Robust Sampled – Data Control of Switched Affine Systems

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***Abstract*—This technical note considers the stabilization problem for switched affine systems with a sampled-data switching law. The switching law is assumed to be a function of the system state at sampling instants and the sampling interval may be subject to variations or uncertainty. We provide a robust switching law design that takes into account the sampled-data implementation and uncertainties. The problem is addressed from the continuous-time point of view. The method is illustrated by numerical examples.**

***Index Terms*—Linear matrix inequalities (LMIs), sampled-data control, switched affine systems, switching control.**

I. Introduction

In the last decade, the design of switching controllers has represented an important problem in the hybrid system community [3], [4], [15], [16], [21], [23], [25]. This problem is very challenging for the case of switched affine systems where, generally, the different subsystems do not share a common equilibrium point. The study is motivated by the wide range of applications to power electronics (see e.g., [5]). Different stabilization solutions exist in the literature based on the existence of stable convex combinations [2], [5], on optimal control methods [12], [20], or on the use of sliding modes [6], [24]. A characterization of the set of attainable equilibrium points using quadratic Lyapunov functions and conic switching laws has been provided in [2], [5].

The next phase toward practical application of switching control is to study its sampled-data implementation. For results on sampled-data control we point to the discrete-time methods in [1], [14], [19] the input delay-approach [8], [17], [22] and the impulsive system method [18]. Recently, increasing attention has been given to the sampled-data control of switched systems [9], [12]. This aspect is crucial in the switched affine system context since, due to sampling, one can no longer drive the state exponentially towards the equilibrium point, but only towards a limit cycle or to some attractive compact set containing the equilibrium. Moreover, in many practical applications timing imperfections due to sampling jitters or delays in switching law may affect the control performances. Recently, a discrete-time *analysis* has been provided in [12] based on control Lyapunov functions and the use of a nonlinear optimization solvers for quadratically constrained quadratic programs. However, it is still a problem to choose the control Lyapunov function so as to optimize the robustness with respect to sampling or the size of the attractive set around the desired equilibrium. Moreover, for an accurate study, it is of interest to exactly describe the continuous-time behavior. For systems with sampled-data switching control, this study is very challenging. The switching control is often described by a discrete-event system with transitions ruled by a partition of the state space. Then the sampling usually induces a delay in the discrete-event system variable. This may imply a mismatch in the control: one system mode may be active in other state zones then the one for which it has been designed. If not appropriately taken into account, the sampling may be a source of poor performance and even may lead to unbounded solutions.

The goal of the technical note is to present a continuous-time approach to sampled-data switching control design that ensures robustness with respect to sampling and to potential implementations imperfections (jitters, uncertainty etc.). Simple criteria are given to optimize the choice of Lyapunov functions.

The technical note is organized as follows: in Section II we formalize the problem under study and we provide simple conditions for practical stabilization based on the existence of stable convex combinations. The case of systems with parametric uncertainties is also presented. In Section III it is shown how the presented methodology may be improved by using switching Lyapunov functions. Numerical examples are given in Section IV. Preliminary results on the quadratic stabilization of uncertainty-free systems were presented in [13].

Notations

We denote the transpose of a matrix M by M^T. The symbol $*$ denotes a block that can be inferred by symmetry. By $\mathbf{I}$ (or $\mathbf{0}$) we denote the identity (or the null) matrix with the appropriate dimension. $|\cdot|$ denotes the Euclidean vector norm. For a square symmetric matrix, $M \succ 0$ ($M \prec 0$) indicates that M is positive (negative) definite. By $eig_{\min}(M)$ we denote the minimum eigenvalue of a square symmetric matrix M. For a given set $\mathcal{F}$, the symbol $co\mathcal{F}$ denotes the convex hull of the set. Given a finite set of index $\mathcal{S}$ and a set of scalars indexed by the elements in $\mathcal{S}$, $a_s, s \in \mathcal{S}$ we denote $\arg\min_{s\in\mathcal{S}} = \{s \in \mathcal{S} : a_s \leq a_r,\ \forall r \in \mathcal{S}\}$.

II. Practical Stabilization Based on Stable Convex Combinations

A. Problem Formulation

Given positive integers n and N, consider N matrices $A_1, A_2, \ldots, A_N \in \mathbb{R}^{n\times n}$ and N vectors $B_1, B_2, \ldots, B_N \in \mathbb{R}^n$. We are interested in the class of switched systems described by

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma,\ \forall t \in \mathbb{R}^+,\ x(0) = x_0 \in \mathbb{R}^n \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ represents the system state and $\sigma : \mathbb{R}^N \to \mathcal{I} = \{1, 2, \ldots, N\}$ a switching control. Consider the simplex $\Lambda = \left\{\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N] \in \mathbb{R}^N,\ \lambda_i \geq 0,\ \sum_{i=1}^N \lambda_i = 1\right\}$, the convex combinations of matrices $A(\lambda) = \sum_{i=1}^N \lambda_i A_i$, $B(\lambda) = \sum_{i=1}^N \lambda_i B_i, \lambda \in \Lambda$, and a subset of Λ associated to Hurwitz matrices: $\Lambda_H = \left\{\lambda \in \Lambda : \exists\, P_\lambda \succ 0, \text{ s.t. } A^T(\lambda)P_\lambda + P_\lambda A(\lambda) \prec 0\right\}$. It has been shown in [2], [5] that associated to each $\lambda \in \Lambda_H$ there exists an equilibrium point $x_e = -A^{-1}(\lambda)\, B(\lambda)$ to which (1) may be exponentially stabilized using a continuous-time switching function.

Without loss of generality, we may consider system's (1) stabilization with respect to the equilibrium point $x_e = \mathbf{0}$, i.e., we may consider that there exists $\lambda \in \Lambda_H$ s.t. $B(\lambda) = \mathbf{0}$. The stabilization with respect to an equilibrium point $x_e = -A^{-1}(\lambda)B(\lambda) \neq \mathbf{0}$, may always be reformulated as a null equilibrium point problem by considering the error state vector $e(t) = x(t) - x_e$ and the model representing the error dynamics $\dot{e}(t) = A_{\tilde{\sigma}(e)}e(t) + \tilde{B}_{\tilde{\sigma}(e)}$, with $\tilde{\sigma}(e) = \sigma(x)$ and

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$\tilde{B}_i = A_i x_e + B_i, \ \forall i \in \mathcal{I}$, for which $\sum_{i\in\mathcal{I}} \lambda_i \tilde{B}_i = \mathbf{0}$. Under the hypothesis that there exists $\lambda \in \Lambda_H$ s.t. $B(\lambda) = \mathbf{0}$, system (1) is stabilized to the origin using a switching law of the form

$$\sigma(x) \in arg\min_{i\in\mathcal{I}} x^T P (A_i x + B_i) \tag{2}$$

where P is a symmetric positive definite matrix satisfying $A(\lambda)^T P + PA(\lambda) \prec \mathbf{0}$.

