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Brief paper

Sliding mode control in the presence of input delay: A singular perturbation approach[☆]

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ABSTRACT

Sliding Mode Control (SMC) in the presence of small, unavoidable input delay as may be present in controller implementation is studied. Linear systems with bounded matched disturbances and uncertain system matrices are considered, where input delay in the SMC will produce oscillations or potentially even unbounded solutions. *Without a priori knowledge of the bounds on the state-dependent terms as required by existing methods*, the design objective is to achieve ultimate boundedness of the closed-loop system with a bound proportional to the delay and disturbance bounds. This is a non-trivial problem because the relay gain depends on the state bound, whereas the latter bound depends on the relay gain. A controller with linear gain proportional to the scalar $\frac{1}{\mu}$ is proposed, which for small enough $\mu > 0$ produces a closed-loop *singularly perturbed system* and yields the desired ultimate bound. A constructive Linear Matrix Inequality (LMI)-based solution for evaluation of both the design parameters and the ultimate bound is derived. The superiority of the proposed control over existing methodologies that ignore input delay within the design is demonstrated through an example.

1. Introduction

Sliding Mode Control is well known for its invariance properties and has received a great deal of attention in the area of robust control. SMC can provide asymptotic stability in the presence of matched uncertainties and disturbances. However, in practical control systems, ideal sliding motion cannot usually be achieved due to model imperfections, time delays etc. The combination of delay phenomenon with relay actuators induces oscillations of finite frequency around the sliding surface and even instability (Fridman, 1997; Fridman, Fridman, & Shustin, 1993; Levaggi & Punta, 2006). The degree to which the robustness of this SMC design paradigm can translate into systems with input delay is thus of considerable interest and an open research question.

Many results are available which apply various control methods to time-delay systems. For example, adaptive control (Wen, Soh, & Zhang, 2000; Zhou, Wen, & Wang, 2009), finite-time stabilization (Karafyllis, 2006), to name a few. Delays are unavoidable in the implementation of any feedback loop. Sampled-data control can be considered as control with a delayed input (Fridman, Seuret, & Richard, 2004). When control engineers approach SMC, the choice of sampling rate is an immediate, and extremely critical design decision (Utkin, 1992). The existing work on sampled-data SMC transforms the system to discrete-time. However, this approach becomes complicated for uncertain or state-delay systems.

While the study of SMC in the presence of state delay has been ongoing (Chou & Cheng, 2003; Shyu, Liu, & Hsu, 2005), results on the effect of input delay in SMC are scarce. It was shown in Akian, Bliman, and Sorine (2002) and Fridman et al. (1993); Fridman, Fridman, and Shustin (2002) that even in the simplest one dimensional delayed relay control system, only oscillatory solutions can occur. Despite the oscillatory performance, relay delay control has advantages over a linear delayed controller in keeping an inverted pendulum upright (Sieber, 2006).

A simple example in Gouaisbaut, Perruquetti, and Richard (2002) pointed out behavioral changes (bifurcations) arising when designing a controller without taking the input delay into consideration. This work motivates the study of specific SMC design methods for systems with input delays. In the existing results (Fridman et al., 2002; Gouaisbaut et al., 2002), *an a priori constant bound is assumed on the state-dependent terms of the system*, which is restrictive. The relay gain is chosen to be greater than this bound. In Fridman, Strygin, and Polyakov (2004) for linear

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time-invariant systems with bounded nonlinear uncertainties, relay delay control was designed to achieve practical and adaptive stabilization provided the signs of the appropriately transformed states are available for measurement.

For the case of known and constant input delay, the predictor-based SMC was designed in Roh and Oh (1999). Stability was achieved without any restriction on the time delay and spectral properties of the open-loop system. However, it was pointed out in Nguang (2001) that the method cannot compensate for matched uncertainties. A summary of some contributions to the field of SMC with relay delay was provided in Richard (2003). A frequency domain method of analysis was given in Boiko (2009). Recently, sampled-data high gain output feedback SMC for linear systems with matched disturbances has been designed via discretization and singular perturbation analysis (Nguyen, Su, & Gajic, 2010). Note that discretization of systems with uncertain matrices may lead to complicated conditions. Also, there may be additional difficulties in the presence of additional input/output delay.

In the present paper, output-feedback SMC for linear systems with bounded disturbances and polytopic type uncertainties is considered under uncertain *time-varying input delays*. The design objective is to achieve ultimate boundedness of the closed-loop system with a bound proportional to the size of the delay and the disturbance. For small enough delay such a controller should have advantages over a corresponding linear controller, because the linear control will produce a bound proportional to the disturbance only.

The main contribution is a general framework for SMC in the presence of input delay *without any a priori knowledge of the bounds on the system states*. The following design difficulty arises, which does not appear in the absence of input delay: the relay gain depends on the ultimate bound on the state, whereas the latter bound depends on the relay gain. To overcome this difficulty, a sliding mode controller is designed with a linear gain proportional to the scalar $\frac{1}{\mu}$, which for small enough $\mu > 0$ produces a closed-loop *singularly perturbed system* and which allows the desired ultimate bound to be achieved for the closed-loop system. The design process seeks to *enlarge μ avoiding a high gain control*. The resulting ultimate bound is proportional to the size of the delay, disturbance and the switching gain. Therefore trade-offs can be made between the linear and discontinuous controller elements in order to minimize delay effects. Preliminary results were presented in Fridman, Han, and Spurgeon (2010).

Notation. Throughout the paper, the superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$. The symbol $\|\cdot\|_\infty$ stands for essential supremum.

2. Problem formulation

Consider the following uncertain dynamical system with time varying input delay $\tau(t)$ and disturbance $w(t)$

$$\dot{x}(t) = Ax(t) + B\left(u(t-\tau(t)) + w(t)\right), \qquad y(t) = Cx(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n, x(t_0) = x_0, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$ with $m < p < n$. It is assumed that $u(t) = 0$ for $t < t_0$. Matrix A may be uncertain with polytopic type uncertainty. The matched disturbance $w(t) \in \mathbb{R}^m$ and the uncertain input delay $\tau(t)$ are assumed to be bounded $\|w(t)\| \le \Delta$, $\tau(t) \in [0, \tau^*]$, where Δ and τ^* are known bounds and τ^* is supposed to be sufficiently small. It is assumed that the delay is either fast varying (without any constraints on the delay derivative) or slowly varying, where the delay-derivative satisfies the bound $\dot{\tau} \le d < 1$. Assuming B and C are both of full rank, a controller will be designed which for sufficiently large t induces the motion of the closed-loop system in the $M\tau^*\Delta$-neighborhood (with $M > 0$ independent of τ^*) of the surface

$$\mathscr{S} = \{x \in \mathbb{R}^n : z_2(t) = FCx(t) = 0\} \tag{2}$$

for some selected matrix $F \in \mathbb{R}^{m\times p}$. The ideal sliding motion $z_2(t) = 0$ is only possible with $\tau = 0$.

Remark 1. Since a static output feedback control is designed, the results are applicable to both input delay, τ_i, and output delay, τ_o, where in the closed-loop system the resulting delay is $\tau = \tau_i + \tau_0$.

3. Sliding manifold design

If $rank(CB) = m$, there exists a change of coordinates $x_r = T_r x$, where $T_r \in \mathbb{R}^{n\times n}$ is non-singular, in which the system has the regular form (Edwards & Spurgeon, 1995)

$$\dot{x}_r(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x_r(t) + \begin{bmatrix} 0 \\ I_m \end{bmatrix} (u(t-\tau(t)) + w(t))$$
$$y(t) = \begin{bmatrix} 0 & T \end{bmatrix} x_r(t) \tag{3}$$

where $x_r(t) = col\{x_1(t), x_2(t)\}$, $T \in \mathbb{R}^{p\times p}$ is invertible, $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$. Given $K \in \mathbb{R}^{m\times(p-m)}$, let $F = \begin{bmatrix} K & I_m \end{bmatrix} T^{-1}$. As a result $F\begin{bmatrix} 0 & T \end{bmatrix} = \begin{bmatrix} KC_1 & I_m \end{bmatrix}$, $C_1 = \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{(p-m)} \end{bmatrix}$. Defining the sliding manifold as

$$z_2(t) = Fy(t) = x_2(t) + KC_1x_1(t) \tag{4}$$

the reduced-order dynamics is governed by the system

$$\dot{x}_1(t) = (A_{11} - A_{12}KC_1)x_1(t) + A_{12}z_2(t) \tag{5}$$

with input z_2. The system triple A_{11}, A_{12}, C_1 is assumed to be stabilizable. In the presence of input delay, z_2 in (5) will not vanish in finite time. Therefore, a K is sought which not only stabilizes (5) (as in the case without delay), but also produces input-to-state stability (with the smallest gain possible). Sufficient conditions for the input-to-state stability of (5) are given by the following lemma:

