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Averaging of linear systems with almost periodic coefficients: A time-delay approach ☆

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ABSTRACT

We study stability of linear systems with fast almost periodic coefficients that are piecewise-continuous in time. The classical averaging method guarantees the stability of such systems for small enough values of parameter provided the corresponding averaged system is stable. However, it is difficult to find an upper bound on the small parameter by using classical tools for asymptotic analysis. In this paper we introduce an efficient constructive method for finding an upper bound on the value of the small parameter that guarantees a desired exponential decay rate. We transform the system into a model with time-delays of the length of the small parameter. The resulting time-delay system is a perturbation of the averaged system. The averaged system is supposed to be exponentially stable. The stability of the time-delay system guarantees the stability of the original one. We construct an appropriate Lyapunov functional for finding sufficient stability conditions in the form of linear matrix inequalities (LMIs). The upper bound on the small parameter that preserves the exponential stability is found from the LMIs. Two numerical examples (stabilization by vibrational control and by time-dependent switching) illustrate the efficiency of the method. Moreover, we apply the time-delay approach to persistently excited systems that leads to a novel quadratic time-independent Lyapunov functional for such systems. We further extend our method to input-to-state stability (ISS) analysis. Finally the results are extended to linear fast-varying systems with time-varying delays.

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1. Introduction

Asymptotic methods for analysis and control of perturbed systems depending on small parameters have led to important qualitative results (see e.g. Bogoliubov & Mitropolsky, 1961; Cheng et al., 2018; Khalil, 2002; Kokotovic & Khalil, 1986; Moreau & Aeyels, 2000; Teel et al., 2003; Tikhonov, 1952; Vasilieva & Butuzov, 1973). However, by using these methods it is difficult to find an efficient bound on the small parameter that preserves the stability of the perturbed system. The direct Lyapunov method may lead to such bounds. Thus, for singularly perturbed systems, such a bound was presented e.g. in Kokotovic and Khalil (1986) and Fridman (2002) by using the direct Lyapunov method.

For the sampled-data systems with fast sampling, the time-delay approach was initiated in the framework of asymptotic methods (Mikheev et al., 1988) and averaging (Fridman, 1992). Later the direct Lyapunov–Krasovskii method to sampled-data systems (Fridman et al., 2004) led to efficient bounds on sampling intervals that preserve the performance of the systems, and to efficient tools for robust sampled-data and networked control (see e.g. Fridman (2014), Hetel and Fridman (2013), Liu et al. (2019)).

In this paper we consider linear systems with fast-varying almost periodic coefficients. Our objective is to propose a constructive time-delay approach with a corresponding Lyapunov–Krasovskii method to the averaging method for these systems. Differently from the classical results (see Chapter 10 of Khalil (2002), where the system coefficients are supposed to be continuous in time, we assume them to be piecewise-continuous. This allows to apply our results to fast switching systems. By taking average of the both sides of the system, we present the resulting system as a perturbation of the averaged system, and model it as a system with time-delays of the length of the small parameter. If the transformed time-delay system is stable, then the original one is also stable. We assume that the averaged system is exponentially stable.

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We suggest a direct Lyapunov–Krasovskii approach, and formulate sufficient exponential stability conditions in the form of LMIs. The upper bound on the small parameter that guarantees a desired decay rate for the original system can be found from LMIs. Two numerical examples (stabilization by vibrational control and by time-dependent switching) illustrate the efficiency of the method. We further apply the time-delay approach to persistently excited systems that leads to a novel quadratic time-independent Lyapunov functional for such systems.

We extend our results to input-to-state stability (ISS) of the perturbed systems and to stability and ISS analysis of linear fast-varying systems with state time-varying delays. Note that classical results for averaging of time-delay systems were presented in Hale and Lunel (1990), Lehman and Weibel (1999), Strygin (1970). We propose a constructive method via appropriate Lyapunov functionals that leads to sufficient stability and ISS analysis conditions in the form of LMIs. By solving these LMIs, upper bounds on the small parameter and on the time-varying delay that preserve the performance can be found.

Note also that stability of linear systems with periodic coefficients and either constant or periodic delays was analyzed by using numerical methods: Chebyshov polynomials and finite element methods (Butcher & Mann, 2009) and semi-discretization method (Insperger & Stépán, 2011). An eigenvalue-based technique for stability analysis of such systems was presented in Michiels and Niculescu (2014). Stability conditions for linear systems with continuous periodic coefficients and constant delays were provided via complete Lyapunov–Krasovskii functional in Gomez et al. (2016), Letyagina and Zhabko (2009). Guaranteed cost control of periodically switching linear systems with constant delay and appropriate dwelling times (that are supposed to be not too small) was studied in Xie and Lam (2018) via piecewise linear in time Lyapunov functionals for delay-dependent stability. Strict time-dependent Lyapunov functionals for nonlinear time-varying systems with delays were presented in Mazenc and Malisoff (2017).

The article is organized as follows. Section 2 presents a time-delay approach to stability by averaging. Section 3 extends the time-delay approach to stability of persistently excited systems. Section 4 extends the results of Sections 2 and 3 to ISS analysis of the perturbed systems. Section 5 deals with the averaging of time-delay systems. Some conclusions are drawn in the last section. A conference version of the results from Section 2 was presented on IFAC World Congress 2020 (Fridman & Zhang, 2020).

1.1. Notations and Jensen's inequalities

Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices with the induced matrix norm $\|\cdot\|$. The superscript T stands for matrix transposition, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$. Denote by $C[-h_M,0]$ the Banach space of continuous functions $\phi:[-h_M,0]\to\mathbb{R}^n$ with the norm $\|\phi\|_C=\max_{\theta\in[-h_M,0]}|\phi(\theta)|$, and by $L_\infty(0,t)$ the space of essentially bounded functions $\phi:(0,t)\to\mathbb{R}^n$ with the norm $\|\phi[0,t]\|_\infty=\operatorname{ess\,sup}_{\theta\in(0,t)}|\phi(\theta)|$.

We will employ extended Jensen's inequalities (Solomon & Fridman, 2013):

Lemma 1.1. *Denote*

$$\mathcal{G}=\int_a^b f(s)x(s)ds,\quad \mathcal{Y}=\int_a^b\int_{t-\theta}^t f(\theta)x(s)dsd\theta,$$

where $a\le b$, $f:[a,b]\to\mathbb{R}$, $x(s)\in\mathbb{R}^n$ and the integrations concerned are well defined. Then for any $n\times n$ matrix $R>0$ the following inequalities hold:

$$\mathcal{G}^TR\mathcal{G}\le\int_a^b|f(\theta)|d\theta\int_a^b|f(s)|x^T(s)Rx(s)ds,\tag{1.1}$$

$$\mathcal{Y}^TR\mathcal{Y}\le\int_a^b|f(\theta)|\theta d\theta\int_a^b\int_{t-\theta}^t|f(\theta)|x^T(s)Rx(s)dsd\theta.\tag{1.2}$$

2. A time-delay approach to stability by averaging

Consider the fast-varying system:

$$\dot{x}(t)=A(\tfrac{t}{\varepsilon})x(t),\quad t\ge 0,\tag{2.1}$$

where $x(t)\in\mathbb{R}^n$, $A:[0,\infty)\to\mathbb{R}^{n\times n}$ is piecewise-continuous, and $\varepsilon>0$ is a small parameter. Similar to the case of general averaging in Section 10.6 of Khalil (2002), assume the following:

A1 There exists $\tau_1\ge 1$ such that the following holds:

$$\begin{array}{l}\frac{1}{\varepsilon}\int_{t-\varepsilon}^t A(\frac{s}{\varepsilon})ds=A_{av}+\Delta A(\frac{t}{\varepsilon}),\\ \|\Delta A(\frac{t}{\varepsilon})\|\le\sigma\quad\forall\frac{t}{\varepsilon}\ge\tau_1\end{array}\tag{2.2}$$

with Hurwitz constant matrix A_{av} and small enough constant $\sigma>0$.

We will say that system (2.1) has almost periodic coefficients if it satisfies **A1**. Matrix $\Delta A(\frac{t}{\varepsilon})$ may stand for system uncertainty whose norm is upper bounded by a known constant σ. Changing variable s in (2.2) to $\theta=\frac{t-s}{\varepsilon}$, we can rewrite the first equation in (2.2) as

$$\int_0^1 A(\tfrac{t}{\varepsilon}-\theta)d\theta=A_{av}+\Delta A(\tfrac{t}{\varepsilon})\quad\forall\tfrac{t}{\varepsilon}\ge\tau_1$$

or, in terms of the fast time $\tau=\frac{t}{\varepsilon}$,

$$\int_0^1 A(\tau-\theta)d\theta=A_{av}+\Delta A(\tau)\quad\forall\tau\ge\tau_1.\tag{2.3}$$

Under **A1**, the averaged system

$$\dot{x}_{av}(t)=[A_{av}+\Delta A(\tfrac{t}{\varepsilon})]x_{av}(t),\quad x_{av}(t)\in\mathbb{R}^n\tag{2.4}$$

is exponentially stable for small enough σ.

Remark 2.1. If $A(\tau)$ is 1-periodic, then in (2.3) we have $\Delta A(\tau)=0$. If $A(\tau)$ is T-periodic with $T>0$, scaling the time $t=T\bar{t}$ and denoting $\bar{x}(\bar{t})=x(T\bar{t})=x(t)$, we can present system (2.1) as

$$\tfrac{d}{d\bar{t}}\bar{x}(\bar{t})=T\cdot A(\tfrac{T\bar{t}}{\varepsilon})\bar{x}(\bar{t})$$

with 1-periodic $A(T\bar{\tau})$, where $\bar{\tau}=\frac{\bar{t}}{\varepsilon}$. In general we can consider almost periodic A (in the sense of (2.3) with non-zero $\Delta A(\tau)$). For example, let $A(\tau)$ in (2.1) have the form

$$A(\tau)=A_1\cos(\tau)+A_2\sin^2(3\tau)+A_3e^{-\tau},\quad\tau=\tfrac{t}{\varepsilon}.\tag{2.5}$$

The $n\times n$ matrices A_1,A_2,A_3 in (2.5) are constant, whereas A_2 is Hurwitz. Then, scaling the time $t=2\pi\bar{t}$ and denoting $\bar{x}(\bar{t})=x(t)$, we arrive at $\dot{\bar{x}}(\bar{t})=2\pi A(\frac{2\pi\bar{t}}{\varepsilon})\bar{x}(\bar{t})$ with

$$\int_0^1 A(2\pi(\tau-\theta))d\theta=0.5A_2+\Delta A(\tau),$$

where

$$\Delta A(\tau)=A_3\int_0^1 e^{-2\pi(\tau-\theta)}d\theta.$$

For all $\tau\ge\tau_1$ the following holds:

$$\|\Delta A(\tau)\|\le\|A_3\|e^{-2\pi\tau_1}\int_0^1 e^{2\pi\theta}d\theta\overset{\Delta}{=}\sigma$$

with small enough $\sigma>0$ for some $\tau_1\ge 1$.

We will now introduce a time-delay approach to averaging, where the original system (2.1) is transformed to a time-delay system. We integrate (2.1) on $[t-\varepsilon,t]$ for $t\ge\varepsilon\tau_1$. Note that similar to Fridman and Shaikhet (2016), we can present

$$\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t\dot{x}(s)ds=\tfrac{x(t)-x(t-\varepsilon)}{\varepsilon}=\tfrac{d}{dt}[x(t)-G(t)],\tag{2.6}$$

where

$$G(t)\overset{\Delta}{=}\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t(s-t+\varepsilon)\dot{x}(s)ds.\tag{2.7}$$

Then, integrating (2.1) and taking into account (2.6) we arrive at

$$\frac{d}{dt}[x(t)-G(t)]=\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}A(\tfrac{s}{\varepsilon})[x(s)\pm x(t)]ds,\quad t\geq\varepsilon\tau_1.$$

By changing variable $\varepsilon\theta=t-s$ in the last integral, we have

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}A(\tfrac{s}{\varepsilon})[x(s)-x(t)]ds\\&=\int_0^1 A(\tfrac{t}{\varepsilon}-\theta)[x(t-\varepsilon\theta)-x(t)]d\theta\\&=-\int_0^1 A(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta}^{t}\dot{x}(s)dsd\theta.\end{aligned}$$

Finally, denoting

$$z(t)=x(t)-G(t) \tag{2.8}$$

and employing (2.2), we transform system (2.1) into a time-delay system for $t\geq\varepsilon\tau_1$

$$\dot{z}(t)=[A_{av}+\Delta A(\tfrac{t}{\varepsilon})]x(t)-\int_0^1 A(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta}^{t}\dot{x}(s)dsd\theta. \tag{2.9}$$

System (2.9) is a kind of neutral type system that depends on the past values of $\dot{x}(s)=A(\frac{s}{\varepsilon})x(s)$, $s\in[t-\varepsilon,t]$. This can be considered as a neutral system in Hale's form (Hale & Lunel, 1993).

