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Brief paper

Improved derivative-dependent control of stochastic systems via delayed feedback implementation[☆]

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ABSTRACT

We study derivative-dependent control of the nth-order stochastic systems where derivatives are not available for measurements. The derivatives are approximated by finite differences giving rise to a delayed feedback. In the deterministic case, an efficient simple LMI-based method for designing of such static output-feedback and its sampled-data implementation was suggested recently. In the present paper, we extend this design to stochastic systems. We present two methods: the direct one that employs a stochastic extension of Lyapunov functionals used previously in the deterministic case, and the method which is based on neutral type model transformation and employs either augmented or simple Lyapunov functionals. Numerical examples illustrate the efficiency of the method.

1. Introduction

Control laws that depend on the output and its derivatives are used to stabilize linear systems. The derivatives are not available, but can be approximated by finite differences giving rise to a delayed feedback. The delay-induced stability can be checked using frequency-domain technique (Kharitonov, Niculescu, Moreno, & Michiels, 2005; Niculescu & Michiels, 2004; Ramírez, Mondié, Garrido, & Sipahi, 2016; Ramírez, Sipahi, Mondié, & Garrido, 2017) and complete Lyapunov–Krasovskii functionals (Gu, Chen, & Kharitonov, 2003), which give necessary and sufficient stability conditions. Simple LMIs for delay-induced stability of the 2nd-order systems were introduced in Fridman and Shaikhet (2016, 2019) and extended to the nth-order systems in Fridman and Shaikhet (2017) and Selivanov and Fridman (2018b), where the delayed terms were represented by Taylor's expansion with the remainders. The results for the nth-order deterministic systems were essentially improved in Selivanov and Fridman (2018a), where the derivative terms were presented as finite differences with remainders and where sampled-data implementation, for the first time, was achieved by using consecutive sampling measurements.

Systems with state multiplicative noise are encountered in many areas of applications, e.g. aircraft engineering, process control, population dynamics (Gershon & Shaked, 2019; Shaikhet, 2013; Yaesh, Shaked, & Yossef, 2004). Multiplicative noise appears due to the system parameters that undergo random perturbations of white noise process and due to nonlinearities (Gershon & Shaked, 2019; Shaikhet, 2013). There are many important results reported on systems with multiplicative noise (see e.g. Fridman & Shaikhet, 2019; Mao, 2007; Wang & Zhu, 2015; Xie & Duan, 2010). It is well-known that for stochastic systems, the design of observer-based controller is very complicated (Gershon & Shaked, 2019). Therefore, a simple static output-feedback is very attractive in the stochastic case.

In this paper, we consider derivative-dependent control of the nth-order stochastic systems where derivatives are not available for measurements. Under assumption of the stabilizability of the system by a state-feedback that depends on the output and its derivatives up to the order $n-1$, a delayed static output-feedback that stabilizes the system is found. Our objective is to present improved LMI-based method for the delayed feedback and its sampled-data implementation in the stochastic case.

We present two methods for continuous-time delayed static output-feedback and its sampled-data implementation (that may be used for practical application of such controller):

(1) The direct method that presents a stochastic extension of Selivanov and Fridman (2018a). Note that Lyapunov functionals of Selivanov and Fridman (2018a) depend on the nth-order derivative, and, thus, are not applicable in the stochastic case. This is because a solution of a stochastic system does not have a derivative. We propose novel

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Lyapunov functionals that depend on the deterministic and stochastic parts of the system.

(2) The method based on the neutral type model transformation, where we present the derivative of the order i $(i = 1, \ldots, n-1)$ in the form of the finite-difference with the remainder in the form of time-derivative of the integral term. The latter term depends on the same ith derivative (and not on $(i+1)$th derivative as in Selivanov & Fridman, 2018a) which is well-defined in the stochastic case. We employ either augmented or simple (i.e. non-augmented) Lyapunov functionals (as was proposed in Fridman & Shaikhet, 2019 for the 2nd-order systems). However, to compensate sampling, we still have to use Lyapunov functionals that depend on the deterministic and stochastic parts of the system.

The efficiency of the method is illustrated by numerical examples. In the continuous-time case, for larger stochastic perturbations, the second method via the simple Lyapunov functional leads to less conservative results with less decision variables in LMIs than the first method. Moreover, the second method with augmented Lyapunov functional improves the results via the simple Lyapunov functional and via the first method (but on the account of computational complexity). In the sampled-data case, the second method leads to more conservative results if the stochastic perturbations are small. However, it allows larger stochastic perturbations even via the simple Lyapunov functional.

Summarizing, we have extended the efficient method (i.e. time-delay implementation of derivative-dependent feedback of Selivanov & Fridman, 2018a) from deterministic to stochastic case, and improved the deterministic results by using neutral type model transformation and augmented Lyapunov functionals. We have also extended results of Fridman and Shaikhet (2019) to the higher-order systems and to sampled-data control. A conference version of this paper confined to the 3rd-order systems was presented in Zhang and Fridman (2019).

Notations. Throughout this paper, I_k is the identity $k \times k$ matrix, the superscript T stands for matrix transposition. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with Euclidean norm $|\cdot|$, $\mathbb{R}^{n\times m}$ denotes the set of all $n \times m$ real matrices with the induced norm $|\cdot|$. Denote by $diag\{\ldots\}$ and $col\{\ldots\}$ the block-diagonal matrix and block-column vector, respectively, $diag\{R_i\}_{i=1}^n$ is the block-diagonal matrix with R_i $(i = 1, \ldots, n)$ being on the diagonal. $X > 0$ means that X is a positive definite symmetric matrix, and for any square matrix X, $sym\{X\}$ denotes $X^T + X$. Denote by $\mathbf{E}X$ the mathematical expectation of stochastic variable X. For matrix S and vector X with appropriate dimensions $|X|_S^2 := X^T S X$.

We now present two useful Lemmas:

Lemma 1 (*Jensen's Inequality, Fridman, 2014; Solomon & Fridman, 2013*)**.** *Denote $G = \int_b^a f(s)x(s)ds$, where $f : [a,b] \to [0,\infty]$, $x : [a,b] \to \mathbb{R}^n$ and the integration concerned is well defined. Then for any $n \times n$ matrix $R > 0$ the following inequality holds:*

$$G^T R G \leq \int_b^a f(s)ds \int_b^a f(s)x^T(s)Rx(s)ds.$$

Lemma 2 (*Exponential Wirtinger's Inequality, Selivanov & Fridman, 2016*)**.** *Let $x(t) : (a,b) \to \mathbb{R}^n$ be absolutely continuous with $\dot{x} \in L_2(a,b)$ and $x(a) = 0$ or $x(b) = 0$. Then for any $\alpha \in \mathbb{R}$ and $n \times n$ matrix $W > 0$ the following inequality holds:*

$$\int_b^a e^{2\alpha t}x^T(s)Wx(s)ds \leq e^{2|\alpha|(b-a)}\frac{4(b-a)^2}{\pi^2}\int_a^b e^{2\alpha t}\dot{x}^T(s)W\dot{x}(s)ds.$$

2. Continuous-time control

Consider the nth-order stochastic system

$$y^{(n)}(t) = \textstyle\sum_{i=0}^{n-1}(A_i + C_i\dot{w}(t))y^{(i)}(t) + Bu(t), \quad (1)$$

where $y(t) = y^{(0)}(t) \in \mathbb{R}^k$ is the measurement, $y^{(i)}(t)$ is the ith derivative of $y(t)$, $u(t) \in \mathbb{R}^m$ is the control input, $w(t)$ is the one-dimensional Brownian motion (Mao, 2007; Shaikhet, 2013), A_i, $C_i \in \mathbb{R}^{k\times k}$ and $B \in \mathbb{R}^{k\times m}$ are constant matrices. Let

$$\begin{aligned}
x(t) &= col\{y^{(0)}(t), \ldots, y^{(n-1)}(t)\} \\
&= col\{x_0(t), \ldots, x_{n-1}(t)\} \in \mathbb{R}^{nk}, \\
A &= \begin{bmatrix} 0 & I_k & 0 & \cdots & 0 \\ 0 & 0 & I_k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I_k \\ A_0 & A_1 & A_2 & \cdots & A_{n-1} \end{bmatrix} \in \mathbb{R}^{nk\times nk}, \\
\bar{B} &= col\{0, B\} \in \mathbb{R}^{nk\times m}, \\
C &= col\{0, \bar{C}\} \in \mathbb{R}^{nk\times nk}, \quad \bar{C} = [C_0, \ldots, C_{n-1}] \in \mathbb{R}^{k\times nk}.
\end{aligned}$$