Lemma 1. *Given tuning parameters $\alpha > 0$, ε, ε_1, b, and $M \in \mathbb{R}^{(p-m)\times(n-p)}$, if there exists an $(n-m) \times (n-m)$ matrix $P > 0$, and matrices $Q_{22} \in \mathbb{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathbb{R}^{(n-p)\times(p-m)}$, $Y \in \mathbb{R}^{m\times(p-m)}$ so that LMI*

$$\Theta = \begin{bmatrix} \theta_{1,1} & \theta_{1,2} & A_{12} \\ * & -\varepsilon Q_2 - \varepsilon Q_2^T & \varepsilon A_{12} \\ * & * & -bI_m \end{bmatrix} < 0$$
$$\theta_{1,1} = A_{11}Q_2 - A_{12}[YM \quad \varepsilon_1 Y] + \alpha P + Q_2^T A_{11}^T - [YM \quad \varepsilon_1 Y]^T A_{12}^T,$$
$$\theta_{1,2} = P - Q_2 + \varepsilon Q_2^T A_{11}^T - \varepsilon [YM \quad \varepsilon_1 Y]^T A_{12}^T \tag{6}$$

holds, where $Q_2 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}M & \varepsilon_1 Q_{22} \end{bmatrix}$, then the solution of (5) with $K = YQ_{22}^{-1}$ and with the initial condition $x_1(t_0)$ at initial time t_0 is bounded by $x_1^T(t)\hat{P}x_1(t) < e^{-\alpha(t-t_0)}x_1^T(t_0)\hat{P}x_1(t_0) + \frac{b}{\alpha}\|z_{2[t_0,t]}\|_\infty^2$ where $\hat{P} = Q_2^{-T}PQ_2{}^{-1}$ (Han, Fridman, & Spurgeon, 2010).

Remark 2. To minimize the ultimate bound on x_1, the following procedure is adopted from Fridman and Dambrine (2009). The $\zeta \in \mathbb{R}$ is minimized subject to LMI (6) and $\begin{bmatrix} -P & Q_2^T \\ * & -\zeta I_{n-m} \end{bmatrix} < 0$, which leads to $\limsup_{t\to\infty} \|x_1(t)\|^2 < \zeta\frac{b}{\alpha} \limsup_{t\to\infty} \|z_2(t)\|^2$.

A delayed sliding mode controller is designed which ensures the closed-loop system is ultimately bounded with bound proportional to the delay, disturbance and switching gain.

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4. Controller design: a singular perturbation approach

Defining $z = \begin{bmatrix} I_{n-m} & 0 \\ KC_1 & I_m \end{bmatrix} x_r = col\{z_1(t), z_2(t)\}$ in (3) it follows that

$$\begin{aligned} \dot{z}_1(t) &= \bar{A}_{11}z_1(t) + \bar{A}_{12}z_2(t) \\ \dot{z}_2(t) &= \bar{A}_{21}z_1(t) + \bar{A}_{22}z_2(t) + u(t-\tau(t)) + w(t) \end{aligned} \tag{7}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1, \bar{A}_{12} = A_{12}, \bar{A}_{21} = KC_1\bar{A}_{11} - A_{22}KC_1 + A_{21}, A_{22} = KC_1\bar{A}_{12} + A_{22}$. For $i = 1, \dots, m$, denote the i-th component z_2 by z_{2_i}. A control law of the form

$$u(t) = -\frac{F}{\mu}y(t) - (1+\delta)\Delta\left[\text{sign}\, z_{2_1}(t)\cdots\text{sign}\, z_{2_m}(t)\right]^T \tag{8}$$

will be designed for (7), where $\mu > 0$ and $\delta > 0$ are tuning parameters. The design objective is to achieve ultimate boundedness of the closed-loop system with a bound *proportional to the delay and the disturbance bounds*. The closed-loop system (7) and (8) has the form

$$\dot{z}_1(t) = \bar{A}_{11}z_1(t) + \bar{A}_{12}z_2(t) \tag{9}$$

$$\begin{aligned} \mu\dot{z}_2(t) \;=\; & \mu\bar{A}_{21}z_1(t) + \mu\bar{A}_{22}z_2(t) - z_2(t-\mu\xi(t)) \\ & + \mu[w(t) - (1+\delta)\Delta \\ & \times [\text{sign}\, z_{2_1}(t-\mu\xi(t))\cdots\text{sign}\, z_{2_m}(t-\mu\xi(t))]^T] \end{aligned} \tag{10}$$

with the initial condition

$$z(t_0) = z_0, \qquad z(t) = 0, \quad t < t_0 \tag{11}$$

where $\mu\xi(t) = \tau(t), 0 \le \xi(t) \le h$ and $\mu h = \tau^*$. For small $\mu > 0$ (7) and (8) is a singularly perturbed system. The delay is scaled by μ in order to guarantee robust (input to state) stability with respect to small enough delay (Fridman, 2002).

Remark 3. For $\xi \equiv 0$, a conventional SMC is designed as follows Edwards and Spurgeon (1995): find $\mu > 0$ so that the linear controller $u_l(t) = -\frac{F}{\mu}y(t)$ asymptotically stabilizes (7) with $w \equiv 0$. For all $\delta > 0$, (8) asymptotically stabilizes (7) with non-zero $\|w\| \le \Delta$. The bound $\|[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(t)\| < \delta\Delta$ is valid for big enough t, which implies finite time convergence of the closed-loop system to $z_2 = 0$. The Lyapunov-based proofs of stability and finite time convergence use the relation $z_{2i}(t)\,\text{sign}\, z_{2_i}(t) \ge 0$. For non-zero $\xi(t)$, the product $z_{2i}(t)\,\text{sign}\, z_{2_i}(t-\mu\xi(t))$ may change sign and the closed-loop system (7) and (8) is not asymptotically stable.

Given $h > 0$, the main problem is *the choice of* $\mu > 0$ *and of* $\delta > 0$ (if any) that ensures the bound

$$\limsup_{t\to\infty} \|[\bar{A}_{21}\ \ \bar{A}_{22}]z(t)\| < \delta\Delta, \quad \forall\xi(t) \in [0, h] \tag{12}$$

holds for solutions of (9), (10). In the following, matrix inequalities are derived for finding μ and δ via a singular perturbation approach, which guarantees the feasibility of these matrix inequalities for small enough μ. The design process seeks to *enlarge* μ *avoiding a high gain control*. Finally, it will be proved that the closed-loop system is ultimately bounded with a bound proportional to $\tau^*\Delta$. For recent results on stability of singularly perturbed systems with small delay, refer to Chen, Yang, Lu, and Shen (2010) and Glizer (2009).

4.1. Input-to-state stability of the time-delay system

Considering the switching component of the SMC as a perturbation, this allows a bound on the state and an appropriate switching gain to be chosen. Denoting

$$\begin{aligned} \bar{w}(t) &= w(t) - (1+\delta) \\ &\quad \times \Delta\left[\text{sign}\, z_{2_1}(t-\mu\xi(t))\cdots\text{sign}\, z_{2_m}(t-\mu\xi(t))\right]^T \end{aligned} \tag{13}$$

the closed-loop system (9), (10) can be presented as

$$\begin{aligned} \dot{z}_1(t) &= \bar{A}_{11}z_1(t) + \bar{A}_{12}z_2(t) \\ \mu\dot{z}_2(t) &= \mu\bar{A}_{21}z_1(t) + \mu\bar{A}_{22}z_2(t) - z_2(t-\mu\xi(t)) + \mu\bar{w}(t) \end{aligned} \tag{14}$$

where $\|\bar{w}(t)\| \le [1 + (1+\delta)\sqrt{m}]\Delta$. Let $P_\mu \in \mathbb{R}^{n\times n}$ be positive definite with structure (Kokotovic, Khalil, & O'Reilly, 1986)

$$P_\mu = \begin{bmatrix} P_1 & \mu P_2^T \\ * & \mu P_3 \end{bmatrix} > 0 \tag{15}$$

where $P_1 \in \mathbb{R}^{n-m}$. For (14), choose the Lyapunov–Krasovskii functional of the form

$$\begin{aligned} V_\mu(t) \;=\; & \int_{t-\mu\xi(t)}^{t} e^{\bar{\alpha}(s-t)}z_2^T(s)Sz_2(s)ds + \int_{t-\mu h}^{t} e^{\bar{\alpha}(s-t)}z_2^T(s)G \\ & \cdot z_2(s)ds + \mu h\int_{-\mu h}^{0}\int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)}\dot{z}_2^T(s)R\dot{z}_2(s)dsd\theta \\ & + z^T(t)P_\mu z(t) \end{aligned} \tag{16}$$

where G, R and $S \in \mathbb{R}^m$ are positive matrices. The inequality

$$W(t) = \frac{d}{dt}V_\mu(t) + \bar{\alpha}V_\mu(t) - \mu^2\bar{b}\bar{w}^T(t)\bar{w}(t) < 0 \tag{17}$$

along the trajectories of (9), (10) for $\|z_0\|^2 + \|\bar{w}_{[t_0,t]}\|_\infty^2 > 0$ guarantees (19) (Fridman & Dambrine, 2009). The following lemma can be stated (for proof see the Appendix) as follows:

Lemma 2. *Given positive tuning scalars μ, h, $\bar{\alpha}$ and $\bar{b}$, let there exist $P_\mu > 0$ in (15) with $(n-m)\times(n-m)$ matrix $P_1 > 0$, $m\times(n-m)$-matrix P_2 and $m\times m$ positive matrices P_3, G, R, S such that the following LMI with its entries*

$$\Theta_\mu = \begin{bmatrix} \tilde{\theta}_{1,1} & \cdots & \tilde{\theta}_{1,6} \\ * & \cdots & \tilde{\theta}_{6,6} \end{bmatrix} < 0 \tag{18}$$

$$\begin{aligned}
&\tilde{\theta}_{1,1} = P_1\bar{A}_{11} + \bar{A}_{11}^TP_1 + \mu P_2^T\bar{A}_{21} + \mu\bar{A}_{21}^TP_2 + \bar{\alpha}P_1, \\
&\tilde{\theta}_{1,2} = P_1\bar{A}_{12} + \mu\bar{A}_{21}^TP_3 + \mu\bar{A}_{11}^TP_2^T + \mu P_2^T\bar{A}_{22} + \bar{\alpha}\mu P_2^T, \\
&\tilde{\theta}_{1,4} = -P_2^T, \qquad \tilde{\theta}_{1,5} = P_2^T, \qquad \tilde{\theta}_{1,6} = h\mu\bar{A}_{21}^TR, \\
&\tilde{\theta}_{2,2} = \mu P_2\bar{A}_{12} + \mu\bar{A}_{12}^TP_2^T + \mu P_3\bar{A}_{22} + \mu\bar{A}_{22}^TP_3 \\
&\qquad + \bar{\alpha}\mu P_3 + G - e^{-\bar{\alpha}\mu h}R + S, \\
&\tilde{\theta}_{2,4} = -P_3 + e^{-\bar{\alpha}\mu h}R, \qquad \tilde{\theta}_{2,5} = P_3, \qquad \tilde{\theta}_{2,6} = h\mu\bar{A}_{22}^TR, \\
&\tilde{\theta}_{3,3} = -e^{-\bar{\alpha}\mu h}G - e^{-\bar{\alpha}\mu h}R, \qquad \tilde{\theta}_{3,4} = e^{-\bar{\alpha}\mu h}R, \\
&\tilde{\theta}_{4,4} = -2e^{-\bar{\alpha}\mu h}R - (1-d)Se^{-\bar{\alpha}\mu h}, \qquad \tilde{\theta}_{4,6} = -hR, \\
&\tilde{\theta}_{5,5} = -\bar{b}I_m, \qquad \tilde{\theta}_{5,6} = hR, \qquad \tilde{\theta}_{6,6} = -R
\end{aligned}$$

is feasible. Then solutions of (9)–(11) satisfy the bound

$$z^T(t)P_\mu z(t) < e^{-\bar{\alpha}(t-t_0)}z^T(t_0)P_\mu z(t_0) + \frac{\mu^2\bar{b}}{\bar{\alpha}}\|\bar{w}_{[t_0,t]}\|_\infty^2 \tag{19}$$

for all $\xi(t) \in [0, h]$ with $\mu\dot{\xi}(t) \le d < 1$ (and thus (9)–(10) is input-to-state stable). Moreover, solutions of (9)–(11) satisfy (19) for all fast-varying delays $\xi(t) \in [0, h]$ if LMI (18) is feasible with $S = 0$.

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4.2. LMIs for the controller design

Conditions will now be derived that guarantee the bound (12) for the solutions of (9), (10). Taking into account (19) and, thus, $\limsup_{t\to\infty} z^T(t)P_\mu z(t) < \frac{\mu^2\bar{b}}{\bar{\alpha}}[1+(1+\delta)\sqrt{m}]^2\Delta^2$, it may be concluded that (12) holds if the inequality $\mu^2 z^T(t)[\bar{A}_{21}\ \bar{A}_{22}]^T[\bar{A}_{21}\ \bar{A}_{22}] z(t) < \frac{\bar{\alpha} z^T(t)P_\mu z(t)\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}$ is satisfied for $t\to\infty$. Hence, the inequality

$$\begin{bmatrix} \frac{-\bar{\alpha}\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}P_1 & \frac{-\mu\bar{\alpha}\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}P_2^T & \mu\bar{A}_{21}^T \\ * & \frac{-\mu\bar{\alpha}\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}P_3 & \mu\bar{A}_{22}^T \\ * & * & -I_m \end{bmatrix} < 0 \tag{20}$$

guarantees that the solutions of (9), (10) satisfy the bound (12). By Schur complements, (20) is feasible if the following matrix inequality is feasible

$$\frac{-\bar{\alpha}\delta^2}{\bar{b}[1+(1+\delta)\sqrt{m}]^2}\begin{bmatrix} P_1 & \mu P_2^T \\ * & \mu P_3 \end{bmatrix} + \mu^2 \begin{bmatrix} \bar{A}_{21}^T \\ \bar{A}_{22}^T \end{bmatrix}\begin{bmatrix} \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} < 0. \tag{21}$$

Matrix inequalities (15), (18) and (20) have been derived for finding the parameters μ and δ of the controller (8). It will now be shown that if the μ-independent LMI

$$\Theta_0 = \Theta_{\mu|\mu=0} < 0 \tag{22}$$

is feasible, then for all $\delta > 0$ inequalities (15), (18) and (20) are feasible for all small enough μ. Let P_1, P_2, P_3 satisfy $\Theta_0 < 0$. Then for small enough $\mu > 0$, (15) and (18) are feasible for the same *μ-independent* matrices P_1, P_2, P_3. Hence, given $\delta > 0$, (21) is feasible for small enough $\mu > 0$.

It is easily seen that $\Theta_0 < 0$ guarantees exponential stability with decay rate $\bar{\alpha}/2$ of the slow subsystem

$$\dot{z}_s(t) = \bar{A}_{11}z_s(t), \quad z_s(t)\in\mathbb{R}^{n-m}$$

and asymptotic stability of the fast subsystem of (14)

$$\mu\dot{z}_f(t) = -z_f(t-\mu\xi(t)), \quad \xi(t)\in[0,h],\ z_f(t)\in\mathbb{R}^m.$$

Since $\bar{A}_{11}$ is Hurwitz, there exists $P_1 > 0$ satisfying $P_1\bar{A}_{11} + \bar{A}_{11}^T P_1 + \bar{\alpha}P_1 < 0$ for small enough $\bar{\alpha} > 0$. Choose next $P_2 = 0$, $G = S = 0$ and $R = P_3 = p_3 I_m$. By using Schur complements, it can be shown that $\Theta_0 < 0$ holds for big enough $p_3 > 0$, $\bar{b}$ and small enough h. The sufficient conditions below for the feasibility of (15), (18) and (20) have been proved:

Proposition 1. (i) *Given positive tuning scalars h, $\bar{\alpha}$ and $\bar{b}$, let there exist $0 < P_1 \in \mathbb{R}^{(n-m)\times(n-m)}$, $P_2 \in \mathbb{R}^{m\times(n-m)}$ and positive $m\times m$-matrices P_3, G, R, S such that LMI (22) is feasible. Then, for all $\delta > 0$ there exists $\mu(\delta) > 0$ such that for all $\mu \in (0, \mu(\delta)]$ LMIs (15), (18) and (20) are feasible and, thus, solutions of (9)–(10) satisfy the bound (12).*