Summarizing, if $x(t)$ is a solution to system (2.1), then it satisfies the time-delay system (2.9). Therefore, the stability of the time-delay system guarantees the stability of the original non-delayed system. Note that (2.9) can be considered as a perturbation of the averaged system (2.4). We will present a Lyapunov–Krasovskii method for (2.9) leading to LMIs for finding an upper bound $\varepsilon^*>0$ on ε that preserves the exponential stability of system (2.1) for all $\varepsilon\in(0,\varepsilon^*]$.

We further assume the following:

A2 All entries $a_{kv}(\tau)$ of $A(\tau)$ are uniformly bounded for $\tau\geq0$ with the values from some finite intervals $a_{kv}(\tau)\in[a_{kv}^m,a_{kv}^M]$ for $\tau\geq\tau_1\geq1$.

Under **A2**, $A(\tau)$ can be presented as a convex combination of the constant matrices A_i with the entries a_{kv}^m or a_{kv}^M:

$$\begin{aligned}&A(\tau)=\sum_{i=1}^N f_i(\tau)A_i\quad\forall\tau\geq\tau_1\geq1,\\&f_i\geq0,\quad\sum_{i=1}^N f_i=1,\quad 1\leq N\leq 2^{n^2}.\end{aligned} \tag{2.10}$$

Note that $f_i\not\equiv0$. For a constant a_{kv}, we have $a_{kv}^m=a_{kv}^M$.

Via (2.10), system (2.9) can be rewritten as

$$\dot{z}(t)=[A_{av}+\Delta A(\tfrac{t}{\varepsilon})]x(t)-\sum_{i=1}^N A_iY_i(t), \tag{2.11}$$

where

$$Y_i(t)\triangleq\int_0^1 f_i\left(\tfrac{t}{\varepsilon}-\theta\right)\int_{t-\varepsilon\theta}^{t}\dot{x}(s)dsd\theta. \tag{2.12}$$

Given $\varepsilon^*>0$, denote by $f_i^*>0$ $(i=1,\dots,N)$ the following bounds:

$$\varepsilon^*\int_0^1\theta f_i(\tau-\theta)d\theta\leq f_i^*\quad\forall\tau\geq\tau_1. \tag{2.13}$$

Note that since $f_i\in[0,1]$ we can always choose $f_i^*\leq\frac{\varepsilon^*}{2}$.

Theorem 2.1. *Assume* **A1** *and* **A2**. *Given matrices A_{av}, A_i $(i=1,\dots,N)$, and constants $\sigma>0$, $\alpha>0$ and $\varepsilon^*>0$, let there exist $n\times n$ matrices $P>0$, $R>0$, $H_i>0$ $(i=1,\dots,N)$ and a scalar $\lambda>0$ that satisfy the following LMIs:*

$$\left[\begin{array}{c|c}\Phi & \begin{array}{c}\sqrt{\varepsilon^*}A_i^T(R+\sum_{j=1}^N H_j)\\ 0_{(N+2)n,n}\end{array}\\\hline *\quad * & -(R+\sum_{j=1}^N H_j)\end{array}\right]<0,\quad i=1,\dots,N. \tag{2.14}$$

Here

$$\Phi=\begin{bmatrix}\Phi_{11} & -A_{av}^TP-2\alpha P & \Phi_{13} & P\\ * & -\frac{4}{\varepsilon^*}e^{-2\alpha\varepsilon^*}R+2\alpha P & \Phi_{23} & -P\\ * & * & \Phi_{33} & 0_{Nn,n}\\ * & * & * & -\lambda I_n\end{bmatrix}, \tag{2.15}$$

$$\begin{aligned}&\Phi_{11}=PA_{av}+A_{av}^TP+2\alpha P+\lambda\sigma^2I_n,\\&\Phi_{13}=-\Phi_{23}=-P[A_1,\dots,A_N],\\&\Phi_{33}=-2e^{-2\alpha\varepsilon^*}diag\{\tfrac{1}{f_1^*}H_1,\dots,\tfrac{1}{f_N^*}H_N\}\end{aligned}$$

with f_i^* $(i=1,\dots,N)$ defined by (2.13). Then system (2.1) is exponentially stable with a decay rate α for all $\varepsilon\in(0,\varepsilon^*]$, meaning that there exists $M_0>0$ such that for all $\varepsilon\in(0,\varepsilon^*]$ the solutions of (2.1) initialized by $x(0)\in\mathbb{R}^n$ satisfy the following inequality:

$$|x(t)|^2\leq M_0e^{-2\alpha t}|x(0)|^2\quad\forall t\geq0. \tag{2.16}$$

Moreover, if the LMIs (2.14) hold with $\alpha=0$, then system (2.1) is exponentially stable with a small enough decay rate $\alpha=\alpha_0>0$ for all $\varepsilon\in(0,\varepsilon^*]$.

Proof. Choose

$$V_P(t)=z^T(t)Pz(t),\quad 0<P\in\mathbb{R}^{n\times n}. \tag{2.17}$$

Differentiating $V_P(t)$ along (2.11) we have

$$\begin{aligned}\frac{d}{dt}V_P(t)=&\,2[x(t)-G(t)]^TP[(A_{av}+\Delta A(\tfrac{t}{\varepsilon}))x(t)\\&-\sum_{i=1}^N A_iY_i(t)].\end{aligned} \tag{2.18}$$

To compensate the $G(t)$-term, we will use as in Fridman and Shaikhet (2016)

$$V_R(t)=\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}e^{-2\alpha(t-s)}(s-t+\varepsilon)^2\dot{x}^T(s)R\dot{x}(s)ds,\quad R>0. \tag{2.19}$$

We have

$$\begin{aligned}&\frac{d}{dt}V_R(t)+2\alpha V_R(t)=\varepsilon\dot{x}^T(t)R\dot{x}(t)\\&-\frac{2}{\varepsilon}\int_{t-\varepsilon}^{t}e^{-2\alpha(t-s)}(s-t+\varepsilon)\dot{x}^T(s)R\dot{x}(s)ds.\end{aligned} \tag{2.20}$$

By Jensen's inequality (1.1)

$$2G^T(t)RG(t)\leq\int_{t-\varepsilon}^{t}(s-t+\varepsilon)\dot{x}^T(s)R\dot{x}(s)ds.$$

Then

$$\frac{d}{dt}V_R(t)+2\alpha V_R(t)\leq\varepsilon\dot{x}^T(t)R\dot{x}(t)-\frac{4}{\varepsilon}e^{-2\alpha\varepsilon}G^T(t)RG(t). \tag{2.21}$$

To compensate the $Y_i(t)$-terms (distributed delay) in (2.18), we employ as in Solomon and Fridman (2013)

$$\begin{aligned}&V_H(t)=\sum_{i=1}^N V_{H_i}(t),\\&V_{H_i}(t)=2\int_0^1\int_{t-\varepsilon\theta}^{t}e^{-2\alpha(t-s)}(s-t+\varepsilon\theta)\dot{x}^T(s)H_i\dot{x}(s)dsd\theta\end{aligned} \tag{2.22}$$

with $H_i>0$. Differentiating $V_{H_i}(t)$, we have

$$\begin{aligned}&\frac{d}{dt}V_{H_i}(t)+2\alpha V_{H_i}(t)\\&=\varepsilon\dot{x}^T(t)H_i\dot{x}(t)-2\int_0^1\int_{t-\varepsilon\theta}^{t}e^{-2\alpha(t-s)}\dot{x}^T(s)H_i\dot{x}(s)dsd\theta\\&\leq\varepsilon\dot{x}^T(t)H_i\dot{x}(t)-2e^{-2\alpha\varepsilon}\int_0^1\int_{t-\varepsilon\theta}^{t}\dot{x}^T(s)H_i\dot{x}(s)dsd\theta.\end{aligned}$$

Applying further Jensen's inequality (1.2)

$$\begin{aligned}&Y_i^T(t)H_iY_i(t)\leq\int_0^1\varepsilon\theta f_i(\tfrac{t}{\varepsilon}-\theta)d\theta\\&\quad\times\int_0^1 f_i(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta}^{t}\dot{x}^T(s)H_i\dot{x}(s)dsd\theta\\&\leq f_i^*\int_0^1\int_{t-\varepsilon\theta}^{t}\dot{x}^T(s)H_i\dot{x}(s)dsd\theta,\end{aligned}$$

where we took into account (2.10) and used (2.13), we arrive at

$$\begin{aligned}&\frac{d}{dt}V_{H_i}(t)+2\alpha V_{H_i}(t)\leq\varepsilon\dot{x}^T(t)H_i\dot{x}(t)\\&-\frac{2}{f_i^*}e^{-2\alpha\varepsilon}Y_i^T(t)H_iY_i(t).\end{aligned} \tag{2.23}$$

Define a Lyapunov functional as

$$V_1(t) = \bar{V}_1(x(t), \dot{x}_t, \varepsilon) = V_P(t) + V_R(t) + V_H(t), \tag{2.24}$$

where $\dot{x}_t = \dot{x}(t+\theta)$ $(\theta \in [-\varepsilon, 0])$, with $V_P(t)$, $V_R(t)$ and $V_H(t)$ given by (2.17), (2.19) and (2.22) respectively. By Jensen's inequality (3.87) in Fridman (2014)

$$\int_{t-\varepsilon}^{t} \phi^T(s)R\phi(s)ds \geq \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \phi^T(s)dsR\int_{t-\varepsilon}^{t} \phi(s)ds$$

with $\phi(s) = (s - t + \varepsilon)\dot{x}(s)$, for all $\varepsilon \in (0, \varepsilon^*]$ we have

$$\begin{aligned} V_1(t) &\geq V_P(t) + V_R(t) \\ &\geq \begin{bmatrix} x(t) \\ G(t) \end{bmatrix}^T \begin{bmatrix} P & -P \\ * & P + e^{-2\alpha\varepsilon^*}R \end{bmatrix} \begin{bmatrix} x(t) \\ G(t) \end{bmatrix} \geq c_1|x(t)|^2 \end{aligned} \tag{2.25}$$

with ε-independent $c_1 > 0$. Thus, $V_1(t)$ is positive-definite.

To compensate $\Delta A(\frac{t}{\varepsilon})x(t)$ in (2.18) we apply S-procedure: we add to $\dot{V}_1(t)$ the left-hand part of

$$\lambda(\sigma^2|x(t)|^2 - |\Delta A(\tfrac{t}{\varepsilon})x(t)|^2) \geq 0 \tag{2.26}$$

with some $\lambda > 0$. Then from (2.18), (2.21), (2.23) and (2.26), we have along (2.11)

$$\begin{aligned} &\tfrac{d}{dt}V_1(t) + 2\alpha V_1(t) \\ &\leq \tfrac{d}{dt}V_1(t) + 2\alpha V_1(t) + \lambda(\sigma^2|x(t)|^2 - |\Delta A(\tfrac{t}{\varepsilon})x(t)|^2) \\ &\leq \xi_1^T \Phi \xi_1 + \varepsilon^* \dot{x}^T(t)(R + \textstyle\sum_{i=1}^N H_i)\dot{x}(t), \end{aligned} \tag{2.27}$$

where Φ is given by (2.15) and

$$\xi_1^T(t) = [x^T(t), G^T(t), Y_1^T(t), \ldots, Y_N^T(t), x^T(t)\Delta A^T(\tfrac{t}{\varepsilon})]. \tag{2.28}$$

Since we are interested in exponential bounds on solutions of (2.11) that satisfy (2.1), we substitute into (2.27)

$$\dot{x}(t) = \textstyle\sum_{i=1}^N f_i(\frac{t}{\varepsilon})A_i x(t).$$

Applying further Schur complements, we conclude that if

$$\left[\begin{array}{c|c} \Phi & \begin{matrix} \sqrt{\varepsilon^*}\sum_{i=1}^N f_i(\frac{t}{\varepsilon})A_i^T(R + \sum_{j=1}^N H_j) \\ 0_{(N+2)n,n} \end{matrix} \\ \hline * \quad * & -(R + \sum_{j=1}^N H_j) \end{array}\right] < 0, \tag{2.29}$$

we have

$$\tfrac{d}{dt}V_1(t) + 2\alpha V_1(t) \leq 0 \quad \forall t \geq \varepsilon\tau_1,$$

yielding for solutions of (2.1) the following bound:

$$c_1|x(t)|^2 \leq V_1(t) \leq e^{-2\alpha(t-\varepsilon\tau_1)}V_1(\varepsilon\tau_1), \quad t \geq \varepsilon\tau_1. \tag{2.30}$$

LMIs (2.14) imply (2.29) since (2.29) is affine in $\sum_{i=1}^N f_i(\frac{t}{\varepsilon})A_i^T$.