Then (1) can be presented as

$$dx(t) = (Ax(t) + \bar{B}u(t))dt + Cx(t)dw(t). \quad (2)$$

Assume that $(A, \bar{B})$ is stabilizable. Then there exist $\bar{K}_i \in \mathbb{R}^{m\times k}$ $(i = 0, \ldots, n-1)$ such that

$$D = A + \bar{B}\bar{K}, \quad \bar{K} = [\bar{K}_0, \ldots, \bar{K}_{n-1}] \quad (3)$$

is Hurwitz. Then system (2) with a small enough stochastic perturbation (i.e. small enough $|C|$) is mean-square stabilized by the state-feedback

$$u(t) = \textstyle\sum_{i=0}^{n-1}\bar{K}_i x_i(t), \quad \bar{K}_i \in \mathbb{R}^{m\times k}. \quad (4)$$

However, differently from the state-feedback case with the full knowledge of the system state, we consider the output-feedback control, where the derivatives $x_i(t)$ $(i = 1, \ldots, n-1)$ in (4) are not available. As in Selivanov and Fridman (2018a), we employ in this paper their finite-difference approximations:

$$\begin{aligned}
&\bar{x}_0(t) = x_0(t), \\
&x_i(t) \approx \bar{x}_i(t) = \tfrac{\bar{x}_{i-1}(t) - \bar{x}_{i-1}(t-h)}{h} \\
&\quad = \tfrac{1}{h^i}\textstyle\sum_{j=0}^{i}\binom{i}{j}(-1)^j x_0(t-jh), \quad i = 1, \ldots, n-1
\end{aligned} \quad (5)$$

with a constant delay $h > 0$ and the binomial coefficients $\binom{i}{j} = \frac{i!}{j!(i-j)!}$. By replacing $x_i(t)$, $(i = 0, \ldots, n-1)$ in (4) with their approximations, we have the following delay-dependent feedback

$$u(t) = \textstyle\sum_{i=0}^{n-1}\bar{K}_i\bar{x}_i(t) = \sum_{i=0}^{n-1}K_i x_0(t-ih), \quad (6)$$

where $x_0(t) = x_0(0)$ for $t < 0$ and

$$K_i = (-1)^i\textstyle\sum_{j=i}^{n-1}\binom{j}{i}\frac{1}{h^j}\bar{K}_j, \quad i = 0, \ldots, n-1. \quad (7)$$

As in the deterministic case (see e.g. French, Ilchmann, and Mueller (2009)), we will show that for small enough stochastic perturbations, if (2) is stabilized by the derivative-dependent feedback (4), then it can be stabilized by static output-feedback (6) with small enough $h > 0$. Note that an alternative output-feedback is an observer-based controller. However, the implementation of such controller is essentially more complicated especially for stochastic systems (see e.g. (7) in Gershon and Shaked (2019)). Here we provide a much simpler static output-feedback (6). Such a feedback can be easily applied e.g. via sampled-data implementation (see Section 3).

Moreover, as mentioned in Ramírez et al. (2016, 2017), the main problem with the derivative-dependent control stems from the noisy measurements of the derivative terms. Here measurement noise problems are mitigated since the finite differences mimic pure derivatives, and controller (6) does not rely on the measurements of the derivatives. Note also that the suggested design method is efficient provided the measurements of $x_0(t)$ are accurate (not noisy).

Following the idea of Selivanov and Fridman (2018a), we present the approximation errors $x_i(t)-\bar{x}_i(t)$ $(i=1,\dots,n-1)$ as

$$\bar{x}_i(t)=x_i(t)-\int_{t-ih}^{t}\varphi_i(t-s)\dot{x}_i(s)ds, \tag{8}$$

where $\varphi_1(v)=\frac{h-v}{h}$, $v\in[0,h]$ and for $i=1,\dots,n-2$

$$\varphi_{i+1}(v)=\begin{cases}\int_0^v \frac{\varphi_i(\lambda)}{h}d\lambda+\frac{h-v}{h}, & v\in[0,h]\\ \int_{v-h}^{v}\frac{\varphi_i(\lambda)}{h}d\lambda, & v\in(h,ih).\\ \int_{v-h}^{ih}\frac{\varphi_i(\lambda)}{h}d\lambda, & v\in[ih,ih+h].\end{cases}$$

The functions $\varphi_i(v)$ $(i=1,\dots,n-1)$ have the following properties:

$$\begin{cases}0\le\varphi_i(v)\le 1, \quad v\in[0,ih],\\ \varphi_i(0)=1, \quad \varphi_i(ih)=0,\\ \int_0^{ih}\varphi_i(v)dv=\frac{ih}{2},\\ \frac{d}{dv}\varphi_i(v)\in[-\frac{1}{h},0], \quad v\in[0,ih].\end{cases} \tag{9}$$

Compared with Selivanov and Fridman (2018a), this paper additionally gives the lower bound of $\frac{d}{dv}\varphi_i(v)$ that can be easily verified since $0\le\varphi_i(v)\le 1$. This property is employed in the stochastic case for the stability analysis under the sampled-data feedback (see e.g. (47) where $\psi_i(v)=-\frac{d}{dv}\varphi_i(v)$).

The system (2), (4) takes the form

$$dx(t)=Dx(t)dt+Cx(t)dw(t), \tag{10}$$

where D is given by (3). Via (8), the system (2), (6) takes the form

$$dx(t)=f_1(t)dt+Cx(t)dw(t) \tag{11}$$

with the same D and

$$\begin{aligned}&f_1(t)=Dx(t)+\textstyle\sum_{i=1}^{n-1}\bar{B}\bar{K}_i\kappa_i(t),\\ &\kappa_i(t)=-\int_{t-ih}^{t}\varphi_i(t-s)\dot{x}_i(s)ds.\end{aligned} \tag{12}$$

2.1. Stability of (11): direct method

For the sake of simplicity, we denote that for $i=1,\dots,n-1$

$$H_i=[0_{k\times ik}\ I_k\ 0_{k\times(n-i-1)k}], \quad \phi_i(\lambda)=\int_\lambda^{ih}\varphi_i(v)dv. \tag{13}$$

The LMI conditions are derived by using Lyapunov functional

$$V_1=V_P+\textstyle\sum_{i=1}^{n-1}\frac{ih}{2}V_{R_i}+V_{F_1}, \tag{14}$$

where

$$\begin{aligned}&V_P=x^T(t)Px(t),\\ &V_{R_i}=\int_{t-ih}^{t}e^{-2\alpha(t-s)}\phi_i(t-s)|x_{i+1}(s)|^2_{R_i}ds, \quad i=1,\dots,n-2,\\ &V_{R_{n-1}}=\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\phi_{n-1}(t-s)|H_{n-1}f_1(s)|^2_{R_{n-1}}ds,\\ &V_{F_1}=\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\phi_{n-1}(t-s)|H_{n-1}Cx(s)|^2_{F_1}ds,\\ &P>0, \quad R_i>0, \quad i=1,\dots,n-1, \quad F_1>0.\end{aligned}$$

Note that the terms V_{R_i} $(i=1,\dots,n-2)$ are from Selivanov and Fridman (2018a), whereas the terms $V_{R_{n-1}}$ and V_{F_1} are stochastic extensions of Lyapunov functionals that depend on $\dot{x}(t)$.

