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Averaging of systems with fast-varying coefficients and non-small delays with application to stabilization of affine systems via time-dependent switching[☆,☆☆]

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ABSTRACT

This paper investigates the stability of systems with fast-varying piecewise-continuous coefficients and non-small delays. Starting from a recent constructive time-delay approach to periodic averaging, that allowed finding upper bound on small parameter $\epsilon > 0$ preserving the stability of the original delay-free systems, here we extend the method to systems with non-small delays and provide their input-to-state stability (ISS) analysis. The original time-delay system is transformed into a neutral type one embedding both initial non-small delay, whose upper bound is essentially larger than ϵ and does not vanish for $\epsilon \to 0$, and an additional induced delay due to transformation, whose length is proportional to ϵ. By exploiting Lyapunov–Krasovskii theory, we derive ISS conditions expressed as Linear Matrix Inequalities (LMIs), whose solution allows evaluating upper bounds both on small parameter ϵ and non-small delays preserving the ISS of the original time-delay system, as well as the resulting ultimate bound of its solutions. We further apply our results to stabilization of delayed affine systems by time-dependent switching. Three numerical examples illustrate the effectiveness of the approach.

1. Introduction

Averaging is one of the most powerful tools to deal with the stability of time-varying systems with a small parameter $\epsilon > 0$. [1–3]. The key idea of averaging relies in the approximation of the solution of a time-varying system by the one of the averaged system. It has been proven that, under the assumption of exponential stability of the averaged system, the asymptotic stability of the original one can be also guaranteed if ϵ is small enough [1]. However, as pointed out in [4], the main drawback of this classical approach is the inability to provide an efficient quantitative upper bound on the small parameter ϵ till which stability is still ensured, thus fixing its proper value on the basis of numerical simulations [5]. To overcome this limitation, a time-delay-based approach to periodic averaging has been recently introduced in [4], where the focus is to present original system as a neutral type system whose delay length is equal to ϵ. This kind of

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representation enables the derivation of input-to-state stability (ISS) conditions in terms of Linear Matrix Inequalities (LMIs) by leveraging Lyapunov-Krasovskii theory for time-delay systems. These LMIs allow finding the upper bound of the small parameter preserving the stability for a certain decay rate. Moreover, different from classical averaging theory, where system coefficients are assumed continuous in time [1], this new approach allows considering them as piecewise-continuous, thus covering also the class of fast switching systems. An extension of this approach can be found in [6], where ISS and L_2-gain analysis are provided both for deterministic and stochastic systems, while in [7] the time-delay approach has been applied for extremum seeking systems.

The classical averaging for systems with delays was presented in [5,8,9]. For instance, the stabilization of the inverted pendulum in the presence of feedback delays and periodic disturbances has been studied in [5] via classical averaging tools, but fixing both ϵ and delay values on the basis of numerical simulations and without providing stability conditions able to find their upper bounds preserving stability performances. The significance of [5] is that it shows that the appropriate averaged equations retain the delay term, as opposed to earlier results which suggest that the delay term can be neglected. The time-delay approach to periodic averaging for systems with small delays of the order of ϵ was developed in [4]. However, constructive conditions with efficient quantitative bounds for systems with non-small delays whose upper bound is essentially larger than ϵ and does not vanish for $\epsilon \to 0$ is still an open problem.

As a subclass of hybrid systems, switched systems have received wide attention and there exist many contributions aiming at stabilizability [10], synchronization [11] and fault estimation [12]. Among switched systems family, the switched affine systems have attracted great interests due to its practical applications including DC–DC power conversion [13,14] and biochemical networks [15]. The control goal is first to find a region of attainable equilibrium points and then designing a proper switching function to drive the state trajectories to the desired one. In general, the switched affine system have several equilibrium points which may not be equilibrium point of any isolated subsystem [16]. Thus, the control is very challenging and requires an appropriate switching rule in order to achieve practical stabilization in the neighborhood of the desirable equilibrium point [14,17–21]. For designing the switching function that stabilizes unstable linear systems, the existence of Hurwitz convex combination guarantees existence of both state- and time-dependent stabilizing switching rules [22]. For stabilization of the switched affine systems, most of the works suggest the state-dependent switching [18,19]. Recently state-dependent switching was extended to affine systems with state delay [23,24]. Furthermore, to enlarge switching frequency, switching control together with event-triggered mechanism have been employed for general LPV systems [25–27].