Note that $V_1(\varepsilon\tau_1)$ defined by (2.24) is upper bounded for all $\varepsilon \in (0, \varepsilon^*]$

$$V_1(\varepsilon\tau_1) \leq c_2\Big[|x(\varepsilon\tau_1)|^2 + \int_{\varepsilon(\tau_1-1)}^{\varepsilon\tau_1}|\dot{x}(s)|^2ds\Big] \tag{2.31}$$

with ε-independent $c_2 > 0$. For $t \in [0, \varepsilon\tau_1]$, $x(t)$ satisfies (2.1), where under **A2** we have $\|A(\tau)\| \leq a$ for some $a > 0$ and all $\tau \geq 0$. Hence, $\frac{d}{dt}|x(t)|^2 \leq 2a|x(t)|^2$ for $t \in [0, \varepsilon\tau_1]$ yielding

$$|x(t)| \leq e^{at}|x(0)|, \quad |\dot{x}(t)| \leq ae^{at}|x(0)|, \quad t \in [0, \varepsilon\tau_1].$$

Therefore, $V_1(\varepsilon\tau_1)$ for all $\varepsilon \in (0, \varepsilon^*]$ can be further upper bounded as

$$\begin{aligned} V_1(\varepsilon\tau_1) &\leq c_2\Big[e^{2a\varepsilon\tau_1}|x(0)|^2 + \int_{\varepsilon(\tau_1-1)}^{\varepsilon\tau_1} a^2e^{2at}|x(0)|^2ds\Big] \\ &\leq c_3e^{-2\alpha\varepsilon\tau_1}|x(0)|^2 \end{aligned} \tag{2.32}$$

with some ε-independent $c_3 > 0$. Then (2.16) follows from (2.30) and (2.32).

The feasibility of the strict LMIs (2.14) with $\alpha = 0$ implies the feasibility of (2.14) with the same decision variables and with a small enough positive $\alpha = \alpha_0$, and thus guarantees a small enough decay rate. □

Remark 2.2. To select the tuning parameters $\alpha > 0$ and $\varepsilon^* > 0$, we suggest the following algorithm: given matrices A_{av} and A_i $(i = 1, \ldots, N)$ and a small constant $\sigma > 0$, we verify the feasibility of the LMIs with ε^* close to zero to enlarge the upper bound α^* on α that preserves the feasibility. We apply here the binary search method (with a high efficiency) to find α^*. Similarly, by choosing $\alpha \in (0, \alpha^*]$, we obtain the upper bound ε^* that preserves the feasibility of the LMIs.

Remark 2.3. LMIs of Theorem 2.1 are always feasible for small enough positive ε^*, α and σ. Indeed, since A_{av} is Hurwitz, there exists a $n \times n$ matrix $P > 0$ such that for small enough $\alpha > 0$ the following holds:

$$\Phi_0 \triangleq PA_{av} + A_{av}^TP + 2\alpha P < 0.$$

We choose $R = N \cdot H_i = e^{2\alpha\varepsilon^*}I_n$ $(i = 1, \ldots, N)$, $\lambda = \frac{1}{\varepsilon^*}$ and $\sigma = \varepsilon^*$. By using Schur complements and further employing $f_i^* \leq \frac{\varepsilon^*}{2}$, we find that LMIs (2.14) are feasible if the following matrix inequalities hold:

$$\begin{aligned} &\begin{bmatrix} \Phi_0 + \varepsilon^*(I_n + 2e^{2\alpha\varepsilon^*}A_i^TA_i) & -A_{av}^TP - 2\alpha P \\ * & -\frac{2}{\varepsilon^*}I_n + 2\alpha P \end{bmatrix} \\ &+\varepsilon^*\begin{bmatrix} P \\ -P \end{bmatrix}(I_n + \tfrac{1}{4}\textstyle\sum_{i=1}^N A_iA_i^T)\begin{bmatrix} P \\ -P \end{bmatrix}^T < 0, \quad i = 1, \ldots, N. \end{aligned}$$

Since $\Phi_0 < 0$, the latter inequalities are always feasible for small enough $\varepsilon^* > 0$.

Note that by similar arguments, all the LMIs presented in the theorems of this paper are feasible for small enough positive ε^* and α (as well as σ, h_M, $b_0^{-1}, \ldots, b_N^{-1}$ in Sections 4 and 5).

Remark 2.4. As it is seen from the proof of Theorem 2.1 (cf. (2.26)), LMIs (2.14) guarantee the stability of a more general system than (2.1) with $A = A(\frac{t}{\varepsilon}, t, \varepsilon)$ provided there exists $\tau_1 > 0$ such that for all small enough $\varepsilon > 0$ and $t \geq \varepsilon\tau_1$ all entries of A are uniformly bounded and

$$\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^{t} A(\tfrac{s}{\varepsilon}, s, \varepsilon)ds = A_{av} + \Delta A(\tfrac{t}{\varepsilon}, t, \varepsilon), \;\; \|\Delta A(\tfrac{t}{\varepsilon}, t, \varepsilon)\| \leq \sigma,$$

where A_{av} is Hurwitz and constant $\sigma > 0$ is small enough.

Example 2.1 (*Khalil (2002), Example 10.10: vibrational control*)**.** Consider the suspended pendulum with the suspension point that is subject to vertical vibrations of small amplitude and high frequency. The linearized at the upper equilibrium position model is given by

$$\dot{x}(t) = \begin{bmatrix} \cos\frac{t}{\varepsilon} & 1 \\ \gamma^2 - \cos^2\frac{t}{\varepsilon} & -\gamma(\beta + \Delta\beta) - \cos\frac{t}{\varepsilon} \end{bmatrix}x(t) \tag{2.33}$$

with $\gamma > 0$ and $\beta > 0$. Here the uncertainty $\Delta\beta$ stems from the uncertainties of friction coefficient and satisfies $|\Delta\beta| \leq \beta_1$ with $\beta_1 \geq 0$. Note that we linearized f given above (10.32) on p. 410 of Khalil (2002) at $x_1 = \pi$, $x_2 = 0$ to derive (2.33). Similar to Remark 2.1, we change the time variable $t = 2\pi\bar{t}$ and define $\bar{x}(\bar{t}) = x(2\pi\bar{t}) = x(t)$, therefore,

$$\dot{\bar{x}}(\bar{t}) = 2\pi\begin{bmatrix} \cos\frac{2\pi\bar{t}}{\varepsilon} & 1 \\ \gamma^2 - \cos^2\frac{2\pi\bar{t}}{\varepsilon} & -\gamma(\beta + \Delta\beta) - \cos\frac{2\pi\bar{t}}{\varepsilon} \end{bmatrix}\bar{x}(\bar{t}). \tag{2.34}$$

Then we obtain

$$A_{av} = 2\pi\begin{bmatrix} 0 & 1 \\ \gamma^2 - 0.5 & -\gamma\beta \end{bmatrix}, \quad \Delta A = -2\pi\gamma\begin{bmatrix} 0 & 0 \\ 0 & \Delta\beta \end{bmatrix}$$

with $\sigma = 2\pi\gamma\beta_1$. It follows from Theorem 10.4 of Khalil (2002) that for $\gamma^2 < 0.5$ and small enough ε, system (2.34) with

$\Delta\beta = 0$ is exponentially stable. By applying Theorem 2.1, we will find upper bounds on ε and uncertainty $\Delta\beta$ that guarantee the exponential convergence of (2.34) with a decay rate $\alpha > 0$ for all ε and $\Delta\beta$ that are not larger than the found upper bounds.

We choose $\gamma = 0.2$ and $\beta = 1$. Note that $\cos\tau \in [-1, 1]$ and $\cos^2\tau \in [0, 1]$. Therefore, system (2.34) can be presented as a system with polytopic type uncertainty, where $A_1, \ldots, A_8$ correspond to the eight vertices:

$$A_i = 2\pi \begin{bmatrix} -1 & 1 \\ -0.46 \pm 0.5 & 0.2(\pm\beta_1 + 4) \end{bmatrix}, \quad i = 1, \ldots, 4,$$
$$A_i = 2\pi \begin{bmatrix} 1 & 1 \\ -0.46 \pm 0.5 & -0.2(\pm\beta_1 + 6) \end{bmatrix}, \quad i = 4, \ldots, 8. \tag{2.35}$$

Here for simplicity we choose $f_i^* = 0.5\varepsilon^*(i = 1, \ldots, 8)$, where f_i^* is defined by (2.13). By verifying the feasibility of LMIs (2.14) in the eight vertices (2.35) (four vertices for $\beta_1 = 0$), we find upper bounds ε^* that guarantee the exponential stability of (2.34) for all $\varepsilon \in (0, \varepsilon^*]$ either with a small enough decay rate (for $\alpha = 0$) or with $\alpha = 0.2$:

$$\begin{aligned} \beta_1 = 0: &\quad \alpha = 0, \quad \varepsilon^* = 0.0031; \\ &\quad \alpha = 0.2, \quad \varepsilon^* = 0.0021; \\ \beta_1 = 0.1: &\quad \alpha = 0, \quad \varepsilon^* = 0.0013; \\ &\quad \alpha = 0.2, \quad \varepsilon^* = 0.0007. \end{aligned}$$

Numerical simulations under an arbitrary initial condition show that the system (2.34) with $\Delta\beta = 0$ preserves the stability till a larger upper bound $\varepsilon^* = 0.47$, which may illustrate the conservatism of the proposed method.

Example 2.2 (*Hetel and Fridman (2013): stabilization by fast switching*). Consider a switched uncertain system

$$\dot{x}(t) = \begin{cases} (\bar{A}_1 + \Delta\bar{A}_1(\frac{t}{\varepsilon}))x(t), & t \in [k\varepsilon, k\varepsilon + \beta\varepsilon), \\ (\bar{A}_2 + \Delta\bar{A}_2(\frac{t}{\varepsilon}))x(t), & t \in [k\varepsilon + \beta\varepsilon, (k+1)\varepsilon), \end{cases} \tag{2.36}$$

where $\varepsilon > 0$, $k = 0, 1, \ldots$ and $\beta \in (0, 1)$, with unstable modes

$$\bar{A}_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.6 & -0.2 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} -0.13 & -0.16 \\ -0.33 & 0.03 \end{bmatrix}$$

and uncertainties

$$\Delta\bar{A}_1(\tfrac{t}{\varepsilon}) = \begin{bmatrix} 0 & 0 \\ 0 & g(\frac{t}{\varepsilon}) \end{bmatrix}, \quad \Delta\bar{A}_2(\tfrac{t}{\varepsilon}) = \begin{bmatrix} g(\frac{t}{\varepsilon}) & 0 \\ 0 & 0 \end{bmatrix},$$
$$|g(\tau)| \le g_1 \quad \forall \tau \ge 0, \tag{2.37}$$

where $g_1 \ge 0$. Then (2.36) can be presented as (2.1) with

$$A(\tau) = \textstyle\sum_{i=1}^{2} \chi_i(\tau)(\bar{A}_i + \Delta\bar{A}_i(\tau)),$$
$$\tau = \tfrac{t}{\varepsilon} \in [k, k+1), \quad k = 0, 1, \ldots, \tag{2.38}$$

where $\chi_1(\tau) = \chi_{[k,k+\beta)}(\tau)$ is the indicator function of $[k, k+\beta)$, $\chi_2(\tau) = 1 - \chi_1(\tau)$. Choose $\beta = 0.4$ that leads to Hurwitz

$$A_{av} = \beta\bar{A}_1 + (1-\beta)\bar{A}_2,$$

and

$$\Delta A(\tau) = \textstyle\int_0^\beta \Delta\bar{A}_1(\tau - \theta)d\theta + \int_\beta^1 \Delta\bar{A}_2(\tau - \theta)d\theta.$$

The latter yields

$$\|\Delta A(\tau)\| \le \textstyle\int_0^\beta \|\Delta\bar{A}_1(\tau - \theta)\|d\theta + \int_\beta^1 \|\Delta\bar{A}_2(\tau - \theta)\|d\theta$$

implying $\sigma = g_1$. Since $A(\tau)$ is not continuous, the classical results with asymptotic methods (e.g. Theorem 10.4 of Khalil (2002) are not applicable here.