Theorem 1. *Given $\bar{K}_i$ $(i=0,\dots,n-1)$ let the derivative-dependent feedback (4) exponentially stabilizes (2), where $C=0$, with a decay rate $\bar{\alpha}>0$.*
(i) Given tuning parameters $h>0$ and $\alpha\in(0,\bar{\alpha})$, let there exist $nk\times nk$ matrix $P>0$, $k\times k$ matrices $R_i>0$ $(i=1,\dots,n-1)$ and $F_1>0$ that satisfy

$$\Phi_1<0, \tag{15}$$

where Φ_1 is the symmetric matrix composed from

$$\begin{aligned}\Phi_{11}&=sym\{PD\}+2\alpha P+C^TPC\\ &\quad+\textstyle\sum_{i=1}^{n-2}\frac{(ih)^2}{4}|H_{i+1}|^2_{R_i}+\frac{(n-1)h}{2}|H_{n-1}C|^2_{F_1},\\ \Phi_{12}&=P\bar{B}[\bar{K}_1,\dots,\bar{K}_{n-1}],\\ \Phi_{14}&=\frac{(n-1)h}{2}D^TH^T_{n-1}R_{n-1},\\ \Phi_{22}&=-diag\{e^{-2\alpha ih}R_i\}_{i=1}^{n-1},\\ \Phi_{23}&=[0_{k\times(n-2)k},-e^{-2\alpha(n-1)h}R_{n-1}]^T,\\ \Phi_{24}&=\frac{(n-1)h}{2}[\bar{K}_1,\dots,\bar{K}_{n-1}]^T\bar{B}^TH^T_{n-1}R_{n-1},\\ \Phi_{33}&=-e^{-2\alpha(n-1)h}(R_{n-1}+F_1), \qquad \Phi_{44}=-R_{n-1}\end{aligned}$$

with D given by (3). Then the delay-dependent feedback (6) with a time-delay $h>0$ and controller gains (7) exponentially mean-square stabilizes (2) with a decay rate $\alpha>0$.
(ii) Given any $\alpha\in(0,\bar{\alpha})$, the LMI of item (i) is always feasible for small enough stochastic perturbations and $h>0$ (meaning that the delay-dependent feedback (6) with controller gains (7) exponentially stabilizes (2) with a decay rate $\alpha>0$).

Proof. (i) Let $\mathcal{L}$ be the generator of the system (11) (Fridman & Shaikhet, 2019; Mao, 2007). Via (9), we have

$$\begin{aligned}&\mathcal{L}V_P+2\alpha V_P=2x^T(t)Pf_1(t)+x^T(t)C^TPCx(t)\\ &\quad+2\alpha x^T(t)Px(t),\\ &\mathcal{L}V_{R_i}+2\alpha V_{R_i}=\frac{ih}{2}|x_{i+1}(t)|^2_{R_i}\\ &\quad-\int_{t-ih}^{t}e^{-2\alpha(t-s)}\varphi_i(t-s)|x_{i+1}(s)|^2_{R_i}ds,\\ &\quad\ i=1,\dots,n-2,\\ &\mathcal{L}V_{R_{n-1}}+2\alpha V_{R_{n-1}}=\frac{(n-1)h}{2}|H_{n-1}f_1(t)|^2_{R_{n-1}}\\ &\quad-\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}f_1(s)|^2_{R_{n-1}}ds,\\ &\mathcal{L}V_{F_1}+2\alpha V_{F_1}=\frac{(n-1)h}{2}|H_{n-1}Cx(t)|^2_{F_1}\\ &\quad-\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}Cx(s)|^2_{F_1}ds.\end{aligned} \tag{16}$$

Using Lemma 1 and via (12), we obtain

$$\begin{aligned}&\frac{ih}{2}\int_{t-ih}^{t}e^{-2\alpha(t-s)}\varphi_i(t-s)|x_{i+1}(s)|^2_{R_i}ds\\ &\ge e^{-2\alpha ih}\kappa_i^T(t)R_i\kappa_i(t), \quad i=1,\dots,n-2,\\ &\frac{nh-h}{2}\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}f_1(s)|^2_{R_{n-1}}ds\\ &\ge e^{-2\alpha(n-1)h}\left|\int_{t-(n-1)h}^{t}\varphi_{n-1}(t-s)H_{n-1}f_1(s)ds\right|^2_{R_{n-1}}\\ &=e^{-2\alpha(n-1)h}|\kappa_{n-1}(t)+\varrho_1(t)|^2_{R_{n-1}},\end{aligned} \tag{17}$$

where

$$\varrho_1(t)=\int_{t-(n-1)h}^{t}\varphi_{n-1}(t-s)H_{n-1}Cx(s)dw(s).$$

By Itô integral properties (see e.g. Fridman & Shaikhet, 2019; Mao, 2007) and via (9), we have for any matrix $F_1>0$

$$\begin{aligned}&\mathbf{E}e^{-2\alpha(n-1)h}\varrho_1^T(t)F_1\varrho_1(t)\\ &=\mathbf{E}e^{-2\alpha(n-1)h}\int_{t-(n-1)h}^{t}\varphi^2_{n-1}(t-s)|H_{n-1}Cx(s)|^2_{F_1}ds\\ &\le\mathbf{E}\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}Cx(s)|^2_{F_1}ds.\end{aligned} \tag{18}$$

In view of (16)–(18), we obtain

$$\mathbf{E}\mathcal{L}V_1+\mathbf{E}2\alpha V_1\le\mathbf{E}\xi_1^T\bar{\Phi}_1\xi_1+\mathbf{E}\frac{(nh-h)^2}{4}|H_{n-1}f_1(t)|^2_{R_{n-1}}, \tag{19}$$

where $\xi_1=col\{x(t),\kappa_1(t),\dots,\kappa_{n-1}(t),\varrho_1(t)\}$ and $\bar{\Phi}_1$ is obtained from Φ_1 given by (15) by taking away the last block-column and block-row. Substituting $f_1(t)$ given by (12) into (19) and further applying Schur's complement, we arrive at $\mathbf{E}\mathcal{L}V_1+\mathbf{E}2\alpha V_1\le 0$ since $\Phi_1<0$, which implies that (7) exponentially mean-square stabilizes (2) with a decay rate $\alpha>0$.
(ii) If (4) exponentially stabilizes (2), where $C=0$, with a decay rate $\bar{\alpha}>0$, then for any $\alpha\in(0,\bar{\alpha})$ there exists $0<P\in\mathbb{R}^{nk\times nk}$ such that $sym\{PD\}+2\alpha P<0$. Thus,

$$sym\{PD\}+2\alpha P+C^TPC<0 \tag{20}$$

for small enough $|C|$. We choose $R_i = F_1 = \frac{1}{\sqrt{h}} I_k$ $(i = 1, \ldots, n-1)$. By using Schur's complement, $\bar{\Phi}_1 < 0$ is equivalent to

$$sym\{PD\} + 2\alpha P + C^T PC + \sqrt{h}(G_1 + hG_2) < 0, \qquad (21)$$

where

$$G_1 = \tfrac{n-1}{2}|H_{n-1}C|^2 + \textstyle\sum_{i=1}^{n-2} e^{2\alpha ih}|P\bar{B}\bar{K}_i|^2 + 2e^{2\alpha(n-1)h}|P\bar{B}\bar{K}_{n-1}|^2,$$
$$G_2 = \textstyle\sum_{i=1}^{n-2} \frac{i^2}{4}|H_{i+1}|^2.$$

Inequality (20) implies (21) for small enough $h > 0$ since $\sqrt{h}(G_1 + hG_2) \to 0$ for $h \to 0$, implying the feasibility of $\bar{\Phi}_1 < 0$ for small enough $h > 0$. Finally, applying Schur's complement to the last block-column and block-row of Φ_1 given by (15), we find that $\Phi_1 < 0$ for small enough $h > 0$ if $\bar{\Phi}_1 < 0$ is feasible. Therefore, the LMI of item (i) is always feasible for small enough $h > 0$ and $|C|$. □