(ii) *LMI (22) is feasible for small enough h, $\bar{\alpha}$ and big enough $\bar{b}$.*

4.3. Main result

Let $\phi(t,t_0,\mu)$ be the fundamental solution of the equation $\mu\dot{\zeta}(t) = -\zeta(t-\mu\xi(t))$, $\zeta(t)\in\mathbb{R}$ with $\phi(t_0,t_0,\mu) = 1$ and $\phi(t,t_0,\mu) = 0$ for $t < t_0$. By using the arguments of Lemma 2 and choosing

$$V_2 = \psi\int_{t-\mu\xi(t)}^{t} e^{\alpha_2(s-t)}\zeta^2(s)ds + q\int_{t-\mu h}^{t} e^{\alpha_2(s-t)}\zeta^2(s)ds + \mu hr\int_{-\mu h}^{0}\int_{t+\theta}^{t} e^{\alpha_2(s-t)}\dot{\zeta}^2(s)dsd\theta + \mu\rho\zeta^2(t)$$

with positive scalars ρ, q, r, ψ, it can be shown that the feasibility of the μ-independent LMI

$$\begin{bmatrix} \psi+q-r & 0 & -\rho+r & 0 \\ * & -q-r & r & 0 \\ * & * & -(1-d)\psi-2r & hr \\ * & * & * & -r \end{bmatrix} < 0 \tag{23}$$

yields the following bound

$$|\phi(t,t_0,\mu)| \le e^{-\frac{\alpha_2(t-t_0)}{\mu}} \tag{24}$$

for small enough $\alpha_2 > 0$ and $\forall\ \mu > 0$, $\xi(t) \le h$, $\mu\dot{\xi} \le d < 1$. Note that (23) is feasible for $h \le 1.414$ if $d = 0$ and for $h \le 1.22$ if d is unknown (i.e. for fast varying delay). The main result is now formulated (for proof see the Appendix) as follows:

Theorem 1. *Let the conditions of Lemma 1 hold. Given positive tuning scalars μ, h, $\bar{\alpha}$, $\bar{b}$ and δ let there exist $0 < P_1 \in \mathbb{R}^{(n-m)\times(n-m)}$, $P_2 \in \mathbb{R}^{m\times(n-m)}$, positive $m\times m$-matrices P_3, G, R, S and positive scalars ρ, q, r, ψ such that LMIs (15), (18), (20) and (23) are feasible. Then for all $\xi\in[0,h]$, $\mu\dot{\xi} \le d < 1$ the solutions $z(t)$ of the closed-loop system (9)–(10) satisfy the following bounds:*

$$\limsup_{t\to\infty} |z_{2_i}(t)| \le 2M_0\mu h, \quad M_0 = (1+\delta)(1+\sqrt{m})\Delta \tag{25}$$

where $i = 1,\ldots,m$ denotes the i-th component of z_2, and

$$\limsup_{t\to\infty} z_1^T(t)\hat{P}z_1(t) \le 4\frac{b}{\alpha}mM_0^2\mu^2h^2. \tag{26}$$

Moreover, the solutions of (9)- (10) satisfy (25) and (26) for all fast varying delays $\xi(t)\in[0,h]$ if the above LMIs are feasible with $S = 0$ and $\psi = 0$.

Remark 4. The singular perturbation approach allows the choice of tuning parameters to occur in two stages:

(i) Given $\bar{b} > 0$, $h = 0$ find the tuning parameter $\bar{\alpha} > 0$ that minimizes $\frac{\bar{b}}{\bar{\alpha}}$ by solving the μ-independent LMI (22) (which corresponds to the slow and the fast subsystems). Increase h arriving to some maximum achievable $h \le 1.22$ which preserves the feasibility of (22).

(ii) With $\bar{\alpha}$, $\bar{b}$ and h as found in (1) search for the remaining tuning parameters $\mu > 0$ and $\delta > 0$ such that the μ-dependent LMIs (15), (18), (20) and (21) (which correspond to the full-order system) are feasible. Start with small μ and big δ for which the above LMIs are feasible (as guaranteed by Proposition 1). Then increase μ (to avoid the high-gain control and to treat bigger delays) by decreasing h and $\bar{\alpha}$ such that μh is maximized and δ is minimized. The latter leads to a smaller ultimate bound.

Note increase in μ leads to increase of the switching parameter δ. Therefore, a trade-off exists between bound minimization and the acceptable control magnitude.

Remark 5. Consider now (1) with the linear controller $u_l(t) = -\frac{F}{\mu}y(t)$. Then the closed-loop system has the form (14) with $\bar{w}(t) = w(t)$. Under the conditions of Lemma 2, the solutions of the resulting closed-loop system satisfy $z^T(t)P_\mu z(t) < e^{-\bar{\alpha}(t-t_0)} z_0^T P_\mu z_0 + \frac{\mu^2\bar{b}}{\bar{\alpha}}\Delta^2$. Given μ and δ satisfying the conditions of Theorem 1, the ultimate bounds under the proposed SMC are given by (25) and (26) and these bounds vanish for $h\to 0$, i.e. the performance under the proposed SMC recovers the performance under the ideal SMC without input delay. Given $\mu > 0$ satisfying Lemma 2, the ultimate bounds under the *linear controller* $u_l(t) = -\frac{F}{\mu}y(t)$ are proportional to the disturbance bound only and do not vanish for $h\to 0$. Therefore, the linear controller leads to vanishing bounds only for $\mu\to 0$, i.e. by using very high gain (even with no input delay).

5. Extension to input and state delay

The following uncertain dynamical system is considered with state and input time varying delay $r(t)$ and $\tau(t)$, respectively, and with matched disturbance $w(t)$

$$\dot{x}(t) = Ax(t) + A_d x(t - r(t)) + B\left(u(t - \tau(t)) + w(t)\right),$$
$$y(t) = Cx(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ with $m < p < n$. The delays and disturbance are bounded by: $r(t) \in [0,\ r^*]$, $\tau(t) \in [0, \tau^*]$ and $\|w(t)\| \le \Delta$. The delays may be either slowly varying with $\dot{r}(t) \le d_1 < 1$, $\dot{\tau}(t) \le d_2 < 1$ or fast varying (with no constraint on the delay derivatives). The input and output matrices B and C are both of full rank. The sliding manifold can be defined by (2).

5.1. Sliding manifold design

In regular form, the system (5) becomes

$$\begin{aligned} \dot{x}_r(t) &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x_r(t) + \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} x_r(t - r(t)) \\ &\quad + \begin{bmatrix} 0 \\ I_m \end{bmatrix} \left(u(t - \tau(t)) + w(t)\right), \qquad (27) \\ y(t) &= \begin{bmatrix} 0 & T \end{bmatrix} x_r(t). \end{aligned}$$

Defining the sliding manifold as in (4), the reduced-order system with inputs $z_2(t)$ and $z_2(t - r(t))$ is

$$\begin{aligned} \dot{x}_1(t) &= (A_{11} - A_{12}KC_1)x_1(t) + (A_{d11} - A_{d12}KC_1)x_1(t - r(t)) \\ &\quad + A_{12}z_2(t) + A_{d12}z_2(t - r(t)) \qquad (28) \end{aligned}$$

where $(A_{11} + A_{d11}, A_{12} + A_{d12}, C_1)$ is assumed stabilizable.

