\ * & * & -\lambda I_p \end{bmatrix}$$

$$\Psi_{11} = PA + A^{\mathrm{T}}P + PB_2k + k^{\mathrm{T}}B_2^{\mathrm{T}}P + (\alpha_3 + \lambda)P$$

and where, for simplicity, we omitted dependence on $x(t)$ of A, B_1, B_2, k and λ.

Setting the right-hand side of (12) for $\dot{x}(t)$ into (9) and applying Schur complements formula, we finally verify that $U < 0$ for $x(t) \neq 0$ if the following NLMI holds:

$$\begin{bmatrix} M_{11}(x) & -rPB_2(x) & PB_1(x) & M_{14}(x) \\ * & M_{22}(x) & 0 & M_{24}(x) \\ * & * & -\lambda(x)I_p & M_{34}(x) \\ * & * & * & -rR \end{bmatrix} < 0,$$

$$\forall 0 \neq x \in R^n,$$

$$\begin{aligned} M_{11}(x) &= P(A(x) + B_2(x)k(x)) + (A(x) + B_2(x)k(x))^{\mathrm{T}}P + (a_3 + \lambda(x))P \\ M_{14}(x) &= r(A(x) + B_2(x)k(x))^{\mathrm{T}}K_x^{\mathrm{T}}(x)R \\ M_{22}(x) &= -rR(1 - r(a_3 + \lambda(x))) \\ M_{24}(x) &= -r^2B_2^{\mathrm{T}}(x)K_x^{\mathrm{T}}(x)R \\ M_{34}(x) &= rB_1^{\mathrm{T}}(x)K_x^{\mathrm{T}}(x)R. \end{aligned} \tag{13}$$

Note that (13) may be considered as a state-dependent LMI, where a_3 and λ are tuning parameters. We proved the following

Theorem 1. *Given a state-feedback (4), where k is a continuously differentiable function, assume that (5) is satisfied with some function a_2 of class $\mathcal{K}$. Then (3)–(4) is ISS with $\gamma^2(s) = \frac{s^2}{\sigma(P)}$ for all piecewise-continuous delays $\tau(t) \in [-r, 0]$ if there exist an $n \times n$-matrix $P > 0$, a scalar-continuous function $\lambda(x) > 0,\ x \neq 0$, a constant $a_3 > 0$, and a constant $m \times m$-matrix R that solve (13).*

Another sufficient condition for ISS (which leads in some cases to less restrictive results) may be derived by using the descriptor approach (Fridman, 2001). We add to U given by (9) the right-hand side of the expression

$$0 = 2[x^{\mathrm{T}}(t)P_2^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t)P_3^{\mathrm{T}}]\delta, \tag{14}$$

where

$$\begin{aligned} \delta =\ & -\dot{x}(t) + A(x(t))x(t) + B_2(x(t))K(x(t)) \\ & - B_2(x(t))\int_{t-\tau(t)}^{t} K_x(x(s))\dot{x}(s)\mathrm{d}s + B_1(x(t))w(t). \end{aligned}$$

We obtain

$$\begin{aligned} U \leq\ & 2x^{\mathrm{T}}(t)P\dot{x}(t) + 2[x^{\mathrm{T}}(t)P_2^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t)P_3^{\mathrm{T}}]\Big[-\dot{x}(t) + A(x(t))x(t) \\ & + B_2(x(t))K(x(t)) - B_2(x(t))\int_{t-\tau(t)}^{t} K_x(x(s))\dot{x}(s)\mathrm{d}s \\ & + B_1(x(t))w(t)\Big] + r\dot{x}^{\mathrm{T}}(t)K_x^{\mathrm{T}}(x(t))RK_x(x(t))\dot{x}(t) \\ & - \int_{t-r}^{t} \dot{x}^{\mathrm{T}}(s)K_x^{\mathrm{T}}(x(s))R[1 - r(a_3 + \lambda(x(t)))]K_x(x(s))\dot{x}(s)\mathrm{d}s \\ & + x^{\mathrm{T}}(t)Px(t)[a_3 + \lambda(x(t))] - \lambda(x(t))w^{\mathrm{T}}(t)w(t). \end{aligned} \tag{15}$$