Remark 1. Based on the analysis LMIs (e.g. LMI (15) of Theorem 1), the LMI-based design can be derived as follows: we first set $P^{-1} = X = [X_{j,i}]_{n\times n}$ and multiply (15) by $diag\left\{X, \sum_{i=1}^n X_{2,i}, \ldots, \sum_{i=1}^n X_{n,i}, \sum_{i=1}^n X_{n,i}, R_{n-1}^{-1}\right\}$ and its transpose from the right and the left, respectively. By denoting

$$Y = \bar{K}X = [Y_0, \ldots, Y_{n-1}],$$
$$\hat{R}_i = \textstyle\sum_{j=1}^n X_{i+1,j} R_i \sum_{j=1}^n X_{j,i+1} \ (i = 1, \ldots, n-1),$$
$$\hat{F}_1 = \textstyle\sum_{j=1}^n X_{n,j} F_1 \sum_{j=1}^n X_{j,n},$$

applying Schur's complement and employing

$$-\textstyle\sum_{j=1}^n X_{j,i+1}\hat{R}_i^{-1}\sum_{j=1}^n X_{i+1,j} \le \hat{R}_i - \sum_{j=1}^n (X_{j,i+1} + X_{i+1,j})$$

and the similar inequality for $\hat{F}_1$-term, we arrive at the following design LMI $\hat{\Phi} < 0$, where $\hat{\Phi}$ is the symmetric matrix composed from

$$\hat{\Phi}_{11} = sym\{AX + \bar{B}Y\} + 2\alpha X,$$
$$\hat{\Phi}_{12} = \bar{B}[Y_1, \ldots, Y_{n-1}],$$
$$\hat{\Phi}_{14} = \tfrac{(n-1)h}{2}(XA^T + Y^T\bar{B}^T)H_{n-1}^T,$$
$$\hat{\Phi}_{15} = X[\tfrac{h}{2}H_2^T, \ldots, \tfrac{(n-2)h}{2}H_{n-1}^T],$$
$$\hat{\Phi}_{16} = XC^T[I_{nk}, \sqrt{\tfrac{(n-1)h}{2}}H_{n-1}^T],$$
$$\hat{\Phi}_{22} = -diag\{e^{-2\alpha ih}\hat{R}_i\}_{i=1}^{n-1},$$
$$\hat{\Phi}_{23} = [0_{k\times(n-2)k}, -e^{-2\alpha(n-1)h}\hat{R}_{n-1}]^T,$$
$$\hat{\Phi}_{24} = \tfrac{(n-1)h}{2}[Y_1, \ldots, Y_{n-1}]^T\bar{B}^T H_{n-1}^T,$$
$$\hat{\Phi}_{33} = -e^{-2\alpha(n-1)h}(\hat{R}_{n-1} + \hat{F}_1),$$
$$\hat{\Phi}_{44} = \hat{R}_{n-1} - \textstyle\sum_{i=1}^n (X_{i,n} + X_{n,i}),$$
$$\hat{\Phi}_{55} = diag\{\hat{R}_j - \textstyle\sum_{i=1}^n (X_{i,j+1} + X_{2,j+1})\}_{j=1}^{n-2},$$
$$\hat{\Phi}_{66} = diag\{-X, \hat{F}_1 - \textstyle\sum_{i=1}^n (X_{i,n} + X_{n,i})\}$$

with the decision variables $X > 0, \hat{R}_i > 0$ $(i = 1, \ldots, n-1), \hat{F}_1 > 0$ and Y to be determined, and tuning scalar parameters $h > 0$ and $\alpha > 0$. If the above design LMI is feasible, the stabilizing controller gain is given by $\bar{K} = YX^{-1}$. Note that differently from the analysis LMI, the feasibility of the design LMI for small enough values of h and α cannot be proved even for $C = 0$. The examples below show that the design LMI is feasible for the 2nd- and 3rd-order systems, but unfeasible for the 4th-order system (see Remark 3).

2.2. Stability via neutral type model transformation

In this section, we derive stability conditions by using neutral type model transformation (Fridman & Shaikhet, 2017, 2019; Niculescu, 2001). First, we show that the integrals $\kappa_i(t)$ $(i = 1, \ldots, n-1)$ given by (12) can be presented as

$$\kappa_i(t) = -\tfrac{d}{dt}\vartheta_i(t), \quad \vartheta_i(t) = \textstyle\int_{t-ih}^t \varphi_i(t-s)x_i(s)ds. \qquad (22)$$

Indeed, differentiating $\vartheta_i(t)$, employing (9) and further integrating by parts we have

$$\tfrac{d}{dt}\vartheta_i(t) = x_i(t) + \mu_i(t) \qquad (23)$$

with

$$\mu_i(t) = -\textstyle\int_{t-ih}^t \frac{d\varphi_i(t-s)}{ds}x_i(s)ds$$
$$= -x_i(t) + \textstyle\int_{t-ih}^t \varphi_i(t-s)\dot{x}_i(s)ds$$

that implies (22).

Then system (11) can be presented as

$$dz(t) = Dx(t)dt + Cx(t)dw(t),$$
$$z(t) = x(t) + \textstyle\sum_{i=1}^{n-1}\bar{B}\bar{K}_i\vartheta_i(t) \qquad (24)$$

with $\vartheta_i(t)$ given by (22). As in Fridman and Shaikhet (2019), we do not need to check further the stability of difference equation $z(t)$.

Before presenting the LMI conditions, we use the following notations

$$\ell_0 = [I_{nk} \ 0_{nk\times lk}],$$
$$\ell_i = [0_{k\times ik} \ I_k \ 0_{k\times(l-i-1)k}], \quad i = 1, \ldots, l,$$
$$\Xi_1 = col\{\ell_0 + \textstyle\sum_{i=1}^{n-1}\bar{B}\bar{K}_i\ell_i, \ell_1, \ldots, \ell_{n-1}\},$$
$$\Xi_2 = col\{D\ell_0, H_1\ell_0 + \ell_n, \ldots, H_{n-1}\ell_0 + \ell_{2n-2}\}, \qquad (25)$$

where the value of l corresponds to the later derived LMIs.

Theorem 2. *Given $\bar{K}_i$ $(i = 0, \ldots, n-1)$ let the derivative-dependent feedback (4) exponentially stabilizes (2), where $C = 0$, with a decay rate $\bar{\alpha} > 0$.*
(i) Given tuning parameters $h > 0$ and $\alpha \in (0, \bar{\alpha})$, let there exist $(2n-1)k \times (2n-1)k$ matrix $P = [P_{ij}]_{n\times n}$, $k \times k$ matrices $R_i > 0$ and $S_i > 0$ $(i = 1, \ldots, n-1)$ that satisfy

$$\Psi > 0, \quad \Phi_{2aug} < 0, \qquad (26)$$

where

$$\Psi = \bar{\Xi}_1^T P\bar{\Xi}_1 + \textstyle\sum_{i=1}^{n-1}\frac{2}{ih}e^{-2\alpha ih}\ell_i^T S_i\ell_i,$$
$$\Phi_{2aug} = \bar{\Phi}_{2aug} + sym\{\Xi_1^T P\Xi_2\} + 2\alpha\Xi_1^T P\Xi_1.$$

Here $\bar{\Xi}_1$ coincides with the first n block-rows of Ξ_1 given by (25) and $\bar{\Phi}_{2aug}$ is the symmetric matrix composed from

$$\Phi_{11} = C^T P_{11}C + \textstyle\sum_{i=1}^{n-1}|H_i|^2_{\frac{(ih)^2}{4}R_i + S_i},$$
$$\Phi_{22} = -diag\{e^{-2\alpha ih}R_i\}_{i=1}^{n-1},$$
$$\Phi_{33} = -diag\{e^{-2\alpha ih}S_i\}_{i=1}^{n-1}.$$

Then the delay-dependent feedback (6) with a time-delay $h > 0$ and controller gains (7) exponentially mean-square stabilizes (2) with a decay rate $\alpha > 0$.
(ii) Given tuning parameters $h > 0$ and $\alpha \in (0, \bar{\alpha})$, let there exist $nk\times nk$ matrices $P_{11} > 0$ and $k\times k$ matrices $R_i > 0$ $(i = 1, \ldots, n-1)$ that satisfy

$$\Phi_{2sim} < 0, \qquad (27)$$

where Φ_{2sim} is obtained from Φ_{2aug} given by (26) by taking away the last $n-1$ block-columns and block-rows and setting $P = diag\{P_{11}, 0\}$. Then the delay-dependent feedback (6) with a time-delay $h > 0$ and controller gains (7) exponentially mean-square stabilizes (2) with a decay rate $\alpha > 0$.
(iii) Given any $\alpha \in (0, \bar{\alpha})$, the LMIs of items (i) and (ii) are always feasible for small enough stochastic perturbations and $h > 0$ (meaning that the delay-dependent feedback (6) with controller gains (7) exponentially stabilizes (2) with a decay rate $\alpha > 0$).