Lemma 3 (Han et al. (2010)). *Given tuning scalars $\alpha > 0$, ε, ε_1, b_1, $b_2 > 0$ and a matrix $M \in \mathbb{R}^{(p-m)\times(n-p)}$, let there exist $(n - m) \times (n - m)$ matrices $P > 0$, $G \ge 0$, $S \ge 0$, $R \ge 0$ and matrices $Q_{22} \in \mathbb{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathbb{R}^{(n-p)\times(p-m)}$, $Y \in \mathbb{R}^{m\times(p-m)}$, $K = YQ_{22}^{-1}$ such that LMI*

$$\hat{\Theta} = \begin{bmatrix} \hat{\theta}_{1,1} & \cdots & \hat{\theta}_{1,6} \\ * & \cdots & \hat{\theta}_{6,6} \end{bmatrix} < 0 \qquad (29)$$

where

$$\begin{aligned} \hat{\theta}_{11} &= A_{11}Q_2 - A_{12}[Y \quad \varepsilon_1 Y] + Q_2^T A_{11}^T \\ &\quad + \alpha P - [YM \quad \varepsilon Y]^T A_{12}^T + G + S - Re^{-\alpha r^*}, \\ \hat{\theta}_{15} &= A_{12}, \qquad \hat{\theta}_{34} = Re^{-\alpha r^*} \end{aligned}$$

$$\hat{\theta}_{12} = P - Q_2 + \varepsilon Q_2^T A_{11}^T - \varepsilon[YM \quad \varepsilon_1 Y]^T A_{12}^T,$$

$$\hat{\theta}_{66} = -b_2 I_m$$

$$\hat{\theta}_{14} = A_{d11}Q_2 - A_{d12}[YM \quad \varepsilon_1 Y] + Re^{-\alpha r^*}, \qquad \hat{\theta}_{16} = A_{d12}$$

$$\hat{\theta}_{22} = -\varepsilon Q_2 - \varepsilon Q_2^T + r^{*2}R, \qquad \hat{\theta}_{33} = -(G + R)e^{-\alpha r^*},$$

$$\hat{\theta}_{24} = \varepsilon A_{d11}Q_2 - \varepsilon A_{d12}[YM \quad \varepsilon_1 Y], \qquad \hat{\theta}_{25} = \varepsilon A_{12},$$

$$\hat{\theta}_{26} = \varepsilon A_{d12}$$

$$\hat{\theta}_{44} = -2e^{-\alpha r^*}R - (1 - d_1)Se^{-\alpha r^*}, \qquad \hat{\theta}_{55} = -b_1 I_m,$$

holds, then for all ultimately bounded z_2, *solutions of* (28) *satisfy the inequality* $\limsup_{t\to\infty} x_1^T(t)\hat{P}x_1(t) < \frac{b_1+b_2}{\alpha} \limsup_{t\to\infty} \|z_2(t)\|^2$, *where $\hat{P}$ and Q_2 are in the form given in Lemma 1.*

5.2. Controller design and the resulting ultimate bound

By similar change of coordinates as in (7) and (27) becomes

$$\begin{aligned} \dot{z}_1(t) &= \bar{A}_{11}z_1(t) + \bar{A}_{d11}z_1(t - r(t)) + \bar{A}_{12}z_2(t) \\ &\quad + \bar{A}_{d12}z_2(t - r(t)) \\ \dot{z}_2(t) &= \bar{A}_{21}z_1(t) + \bar{A}_{d21}z_1(t - r(t)) + \bar{A}_{22}z_2(t) \\ &\quad + \bar{A}_{d22}z_2(t - r(t)) + u(t - \tau(t)) + w(t). \end{aligned}$$

With the controller given in (8) the closed-loop system is

$$\begin{aligned} \dot{z}_1(t) &= \bar{A}_{11}z_1(t) + \bar{A}_{d11}z_1(t - r(t)) + \bar{A}_{12}z_2(t) \\ &\quad + \bar{A}_{d12}z_2(t - r(t)) \\ \mu\dot{z}_2(t) &= \mu\bar{A}_{21}z_1(t) + \mu\bar{A}_{d21}z_1(t - r(t)) + \mu\bar{A}_{22}z_2(t) \\ &\quad + \mu\bar{A}_{d22}z_2(t - r(t)) - z_2(t - \mu\xi(t)) + \bar{w}(t) \qquad (30) \end{aligned}$$

where $\bar{w}(t)$ is given by (13) with $\|\bar{w}(t)\| \le [1 + (1 + \delta)\sqrt{m}]\Delta$, $\mu\xi(t) = \tau(t)$, $0 \le \xi(t) \le h$, $z(t) = col\{z_1(t), z_2(t)\}$. Let P_μ be of the same structure as (15), then input-to-state stability can be derived using the Lyapunov–Krasovskii functional

$$\begin{aligned} V_\mu(t) &= z^T(t)P_\mu z(t) + \int_{t-r^*}^{t} e^{\bar{\alpha}(s-t)} z_1^T(s)G_1 z_1(s)ds \\ &\quad + \int_{t-r(t)}^{t} e^{\bar{\alpha}(s-t)} z_1^T(s)S_1 z_1(s)ds \\ &\quad + \int_{t-\mu h}^{t} e^{\bar{\alpha}(s-t)} z_2^T(s)G_2 z_2(s)ds \\ &\quad + \int_{t-\mu\xi(t)}^{t} e^{\bar{\alpha}(s-t)} z_2^T(s)S_2 z_2(s)ds \\ &\quad + \int_{t-r(t)}^{t} e^{\bar{\alpha}(s-t)} z_2^T(s)S_3 z_2(s)ds \\ &\quad + \int_{t-r^*}^{t} e^{\bar{\alpha}(s-t)} z_2^T(s)S_4 z_2(s)ds \\ &\quad + r^* \int_{-r^*}^{0} \int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)} \dot{z}_1^T(s)R_1 \cdot \dot{z}_1(s)dsd\theta \\ &\quad + \mu h \int_{-\mu h}^{0} \int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)} \dot{z}_2^T(s)R_2 \dot{z}_2(s)dsd\theta \\ &\quad + \mu^2 r^* \int_{-r^*}^{0} \int_{t+\theta}^{t} e^{\bar{\alpha}(s-t)} \dot{z}_2^T(s)R_3 \dot{z}_2(s)dsd\theta \end{aligned}$$

with positive matrices $G_1, G_2, S_1, S_2, S_3, S_4, R_1, R_2$ and R_3.

Lemma 4. *Given positive tuning scalars r^*, μ, h, $\bar{\alpha}$ and $\bar{b}_1$, let there exist $P_\mu > 0$ in (15) with $(n - m) \times (n - m)$ matrix $P_1 > 0$, $m \times (n - m)$-matrix P_2, $m \times m$ positive matrix P_3, $(n - m) \times (n - m)$ positive matrices G_1, S_1, R_1, $m \times m$ positive matrices G_2, S_2, S_3, S_4, R_2 and R_3 such that the LMI*

$$\Theta_\mu = \begin{bmatrix} \bar{\theta}_{1,1} & \cdots & \bar{\theta}_{1,12} \\ * & \cdots & \bar{\theta}_{12,12} \end{bmatrix} < 0 \qquad (31)$$

with entries

$$\begin{aligned} \bar{\theta}_{1,1} &= \bar{\alpha}P_1 + P_1\bar{A}_{11} + \bar{A}_{11}^T P_1 + \mu P_2^T \bar{A}_{21} + \mu\bar{A}_{21}P_2 + G_1 \\ &\quad + S_1 - R_1 e^{-\bar{\alpha}r^*}, \\ \bar{\theta}_{1,2} &= P_1\bar{A}_{d11} + R_1 e^{-\bar{\alpha}r^*} + \mu P_2^T \bar{A}_{d21}, \end{aligned}$$

$$\bar{\theta}_{1,4} = P_1\bar{A}_{12} + \mu\bar{A}_{21}^T P_3 + \mu\bar{A}_{11}^T P_2^T + \mu P_2^T \bar{A}_{22} + \alpha\mu P_2^T,$$

$$\bar{\theta}_{1,5} = P_1\bar{A}_{d12} + \mu P_2^T \bar{A}_{d22}, \qquad \bar{\theta}_{1,7} = -P_2^T, \qquad \bar{\theta}_{1,9} = P_2^T,$$

$$\bar{\theta}_{1,10} = r^*\bar{A}_{11}^T R_1, \qquad \bar{\theta}_{1,11} = \mu h\bar{A}_{21}^T R_2,$$