In a sampled-data implementation, the values of system state are available at sample times $0 = t_0 < t_1 < \cdots < t_k < \cdots$, with $\lim_{k\to\infty} t_k = \infty$. The sampling interval $T_k := t_{k+1} - t_k$ may be unknown and time-varying, with $0 < T_k \le T_{\max}$ where $T_{\max}$ is a known bound. In this technical note we consider that the implemented switching law is a function of x_k, although an implementation based on x_{k-1} may also be of interest in practical applications. With a sampled-data implementation, the switched system becomes

$$\dot{x}(t) = A_{\sigma(x_k)} x(t) + B_{\sigma(x_k)}, \quad \forall t \in [t_k, t_{k+1}) \tag{3}$$

where $\sigma(x_k)$ is constant for all $t \in [t_k, t_{k+1})$ and represents the discrete-time implementation of the switching law (2), i.e.

$$\sigma(x_k) \in arg\min_{i\in\mathcal{I}} x_k^T P (A_i x_k + B_i), \quad x_k := x(t_k). \tag{4}$$

Note that with the sampled-data control sliding modes may not occur since $T_k > 0$ and only one system mode is actif for $t \in [t_k, t_{k+1})$. However, due to the sampling, for $x_0 = x_e = 0$ and $T \in [0, t_1)$ we generally have $x_e \neq x(T) = \int_0^T e^{A_{\sigma(0)} s} ds B_{\sigma(0)}$, i.e., the equilibrium point x_e is no longer invariant. Moreover, the system state cannot be driven to the equilibrium point of the continuous-time switched system, but only to a neighborhood of the equilibrium, whose size may grow with the sampling interval. Our goal is to provide methods for the design of sampled-data switching laws $\sigma(x_k)$ that are practically stabilizing the system (3) to a ball, i.e., to find switching laws of the form (4) that guarantee that $x(t)$ is exponentially converging to the ball $|x|^2 < CT_{\max}$ when $t \to \infty$, where $C = C(T_{\max})$ is positive and satisfies $C(T_{\max})T_{\max} \to 0$ as $T_{\max} \to 0$. The latter recovers the exponential stability of the system under the continuous-time switching, where $T_{\max} = 0$.

B. Design Conditions

In this subsection, simple design conditions for switching law (4) are provided by requiring the decrease of the function $V(x(t)) = x^T(t)Px(t)$ with respect to its value at sampling instants. The idea is to use a continuous function $w : \mathbb{R}^+ \to \mathbb{R}$, $k \in \mathbb{N}$, differentiable over $[t_k, t_{k+1})$, with $w(t_k) = 0$ and $w(t) \ge 0, \ \forall t \in (t_k, t_{k+1}), \forall\, k \in \mathbb{N}$, satisfying the following condition:

$$\dot{V}(x(t)) + \dot{w}(t) + 2\gamma (V(x(t)) + w(t)) < \beta T_{\max} \tag{5}$$

$\forall t \in [t_k, t_{k+1})$ for some $\gamma > 0, \beta > 0$. By the comparison principle, (5) yields

$$V(x(t)) < e^{-2\gamma t}\left(V(x(0)) + w(0) - \frac{\beta}{2\gamma}T_{\max}\right) + \frac{\beta}{2\gamma}T_{\max} - w(t), \quad \forall\, t > 0$$

i.e.

$$V(x(t)) < e^{-2\gamma t} V(x(0)) + \frac{\beta}{2\gamma}T_{\max}, \quad \forall t > 0 \tag{6}$$

which means that $x(t)$ exponentially converges to the attractive ellipsoid given by

$$\mathcal{E}^{T_{\max}} := \left\{ x \in \mathbb{R}^n : x^T P x \le T_{\max}\frac{\beta}{2\gamma} \right\}. \tag{7}$$

By S-procedure, (5) leads to the following implication:

$$V(x(t)) + w(t) - \frac{\beta}{2\gamma}T_{\max} > 0 \Rightarrow \dot{V}(x(t)) + \dot{w}(t) < 0. \tag{8}$$

Hence, (5) guarantees that $V(x_{k+1}) < V(x_{k+1}) + w(t_{k+1}^-) < V(x_k)$ whenever $x(t) \in \mathbb{R}^n - \mathcal{E}^{T_{\max}}, \ \forall t \in [t_k, t_{k+1})$. Then the function $V(x) = x^T P x$ may be related to a Lyapunov function for the discrete-time system representation. For the particular case where

$$w(t) = (t_{k+1} - t)\int_{t_k}^{t} e^{2\gamma(s-t)} \dot{x}^T(s) U_{\sigma(x_k)} \dot{x}(s) ds \ge 0 \tag{9}$$

$\forall t \in [t_k, t_{k+1})$ with $U_i \succ 0, \ \forall i \in \mathcal{I}, \forall t \in [t_k, t_{k+1})$, the function $\bar{V} = V + w$ becomes a Lyapunov–Krasovskii functional as in [8]. Stabilization conditions are presented in the following theorem, as shown in (10) at the bottom of the page.

Theorem 1: Consider system (3), (4) with $T_k \le T_{\max}$ and a given scalar tuning parameter $\gamma > 0$. Assume that $\Lambda_H \neq \{\varnothing\}$, and that there exists a $\lambda \in \Lambda_H$ s.t. $B(\lambda) = 0$.

$$\begin{aligned}
&\begin{bmatrix} T_{\max}\left(U_i - P_3 - P_3^T\right) & T_{\max}\left(P_3^T \tilde{A}_i^j - P_2\right) & T_{\max} P_3^T \tilde{B}_i^j \\ * & K_i^j(\lambda) & T_{\max} P_2^T \tilde{B}_i^j + P\bar{B}_i^j \\ * & * & -\beta T_{\max} - \beta_0 \end{bmatrix} \\
&\prec \mathbf{0}, \\
&\begin{bmatrix} -T_{\max}\left(P_3 + P_3^T\right) & T_{\max}\left(P_3^T \tilde{A}_i^j - P_2\right) & T_{\max} P_3^T \tilde{B}_i^j & \mathbf{0} \\ * & K_i^j(\lambda) & T_{\max} P_2^T \tilde{B}_i^j + P\bar{B}_i^j & -T_{\max}\Psi_i(\lambda) \\ * & * & -\beta T_{\max} - \beta_0 & T_{\max} B_i^T P \\ * & * & * & -T_{\max} U_i e^{-2\gamma T_{\max}} + T_{\max}^2 \Psi_i(\lambda) \end{bmatrix} \\
&\prec \mathbf{0}, \\
&K_i^j(\lambda) = \left(\left(\tilde{A}_i^j\right)^T P_2 + P_2^T \tilde{A}_i^j\right) T_{\max} + \left(A(\lambda) + \bar{A}_i^j\right)^T P + P\left(A(\lambda) + \bar{A}_i^j\right) + 2\gamma P, \\
&\tilde{A}_i^j = A_i + \bar{A}_i^j, \ \tilde{B}_i^j = B_i + \bar{B}_i^j, \\
&\Psi_i(\lambda) = (A(\lambda) - A_i)^T P + P(A(\lambda) - A_i), \quad \forall i \in \mathcal{I}, \ j = 1, \ldots, m
\end{aligned} \tag{10}$$