Differently from the state-dependent switching, the time-dependent switching law does not need to perform measurements and calculations. The time-dependent switching law of linear systems with Hurwitz convex combination and uncertainties can be designed by using periodic averaging as was suggested recently in [4], where the switching period can be found from LMIs. However, results of [4] were confined to linear uncertain systems (without the affine terms), whereas the delay was of the order of $\mathcal{O}(\epsilon)$.

The aim of this work is to extend the time-delay approach to periodic averaging to the class of linear systems with fast-varying coefficients in the presence of non-small delays. For this class of systems we consider the ISS analysis, which allows providing the explicit expression of the Ultimate Bound (UB) for the solutions of the original systems. The result of the proposed procedure are LMI-based conditions for finding upper bounds both on small parameter ϵ and non-small delays preserving ISS for desired decay rate. Therefore, the main contributions of the work can be summarized as follows: (*i*) different from [4] (see Sect. 5), where state delays upper bound is of order of $\mathcal{O}(\epsilon)$, through this paper we relax this assumption on delays size by considering non-small delays; (*ii*) the ISS analysis for this class of systems via Lyapunov-Krasovskii theory leads to bounds on both delay and ϵ ensuring stability, as well as an explicit form of the UB; (*iii*) the results are applied to delayed switched affine systems allowing simple time-dependent switching in the presence of delays and system uncertainties. As pointed out in [5], extension of averaging to delays whose upper bound is essentially larger than ϵ is important in many practical applications due to non-small delays that appear in feedback controllers or internal dynamics latencies. To achieve this goal, we use a novel neutral-type transformation with respect to [4] which implies also the need of a novel Lyapunov-Krasovskii functional. This latter leads for the first time to stability conditions that allows analytically finding upper bounds on ϵ and non-small delays, whose value is essentially larger than ϵ.

The structure of the paper is given as follows. The time-delay approach to periodic averaging for fast-varying systems with non-small delays and its ISS analysis are presented in Section 2 and Section 2.2, respectively, while a numerical example is carried out in Section 2.3. In Section 3 the results are applied to delayed switched affine systems, with simulations presented in Section 3.1. Finally, conclusions are drawn in Section 4.

1.1. Notation and useful lemma

Throughout the manuscript $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $|\cdot|$, while $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices with induced matrix norm $\|\cdot\|$. The superscript $\top$ stands for matrix transposition, while the notation $P > 0$ with $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite, where its symmetric elements are denoted by $\star$. $C[-h_M, 0]$ is the Banach space of continuous functions $\phi : [-h_M, 0] \to \mathbb{R}^n$ with the norm $\|\phi\|_C = \max_{\theta\in[-h_M,0]} |\phi(\theta)|$. We also denote by $W[-h_M, 0]$ the space of absolutely continuous functions $\phi : [-h_M, 0] \to \mathbb{R}^n$, with $\frac{d\phi}{d\theta} \in L_2(-h_M, 0)$ and with the norm $\|\phi\|_W = \|\phi\|_C + \left\|\frac{d\phi}{d\theta}\right\|_{L_2}$. $L_\infty(0, t)$ is the space of essentially bounded functions $\phi : (0, t) \to \mathbb{R}^n$ with the norm $\|\phi[0,t]\|_\infty = ess\,\sup_{\theta\in(0,t)}|\phi(\theta)|$. Moreover, a vector function $h(\epsilon)$ is of the order of ϵ, i.e. $h(\epsilon)\sim\mathcal{O}(\delta(\epsilon))$, if there exist positive constants k and c such that $|h(\epsilon)|\leq k|\delta(\epsilon)|,\ \forall|\epsilon|<c$ (see Definition 10.1 on p. 383 of [1]).

The following useful lemma on Jensen's inequality and its extended version is given in the sequel that is instrumental through the paper [28].