$$\bar{\theta}_{1,12} = \mu r^*\bar{A}_{21}^T R_3,$$

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$\bar\theta_{2,2} = -2R_1e^{-\bar\alpha r^*} - (1-d_1)S_1e^{-\bar\alpha r*}, \qquad \bar\theta_{2,3} = R_1e^{-\bar\alpha r^*},$
$\bar\theta_{2,4} = \mu\bar A_{d21}^TP_3 + \mu\bar A_{d11}^TP_2^T, \qquad \bar\theta_{2,10} = r^*\bar A_{d11}^TR_1,$
$\bar\theta_{2,11} = \mu h\bar A_{d21}^TR_2, \qquad \bar\theta_{2,12} = \mu r^*\bar A_{d21}^TR_3,$
$\bar\theta_{5,10} = r^*\bar A_{d12}^TR_1,$
$\bar\theta_{3,3} = -e^{-\bar\alpha r^*}(R_1+G_1), \qquad \bar\theta_{4,7} = -P_3 + R_2e^{-\bar\alpha\mu h},$
$\bar\theta_{4,4} = \mu P_3\bar A_{22} + \mu\bar A_{22}^TP_3 + \mu P_2\bar A_{12} + \mu\bar A_{12}^TP_2^T + \mu\bar\alpha P_3 - R_2e^{-\bar\alpha\mu h} - \mu^2e^{-\bar\alpha r^*}R_3 + G_2 + S_2 + S_3 + S_4,$
$\bar\theta_{4,5} = \mu P_3\bar A_{d22} + \mu P_2\bar A_{d12} + \mu^2e^{-\bar\alpha r^*}R_3,$
$\bar\theta_{4,9} = P_3, \qquad \bar\theta_{4,10} = r^*\bar A_{12}^TR_1, \qquad \bar\theta_{4,11} = \mu h\bar A_{22}^TR_2,$
$\bar\theta_{5,5} = -(1-d_1)S_3e^{-\bar\alpha r^*} - 2\mu^2e^{-\bar\alpha r^*}R_3,$
$\bar\theta_{5,6} = \mu^2e^{-\bar\alpha r^*}R_3,$
$\bar\theta_{5,11} = \mu h\bar A_{d22}^TR_2, \qquad \bar\theta_{5,12} = \mu r^*\bar A_{d22}^TR_3,$
$\bar\theta_{4,12} = \mu r^*\bar A_{22}^TR_3,$
$\bar\theta_{6,6} = -e^{-\bar\alpha r^*}(\mu^2R_3 + S_4), \qquad \bar\theta_{9,9} = -\bar b_1I_m,$
$\bar\theta_{12,12} = -R_3$
$\bar\theta_{7,7} = -2R_2e^{-\bar\alpha\mu h} - (1-d_2)S_2e^{-\bar\alpha\mu h}, \qquad \bar\theta_{7,8} = R_2e^{-\bar\alpha\mu h},$
$\bar\theta_{7,11} = -hR_2, \qquad \bar\theta_{7,12} = -r^*R_3,$
$\bar\theta_{8,8} = -(R_2+G_2)e^{-\bar\alpha\mu h},$
$\bar\theta_{9,11} = hR_2, \qquad \bar\theta_{9,12} = r^*R_3, \qquad \bar\theta_{10,10} = -R_1,$
$\bar\theta_{11,11} = -R_2.$

Then solutions of (30) satisfy the bound

$$\limsup_{t\to\infty} z^T(t)P_\mu z(t) < \frac{\mu^2\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2}{\bar\alpha}\Delta^2 \tag{32}$$

for all $r(t)\in[0,r^*]$ and $\xi(t)\in[0,h]$ with $\dot r(t)\le d_1<1$ and $\mu\dot\xi\le d_2<1$. Moreover, solutions of (30) satisfy (32) for all fast-varying delays $r(t)\in[0,r^*]$ or $\xi(t)\in[0,h]$ if LMI (31) is feasible with $S_1=S_3=0$ or $S_2=0$ respectively.

Conditions will be derived that guarantee

$$\begin{aligned}&\limsup_{t\to\infty}\|[\bar A_{21}\ \bar A_{22}]z(t)\| < \kappa_1\delta\Delta,\\ &\limsup_{t\to\infty}\|[\bar A_{d21}\ \bar A_{d22}]z(t-r(t))\| < \kappa_2\delta\Delta\end{aligned} \tag{33}$$

for solutions of (30), where $\kappa_1+\kappa_2\le 1$. Given (32) and

$$\limsup_{t\to\infty} z^T(t-r(t))P_\mu z(t-r(t)) < \frac{\mu^2\bar b_1\left(1+(1+\delta)\sqrt{m}\right)^2}{\bar\alpha}\Delta^2$$

are true, (33) holds if the following inequalities are satisfied

$$z^T(t)[\bar A_{21}\ \bar A_{22}]^T[\bar A_{21}\ \bar A_{22}]z(t) < \frac{\bar\alpha\kappa_1^2\delta^2z^T(t)P_\mu z(t)}{\mu^2\bar b_1(1+(1+\delta)\sqrt{m})^2}$$
$$z^T(t-r(t))[\bar A_{d21}\ \bar A_{d22}]^T[\bar A_{d21}\ \bar A_{d22}]z(t-r(t)) < \frac{\bar\alpha\kappa_2^2\delta^2z^T(t-r(t))P_\mu z(t-r(t))}{\mu^2\bar b_1(1+(1+\delta)\sqrt{m})^2}$$

for $t\to\infty$. Hence, the inequalities

$$\begin{aligned}&\begin{bmatrix}\kappa_1^2\varpi P_1 & \mu\kappa_1^2\varpi P_2^T & \mu\bar A_{21}^T\\ * & \mu\kappa_1^2\varpi P_3 & \mu\bar A_{22}^T\\ * & * & -I_m\end{bmatrix} < 0,\\ &\begin{bmatrix}\kappa_2^2\varpi P_1 & \mu\kappa_2^2\varpi P_2^T & \mu\bar A_{d21}^T\\ * & \mu\kappa_2^2\varpi P_3 & \mu\bar A_{d22}^T\\ * & * & -I_m\end{bmatrix} < 0\end{aligned} \tag{34}$$

where $\varpi = -\frac{\bar\alpha\delta^2}{\bar b_1(1+(1+\delta)\sqrt{m})^2}$, guarantee that the solutions of (30) satisfy the bound (33).

Proposition 2. *Given positive tuning scalars $r^*, \mu, h, \bar\alpha, \bar b_1, \kappa_1, \kappa_2, \delta$ let there exist $0 < P_1\in\mathbb{R}^{(n-m)\times(n-m)}$, $P_2\in\mathbb{R}^{m\times(n-m)}$ and positive $(n-m)\times(n-m)$ matrices G_1, S_1, R_1, positive $m\times m$ matrices $P_3, G_2, S_2, S_3, S_4, R_2$ and R_3 such that LMI $\Theta_0<0$ is feasible, where Θ_0 is given by (31) with $\mu=0$. Then, for positive scalars κ_1,κ_2, where $\kappa_1+\kappa_2\le 1$ and all $\delta>0$, there exists $\mu(\delta)>0$ such that for all $\mu\in(0,\mu(\delta)]$ LMIs (15), (31) and (34) are feasible and, thus, solutions of (30) satisfy the bound (33).*

Theorem 2. *Let the conditions of Lemma 4 hold. Given positive tuning scalars r^*, μ, $h\le 1.22$, $\bar\alpha$, $\bar b_1$, κ_1,κ_2,δ let there exist $0 < P_1\in\mathbb{R}^{(n-m)\times(n-m)}$, $P_2\in\mathbb{R}^{m\times(n-m)}$ and positive $(n-m)\times(n-m)$ matrices G_1, S_1, R_1, positive $m\times m$ matrices $P_3, G_2, S_2, S_3, S_4, R_2$ and R_3 such that LMIs (15), (31) and (34) are feasible. Then for all $\xi\in[0,h]$, $\mu\dot\xi\le d_1<1$, $r(t)\in[0,r^*]$, $\dot r(t)\le d_2<1$, the solutions of the closed-loop system (30) satisfy (25) and (26). Moreover, the solutions of (30) satisfy (25) and (26) for all fast varying delays $r(t)\in[0,r^*]$ or $\xi(t)\in[0,h]$ if the above LMIs are feasible with $S_1=S_3=0$ or $S_2=0$ respectively.*

Remark 6. Since LMIs (15), (18), (20) and (23), as well as (29), (31) and (34), are affine in the system matrices, the results are applicable where these matrices have polytopic type uncertainties.