(i) Let there exist matrices $P, U_i \succ 0$ in $\mathbb{R}^{n\times n}, i \in \mathcal{I}$, a scalar $\beta > 0$ such that

$$\begin{bmatrix} \Omega_i^1(\lambda,0) & \Omega_i^2(0) \\ * & \Omega_i^3(0) \end{bmatrix} \prec \mathbf{0} \tag{11}$$

$$\begin{bmatrix} \Omega_i^1(\lambda,T_{\max}) & \Omega_i^2(T_{\max}) & -T_{\max}\Psi_i(\lambda) \\ * & \Omega_i^3(T_{\max}) & T_{\max}B_i^T P \\ * & * & \Omega_i^4(T_{\max},\lambda) \end{bmatrix} \prec \mathbf{0} \tag{12}$$

$\forall i \in \mathcal{I}$, where

$$\begin{aligned}
\Omega_i^1(\lambda,\tau) &= A^T(\lambda)P + PA(\lambda) + 2\gamma P + (T_{\max}-\tau)A_i^T U_i A_i,\\
\Omega_i^2(\tau) &= (T_{\max}-\tau)A_i^T U_i B_i,\\
\Omega_i^3(\tau) &= (T_{\max}-\tau)B_i^T U_i B_i - \beta T_{\max}\mathbf{I},\\
\Omega_i^4(\tau,\lambda) &= -\tau U_i e^{-2\gamma T_{\max}} + \tau^2\Psi_i(\lambda),\\
\Psi_i(\lambda) &= (A(\lambda)-A_i)^T P + P(A(\lambda)-A_i).
\end{aligned}$$

Then (3), (4) is practically stabilizable to the set $\mathcal{E}^{T_{\max}}$, that is $x(t)$ is exponentially attracted to the ellipsoid $\mathcal{E}^{T_{\max}}$ for $t\to\infty$.

(ii) Under conditions of (i), if for some $k\in\mathbb{N}$, $x(t_k)\in\mathcal{E}^{T_{\max}}$ then $x(t)\in\mathcal{E}^{T_{\max}}$ for all $t\geq t_k$.

Proof:

(i) We will show that (11), (12) imply (5) with $V = x^T P x$ and $w(t)$ given by (9).

Assume that $\sigma(x_k) = i$ and denote $\theta(t) = (x(t)-x_k)\,\tau^{-1}(t)$, with $\tau(t) := t - t_k$, for all $t\in[t_k,t_{k+1})$. Using the Jensen inequality it is seen that

$$\dot{w}(t) + 2\gamma w(t) \leq (T_k-\tau(t))\,(A_i x(t)+B_i)^T U_i(A_i x(t) + B_i) - \tau(t)\theta^T(t)U_i\theta(t)e^{-2\gamma T_{\max}}.$$

Furthermore, $\dot{V}(x(t)) = 2x^T(t)P(A_i x(t)+B_i),\ \forall t\in[t_k,t_{k+1})$. Therefore, (5) holds with $0<T_k\leq T_{\max}$ and $\sigma = i$ if

$$\begin{aligned}
&x^T\left(A_i^T P + PA_i + 2\gamma P + (T_{\max}-\tau)A_i^T U_i A_i\right)x\\
&+ 2x^T\left(PB_i + (T_{\max}-\tau)A_i^T U_i B_i\right) + (T_{\max}-\tau)B_i^T U_i B_i\\
&- \beta T_{\max}\mathbf{I} - \tau\theta^T U_i e^{-2\gamma T_{\max}}\theta < 0
\end{aligned} \tag{13}$$

for all $\tau\in[0,T_{\max}]$, where $x = x(t), \theta = \theta(t)$. Note that $\sigma = i$ for all $t\in[t_k,t_{k+1})$ if x_k satisfies the relation $2x_k^T P(A_j x_k + B_j) - 2x_k^T P(A_i x_k + B_i) \geq 0,\ \forall j\in\mathcal{I}$. Multiplying the latter inequality by λ_j, and summing for $j\in\mathcal{I}$ yields $2x_k^T P(A(\lambda)-A_i)x_k - 2x_k^T PB_i \geq 0$ i.e.

$$\begin{bmatrix} x\\ 1\\ \theta\end{bmatrix}^T \begin{bmatrix} \Psi_i(\lambda) & -PB_i & -\tau(t)\Psi_i(\lambda)\\ * & \mathbf{0} & \tau(t)B_i^T P\\ * & * & \tau^2(t)\Psi_i(\lambda)\end{bmatrix}\begin{bmatrix} x\\ 1\\ \theta\end{bmatrix} \geq 0 \tag{14}$$

where the relation $x_k = x - \tau\theta$ has been used. Since $\sigma = i$ whenever (14) holds, using the S – procedure by adding the left side of (14) to the left side of (13), we arrive to $z^T\Omega_i(\tau,\tau^2)z < 0,\ \forall\tau\in[0,T_{\max}]$ where $z = [\,x^T \quad 1 \quad \theta^T\,]^T$

$$\Omega_i(\tau,\tau^2) = \begin{bmatrix} \Omega_i^1(\lambda,\tau) & \Omega_i^2(\tau) & -\tau\Psi_i(\lambda)\\ * & \Omega_i^3(\tau) & \tau B_i^T P\\ * & * & \Omega_i^4(T_{\max},\lambda)\end{bmatrix}.$$

Since

$$\Omega_i(\tau,\tau^2) \in co\left\{\Omega_i(0,0), \Omega_i(T_{\max},0), \Omega_i(T_{\max},T_{\max}^2)\right\},\quad \forall\tau\in[0,T_{\max}]$$

$i\in\mathcal{I}$, then

$$z^T\Omega_i(0,0)z < 0 \tag{15}$$

$$z^T\Omega_i(T_{\max},0)z < 0,\ z^T\Omega_i(T_{\max},T_{\max}^2)z < 0 \tag{16}$$

$i\in\mathcal{I}$ are sufficient to guarantee that the function $V(x)$ satisfies the condition (5) along the solutions of system (3) with the switching law (4). Note that $\Omega_i(T_{\max},T_{\max}^2)\prec 0$ implies $\Omega_i(T_{\max},0)\prec 0$. Therefore (11), (12) yield (15), (16).