Applying further Jensen's inequality, we arrive to (12), where

$$\zeta^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t), \dot{x}^{\mathrm{T}}(t), \frac{1}{r}\int_{t-\tau(t)}^{t}[K_x(x(s))\dot{x}(s)]^{\mathrm{T}}\,\mathrm{d}s, w^{\mathrm{T}}(t)\right],$$

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{22} & rP_2^{\mathrm{T}}B_2 & P_2^{\mathrm{T}}B_1 \\ * & -P_3^{\mathrm{T}} - P_3 & rP_3^{\mathrm{T}}B_2 & P_3^{\mathrm{T}}B_1 \\ * & * & -rR(1 - r(a_3 + \lambda)) & 0 \\ * & * & * & -\lambda I_p \end{bmatrix}$$

and where

$$\begin{aligned} \Psi_{11} &= P_2^{\mathrm{T}}(A + B_2k) + (A^{\mathrm{T}} + k^{\mathrm{T}}B_2^{\mathrm{T}})P_2 + (a_3 + \lambda)P, \\ \Psi_{22} &= P - P_2^{\mathrm{T}} + (A^{\mathrm{T}} + k^{\mathrm{T}}B_2^{\mathrm{T}})P_3. \end{aligned} \tag{16}$$

From Schur complements formula, it follows that $U < 0$ for $x(t) \neq 0$ if the following NLMI holds:

$$\begin{bmatrix} \Psi_{11} & \Psi_{22} & rP_2^{\mathrm{T}}B_2 & P_2^{\mathrm{T}}B_1 & 0 \\ * & -P_3^{\mathrm{T}} - P_3 & rP_3^{\mathrm{T}}B_2 & P_3^{\mathrm{T}}B_1 & rK_x^{\mathrm{T}}R \\ * & * & -rR(1 - r(a_3 + \lambda)) & 0 & 0 \\ * & * & * & -\lambda I_p & 0 \\ * & * & * & * & -rR \end{bmatrix}$$

$$< 0, \quad \forall 0 \neq x \in R^n. \tag{17}$$

Theorem 2. *Given a state-feedback (4), where k is a continuously differentiable function, assume that (5) is satisfied with some function a_2 of class $\mathcal{K}$. Then (3)–(4) is ISS with $\gamma^2(s) = \frac{s^2}{\sigma(P)}$ for all piecewise-continuous delays $\tau(t) \in [-r, 0]$ if there exist an $n \times n$-matrix $P > 0$, $n \times n$-matrices P_2, P_3, an $m \times m$-matrix R, a scalar-continuous function $\lambda(x) > 0,\ x \neq 0$ and a constant $a_3 > 0$ that solve (17) with the notations of (16).*

Choosing $\lambda > 0$ to be constant, we obtain the following bound on the solution of (1):

Proposition 1. *Assume that conditions of Theorem 1 (or Theorem 2) hold with a constant $\lambda > 0$. Then the solution of (3) and (4) with the initial condition $x_{t_0} \in W$ satisfies the following inequality*

$$\underline{\sigma}(P)|x(t)|^2 \leq \mathrm{e}^{-(a_3+\lambda)(t-t_0)}\left[x^{\mathrm{T}}(t_0)Px(t_0) + \int_{t_0-r}^{t_0} D(s, x_{t_0}, \dot{x}_{t_0})\mathrm{d}s\right] + \frac{\lambda}{a_3+\lambda}|w_{[t_0,t)}|^2_{\infty}. \tag{18}$$