Taking into account the uncertainties given by (2.37), system (2.36) can be presented as a system with polytopic type uncertainty (2.10), where $A_1, \ldots, A_4$ correspond to the four vertices:

$$A_i = \begin{bmatrix} 0.1 & 0.3 \\ 0.6 & -0.2 \pm g_1 \end{bmatrix}, \quad f_i(\tau) = \chi_1(\tau), \quad i = 1, 2,$$
$$A_i = \begin{bmatrix} -0.13 \pm g_1 & -0.16 \\ -0.33 & 0.03 \end{bmatrix}, \quad f_i(\tau) = \chi_2(\tau), \quad i = 3, 4. \tag{2.39}$$

The bounds (2.13) in this example can be found as follows:

$$\begin{aligned} \varepsilon^* \textstyle\int_0^1 \theta\chi_1(\tau - \theta)d\theta &\le \varepsilon^* \textstyle\int_{1-\beta}^1 \theta d\theta \\ &= 0.5\varepsilon^*[1 - (1-\beta)^2] \triangleq f_i^*, \quad i = 1, 2, \\ \varepsilon^* \textstyle\int_0^1 \theta\chi_2(\tau - \theta)d\theta &\le \varepsilon^* \textstyle\int_\beta^1 \theta d\theta \\ &= 0.5\varepsilon^*(1 - \beta^2) \triangleq f_i^*, \quad i = 3, 4. \end{aligned} \tag{2.40}$$

By verifying the feasibility of LMIs (2.14) in the four vertices (2.39) (two vertices for $g_1 = 0$), we find the upper bounds ε^* that guarantee the exponential stability of (2.36) for all $\varepsilon \in (0, \varepsilon^*]$ either with a small enough decay rate (for $\alpha = 0$) or with a decay rate $\alpha = 0.005$:

$$\begin{aligned} g_1 = 0: &\quad \alpha = 0, \quad \varepsilon^* = 0.1363; \\ &\quad \alpha = 0.005, \quad \varepsilon^* = 0.0930; \\ g_1 = 0.01: &\quad \alpha = 0, \quad \varepsilon^* = 0.0244; \\ &\quad \alpha = 0.005, \quad \varepsilon^* = 0.0033. \end{aligned} \tag{2.41}$$

Numerical simulations show that system (2.36) with $\Delta\bar{A}_i(\tau) = 0$ $(i = 1, 2)$ is stable for a much larger upper bound $\varepsilon^* = 37.8$.

Remark 2.5. As it is seen from the examples, the theoretical upper bounds on ε are essentially smaller than the values found from simulations. This often happens for systems with two time-scales (e.g. singularly perturbed systems), where the theory is aimed for comparatively small values of ε that preserve the two scales, whereas in the numerical examples it may happen that the stability is guaranteed also for large values of ε. Note that by using classical tools for asymptotic analysis, it is difficult to find an upper bound on ε that preserves the stability. We propose a constructive time-delay approach with a positive bound on ε that guarantees a desired performance and that is found from easily verifiable LMIs. This is the first paper on time-delay approach to averaging. As it happened with the first results for delay-dependent stability conditions (see Sections 3.6–3.10 of Fridman (2014)), we believe that our method will be improved in the future.

3. A time-delay approach to persistently excited systems

In this section, we consider the following persistently excited (PE) system

$$\dot{\bar{x}}(\tau) = -\varepsilon p(\tau)p^T(\tau)\bar{x}(\tau), \quad \tau \ge 0, \tag{3.1}$$

where $\bar{x}(\tau) \in \mathbb{R}^n$, $p : [0, \infty) \to \mathbb{R}^n$ is measurable and $\varepsilon > 0$ is a small parameter. Similar to Pogromsky and Matveev (2017), we assume that function p has the following properties:

A3 Boundedness: there exists a constant M such that for almost all $\tau \ge 0$

$$p(\tau)p^T(\tau) \le M^2 I_n. \tag{3.2}$$

A4 Persistency of excitation: there is a constant $\rho > 0$ such that

$$\int_0^1 p(\tau-\theta)p^T(\tau-\theta)d\theta \geq \rho I_n \quad \forall \tau \geq 1. \tag{3.3}$$

From **A3** it follows that

$$\int_0^1 p(\tau-\theta)p^T(\tau-\theta)d\theta \leq M^2 I_n \quad \forall \tau \geq 1. \tag{3.4}$$

Remark 3.1. The system (3.1) has been studied in Pogromsky and Matveev (2017), Zhang et al. (2019), where sufficient conditions were provided to guarantee the stability. Particularly, in Zhang et al. (2019), the following condition $2\varepsilon M^2 > \sqrt{\varepsilon M^2 - \varepsilon\rho}$ guarantees the asymptotic stability of (3.1). Clearly that given $M > 0$ and $\rho > 0$, the latter condition does not hold for small enough ε. This is different from our results that guarantee stability for small ε. In Pogromsky and Matveev (2017), a bound on the decay rate has been derived by introducing a novel non-quadratic Lyapunov functional. Time-varying Lyapunov functions for PE were considered in Efimov and Fradkov (2015), Verrelli and Tomei (2020). We have proposed explicit time-independent Lyapunov functional with matrices found from LMIs (to be compared with time-varying Lyapunov functions found from time-varying differential Lyapunov equations with some pre-chosen initial conditions). Since our functional leads to exponential stability conditions in terms of LMIs, these conditions can be easily extended further to ISS conditions (see Section 4) in terms of simply verifiable LMIs.

By changing the time $\tau = \frac{t}{\varepsilon}$ and defining $\bar{x}(\tau) = x(\varepsilon\tau) = x(t)$, we can rewrite the PE system (3.1) in the slow time as

$$\dot{x}(t) = -p(\tfrac{t}{\varepsilon})p^T(\tfrac{t}{\varepsilon})x(t), \quad t \geq 0. \tag{3.5}$$

Following the time-delay approach to averaging of Section 2, we integrate (3.5) on $[t-\varepsilon, t]$ for $t \geq \varepsilon$, and employ (2.6) with notation (2.7). Then we transform (3.5) to the following time-delay system:

$$\dot{z}(t) = -A_{av}\left(\tfrac{t}{\varepsilon}\right)x(t) + Y(t), \quad t \geq \varepsilon, \tag{3.6}$$

where $z(t) = x(t) - G(t)$ with $G(t)$ defined by (2.7). Here

$$\begin{aligned} A_{av}\left(\tfrac{t}{\varepsilon}\right) &\triangleq \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t p(\tfrac{s}{\varepsilon})p^T(\tfrac{s}{\varepsilon})ds = \int_0^1 p(\tfrac{t}{\varepsilon}-\theta)p^T(\tfrac{t}{\varepsilon}-\theta)d\theta, \\ Y(t) &\triangleq \int_0^1 p(\tfrac{t}{\varepsilon}-\theta)p^T(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta}^t \dot{x}(s)dsd\theta. \end{aligned} \tag{3.7}$$

Under **A3** and **A4**, the following holds:

$$\rho I_n \leq A_{av}\left(\tfrac{t}{\varepsilon}\right) \leq M^2 I_n. \tag{3.8}$$

As in Section 2, the original system (3.1) is stable if the time-delay system (3.6) is stable. Moreover, (3.6) can be considered as a perturbation of the exponentially stable system

$$\dot{x}_{av}(t) = -A_{av}\left(\tfrac{t}{\varepsilon}\right)x_{av}(t), \quad x_{av}(t) \in \mathbb{R}^n.$$

For the stability analysis of (3.6), we first consider (2.17) with a scalar matrix $P = pI_n$, i.e.

$$V_p(t) = pz^T(t)z(t), \tag{3.9}$$

where $p > 0$ is a scalar. Differentiating $V_p(t)$ along (3.6) we have

$$\tfrac{d}{dt}V_p(t) = 2p[x(t) - G(t)]^T[-A_{av}\left(\tfrac{t}{\varepsilon}\right)x(t) + Y(t)]. \tag{3.10}$$

To compensate the $G(t)$-term in (3.10), we use (2.19) with $R = rI_n$, i.e.

$$V_r(t) = r\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t e^{-2\alpha(t-s)}(s-t+\varepsilon)^2\dot{x}^T(s)\dot{x}(s)ds, \tag{3.11}$$

where r is a positive scalar. Then (2.21) holds, where due to (3.5) and **A3** we obtain

$$\dot{x}^T(t)R\dot{x}(t) \leq r\dot{x}^T(t)\dot{x}(t) \leq rM^4x^T(t)x(t).$$

Hence,

$$\tfrac{d}{dt}V_r(t) + 2\alpha V_r(t) \leq \varepsilon rM^4x^T(t)x(t) - \tfrac{4}{\varepsilon}e^{-2\alpha\varepsilon}rG^T(t)G(t). \tag{3.12}$$

To compensate the $Y(t)$-term in (3.10), we employ

$$V_\eta(t) = 2\eta\int_0^1\int_{t-\varepsilon\theta}^t e^{-2\alpha(t-s)}(s-t+\varepsilon\theta)\dot{x}^T(s)\dot{x}(s)dsd\theta \tag{3.13}$$

with a scalar $\eta > 0$. Differentiating $V_\eta(t)$ and using **A3**, we have

$$\begin{aligned} &\tfrac{d}{dt}V_\eta(t) + 2\alpha V_\eta(t) \leq \varepsilon\eta M^4x^T(t)x(t) \\ &\quad -2e^{-2\alpha\varepsilon}\eta\int_0^1\int_{t-\varepsilon\theta}^t \dot{x}^T(s)\dot{x}(s)dsd\theta. \end{aligned} \tag{3.14}$$

Applying further Jensen's inequality (1.2)

$$\begin{aligned} &Y^T(t)Y(t) \leq \int_0^1\int_{t-\varepsilon\theta}^t p(\tfrac{t}{\varepsilon}-\theta)p^T(\tfrac{t}{\varepsilon}-\theta)dsd\theta \\ &\quad\times\int_0^1 p(\tfrac{t}{\varepsilon}-\theta)p^T(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta}^t \dot{x}^T(s)\dot{x}(s)dsd\theta \\ &\leq \tfrac{\varepsilon}{2}M^4\int_0^1\int_{t-\varepsilon\theta}^t \dot{x}^T(s)\dot{x}(s)dsd\theta, \end{aligned} \tag{3.15}$$

we arrive at

$$\begin{aligned} &\tfrac{d}{dt}V_\eta(t) + 2\alpha V_\eta(t) \leq \varepsilon M^4\eta x^T(t)x(t) \\ &\quad -\tfrac{4}{\varepsilon M^4}e^{-2\alpha\varepsilon}\eta Y^T(t)Y(t). \end{aligned} \tag{3.16}$$

Consider a Lyapunov functional as

$$V_2(t) = V_p(t) + V_r(t) + V_\eta(t), \tag{3.17}$$

where $V_p(t)$, $V_r(t)$ and $V_\eta(t)$ are given by (3.9), (3.11) and (3.13) respectively. From (2.25), it follows that $V_2(t)$ is positive-definite since $V_2(t) \geq V_p(t) + V_r(t) > c_1|x(t)|^2$ for some $c_1 > 0$. By combining (3.10), (3.12) and (3.16), and treating the term $A_{av}\left(\frac{t}{\varepsilon}\right)$ in (3.10) as the one from polytope with two vertices $A_1 = \rho I_n$ and $A_2 = M^2 I_n$ (cf. (3.8)), we arrive at the following result:

Theorem 3.1. *Assume* **A3** *and* **A4**. *Given constants $M^2 > \rho > 0$, $\alpha > 0$ and $\varepsilon^* > 0$, let there exist positive scalars p, r and η that satisfy the following two LMIs:*

$$\Xi_i < 0, \quad i = 1, 2, \tag{3.18}$$

where

$$\begin{aligned} &\Xi_i = \begin{bmatrix} \Xi_{i11} & pa_i - 2\alpha p & p \\ * & -\frac{4}{\varepsilon^*}e^{-2\alpha\varepsilon^*}r + 2\alpha p & -p \\ * & * & -\frac{4}{\varepsilon^* M^4}e^{-2\alpha\varepsilon^*}\eta \end{bmatrix}, \\ &\Xi_{i11} = -2pa_i + 2\alpha p + \varepsilon^* M^4(r+\eta), \\ &a_1 = \rho, \quad a_2 = M^2. \end{aligned} \tag{3.19}$$

Then system (3.5) is exponentially stable with a decay rate α for all $\varepsilon \in (0, \varepsilon^*]$, meaning that there exists $M_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the solutions of (3.5) initialized by $x(0)$ satisfy (2.16). Moreover, if the LMIs are feasible with $\alpha = 0$, then system (3.5) is exponentially stable with a small enough decay rate $\alpha = \alpha_0 > 0$ for all $\varepsilon \in (0, \varepsilon^*]$.