(ii) If (11), (12) are feasible, then (5) and, thus, (8) are satisfied. The inequality $V(x(t)) \leq \beta T_{\max}/2\gamma - w(t)$ implies $x(t)\in\mathcal{E}^{T_{\max}}$ since $w(t)\geq 0$. Let $x(t_k)\in\mathcal{E}^{T_{\max}}$. Assume that for some $t_{lim}\in[t_k,t_{k+1})$ we have $V(x(t_{lim})) = \beta T_{\max}/2\gamma - w(t_{lim})$ and that $V(x(s))\geq\beta T_{\max}/2\gamma - w(s)$ for all $s\in(t_{lim},t]\subset(t_k,t_{k+1})$. Then $\dot{V}(x(s)) + \dot{w}(s) < 0$ for $s\in(t_{lim},t]$, which after integration yields $V(x(t)) + w(t) < V(x(t_{lim})) + w(t_{lim}) = \beta T_{\max}/2\gamma$. The latter implies that $x(t)\in\mathcal{E}^{T_{\max}}$, since $w(t)\geq 0$. Hence, the system solution cannot exit $\mathcal{E}^{T_{\max}}$ between the sampling instants. □

Remark 1: The parameter γ from Theorem 1 corresponds to the system decay rate. For fixed γ, conditions (11), (12) represent LMIs. The optimization of the decay rate may be addressed by combining LMI-based methods for (11), (12) with a line search on γ.

Remark 2: For the case of $B_i = 0,\ \forall i\in\mathcal{I}$, the conditions from the previous theorem are reduced to the existence of $P, U_i\succ 0$ in $\mathbb{R}^{n\times n},\ i\in\mathcal{I}$ such that

$$A^T(\lambda)P + PA(\lambda) + 2\gamma P + T_{\max}A_i^T U_i A_i \prec \mathbf{0},$$

$$\begin{bmatrix} \Omega_i^1(\lambda,T_{\max}) & -T_{\max}\Psi_i(\lambda)\\ * & \Omega_i^4(\lambda,T_{\max})\end{bmatrix} \prec \mathbf{0} \tag{17}$$

$\forall i\in\mathcal{I}$, are feasible. In this case the zero solution of (3), (4) is exponentially stable.

Remark 3: Note that solutions of (11), (12) depend on $T_{\max}$. Given $T_{\max}$, the feasibility of (11), (11) with some $P(T_{\max}), \gamma(T_{\max}), \beta(T_{\max})$ guarantees that for $t\to\infty$ the trajectories of the resulting system approach to the ball $|x|^2 < C(T_{\max})T_{\max}$, where

$$C(T_{\max}) = \beta(T_{\max})\left[2eig_{\min}(P(T_{\max}))\gamma(T_{\max})\right]^{-1}.$$

LMIs (11), (12) for $T_{\max}\to 0$ are reduced to

$$A^T(\lambda)P + PA(\lambda) + 2\gamma P \prec \mathbf{0} \tag{18}$$

which guarantees the exponential stability of the continuous-time system. For small enough $T_{\max}$, the feasibility of (18) implies the feasibility of (11), (12):

Corollary 1: Given $\lambda\in\Lambda_H$ and $\gamma_0 > 0$, let $P_0\succ 0$ be the solution of (18). Then for any $\beta_0 > 0$ there exists a sufficiently small $T_{\max}^0 > 0$ such that (11), (12) are feasible with $P = P_0, U_i = \mathbf{I},\ \forall i\in\mathcal{I},\ \gamma = \gamma_0$ and $\beta = \beta_0 > 0$ for all $T_{\max}\leq T_{\max}^0$. Thus, for all $T_{\max}\leq T_{\max}^0$ the solutions $x(t)$ of (3), (4) are exponentially attracted to the ball $|x(t)|^2 < C_0 T_{\max}$ as $t\to\infty$, where (the $T_{\max}$ – independent) constant C_0 is given by $C_0 = \beta_0\left[2eig_{\min}(P_0)\gamma_0\right]^{-1}$.

Proof: Given P_0, there exists $\beta_0 > 0$ s.t. $B_i^T B_i < \beta_0\mathbf{I}$ and $B_i^T P_0 P_0 B_i < \beta_0\mathbf{I},\ \forall i\in\mathcal{I}$. When (18) holds, there exists a sufficiently small $T_{\max}^0 > 0$ such that

$$\begin{aligned}
E_i^1 &= A^T(\lambda)P_0 + P_0 A(\lambda) + 2\gamma_0 P_0\\
&\quad + T_{\max}\left(A_i^T A_i - A_i^T B_i\left(B_i^T B_i - \beta_0\mathbf{I}\right)^{-1} B_i^T A_i\right) \prec \mathbf{0}
\end{aligned}$$

$-Ie^{-2\gamma_0 T_{\max}} + T_{\max}\Psi_i + P_0 B_i \beta_0^{-1} B_i^T P_0 \prec \mathbf{0}$ and

$$E_i^2 = A^T(\lambda)P_0 + P_0 A(\lambda) + 2\gamma_0 P_0 - T_{\max} \cdot \begin{bmatrix} \Psi_i \\ \mathbf{0} \end{bmatrix}^T \begin{bmatrix} -Ie^{-2\gamma_0 T_{\max}} + T_{\max}\Psi_i & P_0 B_i \\ * & -\beta_0 \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \Psi_i \\ \mathbf{0} \end{bmatrix} \prec \mathbf{0}.$$

$\forall T_{\max} \leq T_{\max}^0$ and $i \in \mathcal{I}$. By Schur complements and classical manipulation, $E_i^1 \prec \mathbf{0}$ and $E_i^2 \prec \mathbf{0}$ lead to (11) and (12), respectively (with $P = P_0, \gamma = \gamma_0, \beta = \beta_0, U_i = \mathbf{I}$). □

Remark 4: Theorem 1 guarantees that V is decreasing outside $\mathcal{E}^{T_{\max}}$ with respect to its values at sampling times only. For the case when x_k is outside $\mathcal{E}^{T_{\max}}$ there may exist time instants $t^a, t^b \in (t_k, t_{k+1}), t^a < t^b$ such that $x(t^a) \in \mathcal{E}^{T_{\max}}$ while $x(t^b) \notin \mathcal{E}^{T_{\max}}$. However, once x_k is in the attractive ellipsoid, the state will stay there as shown by (ii) of Theorem 1.