Proof. (i) Inspired by Fridman and Shaikhet (2019), we consider the augmented functional

$$V_2 = V_P + \sum_{i=1}^{n-1}(\tfrac{ih}{2}V_{R_i} + V_{S_i}), \tag{28}$$

where

$$\begin{aligned}&V_P = \zeta^T(t)P\zeta(t),\\ &V_{R_i} = \int_{t-ih}^{t} e^{-2\alpha(t-s)}\phi_i(t-s)|x_i(s)|^2_{R_i}dvds,\\ &V_{S_i} = \int_{t-ih}^{t} e^{-2\alpha(t-s)}\varphi_i(t-s)|x_i(s)|^2_{S_i}ds,\\ &\zeta(t) = col\{z(t), \vartheta_1(t), \dots, \vartheta_{n-1}(t)\},\\ &R_i > 0,\quad S_i > 0,\quad i = 1,\dots,n-1.\end{aligned}$$

Compared with V_1 of (14), the functional V_2 does not depend on the deterministic or stochastic terms due to the neutral type model transformation. Notice that the terms V_{R_i} compensate $\vartheta_i(t)$, whereas the terms V_{S_i} compensate $\mu_i(t)$. Let $\mathcal{L}$ be the generator of the system (24) (Fridman & Shaikhet, 2019; Mao, 2007). Via (9), (23) and (24), we obtain

$$\begin{aligned}&\mathcal{L}V_P + 2\alpha V_P = 2\zeta^T(t)P\bar{\zeta}(t)\\ &\quad + x^T(t)C^TP_{11}Cx(t) + 2\alpha\zeta^T(t)P\zeta(t),\\ &\mathcal{L}V_{R_i} + 2\alpha V_{R_i} = \tfrac{ih}{2}|x_i(t)|^2_{R_i}\\ &\quad - \int_{t-ih}^{t} e^{-2\alpha(t-s)}\varphi_i(t-s)|x_i(s)|^2_{R_i}ds,\\ &\mathcal{L}V_{S_i} + 2\alpha V_{S_i} = |x_i(t)|^2_{S_i}\\ &\quad - \int_{t-ih}^{t} e^{-2\alpha(t-s)}\psi_i(t-s)|x_i(s)|^2_{S_i}ds,\end{aligned} \tag{29}$$

where $\bar{\zeta}(t) = \Xi_2\xi_2$ with $\xi_2 = col\{\bar{\xi}_2, \mu_1(t), \dots, \mu_{n-1}(t)\}$, $\bar{\xi}_2 = col\{x(t), \vartheta_1(t), \dots, \vartheta_{n-1}(t)\}$. Using Lemma 1, we obtain

$$\begin{aligned}&\tfrac{ih}{2}\int_{t-ih}^{t} e^{-2\alpha(t-s)}\varphi_i(t-s)|x_i(s)|^2_{R_i}ds \geq e^{-2\alpha ih}\vartheta_i^T(t)R_i\vartheta_i(t),\\ &\int_{t-ih}^{t} e^{-2\alpha(t-s)}\psi_i(t-s)|x_i(s)|^2_{S_i}ds \geq e^{-2\alpha ih}\mu_i^T(t)S_i\mu_i(t).\end{aligned} \tag{30}$$

From (29) and (30), it follows that

$$\mathcal{L}V_2 + 2\alpha V_2 \leq \xi_2^T\Phi_{2aug}\xi_2, \tag{31}$$

where Φ_{2aug} is given by (26) and ξ_2 is given below (29). Moreover, by Lemma 1 and via (9)

$$V_{S_i} \geq \tfrac{2}{ih}e^{-2\alpha ih}\vartheta_i(t)^TS_i\vartheta_i(t), \tag{32}$$

then V_2 is positive definite since

$$V_2 \geq V_P + \sum_{i=1}^{n-1}V_{S_i} \geq \bar{\xi}_2^T\Psi\bar{\xi}_2 \tag{33}$$

with Ψ given by (26). Therefore, the delay-dependent feedback (6) with a time-delay $h > 0$ and controller gains (7) exponentially mean-square stabilizes (2) with a decay rate $\alpha > 0$.
(ii) If LMI $\Phi_{2sim} < 0$ holds with $P_{11} > 0$, $R_i > 0$ $(i = 1,\dots,n-1)$, then for $S_i = \rho_iI$ $(i = 1,\dots,n-1)$ with small enough $\rho_i > 0$ LMIs $\Psi > 0$ and $\Phi_{2aug} < 0$ hold with the same P_{11}, R_i $(i = 1,\dots,n-1)$ and others blocks of P being 0. Therefore, the result follows from (i).
(iii) The proof of (iii) is similar to (ii) of Theorem 1. □

Remark 2. Note that (ii) of Theorem 2 can be derived directly by using the simple (non-augmented) Lyapunov functional defined by (28) with $P = diag\{P_{11}, 0\}$ and following arguments of (i). As shown in the examples below, for larger stochastic perturbations (which is the main interest in this paper), the simple Lyapunov functional gives almost the same results as the augmented Lyapunov functional, but by much lower computational price.

2.3. Examples: chains of three and four integrators

To illustrate the efficiency, we consider chains of three and four integrators. The deterministic version of these examples was considered in Selivanov and Fridman (2018a). However, we choose the controller gains below (which are different from Selivanov & Fridman, 2018a) that allow to treat essentially larger stochastic perturbations.

Example 1 (*Chain of Three Integrators*)**.** Consider (1) with

$$A_i = 0,\quad B = 1,\quad C_i = \sigma \in \mathbb{R}. \tag{34}$$

Using the pole placement, we find that for (4) with

$$\bar{K}_0 = -1.32,\quad \bar{K}_1 = -3.62,\quad \bar{K}_2 = -3.3, \tag{35}$$

the eigenvalues of D are $-1, -1.1, -1.2$.

Example 2 (*Chain of Four Integrators*)**.** Consider (1) with (34). Using the pole placement, we find that for (4) with

$$\bar{K}_0 = -1.716,\ \bar{K}_1 = -6.026,\ \bar{K}_2 = -7.91,\ \bar{K}_3 = -4.6, \tag{36}$$

the eigenvalues of D are $-1, -1.1, -1.2, -1.3$.

It is clear that with the above gains, matrix D defined by (3) is Hurwitz. Therefore, the derivative-dependent feedback (4) with these gains given by (35) and (36) stabilizes chains of three and four integrators, respectively, for small enough stochastic perturbations.

For different values of σ and $\alpha = 0.01$, the maximum values of delay h that preserve the exponential mean-square stability are presented in Tables 1 and 2. In the deterministic case ($\sigma = 0$), LMIs $\Phi_1 < 0$ and $\Phi_{2sim} < 0$ give the same results whereas LMIs $\Psi > 0$ and $\Phi_{2aug} < 0$ (for brevity we write them as $\Phi_{2aug} < 0$) admit a slightly better result. Indeed, this improvement is achieved on the account of computational complexity (see the number of decision variables in Table 3). With the larger h, one can obtain via (7) the smaller gains K_i $(i = 0,\dots,n-1)$. For large stochastic perturbations, LMI $\Phi_{2sim} < 0$ leads to efficient results that are close to the results via LMI $\Phi_{2aug} < 0$ and that essentially show improvements compared with the results via LMI $\Phi_1 < 0$.

If the derivative-dependent feedback (4) is chosen with the gains of Selivanov and Fridman (2018a), both LMIs $\Phi_1 < 0$ and $\Phi_{2sim} < 0$ with $\sigma = \alpha = 0$ lead to 2.529 and 0.169 (respectively, for Examples 1 and 2) as in Selivanov and Fridman (2018a). However, the resulting maximal $\sigma = 0.004$ that preserves the stability is essentially smaller than $\sigma = 2$ in Tables 1 and 2.

Remark 3. By solving the design LMI of Remark 1 and letting $\alpha = 0.01$, $\sigma = 0.4$ and $h = 0.2$, we manage to find for the 3rd- and 2nd-order systems in Examples 1 and 3 (see Section 3.3) the stabilizing controller gains

$$\bar{K} = -[1.4328\ 1.7076\ 2.3288]$$

and

$$\bar{K} = [3.1359\ 0.0697\ 0.4147\ 0.1407]$$

respectively. However, for Example 2 (the 4th-order system), the design LMI becomes unfeasible with any $h > 0$, $\alpha \geq 0$ and $\sigma \geq 0$. This conservatism stems from the transformation of the nonlinear terms. A more efficient LMI design may be a topic for future research.