Example 3.1. Consider the PE system (3.1) subject to (3.2) and (3.3). We choose $\alpha = 0.5$. By verifying the feasibility of LMIs (3.18) with $M = 1$ and $\rho = 0.55$, we find an upper bound $\varepsilon^* = 0.0645$ that preserves the exponential stability of (3.1), where $\varepsilon = \varepsilon^*$, with a decay rate $\varepsilon^*\alpha = 0.03225$. For $\varepsilon^* = 0.0645$, the resulting decay rate $\frac{\varepsilon^*\rho}{(1+\varepsilon^* M^2)^2} = 0.0313$ of Pogromsky and Matveev (2017) is a bit smaller.

4. ISS analysis of fast-varying linear systems

In this section we will extend the stability analysis of Sections 2 and 3 to ISS analysis of the perturbed systems.

4.1. ISS analysis by averaging

Consider the fast-varying perturbed system

$$\dot{x}(t) = A(\tfrac{t}{\varepsilon})x(t) + B(\tfrac{t}{\varepsilon})w(t), \quad t \geq 0, \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$, $A : [0,\infty) \to \mathbb{R}^{n\times n}$ and $B : [0,\infty) \to \mathbb{R}^{n\times n_w}$ are piecewise-continuous, $\varepsilon > 0$ is a small parameter, and $w(t) \in \mathbb{R}^{n_w}$ is a disturbance. The disturbance is supposed to be locally essentially bounded meaning that $w \in L_\infty(0,t)$ for all $t > 0$.

We assume that **A1** and **A2** and relation (2.10) hold. Assume additionally

A5 All entries $b_{kv}(\tau)$ of $B(\tau)$ are uniformly bounded for $\tau \ge 0$ with the values from some finite intervals $b_{kv}(\tau) \in [b^m_{kv}, b^M_{kv}]$ for $\tau \ge \tau_1 \ge 1$.

In this paper $B(\tau)$ is treated as a matrix from the time-varying polytope. Under **A5**, $B(\tau)$ can be presented as a convex combination of the constant matrices B_l with the entries b^m_{kv} or b^M_{kv}:

$$\begin{aligned} &B(\tau) = \textstyle\sum_{l=1}^{\bar N} \bar f_l(\tau) B_l \quad \forall \tau \ge \tau_1 \ge 1,\\ &\bar f_l \ge 0, \quad \textstyle\sum_{l=1}^{\bar N} \bar f_l = 1, \quad 1 \le \bar N \le 2^{n\times n_w}. \end{aligned} \tag{4.2}$$

Following the time-delay approach to averaging, we integrate (4.1) on $[t-\varepsilon, t]$ for $t \ge \varepsilon\tau_1$. Then we arrive at

$$\begin{aligned} \dot z(t) = &[A_{av} + \Delta A(\tfrac{t}{\varepsilon})]x(t) - \textstyle\int_0^1 A(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta}^t \dot x(s) ds d\theta\\ &+ \textstyle\int_0^1 B(\tfrac{t}{\varepsilon}-\theta) w(t-\theta\varepsilon) d\theta, \quad t \ge \varepsilon\tau_1, \end{aligned} \tag{4.3}$$

where $z(t)$ is given by (2.8). Compared to (2.9), the latter system has an additional term $\int_0^1 B(\frac{t}{\varepsilon}-\theta)w(t-\theta\varepsilon)d\theta$. From (4.2), we have

$$\textstyle\int_0^1 B(\tfrac{t}{\varepsilon}-\theta)w(t-\theta\varepsilon)d\theta = \sum_{l=1}^{\bar N} B_l w_l(t), \tag{4.4}$$

where

$$w_l(t) \triangleq \textstyle\int_0^1 \bar f_l(\tfrac{t}{\varepsilon}-\theta) w(t-\theta\varepsilon) d\theta. \tag{4.5}$$

Since $0 \le \bar f_l \le 1$, we have

$$\begin{aligned} |w_l(t)| &= |\textstyle\int_0^1 \bar f_l(\tfrac{t}{\varepsilon}-\theta) w(t-\theta\varepsilon)d\theta|\\ &\le \textstyle\int_0^1 |\bar f_l(\tfrac{t}{\varepsilon}-\theta)||w(t-\theta\varepsilon)|d\theta\\ &\le \|w[0,t]\|_\infty, \quad l = 1,\dots,\bar N, \quad t \ge \varepsilon\tau_1. \end{aligned} \tag{4.6}$$

Then system (4.3) has the following form

$$\begin{aligned} \dot z(t) = &[A_{av} + \Delta A(\tfrac{t}{\varepsilon})]x(t) - \textstyle\sum_{i=1}^N A_i Y_i(t)\\ &+ \textstyle\sum_{l=1}^{\bar N} B_l w_l(t), \quad t \ge \varepsilon\tau_1, \end{aligned} \tag{4.7}$$

where $Y_i(t)$ and $w_l(t)$ are given by (2.12) and (4.5), respectively. Note that system (4.1) is ISS if the time-delay system (4.7) is ISS. We now present the ISS conditions for system (4.7):

Theorem 4.1. *Assume* **A1**, **A2** *and* **A5**. *Given matrices A_{av}, A_i ($i = 1,\dots,N$), B_l ($l = 1,\dots,\bar N$), and constants $\sigma > 0$, $\alpha > 0$ and $\varepsilon^* > 0$, let there exist $n \times n$ matrices $P > 0$, $R > 0$, $H_i > 0$ ($i = 1,\dots,N$), and scalars $\lambda > 0$ and $b_l > 0$ ($l = 0,\dots,\bar N$) that satisfy the following LMIs:*

$$\left[\begin{array}{c|c} \bar\Phi & \begin{matrix}\sqrt{\varepsilon^*}A_i^T(R+\sum_{j=1}^N H_j)\\ 0_{(N+2)n+\bar N n_w, n}\\ \sqrt{\varepsilon^*}B_l^T(R+\sum_{j=1}^N H_j)\end{matrix}\\ \hline * \quad * \quad * & -(R+\sum_{j=1}^N H_j)\end{array}\right] < 0, \quad i = 1,\dots,N, \quad l = 1,\dots,\bar N, \tag{4.8}$$

where

$$\begin{aligned} &\bar\Phi = \begin{bmatrix} \Phi & \bar\Phi_{12}\\ * & \bar\Phi_{22}\end{bmatrix},\\ &\bar\Phi_{12} = \begin{bmatrix} P & -P & 0_{n,(N+1)n}\end{bmatrix}^T \begin{bmatrix} B_1 & \dots & B_{\bar N} & 0_{n,n_w}\end{bmatrix},\\ &\bar\Phi_{22} = -diag\{b_1 I_{n_w},\dots, b_{\bar N} I_{n_w}, b_0 I_{n_w}\}, \end{aligned} \tag{4.9}$$

and Φ is given by (2.15). Then system (4.1) is ISS for all $\varepsilon \in (0,\varepsilon^*]$, meaning that there exists $M_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon^*]$ and locally essentially bounded w, the solutions of system (4.1) initialized by $x(0) \in \mathbb{R}^n$ satisfy the following inequality:

$$\begin{aligned} &|x(t)|^2 \le M_0 e^{-2\alpha t}|x(0)|^2\\ &+ \left[M_0 e^{-2\alpha t} + \tfrac{b_0+\dots+b_{\bar N}}{2\alpha c_1}\right]\|w[0,t]\|^2_\infty \quad \forall t \ge 0, \end{aligned} \tag{4.10}$$

where c_1 denotes the smallest eigenvalue of matrix $\begin{bmatrix} P & -P\\ * & P+e^{-2\alpha\varepsilon^*}R\end{bmatrix}$. Moreover, given $\Delta > 0$, the ellipsoid

$$\mathfrak{X} = \{x \in \mathbb{R}^n : |x|^2 \le \tfrac{b_0+\dots+b_{\bar N}}{2\alpha c_1}\Delta^2\} \tag{4.11}$$

is exponentially attractive with a decay rate α for all $x(0) \in \mathbb{R}^n$ and essentially bounded w with ess $\sup_{t\ge0}|w(t)| \le \Delta$. In addition, if LMIs (4.8) are feasible for $\alpha = 0$, then system (4.1) is ISS for all $\varepsilon \in (0,\varepsilon^*]$ and (4.10) holds with a small enough decay rate $\alpha = \alpha_0 > 0$.

Proof. Differentiating $V_1(t)$ given by (2.24) along (4.7) and following arguments of Theorem 2.1, we arrive at

$$\begin{aligned} &\tfrac{d}{dt}V_1(t) + 2\alpha V_1(t) - b_0|w(t)|^2 - \textstyle\sum_{l=1}^{\bar N} b_l|w_l(t)|^2\\ &\le \bar\xi_1^T(t)\bar\Phi\bar\xi_1(t) + \varepsilon^*\dot x^T(t)(R+\textstyle\sum_{i=1}^N H_i)\dot x(t), \end{aligned} \tag{4.12}$$

where

$$\bar\xi_1^T(t) = [\xi_1^T(t), \xi_2^T(t)], \;\; \xi_2^T(t) = [w_1^T(t),\dots,w_{\bar N}^T(t), w^T(t)],$$

and $\xi_1(t)$ and $\bar\Phi$ is given by (2.28) and (4.9) respectively. Via (2.10) and (4.2), we can present system (4.1) as follows

$$\dot x(t) = \textstyle\sum_{i=1}^N f_i(\tfrac{t}{\varepsilon})A_i x(t) + \sum_{l=1}^{\bar N}\bar f_l(\tfrac{t}{\varepsilon})B_l w(t). \tag{4.13}$$

Note that the term $-b_0|w(t)|^2$ in (4.12) compensates $w(t)$ that stems from the substitution of $\dot x$ given by (4.13) into (4.12). By Schur complements, if

$$\left[\begin{array}{c|c} \bar\Phi & \begin{matrix}\sqrt{\varepsilon^*}\sum_{i=1}^N f_i(\frac{t}{\varepsilon})A_i^T(R+\sum_{j=1}^N H_j)\\ 0_{(N+2)n+\bar N n_w, n}\\ \sqrt{\varepsilon^*}\sum_{l=1}^{\bar N}\bar f_l(\frac{t}{\varepsilon})B_l^T(R+\sum_{j=1}^N H_j)\end{matrix}\\ \hline * \quad * \quad * & -(R+\sum_{j=1}^N H_j)\end{array}\right] < 0, \tag{4.14}$$

then for $t \ge \varepsilon\tau_1$ the following holds:

$$\tfrac{d}{dt}V_1(t) + 2\alpha V_1(t) - b_0|w(t)|^2 - \textstyle\sum_{l=1}^{\bar N} b_l|w_l(t)|^2 \le 0.$$

By comparison principle, the latter implies

$$V_1(t) \le e^{-2\alpha(t-\varepsilon\tau_1)}V_1(\varepsilon\tau_1) + \tfrac{b_0+\dots+b_{\bar N}}{2\alpha}\|w[0,t]\|^2_\infty, \quad t \ge \varepsilon\tau_1. \tag{4.15}$$

LMIs (4.8) yield (4.14) and thus (4.15).