Proof. From (9) it follows that

$$\dot{V} + (a_3+\lambda)V - \lambda|w|^2 < 0,$$
$$V_{|t=t_0} = x^{\mathrm{T}}(t_0)Px(t_0) + \int_{t_0-r}^{t_0} D(s, x_{t_0}, \dot{x}_{t_0})\mathrm{d}s.$$

By applying the comparison principle (Khalil, 2002), we have

$$x^{\mathrm{T}}(t)Px(t) \leq V(x_t, \dot{x}_t) \leq \mathrm{e}^{-(a_3+\lambda)(t-t_0)}V(x_{t_0}, \dot{x}_{t_0}) + \int_{t_0}^{t} \mathrm{e}^{-(a_3+\lambda)(t-s)}\lambda|w(s)|^2\mathrm{d}s,$$

which implies (18).

3.2. On ISS of systems without delay

Consider now system (6) without delay:

$$\dot{x} = A(x)x + B_2(x)K(x) + B_1(x)w. \tag{19}$$

For (19), Lemma 1 holds with $V : R^n \to R^+$ and where $\|\phi\|_W$ is changed by $|x(t_0)|$. We use arguments of Theorem 1, where a_3 may be taken either constant or as a function of $|x|$ of class $\mathcal{K}$, since in the latter case

$$\dot{V}(x) \leq -a_3(|x|)V(x) \leq -a_3(\alpha_2^{-1}V(x))V(x) \stackrel{\Delta}{=} -\alpha_3(V(x))$$

and α_3 is of class $\mathcal{K}$. We obtain

Corollary 1. *Given a state-feedback (4), where k is a continuously differentiable function. Then (19) is ISS with $\gamma^2(s) = \frac{s^2}{\underline{\sigma}(P)}$ if there exists a constant $n \times n$-matrix $P > 0$, a scalar-continuous functions $\lambda(x) > 0$ and a function $a_3(|x|) > 0$, either constant or of class $\mathcal{K}$, that solve*

$$\begin{bmatrix} \Psi_{11} & PB_1(x) \\ * & -\lambda(x) \end{bmatrix} < 0, \quad \forall 0 \neq x \in R^n, \tag{20}$$

where

$$\Psi_{11} = P(A(x) + B_2(x)k(x)) + (A(x) + B_2(x)k(x))^{\mathrm{T}}P + (a_3(|x|) + \lambda(x))P.$$

Remark 1. If (20) is feasible with constant a_3 (and thus the system without delay (19) is ISS), then (13) is feasible for small enough r. Therefore, (1) is ISS for all small enough piecewise-continuous delays $\tau(t) \in [-r, 0]$.

The corresponding condition, which follows from Theorem 2, can be obtained similarly.

Example 1. Consider the system without delay

$$\dot{x}_1 = -3x_1^3 - 2x_2^2x_1,$$
$$\dot{x}_2 = -x_1^2x_2 - x_2^3 + (x_1^2 + x_2^2)w.$$

In this example, $B_2 = 0, B_1(x) = [0 \ x_1^2 + x_2^2]^{\mathrm{T}}$. Choosing $A(x) = -\mathrm{diag}\{3x_1^2 + 2x_2^2, x_1^2 + x_2^2\}$, $a_3 = \lambda = 0.1(x_1^2 + x_2^2)$ and $P = \mathrm{diag}\{p, p\}$ with $p > 0$ we obtain (20) of the form

$$\begin{bmatrix} \psi & 0 & 0 \\ * & -1.8p(x_1^2+x_2^2) & p(x_1^2+x_2^2) \\ * & * & -0.1(x_1^2+x_2^2) \end{bmatrix} < 0,$$

where $\psi = -2p(2.9x_1^2 + 1.9x_2^2)$. The latter inequality is feasible if $\frac{0.18}{p} > 1$ and thus the system is ISS with $\gamma^2 = \frac{1}{p}s^2 = 5.56s^2$.