Remark 4. To select the tuning parameters α, σ and h, we suggest the following algorithm: choose $\bar{K}_i$ $(i = 0,\dots,n-1)$ via pole-placement such that the derivative-dependent feedback (4) with controller gains $\bar{K}_i$ $(i = 0,\dots,n-1)$ exponentially stabilizes (2), where $C = 0$, with a decay rate $\bar{\alpha} > 0$. For a high efficiency, when solving the LMIs with $\sigma = 0$ and small enough $h > 0$, we apply the binary search method to find a critical maximal value of α as $\alpha^* < \bar{\alpha}$. Similarly, by choosing $\alpha \in [0, \alpha^*]$ with small enough $h > 0$, we find a critical maximum value of σ as σ^*. Then for $\alpha \in [0, \alpha^*]$ and $\sigma \in [0, \sigma^*]$, we can obtain a critical maximal value of $h = h^*$ such that for $h > h^*$ the LMI becomes unfeasible. We choose this maximal $h = h^*$ that leads to smaller gains K_i $(i = 0,\dots,n-1)$.

Table 1
Max. delay h for different σ (Example 1).

σ	0	0.2	0.5	1	1.5	2
$\Phi_1<0$	0.289	0.271	0.184	0.051	0.012	–
$\Phi_{2aug}<0$	0.317	0.313	0.293	0.233	0.140	0.008
$\Phi_{2sim}<0$	0.289	0.284	0.262	0.206	0.126	0.008

Table 2
Max. delay h for different σ (Example 2).

σ	0	0.2	0.5	1	1.5	2
$\Phi_1<0$	0.128	0.117	0.090	0.030	0.011	0.002
$\Phi_{2aug}<0$	0.138	0.136	0.130	0.107	0.077	0.040
$\Phi_{2sim}<0$	0.128	0.125	0.116	0.093	0.066	0.035

Table 3
Complexity of LMI conditions.

LMIs	Decision variables
$\Phi_1<0$	$0.5(n^2k^2+nk^2+2nk)$
$\Phi_{2aug}<0$	$0.5(4n^2k^2-2nk^2+4nk-k^2-3k)$
$\Phi_{2sim}<0$	$0.5(n^2k^2+nk^2+2nk-k^2-k)$

3. Sampled-data control

For practical application of the delayed feedback (6), in this section we suggest its sampled-data implementation. We assume that the measurement $y(t)=x_0(t)$ is available only at the discrete-time instants $t_k=kh$, $k\in\mathbb{N}_0$. Here $h>0$ is the sampling period. Then the derivative-dependent feedback (4) is approximated by the sampled-data feedback

$$u(t)=\sum_{i=0}^{n-1}\bar K_i\bar x_i(t_k)=\sum_{i=0}^{n-1}K_ix_0(t_{k-i}),\quad t\in[t_k,t_{k+1}),\quad k\in\mathbb{N}_0,\tag{37}$$

where $x_0(t)=x_0(0)$ for $t<0$, $\bar x_i(t)$ and K_i are given by (5) and (7), respectively. Note that the feedback (37) depends only on n discrete-time measurements $y(t_{k-n+1}),\dots,y(t_k)$, which is easy to implement. One may store $n-1$ measurements $y(t_{k-n+1}),\dots,y(t_{k-1})$ in the buffer.

Introduce the errors due to sampling

$$\bar x_0(t_k)=x_0(t)-\int_{t_k}^t\dot x_0(s)ds,$$
$$\bar x_i(t_k)=\bar x_i(t)-\int_{t_k}^t\dot{\bar x}_i(s)ds,\quad i=1,\dots,n-1.$$

Via (8), we have

$$\bar x_i(t_k)=x_i(t)-\int_{t-ih}^t\varphi_i(t-s)\dot x_i(s)ds-\int_{t_k}^t\dot{\bar x}_i(s)ds,\quad i=1,\dots,n-1.$$

Then the system (2), (37) takes the form

$$dx(t)=f_2(t)dt+Cx(t)dw(t),\tag{38}$$

where

$$f_2(t)=Dx(t)+\bar B\bar K_0\delta_0(t)+\sum_{i=1}^{n-1}\bar B\bar K_i(\kappa_i(t)+\delta_i(t)),\quad \delta_i(t)=-\int_{t_k}^t\dot{\bar x}_i(s)ds,\quad i=0,\dots,n-1\tag{39}$$

with D, $\kappa_i(t)$ given by (3), (12), respectively.

3.1. Stability of (38): direct method

To compensate $\delta_i(t)$ $(i=0,\dots,n-1)$ in the stability analysis of (38) we follow Selivanov and Fridman (2018a) and consider

$$V_{W_i}=h^2\int_{t_k}^t e^{-2\alpha(t-s)}|\dot{\bar x}_i(s)|^2_{W_i}ds-\frac{\pi^2}{4}e^{-2\alpha h}\int_{t_k}^t e^{-2\alpha(t-s)}|\delta_i(s)|^2_{W_i}ds,\quad W_i>0,\quad i=0,\dots,n-1,\quad t\in[t_k,t_{k+1}).\tag{40}$$

Since $\dot\delta_i(t)=-\bar B\bar K_i\dot{\bar x}_i(t)$ and $\delta_i(t_k)=0$, Lemma 2 implies $V_{W_i}\ge 0$. Using the generator $\mathcal{L}$ (Fridman & Shaikhet, 2019; Mao, 2007), we obtain

$$\mathcal{L}V_{W_i}+2\alpha V_{W_i}=h^2|\dot{\bar x}_i(t)|^2_{W_i}-\frac{\pi^2}{4}e^{-2\alpha h}\delta_i^T(t)W_i\delta_i(t),\quad i=0,\dots,n-1.\tag{41}$$

Differentiating (8) and via (5), (38), we have $\dot{\bar x}_0(t)=x_1(t)$ and

$$\begin{aligned}\dot{\bar x}_i(t)&=\int_{t-ih}^t\psi_i(t-s)x_{i+1}(s)ds,\quad i=1,\dots,n-2,\\ \dot{\bar x}_{n-1}(t)&=\int_{t-(n-1)h}^t\psi_{n-1}(t-s)H_{n-1}f_2(s)ds\\&\quad+\int_{t-(n-1)h}^t\psi_{n-1}(t-s)H_{n-1}Cx(s)dw(s).\end{aligned}\tag{42}$$

To compensate the terms $\bar x_i(t)$, $i=1,\dots,n-2$, we consider

$$\bar V_{W_i}=\int_{t-ih}^t e^{-2\alpha(t-s)}\varphi_i(t-s)|x_{i+1}(s)|^2_{W_i}ds,\quad i=1,\dots,n-2.\tag{43}$$

Via (9) and using Lemma 1, we have

$$\mathcal{L}\bar V_{W_i}+2\alpha\bar V_{W_i}\le|x_{i+1}(t)|^2_{W_i}-e^{-2\alpha ih}|\dot{\bar x}_i(t)|^2_{W_i}.\tag{44}$$

For the deterministic part of $\dot{\bar x}_{n-1}(t)$, we consider

$$V_Q=\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}f_2(s)|^2_Q ds\tag{45}$$

with $Q>0$. Lemma 1 leads to

$$\mathcal{L}V_Q+2\alpha V_Q\le|H_{n-1}f_2(t)|^2_Q-e^{-2\alpha(n-1)h}\varrho_2^T(t)Q\varrho_2(t),\tag{46}$$

where

$$\varrho_2(t)=\int_{t-(n-1)h}^t\psi_{n-1}(t-s)H_{n-1}f_2(s)ds.$$

Similarly, we define

$$\varrho_3(t)=\int_{t-(n-1)h}^t\psi_{n-1}(t-s)H_{n-1}Cx(s)dw(s).$$

By Itô integral properties and via (9), we obtain

$$\begin{aligned}&\mathbf{E}he^{-2\alpha(n-1)h}\varrho_3^T(t)F_2\varrho_3(t)\\&=\mathbf{E}he^{-2\alpha(n-1)h}\int_{t-(n-1)h}^t\psi_{n-1}^2(t-s)|H_{n-1}Cx(s)|^2_{F_2}ds\\&\le\mathbf{E}\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\psi_{n-1}(t-s)|H_{n-1}Cx(s)|^2_{F_2}ds.\end{aligned}\tag{47}$$

Using the additional term

$$V_{F_2}=\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}Cx(s)|^2_{F_2}ds\tag{48}$$

with $F_2>0$ to compensate $\varrho_3(t)$, we have

$$\mathbf{E}\mathcal{L}V_{F_2}+\mathbf{E}2\alpha V_{F_2}\le\mathbf{E}|H_{n-1}Cx(t)|^2_{F_2}-\mathbf{E}he^{-2\alpha(n-1)h}\varrho_3^T(t)F_2\varrho_3(t).\tag{49}$$