C. LMI Conditions for Systems With Parametric Uncertainties

The conditions (11), (12) presented in the previous subsection are not affine in the systems parameters. This makes difficult their use for time-varying parametric uncertainties. The conditions may be modified to cope with this important case using the descriptor method [7]. Consider the system

$$\dot{x} = \tilde{A}_{\sigma(x_k)}x + \tilde{B}_{\sigma(x_k)}, \quad \forall\, t \in [t_k, t_{k+1}) \tag{19}$$

where for all $i \in \mathcal{I}$ the system matrices have the form $\tilde{A}_i = A_i + \sum_{j=1}^m \alpha_i^j(t)\bar{A}_i^j$, $\tilde{B}_i = B_i + \sum_{j=1}^m \alpha_i^j(t)\bar{B}_i^j$ with $\alpha_i^j(t) \geq 0$, $\sum_{j=1}^m \alpha_i^j(t) = 1$ and $m \in \mathbb{N}^+$. A_i, B_i represent the nominal parameters while $\bar{A}_i^j, \bar{B}_i^j$ represent the vertices of perturbations with respect to the nominal values. Note that for the uncertain case, even in the continuous-time case, we only have practical stabilization to a ball around the origin and not exponential stabilization. Robust practical stabilization conditions for the sampled-data case are given as shown in (20) at the bottom of the page.

Corollary 2: Consider (19), (4) with $T_k \leq T_{\max}$. Assume that $\Lambda_H \neq \{\varnothing\}$ and that there exists a $\lambda \in \Lambda_H$ such that $B(\lambda) = 0$. Given the scalar tuning parameter $\gamma > 0$, let there exist matrices $P, U_i \succ 0, i \in \mathcal{I}$, $P_2, P_3 \in \mathbb{R}^{n\times n}$ and scalars $\beta, \beta_0 > 0$ such that conditions (10) are feasible. Then for $t \to \infty$ solutions $x(t)$ of (19), (4) are exponentially attracted to the ellipsoid

$$\bar{\mathcal{E}}^{T_{\max}} := \left\{ x \in \mathbb{R}^n : x^T P x \leq \frac{\beta T_{\max} + \beta_0}{2\gamma} \right\}. \tag{21}$$

Proof: Following the arguments in the proof of Theorem 1 with $V(x) = x^T P x$ and w given by (9), a sufficient condition for (5) is

$$2x^T P\left(\tilde{A}_i x + \tilde{B}_i\right) + 2\gamma x^T P x + (T_{\max} - \tau)\dot{x}^T U_i \dot{x} - \tau\theta^T U_i \theta e^{-2\gamma T_{\max}} - \beta T_{\max} - \beta_0 < 0 \tag{22}$$

whenever $\sigma(x_k) = i$. Using the descriptor method

$$0 = 2T_{\max}\left(x^T P_2^T + \dot{x}^T P_3^T\right)\left(\tilde{A}_i x + \tilde{B}_i - \dot{x}\right). \tag{23}$$

Summing (14), (22), (23) and using convexity arguments, the conditions (10) are obtained. □

Remark 5: The set of matrix inequalities (10) for $T_{\max} \to 0$ are reduced to the conditions

$$\begin{bmatrix} \left(A(\lambda) + \bar{A}_i^j\right)^T P + P\left(A(\lambda) + \bar{A}_i^j\right) + 2\gamma P & P\bar{B}_i^j \\ * & -\beta_0 \end{bmatrix} \prec \mathbf{0} \tag{24}$$

$\forall i \in \mathcal{I},\ j \in \{1, \ldots, m\}$, which ensures that under the continuous-time switching law, the (uncertain) system state $x(t)$ is exponentially attracted to the ellipsoid $\bar{\mathcal{E}}^0 := \left\{x \in \mathbb{R}^n : x^T P x \leq \beta_0/2\gamma\right\}$ as $t \to \infty$ (and not to the equilibrium point).

III. Practical Stabilization Based on Switched Lyapunov Functions

In this section we show how the presented methodology may be generalized for systems that do not admit a stable convex combination by using switched Lyapunov functions. We extend the methodology in [15], where the switched linear systems were studied, to the case of

$$\begin{bmatrix} A^T(\lambda^j)P_j + P_j A(\lambda^j) + 2\gamma P_j + T_{\max} A_i^T U_i A_i + \sum_{r=1}^M \mu_{j,r}\left(P_r - P_j\right) & T_{\max} A_i^T U_i B_i \\ * & T_{\max}\left(B_i^T U_i B_i - \beta \mathbf{I}\right) \end{bmatrix} \prec \mathbf{0},$$

$$\begin{bmatrix} A^T(\lambda^j)P_j + P_j A(\lambda^j) + 2\gamma P_j + \sum_{r=1}^M \mu_{j,r}\left(P_r - P_j\right) & \mathbf{0} & -T_{\max}\left(\Psi_{i,j} + \sum_{r=1}^M \mu_{j,r}\left(P_r - P_j\right)\right) \\ * & -T_{\max}\beta \mathbf{I} & T_{\max} B_i^T P_j \\ * & * & \Theta_{i,j} + \sum_{r=1}^M \mu_{j,r}\left(P_r - P_j\right) T_{\max}^2 \end{bmatrix} \prec \mathbf{0},$$

$$\Psi_{i,j} = \left(A(\lambda^j) - A_i\right)^T P_j + P_j\left(A(\lambda^j) - A_i\right), \quad \Theta_{i,,j} = -T_{\max} U_i e^{-2\gamma T_{\max}} + T_{\max}^2 \Psi_{i,j}, \quad (i,j) \in \mathcal{I} \times \mathcal{J} \tag{20}$$

switched affine systems under the sampled-data measurements. Consider the sets

$$\Lambda_e = \{\lambda \in \Lambda : \exists\, x_e \in \mathbb{R}^n,\ \ s.t.\ A(\lambda)x_e = -B(\lambda)\},$$
$$X_e = \{x_e \in \mathbb{R}^n : \exists\, \lambda \in \Lambda_e,\ \ s.t.\ A(\lambda)x_e = -B(\lambda)\}.$$

Similarly to the case of quadratic stabilization, it may be shown that the stabilization to an equilibrium point $x_e \in X_e$ may always be reformulated as a stabilization problem with a null equilibrium point by considering the dynamics of the error vector $e = x - x_e$, $\dot{e}(t) = A_{\tilde{\sigma}(e(t_k))}e(t) + \tilde{B}_{\tilde{\sigma}(e(t_k))}$, $\forall t \in [t_k, t_{k+1})$ with $\tilde{\sigma}(e(t_k)) = \sigma(x_k)$ and $\tilde{B}_i = A_i x_e + B_i$, $\forall i \in \mathcal{I}$, for which $\sum_{i\in\mathcal{I}} \lambda_i \tilde{B}_i = \mathbf{0}$. For the case when the barycentric coordinates λ are not unique, i.e., there exist M vectors $\lambda^j \in \Lambda_e$, $j \in \mathcal{J} = \{1, \ldots, M\}$ such that $B(\lambda^j) = 0$, $j \in \mathcal{J}$, we may use switched Lyapunov functions. To each $j \in \mathcal{J}$ we associate a quadratic form $x^T P_j x$, where $P_j = P_j^T \succ 0$. For $x_k \in \mathbb{R}^n$ the quadratic forms with the minimum value are indicated by the index set

$$J_{\min}(x_k) = \left\{ j \in \mathcal{J} : x_k^T P_j x_k \le x_k^T P_r x_k,\ \forall r \in \mathcal{J} \right\}. \tag{25}$$