Lemma 1 *([29]). For any $n\times n$ matrix $R>0$, scalars $\alpha\leq\beta$, functions $f:[\alpha,\beta]\to\mathbb{R}$ and $\phi:[\alpha,\beta]\to\mathbb{R}^n$ such that the integration concerned are well defined, the following Jensen's inequality*

$$\int_\alpha^\beta \phi^\top(s)\,ds R\int_\alpha^\beta \phi(s)\,ds \leq (\beta-\alpha)\int_\alpha^\beta \phi^\top(s)R\phi(s)\,ds, \tag{1}$$

as well as the extended Jensen's inequalities hold:

$$\int_\alpha^\beta f(s)\phi^\top(s)\,ds R\int_\alpha^\beta f(s)\phi(s)\,ds \leq \int_\alpha^\beta |f(\theta)|\,d\theta\int_\alpha^\beta |f(s)|\phi^\top(s)R\phi(s)\,ds, \tag{2}$$

$$\int_\alpha^\beta\int_s^\beta \phi^\top(\theta)\,d\theta\,ds R\int_\alpha^\beta\int_s^\beta \phi(\theta)\,d\theta\,ds \leq \frac{(\beta-\alpha)^2}{2}\int_\alpha^\beta\int_s^\beta \phi^\top(\theta)R\phi(\theta)\,d\theta\,ds. \tag{3}$$

2. Periodic averaging of systems with non-small delays

Given piecewise-continuous $A:[0,\infty)\to\mathbb{R}^{n\times n}$ and $B:[0,\infty)\to\mathbb{R}^{n\times n_w}$, a constant matrix $A_d\in\mathbb{R}^{n\times n}$ and a small parameter $\epsilon>0$, we consider the following class of fast-varying systems (see [8]):

$$\dot{x}(t) = A(\tfrac{t}{\epsilon})x(t) + (A_d+\Delta A_d(t))x(t-h(t)) + B(\tfrac{t}{\epsilon})w(t), \qquad t\geq 0, \tag{4}$$

where $x(t)\in\mathbb{R}^n$ is the system state and $w(t)\in\mathbb{R}^{n_w}$ is the disturbance, assumed to be locally essentially bounded, i.e. $w(t)\in L_\infty(0,t),\forall t>0$. Moreover, $\Delta A_d(t)$ stands for parameter uncertainties affecting the delayed part such that $\|\Delta A_d(t)\|\leq\kappa_d$, with $\kappa_d>0$ a known constant. The function $h(t)$ is the delay, assumed to be time-varying and bounded, i.e. $0\leq h(t)\leq h_M$. Initial conditions of system (4) are given as $x(\theta)=\phi(\theta),\theta\in[-h_M,0]$ and ϕ absolutely continuous with $\dot{\phi}\in L_2[-h_M,0]$.

Remark 1. System (4) contains both fast time $\frac{t}{\epsilon}$ and slow time t. To deal with the interaction of slow and fast variables, classical averaging procedure has been deeply exploited [1,8]. Note that, compared to classical LPV system $\dot{x}(t)=A(t)x(t)$, the introduction of small parameter $\epsilon>0$ rescales this latter to the fast-time $\frac{t}{\epsilon}$. Thus, for ϵ small enough, $A(\frac{t}{\epsilon})$ varies faster than $A(t)$.

Let us introduce the following assumptions that are instrumental through the paper.

Assumption 1. The following holds:

$$\frac{1}{\epsilon T}\int_{t-\epsilon T}^t A(\frac{s}{\epsilon})ds = A_{av}+\Delta A(\tfrac{t}{\epsilon}), \quad \|\Delta A(\tfrac{t}{\epsilon})\|\leq\kappa, \qquad \forall\tfrac{t}{\epsilon}\geq T, \tag{5}$$

where $A_{av}+A_d$ is Hurwitz matrix, T is the averaging period and $\kappa>0$ is a small enough constant.