We now consider the functional

$$V_3=V_1+\bar V_3,\tag{50}$$

where V_1 is defined by (14) with $f_1(t)$ changed by $f_2(t)$ and

$$\bar V_3=V_{W_0}+\sum_{i=1}^{n-2}(V_{W_i}+h^2e^{2\alpha ih}\bar V_{W_i})+V_{W_{n-1}}+\frac{(nh-h)^2}{4}V_Q+\frac{(n-1)h}{2}V_{F_2}.\tag{51}$$

Then LMI conditions are derived as follows:

Theorem 3. *Given $\bar{K}_i$ $(i = 0, \dots, n-1)$ let the derivative-dependent feedback (4) exponentially stabilizes (2), where $C = 0$, with a decay rate $\bar{\alpha} > 0$.*
(i) Given tuning parameters $h > 0$ and $\alpha \in (0, \bar{\alpha})$, let there exist $nk \times nk$ matrix $P > 0$, and $k \times k$ matrices $W_0 > 0$, $R_i > 0$, $W_i > 0$ $(i = 1, \dots, n-1)$, $F_1 > 0$, $F_2 > 0$ and $Q > 0$ that satisfy

$$\Phi_3 < 0, \quad \Omega < 0, \tag{52}$$

where Φ_3 and Ω are the symmetric matrices composed from

$$\begin{aligned}
&\Phi_{11} = sym\{PD\} + 2\alpha P + C^T PC + h^2 |H_1|^2_{W_0} \\
&\quad + \textstyle\sum_{i=1}^{n-2} |H_{i+1}|^2_{\frac{(ih)^2}{4} R_i + h^2 e^{2\alpha ih} W_i} + \frac{(n-1)h}{2} |H_{n-1}C|^2_{F_1+F_2}, \\
&\Phi_{12} = P\bar{B}[\bar{K}_1, \dots, \bar{K}_{n-1}], \qquad \Phi_{14} = P\bar{B}\bar{K}, \\
&\Phi_{15} = \tfrac{(n-1)h}{2} D^T H^T_{n-1}(R_{n-1} + Q), \\
&\Phi_{22} = -diag\{e^{-2\alpha ih} R_i\}_{i=1}^{n-1}, \\
&\Phi_{23} = [0_{k\times(n-2)k}, -e^{-2\alpha(n-1)h} R_{n-1}]^T, \\
&\Phi_{25} = \tfrac{(n-1)h}{2} [\bar{K}_1, \dots, \bar{K}_{n-1}]^T \bar{B}^T H^T_{n-1}(R_{n-1} + Q), \\
&\Phi_{33} = -e^{-2\alpha(n-1)h}(R_{n-1} + F_1), \\
&\Phi_{44} = -\tfrac{\pi^2}{4} e^{-2\alpha h} diag\{W_i\}_{i=0}^{n-1}, \\
&\Phi_{45} = \tfrac{(n-1)h}{2} \bar{K}^T \bar{B}^T H^T_{n-1}(R_{n-1} + Q), \\
&\Phi_{55} = -R_{n-1} - Q, \\
&\Omega_{11} = W_{n-1} - \tfrac{(n-1)^2}{4} e^{-2\alpha(n-1)h} Q, \qquad \Omega_{12} = W_{n-1}, \\
&\Omega_{22} = W_{n-1} - \tfrac{n-1}{2} e^{-2\alpha(n-1)h} F_2
\end{aligned}$$

with D and $\bar{K}$ given by (3). Then the sampled-data feedback (37) with a sampling period $h > 0$ and controller gains (7) exponentially mean-square stabilizes (2) with a decay rate $\alpha > 0$.
(ii) Given any $\alpha \in (0, \bar{\alpha})$, the LMI of item (i) is always feasible for small enough stochastic perturbations and $h > 0$ (meaning that the sampled-data feedback (37) with controller gains (7) exponentially stabilizes (2) with a decay rate $\alpha > 0$).

Proof. (i) Following arguments of Theorem 1 and in view of (40)–(49), we obtain

$$\begin{aligned}
&\mathbf{E}\mathcal{L}V_3 + \mathbf{E}2\alpha V_3 \le \mathbf{E}\xi_3^T \bar{\Phi}_3 \xi_3 + \mathbf{E}h^2 \eta^T \Omega \eta \\
&\quad + \mathbf{E}\tfrac{(nh-h)^2}{4} |H_{n-1} f_2(t)|^2_{R_{n-1}+Q},
\end{aligned} \tag{53}$$

where $\xi_3 = col\{\xi_1, \delta_0(t), \dots, \delta_{n-1}(t)\}$ with ξ_1 given by (19), $\eta = col\{\varrho_2(t), \varrho_3(t)\}$ and $\bar{\Phi}_3$ is obtained from Φ_3 given by (52) by taking away the last block-column and block-row. Therefore, by substituting $f_2(t)$ given by (39) and applying Schur's complement the system (38) is exponentially mean-square stable with a decay rate α.
(ii) The proof of (ii) is similar to (ii) of Theorem 1. □

3.2. Stability via neutral type model transformation

As in the previous section, we also use neutral type model transformation. Then the system (38) is equivalent to

$$dz(t) = (Dx(t) + \textstyle\sum_{i=0}^{n-1} \bar{B}\bar{K}_i \delta_i(t))dt + Cx(t)dw(t) \tag{54}$$

with D, $z(t)$ and $\delta_i(t)$ given by (3), (24) and (39), respectively. For (54), we consider the functional

$$V_4 = V_2 + \bar{V}_3, \tag{55}$$

where V_2 and $\bar{V}_3$ are given by (28) and (51), respectively. For the term V_P of V_2, we have

$$\begin{aligned}
&\mathcal{L}V_P + 2\alpha V_P = 2\zeta^T(t) P \tilde{\zeta}(t) \\
&\quad + x^T(t) C^T P_{11} C x(t) + 2\alpha \zeta^T(t) P \zeta(t),
\end{aligned} \tag{56}$$

where $\tilde{\zeta}(t) = \bar{\Xi}_2 \xi_2$ with ξ_2 given below (29) and

$$\bar{\Xi}_2 = \Xi_2 + col\{\textstyle\sum_{i=0}^{n-1} \bar{B}\bar{K}_i \ell_{2n+i-1}, 0_{nk\times lk}\}. \tag{57}$$

Here Ξ_2 is defined by (25). When we differentiate V_4, we will employ (46) that depends on f_2. If the definition (39) of $f_2(t)$ is directly applied, then the terms $\kappa_i(t)$ $(i = 1, \dots, n-1)$ should be compensated that complicates the stability analysis. To avoid this, we use (23) to obtain a different representation

$$\begin{aligned}
&f_2(t) = \bar{D}x(t) + \bar{B}\bar{K}_0 \delta_0(t) + \textstyle\sum_{i=1}^{n-1} \bar{B}\bar{K}_i(\delta_i(t) - \mu_i(t)), \\
&\bar{D} = D - \textstyle\sum_{i=1}^{n-1} \bar{B}\bar{K}_i H_i.
\end{aligned} \tag{58}$$

By using V_4 of (55) and in view of (56), (58), we arrive at the following LMI conditions for the system (54):

Theorem 4. *Given $\bar{K}_i$ $(i = 0, \dots, n-1)$ let the derivative-dependent feedback (4) exponentially stabilizes (2), where $C = 0$, with a decay rate $\bar{\alpha} > 0$.*
(i) Given tuning parameters $h > 0$ and $\alpha \in (0, \bar{\alpha})$, let there exist $(2n-1)k \times (2n-1)k$ matrix $P = [P_{ij}]_{n\times n}$, $k \times k$ matrices $W_0 > 0$, $R_i > 0$, $S_i > 0$, $W_i > 0$ $(i = 1, \dots, n-1)$, $F_2 > 0$ and $Q > 0$ that satisfy

$$\Psi > 0, \quad \Phi_{4aug} < 0, \quad \Omega < 0, \tag{59}$$

where Ψ and Ω are given by Theorems 2 and 3, and

$$\Phi_{4aug} = \bar{\Phi}_{4aug} + sym\{\Xi_1^T P \bar{\Xi}_2\} + 2\alpha \Xi_1^T P \Xi_1.$$