Consider a pair

$$(\sigma(x_k), j^*(x_k)) \in \arg \min_{(i,j)\in\mathcal{I}\times\mathcal{J}_{\min}(x_k)} x_k^T P_j (A_i x_k + B_i). \tag{26}$$

We define $\sigma(x_k)$ as a switching function. The function $j^*(x_k)$ associated to $\sigma(x_k)$ is used for indicating the active index in the switched Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$

$$V(x,t) = x^T P_{j^*(x_k)} x,\ \ \forall\, t \in [t_k, t_{k+1}). \tag{27}$$

The following theorem provides practical stabilization conditions:

Theorem 2: Consider system (3), (26) with $T_k \le T_{\max}$ and scalar tuning parameters $\gamma > 0, \mu_{j,r} > 0,\ j, r \in \mathcal{J}$. Assume that $\Lambda_e \ne \{\varnothing\}$ and that there exists vectors $\lambda^j \in \Lambda_e$, such that $B(\lambda^j) = \mathbf{0}$, $j \in \mathcal{J}$. Let there exist matrices $U_i, P_j \succ 0$, $(i,j) \in \mathcal{I} \times \mathcal{J}$ in $\mathbb{R}^{n\times n}$ and a scalar $\beta > 0$ such that the set of matrix inequalities (20) (given at the bottom of the page) hold. Then for $t \to \infty$ solutions $x(t)$ of (3), (26) are exponentially attracted to the ball

$$\mathcal{B}^{T_{\max}} := \left\{ x \in \mathbb{R}^n : \min_{j\in\mathcal{J}} \{eig_{\min}(P_j)\}|x|^2 \le \frac{\beta T_{\max}}{2\gamma} \right\}. \tag{28}$$

Proof: The definition of switching and Lyapunov functions in (26) and (27) respectively guarantees, for the fixed $\sigma(x_k)$, a unique definition for the Lyapunov function as the function $x^T P_j x$ with the index $j = j^*(x_k)$ associated to $\sigma(x_k)$. V is piecewise differentiable for any $t \in [t_k, t_{k+1})$ and does not grow in the jumps, i.e.

$$V(x_k, t_k^+) = x_k^T P_{j^*(x_k)} x_k \le x_k^T P_{j^*(x_{k-1})} x_k = V(x_k, t_k^-). \tag{29}$$

Moreover, it is positive definite, radially unbounded and zero at $x = \mathbf{0}$. As in the quadratic case, we use a continuous function $w(t)$ given by (9) and show that (20) imply (5), where $V(x(t))$ is changed by $V(x(t), t)$. From (5), (29) it follows that for $t \in [t_k, t_{k+1})$

$$\begin{aligned}
&V(x(t), t)\\
&\le e^{-2\gamma(t-t_k)} V\left(x_k, t_k^+\right) + \beta T_{\max} \int_{t_k}^{t_{k+1}} e^{-2\gamma(t-s)} ds\\
&\le e^{-2\gamma(t-t_k)} V\left(x_k, t_k^-\right) + \beta T_{\max} \int_{t_k}^{t_{k+1}} e^{-2\gamma(t-s)} ds\\
&\le e^{-2\gamma(t-t_{k-1})} V\left(x(t_{k-1}), t_{k-1}\right) + \beta T_{\max} \int_{t_{k-1}}^{t} e^{-2\gamma(t-s)} ds\\
&\le \ldots \le e^{-2\gamma(t-t_0)} V(x_0, 0) + \beta T_{\max} \int_0^t e^{-2\gamma(t-s)} ds
\end{aligned}$$

i.e., that $x(t)$ is exponentially attracted to $\mathcal{B}^{T_{\max}}$ as $t \to \infty$.

The inequality (5) is satisfied for $(\sigma(x_k), j^*(x_k)) = (i,j) \in \mathcal{I} \times \mathcal{J}$ if

$$2x^T(t)P_j\left(A_i x(t) + B_i\right) + 2\gamma x^T(t) P_j x(t) + \dot{w}(t) + 2\gamma w(t) \le \beta T_{\max}$$

for all $t \in [t_k, t_{k+1})$ whenever

$$x_k^T (P_j - P_r) x_k \le 0,\ \ \forall r \in \mathcal{J} \tag{30}$$
$$2x_k^T P_j \left(A_i x_k + B_i\right) \le 2x_k^T P_j \left(A_l x_k + B_l\right),\ \forall l \in \mathcal{I}. \tag{31}$$

Following arguments of Theorem 1, a sufficient condition for (5) to hold with $0 < t_{k+1} - t_k \le T_{\max}$ and $(\sigma(x_k), j^*(x_k)) = (i,j)$ is:

$$\begin{aligned}
&x^T\left(A_i^T P_j + P_j A_i + 2\gamma P_j + (T_{\max} - \tau) A_i^T U_i A_i\right) x\\
&\quad + 2x^T\left(P_j B_i + (T_{\max} - \tau) A_i^T U_i B_i\right) + (T_{\max} - \tau) B_i^T U_i B_i\\
&\quad - \beta T_{\max}\mathbf{I} - \tau\theta^T U_i \theta e^{-2\gamma T_{\max}} < 0
\end{aligned} \tag{32}$$

for all $\tau \in [0, T_{\max}]$, with $\theta(t) = (x(t) - x_k)\,\tau^{-1}(t)$, $x = x(t)$. Multiplying (31) by λ_l^j and (30) by $\mu_{j,r}$, summing and expressing $x_k = x(t) - \tau(t)\theta(t)$ leads to

$$\begin{aligned}
&x^T\left(\Psi_{i,j} + \sum_{r=1}^{M} \mu_{j,r}(P_r - P_j)\right) x\\
&\quad - 2x^T P_j B_i - 2\tau x^T \left(\Psi_{i,j} + \sum_{r=1}^{M} \mu_{j,r}(P_r - P_j)\right)\theta\\
&\quad + 2\tau B_i^T P_j \theta + \tau^2 \theta^T \left(\Psi_{i,j} + \sum_{r=1}^{M} \mu_{j,r}(P_r - P_j)\right)\theta \ge 0.
\end{aligned} \tag{33}$$

To end the proof, add (33) to (32) and use convexity arguments. □

Remark 6: Theorem 2 allows to reduce the conservatism of the stabilization conditions by using switching Lyapunov functions (27). It may be used to stabilize a switched affine system to equilibrium points x_e in the set X_e for which $A(\lambda)$ is not Hurwitz. However, the resulting stabilization conditions represent essentially more complicated matrix inequalities with many additional tuning parameters $\mu_{j,r} > 0,\ j, r \in \mathcal{J}$ comparatively to the conditions of Theorem 1 and Corollary 2. They represent a non-convex optimization problem due to the products of variables P_j and $\mu_{j,r}$. It is similar to the problem of choosing the Metzler matrix in the design conditions related to $\min$ switching strategies [10] for switched linear systems with continuous-time switching laws. Such Bilinear Matrix Inequalities (BMIs) may be addressed via LMI-based numerical approaches by combining the path-following method in [11] and the direct iteration with P_j as variables [15]. However, for the moment, there is no guarantee to find the global optimal solution.