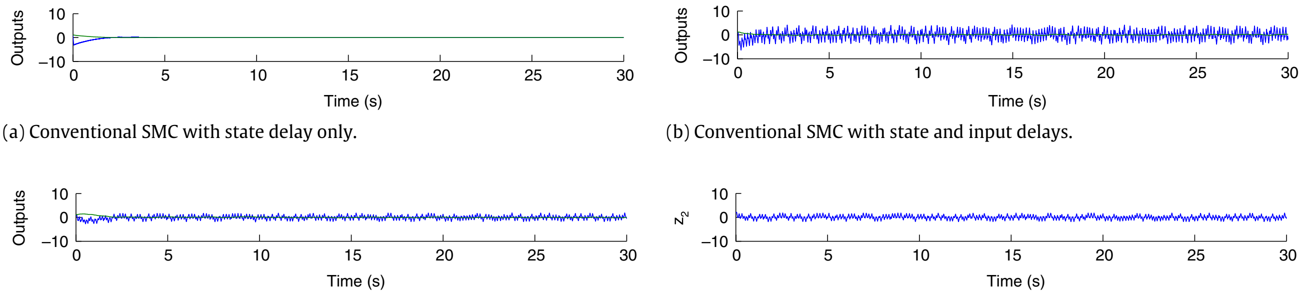
6. Example

The following model of combustion in a liquid monopropellant rocket motor has been considered in Zheng, Cheng, and Gao (1995), where the system is given by (5) with

$$A = \begin{bmatrix}0.2\rho(t) & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & -1 & 1\\ 0 & 1 & -1 & 0\end{bmatrix},$$
$$A_d = \begin{bmatrix}-1-0.2\rho(t) & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$
$$C = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \qquad B = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^T. \tag{35}$$

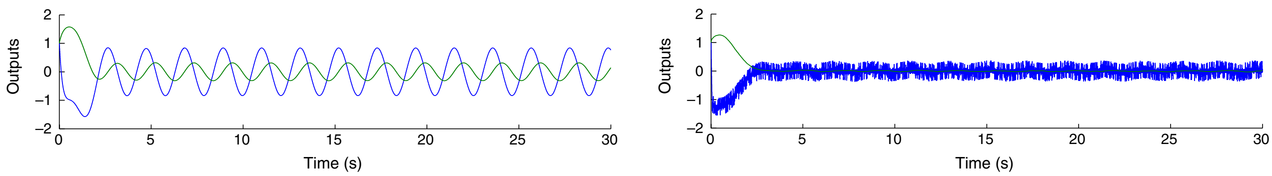
Here $\rho(t)=\sin(t)$ and the exogenous disturbance $w(t)$ satisfy $\|w(t)\|\le\Delta=5$. Time-varying delays in the states and input are due to pressure force propagation in the combustion chamber and gas injector respectively.

LMI solutions for the controller design incorporate matrices A and A_d with two vertices corresponding to $\rho=\pm1$. The controller is designed for fast varying state delay $r(t)\le 0.2$ s and fast varying input delay $\tau(t)\le 0.05$ s. Setting $r^*=0.2$, $\alpha=0.9$, $b_1=0.0002$, $b_2=0.0001$, $\varepsilon=1.5$, $\varepsilon_1=3.5$, $M=[4\ 2.4]$ in LMI (29) and $\zeta=147000$ in the LMI in Remark 4, the reduced order system (28) is ultimately bounded with $K=1.015$. Choosing LMI tuning parameters according to the algorithm in Remark 4, it is obtained by solving the μ-independent LMI (22) (with $\mu=0$), that $\bar\alpha=0.44$, $\bar b_1=0.000005$ and maximum $h=0.7$. Substituting the above $\bar\alpha$, $\bar b_1$ and h into the μ-dependent LMI (31), we find that LMIs (31) and (34) are feasible for $\mu=0.17$, $\mu h=0.05$ s with a smaller $\bar\alpha=0.28$, $h=0.29$ and $\kappa_1=0.9999$, $\kappa_2=0.0001$, $\delta=5.03$. Therefore, the system is ultimately bounded under the linear controller $u(t)=-\frac{F}{\mu}y(t)$ with $F=[1\ 1.015]$. The controller (8) has been fully synthesized to guarantee the bound $\|z_2(t)\|_{t\to\infty}\le 6.03$ according to (25) for all fast varying state delays $r(t)\le 0.2$ s



(a) Conventional SMC with state delay only.
(b) Conventional SMC with state and input delays.
(c) Proposed SMC with both state and input delays.
(d) Proposed SMC with both state and input delays.

Fig. 1. Conventional and proposed SMC under slowly varying state delay $r(t) \le 0.2$ s and fast varying input delay $\tau(t) \le 0.05$ s.



(a) Using linear controller.
(b) Using proposed SMC.

Fig. 2. Comparison of proposed SMC and the linear controller of the SMC under fast varying delays $r(t) \le 0.2$ s and $\tau(t) \le 0.01$ s.

and input delays $\tau(t) \le 0.05$ s. The control input does not produce high gain (here$\|F/\mu\| \le 8.4$ since $\mu = 0.17$ has been chosen large enough). The designed controller is simulated, where the disturbance is $w(t) = 5 \sin 3t$.

The advantages of the designed SMC over the conventional SMC from Han et al. (2010) (which ignores the input delay)

$$u(t) = -[15.8\ 43]y(t) - 67\frac{Fy(t)}{\|Fy(t)\|}, \quad F = [1\ 3.1] \tag{36}$$

are first demonstrated. Note that (36) has been designed for (35) with the slowly varying state delay $r(t) \le 0.2$ s, $\dot{r} \le 0.5$. For simulation, the slowly varying state delay is chosen as $r(t) = 0.1 \sin(5t) + 0.1 \le 0.2$ s and the fast varying input delay is $\tau(t) = 0.025 \sin(40t) + 0.025 \le 0.05$ s. The conventional SMC (36) achieves asymptotic stability in the presence of the state delay (see Fig. 1(a), where $\tau \equiv 0$). In the presence of the input delay the controller leads to the outputs bounded by $\|y\|_{t\to 30\text{ s}} \le 4$ (see Fig. 1(b)). In comparison the proposed SMC leads to a smaller output bounds with $\|y(t)\|_{t\to 30\text{ s}} \le 1.6$ as shown in Fig. 1(c). The resulting switching variable is bounded by $\|z_2\|_{t\to 30\text{ s}} \le 1.7$ as shown in Fig. 1(d), which is in line with the theoretical estimation. In the simulations under both SMC methods, chattering of high frequency is observed due to the delayed switching component sign $u(t-\tau(t))$. In the real implementations, many methods are introduced for reducing the chattering. A comprehensive review is given in Young, Utkin, and Özgüner (1999).

For sufficiently small input delay the proposed SMC should have advantage over its linear control component leading to smaller bounds on the system outputs. Consider next the fast varying state delay $r(t) = 0.1 \sin(15t) + 0.1 \le 0.2$ s and the fast varying input delay $\tau(t) = 0.005 \sin(200t) + 0.005 \le 0.01$ s, where the disturbance was kept the same. The outputs of the system under the linear controller $u(t) = -\frac{F}{\mu}y(t)(\mu = 0.17)$ and under the proposed SMC are shown in Fig. 2(a) and (b) respectively. The bound on the outputs produced by the linear controller is $\|y(t)\|_{t\to 30\text{ s}} \le 0.85$, whereas the bound obtained by the proposed SMC is smaller with $\|y(t)\|_{t\to 30\text{ s}} \le 0.35$. Note that the linear controller leads to the same bound $\|y(t)\|_{t\to 30\text{ s}} \le 0.35$ as SMC by more than twice higher gain with $\mu = 0.07$.

Remark 7. Without input delay, the new SMC design method has advantages over existing methods (Edwards & Spurgeon, 1995; Han et al., 2010). The matrix P_μ for the analysis of the closed-loop system is full and not diagonal as in existing methods. The conservativeness of the diagonally structured P_μ was verified by setting $P_2 = 0$ in (15) for the above example while keeping all the other tuning parameters in the LMIs unchanged. The bound on the feasible input delay in this case was found to be $\mu h = 0.02$ s, which is smaller than $\mu h = 0.05$ s obtained using the full P_μ.

7. Conclusion

Sliding mode control for systems with matched bounded disturbances in the presence of input time-varying delay has been studied using a singular perturbation approach. Unlike existing results on relay control with input delay (Fridman et al., 2002; Gouaisbaut et al., 2002) a priori knowledge of the *bounds on the system states* is not needed. Ultimately bounded solutions of the delayed system are found based on LMI formulations and various Lyapunov-based methods. The ultimate bound is proportional to the delay, the disturbances and the switching gain. The proposed SMC brings the input delay analysis into the design phase which is shown in the example to have key advantages when compared with an existing SMC that ignores the input delay and with a linear control (for sufficiently small input delay). The method is applicable to linear systems with polytopic uncertainties in all blocks of the system matrices. In the extension to state delays, for the first time a static output feedback, a SMC is designed via the Krasovskii method for systems with fast varying delays.