For $t \in [0,\varepsilon\tau_1]$, $x(t)$ satisfies (4.1), where under **A2** and **A5** there exist $a > 0$ and $b > 0$ such that $\|A(\tau)\| \le a$ and $\|B(\tau)\| \le b$ for all $\tau \ge 0$. We obtain

$$|\dot x(t)| \le a|x(t)| + b\|w[0,t]\|_\infty, \quad t \in [0,\varepsilon\tau_1]$$

implying

$$2|x(t)||\dot x(t)| \le 2a|x(t)|^2 + 2b|x(t)|\|w[0,t]\|^2_\infty, \quad t \in [0,\varepsilon\tau_1],$$

or, by Young's inequality,

$$\tfrac{d}{dt}|x(t)|^2 \le (2a+b)|x(t)|^2 + b\|w[0,t]\|^2_\infty, \quad t \in [0,\varepsilon\tau_1].$$

By comparison principle, the latter yields

$$|x(t)|^2 \le e^{(2a+b)t}|x(0)|^2 + \tfrac{be^{(2a+b)t}}{2a+b}\|w[0,t]\|^2_\infty, \quad t \in [0,\varepsilon\tau_1]. \tag{4.16}$$

Then

$$\begin{aligned}&|\dot{x}(t)|^2 \le 2a^2|x(t)|^2 + 2b^2\|w[0,t]\|^2_\infty\\ &\le 2a^2e^{(2a+b)t}|x(0)|^2 + 2b\left(a^2\frac{e^{(2a+b)t}}{2a+b}+b\right)\|w[0,t]\|^2_\infty,\\ &\quad t\in[0,\varepsilon\tau_1].\end{aligned} \tag{4.17}$$

From (2.31), (4.16) and (4.17), it follows that

$$\begin{aligned}V_1(\varepsilon\tau_1) \le c_3e^{-2\alpha\varepsilon\tau_1}\left[|x(0)|^2 + \|w[0,t]\|^2_\infty\right],\\ t\in[0,\varepsilon\tau_1],\quad \varepsilon\in(0,\varepsilon^*]\end{aligned} \tag{4.18}$$

with some ε-independent $c_3 > 0$. The latter inequality together with (2.25) and (4.15) implies (4.10).

The feasibility of the strict LMIs (4.8) with $\alpha = 0$ implies their feasibility with the same decision variables and with a small enough $\alpha = \alpha_0 > 0$, and thus guarantees ISS. □

4.2. ISS analysis of persistently excited systems

In this section, we consider the PE system (3.5) in the presence of perturbations

$$\dot{x}(t) = -p(\tfrac{t}{\varepsilon})p^T(\tfrac{t}{\varepsilon})x(t) + B(\tfrac{t}{\varepsilon})w(t),\quad t\ge 0, \tag{4.19}$$

where $\varepsilon > 0$, $B : [0,\infty) \to \mathbb{R}^{n\times n_w}$ is piecewise-continuous, and $w(t)\in\mathbb{R}^{n_w}$ is the essentially bounded disturbance. Assume that **A3** - **A5** and presentation (4.2) hold. Following the time-delay approach to averaging of Section 2, we integrate (4.19) on $[t-\varepsilon, t]$ for $t \ge \varepsilon$. Using the notations given by (2.7), (2.8), (3.7) and (4.4), we rewrite system (4.19) as

$$\dot{z}(t) = -A_{av}(\tfrac{t}{\varepsilon})x(t) + Y(t) + \textstyle\sum_{l=1}^{\bar{N}} B_l w_l(t),\quad t\ge\varepsilon. \tag{4.20}$$

Note that the ISS of the time-delay system (4.20) implies the ISS of (4.19). To derive stability conditions of system (4.20), we consider Lyapunov functional $V_2(t)$ given by (3.17). Differentiating $V_2(t)$ along (4.20) via (3.12) and (3.16) we arrive at

$$\begin{aligned}&\tfrac{d}{dt}V_2(t) + 2\alpha V_2(t) - b_0|w(t)|^2 - \textstyle\sum_{l=1}^{\bar{N}} b_l|w_l(t)|^2\\ &\le 2p[x(t)-G(t)]^T[-A_{av}(\tfrac{t}{\varepsilon})x(t) + Y(t) + \textstyle\sum_{l=1}^{\bar{N}} B_l w_l(t)]\\ &\quad +2\alpha p x^T(t)x(t) + \varepsilon(r+\eta)\dot{x}^T(t)\dot{x}(t)\\ &\quad -\tfrac{4}{\varepsilon}e^{-2\alpha\varepsilon}rG^T(t)G(t) - \tfrac{4}{\varepsilon M^4}e^{-2\alpha\varepsilon}\eta Y^T(t)Y(t)\\ &\quad -b_0|w(t)|^2 - \textstyle\sum_{l=1}^{\bar{N}} b_l|w_l(t)|^2.\end{aligned} \tag{4.21}$$

Further by substituting (4.19) and applying Young's inequality, we obtain due to **A3**

$$\begin{aligned}&\dot{x}^T(t)\dot{x}(t) \le 2x^T(t)p(\tfrac{t}{\varepsilon})p^T(\tfrac{t}{\varepsilon})p(\tfrac{t}{\varepsilon})p^T(\tfrac{t}{\varepsilon})x(t)\\ &\quad +2w^T(t)B^T(\tfrac{t}{\varepsilon})B(\tfrac{t}{\varepsilon})w^T(t)\\ &\le 2M^4x^T(t)x(t) + 2w^T(t)B^T(\tfrac{t}{\varepsilon})B(\tfrac{t}{\varepsilon})w^T(t).\end{aligned} \tag{4.22}$$

Then for all $t\ge\varepsilon$

$$\tfrac{d}{dt}V_2(t) + 2\alpha V_2(t) - b_0|w(t)|^2 - \textstyle\sum_{l=1}^{\bar{N}} b_l|w_l(t)|^2 \le 0 \tag{4.23}$$

if the two matrix inequalities

$$\left[\begin{array}{c|c}\bar{\Xi}_i & \begin{matrix}0_{3n+\bar{N}n_w,n}\\ \sqrt{2\varepsilon^*}(r+\eta)\sum_{l=1}^{\bar{N}}\bar{f}_l(\tfrac{t}{\varepsilon})B_l^T\end{matrix}\\ \hline * \quad * & -(r+\eta)I_n\end{array}\right] < 0,\quad i=1,2 \tag{4.24}$$

are feasible, where

$$\bar{\Xi}_i = \begin{bmatrix}\left(\Xi_i + \mathrm{diag}\{\varepsilon M^4(r+\eta), 0_{2,2}\}\right)\otimes I_n & \tilde{\Phi}_{12}\\ * & \bar{\Phi}_{22}\end{bmatrix}. \tag{4.25}$$

Here Ξ_i and $\bar{\Phi}_{22}$ are given by (3.19) and (4.9) respectively, and

$$\tilde{\Phi}_{12} = p\begin{bmatrix}I_n & -I_n & 0_{n,n}\end{bmatrix}^T\begin{bmatrix}B_1 & \dots & B_{\bar{N}} & 0_{n,n_w}\end{bmatrix}.$$

Since (4.24) is affine in $\sum_{l=1}^{\bar{N}}\bar{f}_l(\frac{t}{\varepsilon})B_l^T$, we obtain the following ISS result:

Theorem 4.2. *Assume* **A3** - **A5**. *Given matrices B_l ($l = 1,\dots,\bar{N}$), and constants $M^2 > \rho > 0$, $\alpha > 0$ and $\varepsilon^* > 0$, let there exist positive scalars p, r, η and b_l ($l = 0,\dots,\bar{N}$) that satisfy the following LMIs:*

$$\left[\begin{array}{c|c}\bar{\Xi}_i & \begin{matrix}0_{3n+\bar{N}n_w,n}\\ \sqrt{2\varepsilon^*}(r+\eta)B_l^T\end{matrix}\\ \hline * \quad * & -(r+\eta)I_n\end{array}\right] < 0,\quad i=1,2,\quad l=1,\dots,\bar{N}, \tag{4.26}$$

where $\bar{\Xi}_i$ is given by (4.25). Then system (4.19) is ISS for all $\varepsilon\in(0,\varepsilon^*]$, meaning that there exists $M_0 > 0$ such that for all $\varepsilon\in(0,\varepsilon^*]$ and locally essentially bounded w, the solutions of system (4.19) initialized by $x(0)\in\mathbb{R}^n$ satisfy (4.10). Moreover, given $\Delta > 0$, the ellipsoid $\mathfrak{X}$ given by (4.11) is exponentially attractive with a decay rate α for all $x(0)\in\mathbb{R}^n$ and essentially bounded w with $\mathrm{ess\,sup}_{t\ge 0}|w(t)| \le \Delta$. In addition, if the LMIs (4.26) hold with $\alpha = 0$, then system (4.19) is ISS for all $\varepsilon\in(0,\varepsilon^*]$ and (4.10) holds with a small enough decay rate $\alpha = \alpha_0 > 0$.

5. Averaging of systems with time-varying delays

In this section, we consider the fast-varying system with a time-varying delay $h(t)$:

$$\dot{x}(t) = A(\tfrac{t}{\varepsilon})x(t) + A_d(\tfrac{t}{\varepsilon})x(t-h(t)),\quad t\ge 0, \tag{5.1}$$

where $x(t)\in\mathbb{R}^n$, A, A_d : $[0,\infty)\to\mathbb{R}^{n\times n}$ are piecewise-continuous, and $\varepsilon > 0$ is a small parameter. The delay $h(t)$ is supposed to be bounded

$$0 \le h(t) \le h_M \tag{5.2}$$

and fast-varying (without any restriction on the delay derivative). The initial condition of system (5.1) is given by $x(\theta) = \phi(\theta)$, $\theta\in[-h_M, 0]$ with $\phi\in C[-h_M, 0]$.

We assume the following:

A6 There exists $\tau_1 \ge 1$ such that (2.2) holds and

$$\begin{aligned}&\tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t A_d(\tfrac{s}{\varepsilon})ds = A_{dav} + \Delta A_d(\tfrac{t}{\varepsilon}),\\ &\|\Delta A_d(\tfrac{t}{\varepsilon})\| \le \sigma_d \quad \forall \tfrac{t}{\varepsilon}\ge\tau_1,\end{aligned} \tag{5.3}$$

where $\sigma_d > 0$ is a small enough constant. Moreover, matrix $A_{av} + A_{dav}$ is Hurwitz.

A7 Let **A2** hold and all entries $a_{dkv}(\tau)$ of $A_d(\tau)$ are uniformly bounded for $\tau \ge 0$ with the values from some finite intervals $a_{dkv}(\tau)\in[a^m_{dkv}, a^M_{dkv}]$ for $\tau\ge\tau_1\ge 1$.

Under **A7**, (2.10) holds and $A_d(\tau)$ can be presented as a convex combination of the constant matrices A_{dj} with the entries a^m_{dkv} or a^M_{dkv}:

$$\begin{aligned}&A_d(\tau) = \textstyle\sum_{j=1}^{N_d} f_{dj}(\tau)A_{dj}\quad \forall\tau\ge\tau_1\ge 1,\\ &f_{dj}\ge 0,\quad \textstyle\sum_{j=1}^{N_d} f_{dj} = 1,\quad 1\le N_d\le 2^{n^2}.\end{aligned} \tag{5.4}$$

For a constant a_{dkv}, we have $a^m_{dkv} = a^M_{dkv}$.

Following the time-delay approach to averaging, we integrate (5.1) on $[t-\varepsilon, t]$ for $t\ge\varepsilon\tau_1 + h_M$. Then we arrive at the following time-delay system for $t\ge\varepsilon\tau_1 + h_M$

$$\begin{aligned}\dot{z}(t) &= [A_{av} + A_{dav} + \Delta A(\tfrac{t}{\varepsilon}) + \Delta A_d(\tfrac{t}{\varepsilon})]x(t)\\ &\quad -\textstyle\int_0^1 A(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta}^t \dot{x}(s)ds d\theta\\ &\quad -\textstyle\int_0^1 A_d(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta-h(t-\varepsilon\theta)}^t \dot{x}(s)ds d\theta,\end{aligned} \tag{5.5}$$

where $z(t)$ is given by (2.8). Compared to system (2.9), the latter system has an additional integral term that via (5.4) can be presented as

$$-\textstyle\int_0^1 A_d(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta-h(t-\varepsilon\theta)}^t \dot{x}(s)ds d\theta = -\sum_{j=1}^{N_d} A_{dj}Y_{dj}(t),$$

where

$$Y_{dj}(t) \triangleq \int_0^1 f_{dj}(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta-h(t-\varepsilon\theta)}^{t}\dot{x}(s)dsd\theta. \tag{5.6}$$

Then system (5.5) can be rewritten as

$$\begin{aligned}\dot{z}(t) = &\left[A_{av}+A_{dav}+\Delta A(\tfrac{t}{\varepsilon})+\Delta A_d(\tfrac{t}{\varepsilon})\right]x(t)\\ &-\sum_{i=1}^{N}A_iY_i(t)-\sum_{j=1}^{N_d}A_{dj}Y_{dj}(t), \quad t\ge \varepsilon\tau_1+h_M,\end{aligned} \tag{5.7}$$

where $Y_i(t)$ and $Y_{dj}(t)$ are given by (2.12) and (5.6), respectively. Note that (5.1) is stable if the time-delay system (5.7) is stable. The latter system is a perturbation of the averaged system

$$\dot{x}_{av}(t) = \left[A_{av}+A_{dav}+\Delta A(\tfrac{t}{\varepsilon})+\Delta A_d(\tfrac{t}{\varepsilon})\right]x_{av}(t), \quad x_{av}(t)\in\mathbb{R}^n,$$

which is exponentially stable for small enough σ and σ_d.