Remark 7: There are two main sources of conservatism for the results in the technical note. The first one stems from the choice of (quadratic or switching) Lyapunov functions. Another is related to the form of $w(t)$. Using additional terms in $w(t)$ may reduce the conservatism of the design conditions, but on the account of the computational complexity.

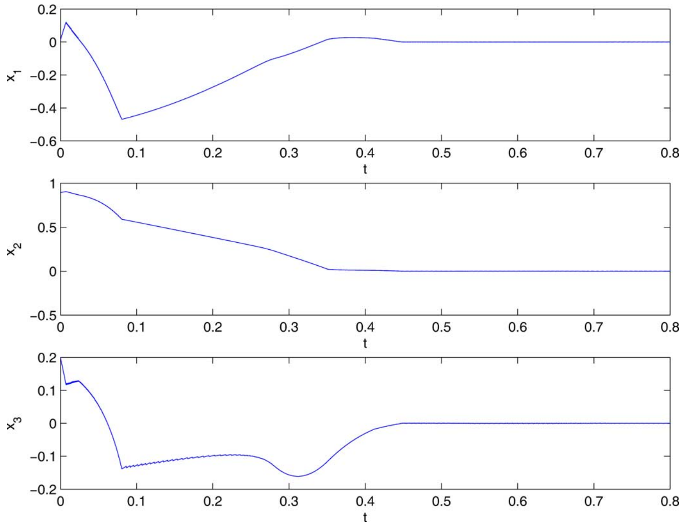

Fig. 1. Example 1: evolution of system states under the control law based on Theorem 1 with a fixed sampling interval $T = 3.2 \cdot 10^{-4}$.

IV. Numerical Examples

Example 1: Quadratic Stabilization: Consider a switched affine system (1) consisting of four affine subsystems with $x(t) \in \mathbb{R}^3$ and the following matrices [2]:

$$A_1 = \begin{bmatrix} 4.15 & -1.06 & -6.7 \\ 5.74 & 4.78 & -4.68 \\ 26.38 & -6.38 & -8.29 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -3.2 & -7.6 & -2 \\ 0.9 & 1.2 & -1 \\ 1 & 6 & 5 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 5.75 & -16.48 & 2.41 \\ 9.51 & -9.49 & 19.55 \\ 16.19 & 4.64 & 14.05 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} -12.38 & 18.42 & 0.54 \\ -11.90 & 3.24 & -16.32 \\ -26.5 & -8.64 & -16.6 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \; B_2 = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \; B_3 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \; B_4 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Each individual subsystem is unstable. For $\lambda_1 = 0.15$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$ and $\lambda_4 = 0.35$, the $A(\lambda)$ is Hurwitz and $B(\lambda) = \mathbf{0}$. Using Theorem 1 we find that the system is practically stabilizable under variable sampling with $T_k \leq T_{\max} \leq 3.2 \cdot 10^{-4}$. Conditions (11), (12) are found to be feasible with

$$P = \begin{bmatrix} 0.1 & -0.02 & 0 \\ -0.02 & 0.15 & 0.02 \\ 0 & 0.02 & 0.11 \end{bmatrix}, \; U_i = \begin{bmatrix} 0.13 & 0 & 0.02 \\ 0 & 0.17 & 0.03 \\ 0.02 & 0.03 & 0.16 \end{bmatrix} \quad (34)$$

$i = 1, \ldots, 4$, $\beta = 3.16$ and $\gamma = 0.022$. An illustration of system evolution with an arbitrary initial condition is shown in Fig. 1. Numerical simulations under *uniform* sampling show that the system is pratically stable for bigger sampling intervals with $T_k = T = 1.1 \cdot 10^{-1}$, which illustrates the conservatism of the proposed method.

Example 2: Uncertain System: We illustrate the applicability of our results on an example from power electronics. Consider the DC-DC converter from [12], where the model has the form $\dot{x}(\mathbf{t}) = \mathcal{A}_\sigma x(\mathbf{t}) + \mathcal{B}_\sigma$ with

$$\mathcal{A}_1 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{(RC)} \end{bmatrix}, \; \mathcal{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{(RC)} \end{bmatrix} \quad (35)$$

$\mathcal{B}_1 = [0 \quad 0]^T$, $\mathcal{B}_2 = [E/L \quad 0]^T$ with $E = 6$ V, $R = 50\ \Omega$, $L = 20$ mH and $C_0 = 220\ \mu$F. For $\lambda_1 = \lambda_2 = 0.5$, the matrix $A(\lambda)$ is Hurwitz and the system may be stabilized to the equilibrium point $x_e = -A(\lambda)^{-1}B(\lambda) = [0.24 - 6]^T$ using a continuous-time switching law. Consider the error $e = x - x_e$ dynamics

$$\frac{de}{d\mathbf{t}} = \mathcal{A}_\sigma e(\mathbf{t}) + \mathcal{A}_\sigma x_e + \mathcal{B}_\sigma.$$

For the numerical tests, the time scale change $t = \epsilon \mathbf{t}$ with $\epsilon = 10^4$ is used to cope with large numerical values in the system matrices and to avoid ill conditioned matrix inequalities. The system of the form (3) is obtained with $A_i = \epsilon^{-1}\mathcal{A}_i$, $B_i = \epsilon^{-1}(\mathcal{A}_i x_e + \mathcal{B}_i)$, $T_{\max} = \epsilon \mathbf{T}_{\max}$, $x = e$. Note that the trajectories are invariant with respect to time scaling. Furthermore, the switching laws are equivalent, since

$$\begin{aligned} &\arg\min (x - x_e)^T P (\epsilon A_i x + \epsilon B_i) \\ &\qquad = \arg\min (x - x_e)^T P (A_i x + B_i). \end{aligned}$$

Concerning the robust switching law design, conditions (11), (12) of Theorem 1 are feasible for any (time-varying) sampling intervals with $\mathbf{T}_{\max} \leq 1.5 \cdot 10^{-3} s$.