Remark 2. A necessary condition for the feasibility of NLMIs (13), (17) and (20) is that the matrix $A(x) + B_2(x)k(x)$ is Hurwitz $\forall x \neq 0$. Since the choice of $A(x)$ in representations (3) and (19) is not unique, $A(x)$ should be chosen in such a way that it satisfies the above necessary condition. Thus, in Example 1, the considered choice of $A(x)$ is the only one that leads to Hurwitz $A(x) \forall x \neq 0$. Choosing e.g. $A(x) = \begin{bmatrix} -3x_1^2 - 2x_2^2 & 0 \\ -x_1x_2 & -x_2^2 \end{bmatrix}$, we see that the latter matrix is not Hurwitz for $x_2 = 0, x_1 \neq 0$.

3.3. On local ISS

For nonlinear system (1), it may happen that (13) holds locally for $x \neq 0$ from the ball $\mathcal{B}_b = \{x \in R^n : |x| \leq b\}$. Then Theorem 1 holds for the initial condition and for the disturbances, which lead to $x(t) \in \mathcal{B}_b$ for all $t \geq t_0$.

Example 2 (*Liberzon, 2006*). Consider the scalar system with delayed and quantized feedback

$$\dot{x} = -x + x^2 + u,$$
$$u = \begin{cases} q(-x^2(t-\tau(t))), & t - \tau(t) \geq t_0, \\ 0, & t - \tau(t) < t_0. \end{cases} \tag{21}$$

The input quantizer is a piecewise constant function $q : R \to Q$, where Q is a finite subset of R and for some $M > \Delta > 0$ the following holds: $|y| \leq M \Rightarrow |q(y) - y| \leq \Delta$. Following Liberzon (2006), we represent (21) in the form

$$\dot{x}(t) = -x(t) + x^2(t) - x^2(t-\tau(t)) + e(t),$$
$$x(t_0) = x_0, \qquad x(t) = 0, \quad t < t_0,$$

where

$$e(t) = \begin{cases} q(-x^2(t-\tau(t))) + x^2(t-\tau(t)), & t - \tau(t) \geq t_0, \\ 0, & t - \tau(t) < t_0, \end{cases}$$

choose $\Delta = 0.05$ and $M = 5$ and consider the set of the initial conditions satisfying $|x_0| \leq 2$.

The latter system has the form of (3) with $A(x) = -1 + x$, $k(x) = -x$, $K_x = -2x$, $B_1 = B_2 = 1$, and $e = w$. If the conditions of Proposition 1 hold, we find

$$|x(t)|^2 < |x_0|^2 + \frac{\lambda}{a_3+\lambda}\frac{|e_{[t_0,t)}|^2_{\infty}}{P} \leq 4.01 < (2.0025)^2$$

if $\frac{|e_{[t_0,t)}|^2_{\infty}}{P} \leq \frac{\Delta^2}{P} = \frac{0.0025}{P} < \frac{a_3+\lambda}{\lambda}0.01$, i.e. if

$$\frac{\lambda}{a_3+\lambda} - 4P < 0. \tag{22}$$

NLMI (17) will take the form

$$\begin{bmatrix} M_{11} & P - P_2 - P_3 & rP_2 & P_2 & 0 \\ * & -2P_3 & rP_3 & P_3 & -2xrR \\ * & * & -rR(1 - r(a_3+\lambda)) & 0 & 0 \\ * & * & * & -\lambda & 0 \\ * & * & * & * & -rR \end{bmatrix} < 0, \quad \forall 0 < |x| \leq 2.0025 \tag{23}$$

with $M_{11} = -2P_2 + (a_3+\lambda)P$. The latter inequality is affine in x. Therefore, considering x inside the polytope with the vertices ± 2.0025, we solve (23) for these two vertices and (22) with the same decision variables $P > 0, P_2, P_3, R$ and choosing $a_3 = 0.7$,

$\lambda = 0.1$. We find that the resulting LMIs are feasible for $r = 0.19$. We thus conclude that for every $0 \le \tau(t) \le 0.19$, the solution starting from $x_0 \in [-2, 2]$ remains in $[-2.0025, 2.0025]$ for all $t \ge t_0$ and asymptotically approaches $[-\gamma(\Delta), \gamma(\Delta)] \subset [-0.1, 0.1]$ (since $\gamma^2(\Delta) = \frac{\lambda}{a_3+\lambda}\frac{\Delta^2}{P} < (0.1)^2$). This result is less restrictive than the one (obtained via Razumikhin approach) in Liberzon (2006), where it was found that the corresponding $\tau \le 0.01$.