Let f_i^* $(i=1,\dots,N)$ be defined by (2.13). Denote by $f_{dj}^*>0$ $(j=1,\dots,N_d)$ the following bounds:

$$\int_0^1(\varepsilon^*\theta+h_M)f_{dj}(\tau-\theta)d\theta \le f_{dj}^* \quad \forall \tau\ge \tau_1+\tfrac{h_M}{\varepsilon}. \tag{5.8}$$

We can always choose $f_{dj}^*\le \frac{\varepsilon^*}{2}+h_M$ since $f_{dj}\in[0,1]$.

We now present the stability conditions for (5.1):

Theorem 5.1. *Assume* **A6** *and* **A7**. *Given matrices A_{av}, A_{dav}, A_i $(i=1,\dots,N)$, A_{dj} $(j=1,\dots,N_d)$, and constants $\sigma>0$, $\sigma_d>0$, $\alpha>0$, $\varepsilon^*>0$ and $h_M>0$, let there exist $n\times n$ matrices $P>0$, $R>0$, $H_i>0$ $(i=1,\dots,N)$, $Q_j>0$ $(j=1,\dots,N_d)$, $S_1>0$, $R_1>0$, U and scalars $\lambda>0$, $\lambda_d>0$ that satisfy the following LMIs:*

$$\begin{bmatrix} R_1 & U\\ * & R_1\end{bmatrix}\ge 0, \tag{5.9}$$

$$\begin{bmatrix} \Omega & \Theta_{ij}\\ * & \Theta_2\end{bmatrix}<0, \quad i=1,\dots,N, \quad j=1,\dots,N_d. \tag{5.10}$$

Here Ω is the symmetric matrix composed of

$$\begin{aligned}
\Omega_{11} &= P(A_{av}+A_{dav})+(A_{av}+A_{dav})^TP+2\alpha P\\
&\quad +\lambda\sigma^2I_n+\lambda_d\sigma_d^2I_n+S_1-\tfrac{1}{h_M}e^{-2\alpha h_M}R_1,\\
\Omega_{12} &= -(A_{av}+A_{dav})^TP-2\alpha P,\\
\Omega_{13} &= -\Omega_{23} = -P[A_1,\dots,A_N],\\
\Omega_{14} &= \Omega_{18} = -\Omega_{24} = -\Omega_{28} = P,\\
\Omega_{15} &= \Omega_{56} = \tfrac{1}{h_M}e^{-2\alpha h_M}(R_1-U),\\
\Omega_{16} &= \tfrac{1}{h_M}e^{-2\alpha h_M}U,\\
\Omega_{17} &= -\Omega_{27} = -P[A_{d1},\dots,A_{dN_d}],\\
\Omega_{22} &= -\tfrac{4}{\varepsilon^*}e^{-2\alpha\varepsilon^*}R+2\alpha P,\\
\Omega_{33} &= -2e^{-2\alpha\varepsilon^*}diag\{\tfrac{1}{f_1^*}H_1,\dots,\tfrac{1}{f_N^*}H_N\},\\
\Omega_{44} &= -\lambda I_n,\\
\Omega_{55} &= -\tfrac{1}{h_M}e^{-2\alpha h_M}(2R_1-U-U^T),\\
\Omega_{66} &= -e^{-2\alpha h_M}S_1-\tfrac{1}{h_M}e^{-2\alpha h_M}R_1,\\
\Omega_{77} &= -2e^{-2\alpha(\varepsilon^*+h_M)}diag\{\tfrac{1}{f_{d1}^*}Q_1,\dots,\tfrac{1}{f_{dN_d}^*}Q_{N_d}\},\\
\Omega_{88} &= -\lambda_dI_n,
\end{aligned} \tag{5.11}$$

and other blocks are zero matrices with f_i^* $(i=1,\dots,N)$ and f_{dj}^* $(j=1,\dots,N_d)$ defined by (2.13) and (5.8) *respectively, and*

$$\begin{aligned}
\Theta_{ij} &= \begin{bmatrix} \sqrt{\varepsilon^*}A_i^T\Lambda_1 & \sqrt{h_M}A_i^T\Lambda_2\\ 0_{(N+2)n,n} & 0_{(N+2)n,n}\\ \sqrt{\varepsilon^*}A_{dj}^T\Lambda_1 & \sqrt{h_M}A_{dj}^T\Lambda_2\\ 0_{(N_d+2)n,n} & 0_{(N_d+2)n,n}\end{bmatrix},\\
\Theta_2 &= -diag\{\Lambda_1,\Lambda_2\},\\
\Lambda_1 &= R+(H_1+\cdots+H_N)+(Q_1+\cdots+Q_{N_d}),\\
\Lambda_2 &= R_1+2(Q_1+\cdots+Q_{N_d}).
\end{aligned} \tag{5.12}$$

Then system (5.1) *is exponentially stable with a decay rate α for all $\varepsilon\in(0,\varepsilon^*]$ and $h(t)\in[0,h_M]$, meaning that there exists $M_0>0$ such that for all $\varepsilon\in(0,\varepsilon^*]$, $h(t)\in[0,h_M]$ the solutions of* (5.1) *initialized by $\phi\in C[-h_M,0]$ satisfy*

$$|x(t)|^2\le M_0e^{-2\alpha t}\|\phi\|_C^2 \quad \forall t\ge 0. \tag{5.13}$$

Moreover, if the LMIs (5.9) and (5.10) *hold with $\alpha=0$, then system* (5.1) *is exponentially stable with a small enough decay rate $\alpha=\alpha_0>0$ for all $\varepsilon\in(0,\varepsilon^*]$ and $h(t)\in[0,h_M]$.*

Proof. Differentiating $V_P(t)$ given by (2.17) along (5.7), we have

$$\begin{aligned}\tfrac{d}{dt}V_P(t) &= 2[x(t)-G(t)]^TP[(A_{av}+A_{dav}+\Delta A(\tfrac{t}{\varepsilon})\\ &+\Delta A_d(\tfrac{t}{\varepsilon}))x(t)-\sum_{i=1}^{N}A_iY_i(t)-\sum_{j=1}^{N_d}A_{dj}Y_{dj}(t)].\end{aligned} \tag{5.14}$$

We use (2.19) and (2.22) to compensate the $G(t)$- and $Y_i(t)$-terms in (5.14). For the $Y_{dj}(t)$-terms, we choose

$$\begin{aligned}
V_Q(t) &= \sum_{j=1}^{N_d}V_{Q_j}(t),\\
V_{Q_j}(t) &= 2\int_0^1\int_{t-\varepsilon\theta-h_M}^{t}e^{-2\alpha(t-s)}\\
&\quad\times(s-t+\varepsilon\theta+h_M)\dot{x}^T(s)Q_j\dot{x}(s)dsd\theta
\end{aligned} \tag{5.15}$$

with $Q_j>0$. Differentiating $V_{Q_j}(t)$, we have

$$\begin{aligned}
\tfrac{d}{dt}V_{Q_j}(t)+2\alpha V_{Q_j}(t) &= 2\int_0^1(\varepsilon\theta+h_M)\dot{x}^T(t)Q_j\dot{x}(t)d\theta\\
&\quad -2\int_0^1\int_{t-\varepsilon\theta-h_M}^{t}e^{-2\alpha(t-s)}\dot{x}^T(s)Q_j\dot{x}(s)dsd\theta\\
&\le(\varepsilon+2h_M)\dot{x}^T(t)Q_j\dot{x}(t)\\
&\quad -2e^{-2\alpha(\varepsilon+h_M)}\int_0^1\int_{t-\varepsilon\theta-h(t-\varepsilon\theta)}^{t}\dot{x}^T(s)Q_j\dot{x}(s)dsd\theta.
\end{aligned} \tag{5.16}$$

By using Jensen's inequality (1.2), taking into account (5.4) and employing the notations (5.8), we obtain

$$\begin{aligned}
&Y_{dj}^T(t)Q_jY_{dj}(t)\\
&\le\int_0^1\int_{t-\varepsilon\theta-h(t-\varepsilon\theta)}^{t}f_{dj}(\tfrac{t}{\varepsilon}-\theta)dsd\theta\\
&\quad\times\int_0^1 f_{dj}(\tfrac{t}{\varepsilon}-\theta)\int_{t-\varepsilon\theta-h(t-\varepsilon\theta)}^{t}\dot{x}^T(s)Q_j\dot{x}(s)dsd\theta\\
&\le\int_0^1(\varepsilon\theta+h(t-\varepsilon\theta))f_{dj}(\tfrac{t}{\varepsilon}-\theta)d\theta\\
&\quad\times\int_0^1\int_{t-\varepsilon\theta-h(t-\varepsilon\theta)}^{t}\dot{x}^T(s)Q_j\dot{x}(s)dsd\theta\\
&\le f_{dj}^*\int_0^1\int_{t-\varepsilon\theta-h(t-\varepsilon\theta)}^{t}\dot{x}^T(s)Q_j\dot{x}(s)dsd\theta.
\end{aligned} \tag{5.17}$$

Then

$$\begin{aligned}\tfrac{d}{dt}V_{Q_j}(t)+2\alpha V_{Q_j}(t) &\le(\varepsilon+2h_M)\dot{x}^T(t)Q_j\dot{x}(t)\\ &\quad -\tfrac{2}{f_{dj}^*}e^{-2\alpha(\varepsilon+h_M)}Y_{dj}^T(t)Q_jY_{dj(t)}.\end{aligned} \tag{5.18}$$

Note that via (2.10) and (5.4), we can present system (5.1) as follows

$$\dot{x}(t) = \sum_{i=1}^{N}f_i(\tfrac{t}{\varepsilon})A_ix(t)+\sum_{j=1}^{N_d}f_{dj}(\tfrac{t}{\varepsilon})A_{dj}x(t-h(t)). \tag{5.19}$$

Substitution of the latter representation into (5.18) leads to the delayed state $x(t-h(t))$. To compensate $x(t-h(t))$, we add the following standard terms for delay-dependent stability (see p. 90 in Fridman (2014)) to Lyapunov functional

$$V_{S_1}(t)=\int_{t-h_M}^{t} e^{-2\alpha(t-s)}x^T(s)S_1x(s)ds,\quad S_1>0, \tag{5.20}$$

$$V_{R_1}(t)=\int_{t-h_M}^{t}(s-t+h_M)e^{-2\alpha(t-s)}\dot{x}^T(s)R_1\dot{x}(s)ds,\\ R_1>0. \tag{5.21}$$

We have

$$\frac{d}{dt}V_{S_1}(t)+2\alpha V_{S_1}(t)\\ =x^T(t)S_1x(t)-e^{-2\alpha h_M}x^T(t-h_M)S_1x(t-h_M). \tag{5.22}$$

Further by Park et al. (2011), we obtain

$$\begin{aligned}&\frac{d}{dt}V_{R_1}(t)+2\alpha V_{R_1}(t)\\ &=h_M\dot{x}^T(t)R_1\dot{x}(t)-\int_{t-h_M}^{t} e^{-2\alpha(t-s)}\dot{x}^T(s)R_1\dot{x}(s)ds\\ &\le h_M\dot{x}^T(t)R_1\dot{x}(t)-\frac{e^{-2\alpha h_M}}{h_M}\begin{bmatrix} x(t)-x(t-h(t))\\ x(t-h(t))-x(t-h_M)\end{bmatrix}^T\\ &\times\begin{bmatrix} R_1 & U\\ * & R_1\end{bmatrix}\begin{bmatrix} x(t)-x(t-h(t))\\ x(t-h(t))-x(t-h_M)\end{bmatrix},\end{aligned} \tag{5.23}$$

where matrix U satisfies (5.9).

We now define a Lyapunov functional for system (5.7):

$$V_3(t)=V_P(t)+V_R(t)+V_H(t)+V_Q(t)+V_{S_1}(t)+V_{R_1}(t), \tag{5.24}$$

where $V_P(t)$, $V_R(t)$, $V_H(t)$, $V_Q(t)$, $V_{S_1}(t)$ and $V_{R_1}(t)$ are, respectively, given by (2.17), (2.19), (2.22), (5.15), (5.20) and (5.21). It is clear from (2.25) that $V_3(t)$ is positive-definite for all $\varepsilon\in(0,\varepsilon^*]$, where due to (2.25) $V_3(t)\ge V_P(t)+V_R(t)>c_1|x(t)|^2$ for some ε-independent $c_1>0$.