If $A_{av}+A_d$ is Hurwitz, then the unperturbed averaged system

$$\dot{x}_{av}(t) = [A_{av}+\Delta A(\tfrac{t}{\epsilon})]x_{av}(t) + (A_d+\Delta A_d(t))x_{av}(t-h(t)) \tag{6}$$

is exponentially stable for small enough $\kappa>0,\ \kappa_d>0$ and $h_M\sim\mathcal{O}(\epsilon)$ [4,29]. Here $\Delta A(\frac{t}{\epsilon})$ involves system uncertainty, whose norm is upper bounded by a known constant $\kappa>0$.

Assumption 2. Following [4], we assume that all the entries $a_{kj}(\frac{t}{\epsilon})$ of $A(\frac{t}{\epsilon})$ in (4) belong to some finite intervals, i.e., $a_{kj}(\frac{t}{\epsilon})\in[\underline{a}_{kj},\bar{a}_{kj}]$ for $\frac{t}{\epsilon}\geq T$, meaning that all a_{kj} are uniformly bounded, $\forall k,j=1,\dots,n$.

If Assumption 2 is fulfilled, then $A(\tau)$ in (4) with $\tau=\frac{t}{\epsilon}$ can be expressed as the following convex combination:

$$A(\tau) = \sum_{i=1}^N \rho_i(\tau)A_i \quad \forall\tau\geq T, \quad \rho_i\geq 0, \quad \sum_{i=1}^N\rho_i = 1, \quad 1\leq N\leq 2^{n^2}, \tag{7}$$

being A_i constant matrices with entries $\underline{a}_{kj}$ or $\bar{a}_{kj}$.

Assumption 3. All the entries $b_{kv}(\frac{t}{\epsilon}),\ v=1,\dots,n_w$, of $B(\frac{t}{\epsilon})$ in (4) belong to some finite intervals, i.e., $b_{kv}(\frac{t}{\epsilon})\in[\underline{b}_{kv},\bar{b}_{kv}]$ for $\frac{t}{\epsilon}\geq T$, meaning that all b_{kv} are uniformly bounded.

If Assumption 3 holds, then $B(\tau)$ can be presented as

$$B(\tau)=\sum_{l=1}^{\bar{N}} f_l(\tau)B_l,\quad \forall\tau\geq T,\quad f_l\geq 0,\quad \sum_{l=1}^{\bar{N}} f_l=1,\quad 1\leq\bar{N}\leq 2^{n\times n_w},\tag{8}$$

where B_l are constant matrices whose entries are $\underline{b}_{kv}$ or $\bar{b}_{kv}$.

2.1. Transformation to a neutral type system

Following the time-delay approach to averaging [4], we will present (4) in the form of neutral type system with additional distributed delays of the length of ϵT. Let us introduce the following notations:

$$g(t,\epsilon)=A(\tfrac{t}{\epsilon})x(t)+B(\tfrac{t}{\epsilon})w(t),\tag{9}$$

$$G(t,\epsilon)\triangleq\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}(s-t+\epsilon T)g(s,\epsilon)\,ds.\tag{10}$$

For shortness, we omit the dependence on ϵ throughout this paper of $g,\ G$ and Y in (14) below.

Then, by exploiting [6,30], it follows:

$$\begin{aligned}\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}\dot{x}(s)\,ds=\frac{1}{\epsilon T}[x(t)-x(t-\epsilon T)]&=\frac{d}{dt}\left[x(t)-G(t)-\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}(A_d+\Delta A_d(s))(s-t+\epsilon T)x(s-h(s))\,ds\right]\\&=\frac{d}{dt}[x(t)-G(t)]+\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}(A_d+\Delta A_d(s))x(s-h(s))\,ds-(A_d+\Delta A_d(t))x(t-h(t)).\end{aligned}\tag{11}$$

Integrating (4) on $[t-\epsilon T,t]$ for $t\geq\epsilon T+h_M$ and denoting $z(t)=x(t)-G(t)$, we obtain:

$$\dot{z}(t)=\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}A(\frac{s}{\epsilon})x(s)\,ds+(A_d+\Delta A_d(t))x(t-h(t))+\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}B(\frac{s}{\epsilon})w(s)\,ds.\tag{12}$$