In order to compare with [12] the estimates of the attraction domains, consider the particular case studied in [12], with $\mathbf{T}_{\max} = 2.5 \cdot 10^{-5} s$. The conditions in Theorem 1 allow to design a switching law that guarantees that for $t \to \infty$, $|e(t)| < 1.9$, which is not very far from the value $|e(t_k)| < e_{\max} \approx 1.25$ for $k \to \infty$ obtained in [12] by using the exact integration over a sampling interval. The obtained numerical values (solutions of (11), (12)) are $\beta = 1.15 \cdot 10^{-2}$, $\gamma = 3.95 \cdot 10^{-3}$

$$P = \begin{bmatrix} 9.62 & 0.177 \\ 0.177 & 0.103 \end{bmatrix}, \; U_i = \begin{bmatrix} 8.425 & 0.182 \\ 0.182 & 0.059 \end{bmatrix}, \; i = 1, 2.$$

To illustrate the use of our method for uncertain systems, choose $\mathbf{T}_{\max} = 2.5 \cdot 10^{-5}$ s and assume that the resistor is subject to unknown time-varying uncertainties $\delta R(t) \in [-15\ \Omega, +15\ \Omega]$. Then each of the matrices $\mathcal{A}_i$ is varying in a polytope corresponding to the two vertices $R \pm 15\ \Omega$. Considering the equivalent error dynamics and using the same time scale change, a model of the form (19) is obtained with two vertices for each subsystem. The conditions in Corollary 2 are feasible with

$$P = \begin{bmatrix} 9.175 & 0.088 \\ 0.088 & 0.1 \end{bmatrix}, \; U_i = \begin{bmatrix} 7.75 & 0.161 \\ 0.161 & 0.048 \end{bmatrix}, \; i = 1, 2,$$

$\beta = 2.69 \cdot 10^{-2}$, $\gamma = 1.9 \cdot 10^{-3}$, which implies that $|e(t)| < 4.23$ as $t \to \infty$. The error system evolution with the initial condition $x(0) = 0$ is shown in Fig. 2. The figure presents the attractive ellipsoids for both the sampled-data case ($\bar{\mathcal{E}}^{\mathbf{T}_{\max}}$, obtained based on Corollary 2) and for the continuous-time switching implementation ($\bar{\mathcal{E}}^0$, representing the limiting set when $\mathbf{T}_{\max} \to 0$). Due to sampling and to parametric uncertainties the system state (in black) does not converge to the equilibrium point (the center of the ellipsoid) but only to a bounded region which is not far from the border of $\bar{\mathcal{E}}^{T_{\max}}$. Numerical simulations under an uniform sampling $\mathbf{T}_k = \mathbf{T} \leq \mathbf{T}_{\max}$ show that the same attractive ellipsoid is achieved for bigger $\mathbf{T}_{\max} = 1.4 \cdot 10^{-3}$, to be compared with $\mathbf{T}_{\max} = 2.5 \cdot 10^{-5}$ proved in theory under the variable sampling. The latter may illustrate the conservatism of the method.

Example 3: Switched Lyapunov Functions: Consider the example adapted from [15] with

$$A_1 = A_3 = \begin{bmatrix} -3 & -6 & 3 \\ 2 & 2 & -3 \\ 1 & 0 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 3 & 3 \\ -1 & -3 & -3 \\ 0 & 0 & -2 \end{bmatrix} \quad (36)$$

$B_1 = -B_3 = [-35 \quad 0 \quad 0]^T$, $B_2 = \mathbf{0}$. All of the matrices A_i, $i = 1$, 2, 3, are neutrally stable and there exists no Hurwitz convex combina-

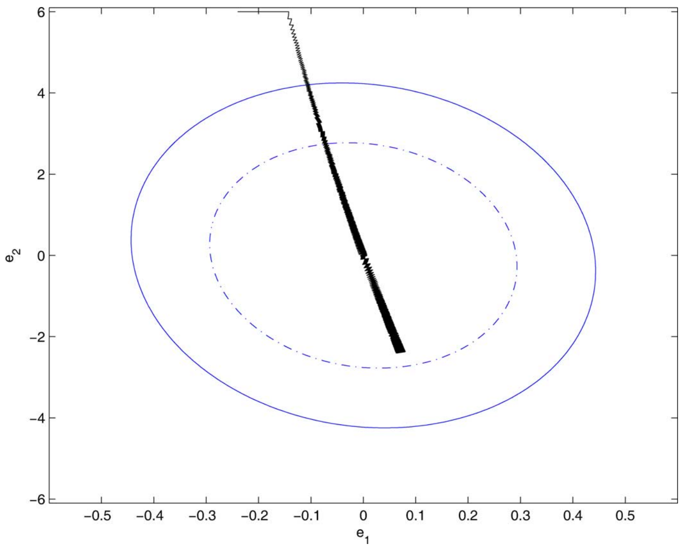

Fig. 2. Example 2: trajectory in the error state space with a control law based on Corollary 2 under variations in the resistor value from 35 Ω to 65 Ω with a fixed sampling interval $\mathbf{T}_{\max} = 2.5 \cdot 10^{-5}$ s (solid black line), the attractive sets obtained for the continuous-time case (dashed line) and for $\mathbf{T}_{\max} = 2.5 \cdot 10^{-5}$ s (solid line).

tion $A(\lambda)$. Therefore the quadratic stabilization methodology cannot be applied. We achieve practical stabilization by using a switching Lyapunov function (Theorem 2) of the form (27), switching among $M = 4$ quadratic forms. BMIs (20) are feasible for $T_{\max} \leq 3.2 \cdot 10^{-2}$ with $\beta = 2.5 \cdot 10^3$, $\gamma = 0.5 \cdot 10^{-3}$ and

$$P_1 = \begin{bmatrix} 1.6148 & 3.8733 & -3.2318 \\ 3.8733 & 10.2608 & -8.7473 \\ -3.2318 & -8.7473 & 8.2276 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 0.7063 & 0.9509 & -0.1946 \\ 0.9509 & 1.6398 & -0.3932 \\ -0.1946 & -0.3932 & 1.0599 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 0.7555 & 1.2625 & -0.9879 \\ 1.2625 & 2.6476 & -2.4533 \\ -0.9879 & -2.4533 & 3.6626 \end{bmatrix},$$
$$P_4 = \begin{bmatrix} 0.9733 & 1.9626 & -1.4013 \\ 1.9626 & 5.0996 & -3.6328 \\ -1.4013 & -3.6328 & 3.4206 \end{bmatrix}.$$

$\lambda^1 = \lambda^4 = [0.5 \quad 0 \quad 0.5]^T$, $\lambda^2 = \lambda^3 = [0 \quad 1 \quad 0]^T$. We used a dichotomy search for $T_{\max}$, at each step alternating P_j and $\mu_{j,r}$ as decision variables. Then, for fixed P_j or $\mu_{j,r}$ conditions (20) are LMIs. The P_j matrices were initialized with the values from [15].

V. Conclusion

This technical note presented a sampled-data switching design method for practical stabilization of switched affine systems. The results are robust with respect to sampling and to potential implementation imperfections such as sampling jitters or parametric uncertainties. The method uses LMI-based methods for the optimization of Lyapunov functions and it is illustrated by numerical examples. Improvement of the presented method by e.g., using sum-of-squares or by modifying the Lyapunov–Krasovskii functional may be topics for the future research.

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