Taking into account (2.21), (2.23), (5.14), (5.18), (5.22) and (5.23), we have

$$\frac{d}{dt}V_3(t)+2\alpha V_3(t)\le \xi_3^T(t)\Omega\xi_3(t)\\ +\dot{x}^T(t)(\varepsilon^*\Lambda_1+h_M\Lambda_2)\dot{x}(t),\quad t\ge\varepsilon\tau_1+h_M, \tag{5.25}$$

where

$$\xi_3^T(t)=[\xi_1^T(t),x^T(t-h(t)),x^T(t-h_M),\\ Y_{d1}^T(t),\ldots,Y_{dN_d}^T(t),x^T(t)\Delta A_d^T(\tfrac{t}{\varepsilon})], \tag{5.26}$$

and $\xi_1(t)$ is given by (2.28), Ω is the symmetric matrix composed of (5.11), and Λ_1 and Λ_2 are given by (5.12). Substituting (5.19) into (5.25) and applying Schur complements, via (5.10) we arrive at

$$\frac{d}{dt}V_3(t)+2\alpha V_3(t)\le 0,\quad t\ge\varepsilon\tau_1+h_M. \tag{5.27}$$

The latter implies

$$c_1|x(t)|^2\le V_3(t)\le e^{-2\alpha(t-\varepsilon\tau_1-h_M)}V_3(\varepsilon\tau_1+h_M),\\ t\ge\varepsilon\tau_1+h_M. \tag{5.28}$$

Denote $x_t(\theta)=x(t+\theta)$, $\theta\in[-h_M,0]$.

$$V_3(\varepsilon\tau_1+h_M)\le c_2\Big[\|x_{\varepsilon\tau_1+h_M}\|_C^2+\int_{\varepsilon(\tau_1-1)}^{\varepsilon\tau_1+h_M}|\dot{x}(s)|^2ds\Big] \tag{5.29}$$

for some ε-independent $c_2>0$. From (5.1), it follows that

$$x_t(\theta)=\begin{cases}\phi(t+\theta), & t+\theta<0,\\ \phi(0)+\int_0^{t+\theta}[A(\tfrac{s}{\varepsilon})x(s) & \\ \quad +A_d(\tfrac{s}{\varepsilon})x(s-h(s))]ds, & t+\theta\ge 0.\end{cases} \tag{5.30}$$

From (5.30), we arrive at

$$\|x_t\|_C\le\|\phi\|_C+\int_0^t c_3\|x_s\|_C ds,\quad t\in[0,\varepsilon\tau_1+h_M]$$

for some ε-independent $c_3>0$. By the Gronwall inequality, the latter implies

$$\|x_t\|_C\le e^{c_3t}\|\phi\|_C,\quad t\in[0,\varepsilon\tau_1+h_M]. \tag{5.31}$$

From (5.1) and (5.31) we find

$$|\dot{x}(t)|^2\le c_4\|\phi\|_C^2,\quad t\in[0,\varepsilon\tau_1+h_M] \tag{5.32}$$

with some ε-independent $c_4>0$. So, from (5.29), (5.31) and (5.32), we obtain

$$\begin{aligned}&V_3(\varepsilon\tau_1+h_M)\\ &\le c_2\Big[e^{2c_3(\varepsilon\tau_1+h_M)}\|\phi\|_C^2+\int_{\varepsilon(\tau_1-1)}^{\varepsilon\tau_1+h_M}c_4\|\phi\|_C^2ds\Big]\\ &\le c_5e^{-2\alpha(\varepsilon\tau_1+h_M)}\|\phi\|_C^2\end{aligned} \tag{5.33}$$

for some ε-independent $c_5>0$. Clearly, (5.28) and (5.33) imply (5.13) for some ε-independent $M_0>0$. □

We further extend our results to the ISS analysis of the perturbed system

$$\dot{x}(t)=A(\tfrac{t}{\varepsilon})x(t)+A_d(\tfrac{t}{\varepsilon})x(t-h(t))+B(\tfrac{t}{\varepsilon})w(t),\quad t\ge 0. \tag{5.34}$$

Assume **A5** - **A7** and let (2.10), (4.2) and (5.4) hold. By using arguments of Theorems 4.1 and 5.1, we arrive at the following ISS result:

Theorem 5.2. *Assume* **A5** - **A7**. *Given matrices A_{av}, A_{dav}, A_i ($i=1,\ldots,N$), A_{dj} ($j=1,\ldots,N_d$), B_l ($l=1,\ldots,\bar{N}$), and constants $\sigma>0$, $\sigma_d>0$, $\alpha>0$, $\varepsilon^*>0$ and $h_M>0$, let there exist $n\times n$ matrices $P>0$, $R>0$, $H_i>0$ ($i=1,\ldots,N$), $Q_j>0$ ($j=1,\ldots,N_d$), $S_1>0$, $R_1>0$, U and scalars $\lambda>0$, $\lambda_d>0$ and $b_l>0$ ($l=0,\ldots,\bar{N}$) that satisfy (5.9) and the following LMIs:*

$$\left[\begin{array}{cc|c}\Omega & \hat{\Phi}_{12} & \bar{\Theta}_{ijl}\\ * & \bar{\Phi}_{22} & \\ \hline & * & \Theta_2\end{array}\right]<0,\quad \begin{array}{l} i=1,\ldots,N,\ j=1,\ldots,N_d,\\ l=1,\ldots,\bar{N}\end{array} \tag{5.35}$$

with

$$\hat{\Phi}_{12}=\begin{bmatrix}P & -P & 0_{n,(N+N_d+4)n}\end{bmatrix}^T\begin{bmatrix}B_1 & \ldots & B_{\bar{N}} & 0_{n,n_w}\end{bmatrix},\\ \bar{\Theta}_{ijl}=\left[\begin{array}{cc}\multicolumn{2}{c}{\Theta_{ij}}\\ \hline 0_{\bar{N}n_w,n} & 0_{\bar{N}n_w,n}\\ \sqrt{\varepsilon^*}B_l^T\Lambda_1 & \sqrt{h_M}B_l^T\Lambda_2\end{array}\right], \tag{5.36}$$

where $\bar{\Phi}_{22}$ is given by (4.9), Ω is the symmetric matrix composed of (5.11), and Θ_{ij}, Θ_2, Λ_1 and Λ_2 are given by (5.12). Then system (5.34) is ISS for all $\varepsilon\in(0,\varepsilon^*]$ and $h(t)\in[0,h_M]$, meaning that there exists $M_0>0$ such that for all $\varepsilon\in(0,\varepsilon^*]$, $h(t)\in[0,h_M]$ and locally essentially bounded w, the solutions of system (5.34) initialized by $\phi\in C[-h_M,0]$ satisfy (4.10) with $|x(0)|^2$ changed by $\|\phi\|_C^2$. Moreover, given $\Delta>0$, the ellipsoid $\mathfrak{X}$ given by (4.11) is exponentially attractive (meaning that $x(t)$ approaches $\mathfrak{X}$ for $t\to\infty$) with a decay rate α for all $\phi\in C[-h_M,0]$ and essentially bounded w with $\mathrm{ess\,sup}_{t\ge0}|w(t)|\le\Delta$. In addition, if the LMIs (5.9) and (5.35) hold with $\alpha=0$, then system (5.34) is ISS for all $\varepsilon\in(0,\varepsilon^*]$ and (4.10) with $|x(0)|^2$ changed by $\|\phi\|_C^2$ holds with a small enough decay rate $\alpha=\alpha_0>0$.

Example 5.1. Consider a delayed version of the switched uncertain system considered in Example 2.2:

$$\dot{x}(t)=\\ \begin{cases}(\bar{A}_{d1}+\Delta\bar{A}_{d1}(\tfrac{t}{\varepsilon}))x(t-h(t)), & t\in[k\varepsilon,k\varepsilon+\beta\varepsilon),\\ (\bar{A}_{d2}+\Delta\bar{A}_{d2}(\tfrac{t}{\varepsilon}))x(t-h(t)), & t\in[k\varepsilon+\beta\varepsilon,(k+1)\varepsilon),\end{cases} \tag{5.37}$$

where $\varepsilon > 0$, $k = 0, 1, \ldots$ and $\beta \in (0, 1)$. Here $\bar{A}_{d1}$, $\Delta\bar{A}_{d1}(\frac{t}{\varepsilon})$, $\bar{A}_{d2}$ and $\Delta\bar{A}_{d2}(\frac{t}{\varepsilon})$ are equal to $\bar{A}_1$, $\Delta\bar{A}_1(\frac{t}{\varepsilon})$, $\bar{A}_2$ and $\Delta\bar{A}_2(\frac{t}{\varepsilon})$ given by (2.37) respectively. Then by using the functions χ_1 and χ_2 defined below (2.38), system (5.37) can be presented as (5.34) with $A(\tau) = 0$ and

$$\begin{array}{l} A_d(\tau) = \sum_{i=1}^{2} \chi_i(\tau)(\bar{A}_{di} + \Delta\bar{A}_{di}(\tau)), \\ \tau = \frac{t}{\varepsilon} \in [k, k+1), \quad k = 0, 1, \ldots, \end{array} \tag{5.38}$$

Choose $\beta = 0.4$ that leads to Hurwitz

$$A_{av} + A_{dav} = \beta\bar{A}_{d1} + (1-\beta)\bar{A}_{d2},$$

and

$$\begin{array}{l} \Delta A(\tau) = 0, \\ \Delta A_d(\tau) = \int_0^{\beta} \Delta\bar{A}_{d1}(\tau-\theta)d\theta + \int_{\beta}^{1} \Delta\bar{A}_{d1}(\tau-\theta)d\theta \end{array}$$

implying $\sigma = 0$ and $\sigma_d = g_1$ with g_1 given by (2.37). Note that $A_{av} = 0$ in this example is not Hurwitz. Since $A_d(\tau)$ is not continuous, the classical results with asymptotic methods (Hale & Lunel, 1990; Lehman & Weibel, 1999) are not applicable here.

The bounds (5.8) in this example can be found as

$$\begin{array}{l} \int_0^1 (\varepsilon^*\theta + h_M)\chi_1(\tau-\theta)d\theta \\ \leq 0.5\varepsilon^*[1-(1-\beta)^2] + h_M\beta \stackrel{\Delta}{=} f_{di}^*, \quad i = 1, 2, \\ \int_0^1 (\varepsilon^*\theta + h_M)\chi_2(\tau-\theta)d\theta \\ \leq 0.5\varepsilon^*(1-\beta^2) + h_M(1-\beta) \stackrel{\Delta}{=} f_{di}^*, \quad i = 3, 4. \end{array} \tag{5.39}$$

We verify the feasibility of LMIs (5.9) and (5.10) in the four vertices given by (2.39) with A_i changed by A_{di} (also here two vertices for $g_1 = 0$). We find the following upper bounds h_M that preserve the exponential stability of (5.37) either with a small enough decay rate (for $\alpha = 0$) or with a decay rate $\alpha = 0.005$ for all $\varepsilon \in (0, \varepsilon^*]$ and $h(t) \in [0, h_M]$ (to be compared with the results given by (2.41) for $h(t) \equiv 0$):

$$\begin{array}{llll} g_1 = 0, & \varepsilon^* = 0.05: & \alpha = 0, & h_M = 0.0516; \\ & & \alpha = 0.005, & h_M = 0.0259; \\ g_1 = 0.01, & \varepsilon^* = 0.0015: & \alpha = 0, & h_M = 0.0140; \\ & & \alpha = 0.005, & h_M = 0.0010. \end{array}$$

Thus, the perturbed switched uncertain system (5.37) is ISS for $\varepsilon \in (0, \varepsilon^*]$ and $h(t) \in [0, h_M]$.

6. Conclusions

This paper has presented a constructive method to averaging of linear systems with piecewise-continuous almost periodic coefficients. The introduced time-delay approach allows, for the first time, to derive efficient LMI-based conditions on the upper bound of the small parameter that preserves the stability. The method has been extended to persistently excited systems and to ISS analysis, as well as to averaging of systems with time-varying delay.

We have suggested some simple Lyapunov functionals for the transformed time-delay system, and we expect that in the future the results may be improved e.g. by using advanced Lyapunov-based methods (for example, by using augmented Lyapunov functionals with appropriate integral inequalities). The time-delay approach may be further extended to more general classes of systems and applied to various control problems that employ averaging. These problems may include vibrational control, stabilization by switching and extremum seeking (Scheinker & Krstić, 2017).

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