Remark 3. In this section we gave some *sufficient* conditions for ISS, which may be conservative. Similar to delay-dependent stability results for linear systems (see e.g. Xu and Lam (2005)), different improvements may be the topic for the future research.

4. Input-to-state stabilization of linear systems

Consider a linear system with a control input $u(t) \in R^m$ and a disturbance $w(t) \in R^p$:

$$\dot{x}(t) = Ax(t) + B_2u(t) + B_1w(t), \tag{24}$$

where $x(t) \in R^n$, A, B_2 and B_1 are constant matrices. Consider a linear state-feedback with time-varying delay

$$u(t) = Lx(t - \tau(t)), \tag{25}$$

where $L \in R^{m\times n}$ is a constant gain and $\tau(t)$ is an unknown piecewise-continuous delay that satisfies $0 \le \tau(t) \le r$.

Theorems 1 and 2 give sufficient conditions for ISS analysis, where $K_x = k = L$. To find the unknown gain L we use Theorem 2. Following Suplin, Fridman, and Shaked (2004), we choose $P_3 = \epsilon P_2$, where ϵ is a tuning scalar parameter (which may be restrictive). Note that P_2 is nonsingular due to the fact that the only matrix which can be negative definite in the second block on the diagonal of (17) is $-\epsilon(P_2 + P_2^T)$. Defining:

$$Q_2 = P_2^{-1}, \quad \bar{P} = Q_2^TPQ_2, \quad \bar{R} = R^{-1}, \quad Y = LQ_2, \tag{26}$$

we multiply (17) by $\mathrm{diag}\{P_2^{-1}, P_2^{-1}, R^{-1}, I_p, R^{-1}\}$ and its transpose, from the right and the left, respectively. We obtain:

Theorem 3. *Consider (24) with a piecewise-continuous delay $\tau(t) \in [0, r]$. Given $\lambda > 0$ and $a_3 > 0$ the system is ISS under the state-feedback law (25) if for some tuning scalar parameter ϵ there exist $n \times n$-matrices $0 < \bar{P}$, Q_2, an $m \times m$-matrix $\bar{R}$ and an $m \times n$-matrix Y such that the following LMI is satisfied*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & rB_2\bar{R} & B_1 & 0 \\ * & -\epsilon Q_2^T - \epsilon Q_2 & \epsilon rB_2\bar{R} & \epsilon B_1 & rY^T \\ * & * & -r\bar{R}(1 - r(a_3+\lambda)) & 0 & 0 \\ * & * & * & -\lambda I_p & 0 \\ * & * & * & * & -r\bar{R} \end{bmatrix} < 0, \tag{27}$$

where

$$\Gamma_{11} = AQ_2 + Q_2^TA^T + B_2Y + Y^TB_2^T + (a_3+\lambda)\bar{P},$$
$$\Gamma_{12} = \bar{P} - Q_2 + \epsilon(Q_2^TA^T + Y^TB_2^T).$$

The resulting γ and L are given by

$$\gamma^2(s) = \frac{1}{\underline{\sigma}(Q_2^{-T}\bar{P}Q_2^{-1})}\cdot\frac{\lambda}{a_3+\lambda}s^2, \qquad L = YQ_2^{-1}.$$