Here Ξ_1 and $\bar{\Xi}_2$ are given by (25) and (57), and $\bar{\Phi}_{4aug}$ is the symmetric matrix composed from

$$\begin{aligned}
&\Phi_{11} = C^T P_{11} C + \textstyle\sum_{i=1}^{n-1} |H_i|^2_{\frac{(ih)^2}{4} R_i + S_i} \\
&\qquad + \textstyle\sum_{i=0}^{n-2} h^2 e^{2\alpha ih} |H_{i+1}|^2_{W_i} + \frac{nh-h}{2} |H_{n-1}C|^2_{F_2}, \\
&\Phi_{15} = \tfrac{(n-1)h}{2} \bar{D}^T H^T_{n-1} Q, \\
&\Phi_{22} = -diag\{e^{-2\alpha ih} R_i\}_{i=1}^{n-1}, \\
&\Phi_{33} = -diag\{e^{-2\alpha ih} S_i\}_{i=1}^{n-1}, \\
&\Phi_{35} = -\tfrac{(n-1)h}{2} [\bar{K}_1, \dots, \bar{K}_{n-1}]^T \bar{B}^T H^T_{n-1} Q, \\
&\Phi_{44} = -\tfrac{\pi^2}{4} e^{-2\alpha h} diag\{W_i\}_{i=0}^{n-1}, \\
&\Phi_{45} = \tfrac{(n-1)h}{2} \bar{K}^T \bar{B}^T H^T_{n-1} Q, \qquad \Phi_{55} = -Q
\end{aligned}$$

with $\bar{K}$ given by (3). Then the sampled-data feedback (37) with a sampling period $h > 0$ and controller gains (7) exponentially mean-square stabilizes (2) with a decay rate $\alpha > 0$.
(ii) Given tuning parameters $h > 0$ and $\alpha \in (0, \bar{\alpha})$, let there exist $nk \times nk$ matrix $P_{11} > 0$, $k \times k$ matrices $W_0 > 0$, $R_i > 0$, $S_i > 0$, $W_i > 0$ $(i = 1, \dots, n-1)$, $F_2 > 0$ and $Q > 0$ that satisfy

$$\Phi_{4sim} < 0, \quad \Omega < 0, \tag{60}$$

where Φ_{4sim} is obtained from Φ_{4aug} given by (59) by setting $P = diag\{P_{11}, 0\}$ and Ω is given by Theorem 3. Then the sampled-data feedback (37) with a sampling period $h > 0$ and controller gains (7) exponentially mean-square stabilizes (2) with a decay rate $\alpha > 0$.
(iii) Given any $\alpha \in (0, \bar{\alpha})$, the LMIs of items (i) and (ii) are always feasible for small enough stochastic perturbations and $h > 0$ (meaning that the sampled-data feedback (37) with controller gains (7) exponentially stabilizes (2) with a decay rate $\alpha > 0$).

Table 4
Max. delay h for different σ (continuous-time).

σ	0	0.05	0.1	0.2	0.3	0.4
$\Phi_1 < 0$	0.508	0.501	0.476	0.353	0.127	0.015
$\Phi_{2aug} < 0$	0.639	0.638	0.637	0.634	0.626	0.358
$\Phi_{2sim} < 0$	0.508	0.508	0.508	0.505	0.500	0.358

Table 5
Max. sampling period h for different σ (sampled-data).

σ	0	0.05	0.1	0.2	0.3	0.4
$\Phi_3 < 0$	0.103	0.102	0.098	0.075	0.028	0.003
$\Phi_{4aug} < 0$	0.099	0.098	0.096	0.088	0.054	0.010
$\Phi_{4sim} < 0$	0.065	0.063	0.060	0.050	0.027	0.006

3.3. Example: Furuta pendulum

Example 3 (*Selivanov & Fridman, 2018a*). Consider a model of Furuta pendulum given by (2) with $x = col\{\theta, \phi, \dot{\theta}, \dot{\phi}\}$ and

$$\left[\, A \,\vdots\, \bar{B} \,\right] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 37.377 & 0 & -0.515 & 0.142 & -35.42 \\ -8.228 & 0 & 0.113 & -0.173 & 43.28 \end{bmatrix}, \quad (61)$$

where θ is the angular position of the pendulum, ϕ is the angle of the rotational arm. The nonlinear model of Furuta pendulum consists of two coupled nonlinear 2nd-order differential equations with respect to θ and ϕ. We consider a model linearized in the unstable equilibrium point 0. Following the classical arguments for stochastic systems (see e.g. Shaikhet, 2013; Yaesh et al., 2004), while considering a linearized system (two coupled 2nd-order differential equations) we add multiplicative noise that models the error due to linearization (this error increases when state becomes larger). We therefore consider (2) with (61) and $C_i = \sigma I_2$ $(i = 0, 1)$, where $\sigma \in \mathbb{R}$. The continuous-time state-feedback is given by (4) with gains from Selivanov and Fridman (2018a):

$$\bar{K}_0 = [1.2826\ 0.0013], \quad \bar{K}_1 = [0.1209\ 0.0086]. \quad (62)$$

Consider now the system under the continuous-time delayed controller (6). For different values of σ and $\alpha = 0.01$, the maximum values of delay h that preserve the exponential mean-square stability are presented in Table 4. Note that the stability conditions of Fridman and Shaikhet (2019) lead to the same results as shown in Table 4. However, they are not applicable to the nth-order systems for $n > 2$ and to sampled-data control for $n \geq 2$.

Consider next the sampled-data feedback given by (37) with $\bar{K}_0$ and $\bar{K}_1$ in (62). For different values of σ and $\alpha = 0.01$, Table 5 presents the maximum values of sampling period h via LMIs $\Phi_3 < 0$ and $\Omega < 0$, LMIs $\Psi > 0$, $\Phi_{4aug} < 0$ and $\Omega < 0$, LMIs $\Phi_{4sim} < 0$ and $\Omega < 0$ (for brevity we write them as $\Phi_3 < 0$, $\Phi_{4aug} < 0$ and $\Phi_{4sim} < 0$, respectively). In the deterministic case ($\sigma = 0$), the result via LMI $\Phi_3 < 0$ coincides with the one of Selivanov and Fridman (2018a). Clearly, LMI $\Phi_{4aug} < 0$ improves the results via $\Phi_{4sim} < 0$, and LMI $\Phi_{4sim} < 0$ leads to efficient results for large stochastic perturbations.

Fig. 1 plots $|x|$ with $x(0) = [\pi, 0, 0, 0]^T$. From Fig. 1, it is clear that the sampled-data static output-feedback (37) leads almost to the same performance as the full state-feedback (4). Compared with the state-feedback (4) that requires the continuously measured full system state $[\theta, \phi, \dot{\theta}, \dot{\phi}]^T$, the sampled-data feedback (37) uses only the sampled measurements $\theta(t_{k-1})$, $\theta(t_k)$ and $\phi(t_{k-1})$, $\phi(t_k)$.

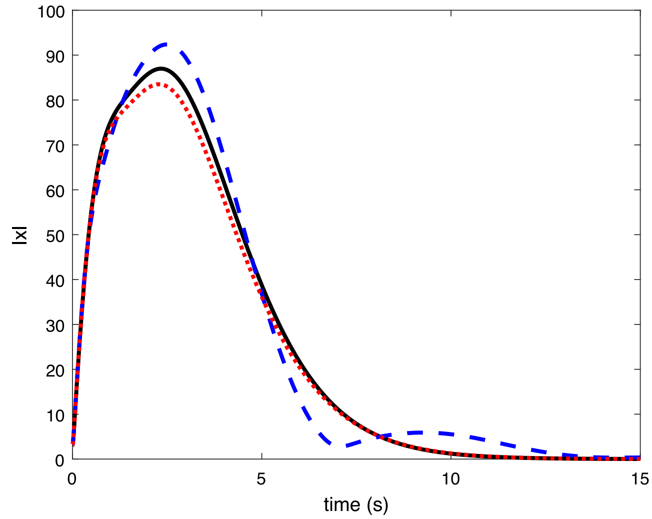

Fig. 1. Dynamics of (2) with (61) and $\sigma = 0.4$ under the state-feedback (4) (black solid line), delay-dependent feedback (6) with $h = 0.358$ (blue dashed line), and sampled-data feedback (37) with $h = 0.010$ (red dotted line).

4. Conclusions

This paper presented the time-delay and sampled-data implementations of derivative-dependent control for the nth-order stochastic systems. This was done by extending the recent efficient results in the deterministic case and by developing two methods (direct method and neutral type model transformation). The presented methods can be extended to PID control of stochastic systems. This may be a topic for future research.

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