Appendix A. Proof of Lemma 2

Differentiating V of the structure (15) and (16) along (14) it follows from (17) that

$$\begin{aligned} W(t) \le\ & 2z_1^T(t)P_1[\bar{A}_{11}\ \bar{A}_{12}]z(t) + 2\mu z_2^T(t)P_2[\bar{A}_{11}\ \bar{A}_{12}]z(t) \\ & + 2z_1^T(t)P_2^T\left(\mu[\bar{A}_{21}\ \bar{A}_{22}]z(t) - z_2(t-\mu\xi(t)) + \mu\bar{w}(t)\right) \\ & + 2z_2^T(t)P_3\left(\mu[\bar{A}_{21}\ \bar{A}_{22}]z(t) - z_2(t-\mu\xi(t)) + \mu\bar{w}(t)\right) \\ & - \mu^2\bar{b}\bar{w}^T(t)\bar{w}(t) + \bar{\alpha}z_1^T(t)P_1z_1(t) + \bar{\alpha}\mu z_2^T(t)P_2z_1(t) \end{aligned}$$

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$$+\bar{\alpha}\mu z_1^T(t)P_2^T z_2(t) + \bar{\alpha}\mu z_2^T(t)P_3 z_2(t) + \mu^2 h^2 \dot{z}_2^T(t)R\dot{z}_2(t)$$
$$-\mu h\int_{t-\mu h}^{t} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds + z_2^T(t)Gz_2(t)$$
$$-e^{-\bar{\alpha}\mu h}z_2^T(t-\mu h)Gz_2(t-\mu h) + z_2^T(t)Sz_2(t)$$
$$-(1-d)e^{-\bar{\alpha}\mu\xi(t)}z_2^T(t-\mu\xi(t))Sz_2(t-\mu\xi(t)).$$

Using the identity

$$\int_{t-\mu h}^{t} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds = \int_{t-\mu h}^{t-\mu\xi(t)} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds + \int_{t-\mu\xi(t)}^{t} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds$$

apply Jensen's inequality

$$-\mu h\int_{t-\mu h}^{t-\mu\xi(t)} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds$$
$$\le -e^{-\bar{\alpha}\mu h}[z_2^T(t-\mu\xi(t)) - z_2^T(t-\mu h)]R[z_2(t-\mu\xi(t))$$
$$-z_2(t-\mu h)] - \mu h\int_{t-\mu\xi(t)}^{t} e^{-\bar{\alpha}\mu h}\dot{z}_2^T(s)R\dot{z}_2(s)ds$$
$$\le -e^{-\bar{\alpha}\mu h}[z_2^T(t) - z_2^T(t-\mu\xi(t))]R[z_2(t) - z_2(t-\mu\xi(t))].$$

Then, setting $\zeta(t) = col\{z_1(t), z_2(t), z_2(t-\mu h), z_2(t-\mu\xi), \mu\bar{w}(t)\}$ and applying Schur complements to the term $\mu^2 h^2 \dot{z}_2^T(t)\bar{R}\dot{z}_2(t)$, where $\dot{z}_2(t)$ is substituted by the right-hand side of (14), it is established that $W(t) < 0$ if $\Theta_\mu < 0$.

Appendix B. Proof of Theorem 1

The i-th component of differential equation (10) with the initial condition (11) can be represented in the form of an integral equation (Kolmanovskii & Myshkis, 1992)

$$z_{2_i}(t) = \phi(t,t_0,\mu)z_{2_i}(t_0) + \int_{t_0}^{t}\phi(t,s,\mu)\Big[[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(s) + w_i(s) - (1+\delta)\Delta\,\mathrm{sign}\,z_{2_i}(s-\mu\xi(s))\Big]ds. \quad \text{(B.1)}$$

The feasibility of (20) implies the bound (12), then the following inequality holds for $t \to \infty$:

$$|[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(s) + w_i(s) - (1+\delta)\Delta\,\mathrm{sign}\,z_{2_i}(s-\mu\xi(s))| < M_0. \quad \text{(B.2)}$$

Taking into account (24) and (B.2), it is established from (B.1) that for $t \to \infty$

$$|z_{2_i}(t+\theta) - z_{2_i}(t)| \le \left|\int_{t+\theta}^{t}\phi(t,s,\mu)\Big([\bar{A}_{21_i}\ \bar{A}_{22_i}]z(s) + w_i(s) - (1+\delta)\Delta\,\mathrm{sign}\,z_{2_i}(s-\mu\xi(s))\Big)ds\right|$$
$$< M_0\int_{t+\theta}^{t} e^{-\frac{\alpha_2(t-s)}{\mu}}ds < \mu M_0\frac{1-e^{-2\alpha_2 h}}{\alpha_2} \le 2M_0\mu h$$

where $\theta \in [-2\mu h,\ 0]$. Therefore,

$$z_{2_i}(t) - 2M_0\mu h < z_{2_i}(t+\theta) < z_{2_i}(t) + 2M_0\mu h$$

for $t \to \infty$ and the following implication holds

$$|z_{2_i}(t)| \ge 2M_0\mu h \Rightarrow \mathrm{sign}\,z_{2_i}(t+\theta) = \mathrm{sign}\,z_{2_i}(t) \quad \text{(B.3)}$$

for large enough t. Thus, from (12), (B.2) and (B.3) for sufficiently large t the following implication follows:

$$|z_{2_i}(t)| \ge 2M_0\mu h \Rightarrow z_{2_i}^T(t)[[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(t+\theta) + w_i(t+\theta) - (1+\delta)\Delta\,\mathrm{sign}\,z_{2_i}(t+\theta)]$$
$$< |z_{2_i}(t)|(|[\bar{A}_{21_i}\ \bar{A}_{22_i}]z(t+\theta)| + \Delta) - (1+\delta)\Delta|z_{2_i}(t)| < 0. \quad \text{(B.4)}$$

It will be shown next that the z_{2_i}-component of the solutions to (10) exponentially converges to the ball (25). Moreover, for sufficiently large t, whenever $z_{2_i}(t)$ achieves the ball (25), it will never leave it. Taking into account (B.4), for sufficiently large t it follows that

$$|z_{2_i}(t)| \ge 2M_0\mu h \Rightarrow \frac{d}{dt}\mu z_{2_i}^2(t) = 2\mu z_{2_i}(t)\dot{z}_{2_i}(t)$$
$$= 2z_{2_i}(t)[-z_{2_i}(t-\mu\xi(t)) + \mu([\bar{A}_{21_i}\ \bar{A}_{22_i}]z(t) + w_i(t) - (1+\delta)\Delta\,\mathrm{sgn}\,z_{2_i}(t))]$$
$$\le -2z_{2_i}(t)\left(z_{2_i}(t) - \int_{t-\mu\xi(t)}^{t}\dot{z}_{2_i}(s)ds\right)$$
$$= -2z_{2_i}^2(t) + 2z_{2_i}(t)\int_{t-\mu\xi(t)}^{t}\left[-\frac{z_{2_i}(s-\mu\xi(s))}{\mu} + [\bar{A}_{21_i}\ \bar{A}_{22_i}]z(s) + w_i(s) - (1+\delta)\Delta\,\mathrm{sgn}\,z_{2_i}(s)\right]ds$$
$$\le -2z_{2_i}^2(t) - 2\frac{z_{2_i}(t)}{\mu}\int_{t-\mu\xi(t)}^{t} z_{2_i}(s-\mu\xi(t))ds.$$

Therefore, given (B.3) holds for large enough t, it follows that $-\int_{t-\mu\xi(t)}^{t} z_{2_i}(t)z_{2_i}(s-\mu\xi(t))ds \le 0$. Hence

$$|z_{2_i}(t)| \ge 2M_0\mu h \Rightarrow \frac{d}{dt}\mu z_{2_i}^2(t) \le -2z_{2_i}^2(t). \quad \text{(B.5)}$$

Assume now that for large enough t_1 the z_{2_i} component of the solution to (10) is outside the ball (25). Then from (B.5) it follows that for all $t \ge t_1$ such that $|z_{2_i}(t)| \ge 2M_0\mu h$ the inequality holds $z_{2_i}^2(t) \le e^{-\frac{2}{\mu}(t-t_1)}z_{2_i}^2(t_1)$, i.e. z_{2_i} exponentially converges to the ball (25). Let $t_2 > t_1$ be the time when $|z_{2_i}(t_2)| = 2M_0\mu h$. Then due to (B.5) $z_{2_i}^2(t_2^+) < z_{2_i}^2(t_2)$. Therefore, whenever $z_{2_i}(t)$ attains the ball (25), it will never leave it. Then (26) follows from Lemma 1 and (25).

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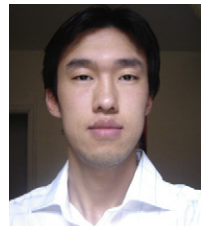

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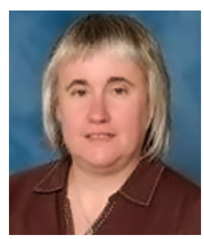

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