To minimize the gain γ, we can consider the following procedure: let $z \in R$ satisfy the LMI

$$\begin{bmatrix} -\bar{P} & Q_2^T \\ * & -zI_n \end{bmatrix} < 0. \tag{28}$$

By the Schur complement, this matrix inequality can be equivalently rewritten as $\bar{P} - z^{-1}Q_2^TQ_2 > 0$ or, $Q_2^{-T}\bar{P}Q_2^{-1} > z^{-1}I_n$, which implies that $\underline{\sigma}(Q_2^{-T}\bar{P}Q_2^{-1}) > z^{-1}$. And so, we obtain the bound $\gamma^2(s) \le \frac{\lambda z}{a_3+\lambda}s^2$. Given positive numbers λ, a_3 and ϵ such that LMI (27) is feasible, one may obtain the least value for the gain γ^2 by finding the minimum value of z such that LMIs (27) and (28) are satisfied. This is an LMI optimization problem that can be solved with now standard numerical tools.

Example 3. We consider the problem of the stabilization of a two cart–pendulum from Section 4 of Ishii and Francis (2003). The system is described by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{29}$$

where $x(t) \in R^8$, $u(t) \in R^2$, and A and B are given by:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -2.5 & -1.962 & 0 & 5 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 10 & 5 & 23.544 & 0 & -10 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 2.5 & 0 & 0 & -5 & -2.5 & -1.962 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & -5 & 0 & 0 & 10 & 5 & 23.544 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0.5 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

The objective is to find a memoryless state-feedback such that, under sampling and quantization, the ball $\mathcal{B}_{b_0}$ (for a given radius b_0) is attractive from R^n (i.e. every trajectory enters it and does not leave). The control law has the form

$$u^T(t) = [q(L_1x(t_k))\ q(L_2x(t_k))],$$
$$t_k \le t < t_{k+1}, \ k = 0, 1, \ldots, \tag{30}$$

where $L = [L_1^T\ L_2^T]^T$ is the state-feedback gain. The sampling may be not uniform with bounded sampling times: $t_{k+1} - t_k \le T_{\max}$, $\lim_{k\to\infty} t_k = \infty$. The quantization function q is defined as in Example 2 (but with $M = \infty$ and with a countable set Q).

Following Fridman, Seuret, and Richard (2004) we represent sampled-data control as delayed control and rewrite system (29) and (30) in the form

$$\dot{x}(t) = Ax(t) + BLx(t - \tau(t)) + Bw(t), \tag{31}$$

where $0 \le \tau(t) = t - t_k \le T_{\max}$ and $w^T(t) = [w_1(t)\ w_2(t)]$, $w_i(t) = q(L_ix(t - \tau(t))) - L_ix(t - \tau(t))$, $i = 1, 2$. Note that for all t, $|w(t)| \le \sqrt{2}\Delta$.

Applying Theorem 3, we choose $a_3 = 10^{-4}$, $\lambda = 0.019$, $\varepsilon = 8$ and verify feasibility of (27). We obtain $T_{\max} = 0.194$ for the system to be ISS, which is to compare with $T_{\max} = 7.91 \times 10^{-3}$ obtained in Ishii and Francis (2003) (for the case of uniform sampling). The

Table 1
Results for example 3

$T_{\max}$	0.002	0.004	0.006
Δ by Theorem 3	105	68.3	50.7
Δ by Ishii and Francis (2003)	0.267	0.242	0.217

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ball $\mathcal{B}_{b_0}$ of radius b_0 is attractive if $\gamma(\sqrt{2}\Delta) \leq b_0$. We use $b_0 = 4$ as in Ishii and Francis (2003). The value of γ is obtained by solving the following optimization problem: minimize z (i.e. minimize γ) under the LMI constraints (27) and (28) (with $B_1 = B_2 = B$ and $r = T_{\max}$). Table 1 gives the values of Δ for several values of $T_{\max}$, that were obtained by our results and by Ishii and Francis (2003).

5. Conclusions

In this paper, we have proposed a new methodology to analyze ISS of nonlinear systems with time-varying delays based on the use of NLMI.

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