By exploiting Assumption 1, the first integral term of (12) can be presented as

$$\begin{aligned}\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}A(\frac{s}{\epsilon})\,[x(s)+x(t)-x(t)]\;ds=&\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}A(\frac{s}{\epsilon})x(t)\,ds+\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}A(\frac{s}{\epsilon})[x(s)-x(t)]\,ds\\=&\left[A_{av}+\Delta A(\tfrac{t}{\epsilon})\right]x(t)-Y(t),\end{aligned}\tag{13}$$

with

$$Y(t)=\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}A(\frac{s}{\epsilon})\int_{s}^{t}\dot{x}(\theta)\,d\theta\,ds,\tag{14}$$

while for the second integral term of (12) we exploit Assumption 3, i.e.,

$$\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}B(\frac{s}{\epsilon})w(s)\,ds=\int_0^1 B(\tfrac{t}{\epsilon}-T\theta)w(t-\epsilon T\theta)\,d\theta=\sum_{l=1}^{\bar{N}}B_l w_l(t),\tag{15}$$

with

$$w_l(t)\triangleq\int_0^1 f_l(\tfrac{t}{\epsilon}-T\theta)w(t-\epsilon T\theta)\,d\theta.\tag{16}$$

Note that

$$|w_l(t)|=\left|\int_0^1 f_l(\tfrac{t}{\epsilon}-T\theta)w(t-\epsilon T\theta)\,d\theta\right|\leq\int_0^1|f_l(\tfrac{t}{\epsilon}-T\theta)||w(t-\epsilon T\theta)|\,d\theta\leq\|w[0,t]\|_\infty,\tag{17}$$

$\forall l=\{1,\ldots,\bar{N}\},\ t\geq\epsilon T$ due to $0\leq f_l\leq 1$. Therefore, system (12) can be finally rewritten as:

$$\dot{z}(t)=\left[A_{av}+\Delta A(\tfrac{t}{\epsilon})\right]x(t)-Y(t)+(A_d+\Delta A_d(t))x(t-h(t))+\sum_{l=1}^{\bar{N}}B_l w_l(t),\quad t\geq\epsilon T+h_M.\tag{18}$$

System (18) is a kind of neutral type system, where $\dot{x}$ is given by (4). If $w(t)=0$, (18) can be considered as a perturbation of the averaged system (6) due to the presence of additional terms $G(t)$ and $Y(t)$, both of them of the order of $\mathcal{O}(\epsilon)$ provided $x(t)$ and $\dot{x}(t)$ are of the order of $\mathcal{O}(1)$. Therefore, any solution $x(t)$ of (4) satisfies (18). Thus, ISS of (18) guarantees ISS of (4).

2.2. ISS analysis via direct Lyapunov method

In the sequel, we leverage Lyapunov-Krasovskii method for time-delay systems in order to find ISS conditions expressed as LMIs. Upper bounds $\epsilon^\star$ on ϵ and h_M on the delay $h(t)$ that ensure ISS of (18) can be found from these LMIs.

Theorem 1. *Let Assumptions 1–3 hold. Given matrices A_{av}, $A_i (i = 1, \dots, N)$, A_d, $B_l (l = 1, \dots, \bar{N})$ and positive constants κ, κ_d, α, $\epsilon^\star$, T and h_M, let there exist positive-definite matrices P, R, H, W, S and $\bar{H} \in \mathbb{R}^{n\times n}$, a matrix $U \in \mathbb{R}^{n\times n}$ and positive scalars $b_0, b_1, \dots, b_{\bar{N}}, \lambda, \lambda_d$ that satisfy the following LMIs:*

$$\begin{bmatrix} P & -P \\ \star & P + e^{-2\alpha\epsilon^\star T} R \end{bmatrix} \geq 0, \tag{19}$$

$$\begin{bmatrix} W & U \\ \star & W \end{bmatrix} \geq 0, \tag{20}$$

$$\frac{1}{T^2}\int_{\tau-T}^{\tau} (\zeta - \tau + T) A^\top(\zeta) H A(\zeta)\, d\zeta \leq \bar{H} \quad \forall \tau \geq T, \tag{21}$$

and

$$\left[\begin{array}{c|ccc} \Omega & \begin{array}{c} \sqrt{\epsilon^\star T}A_i^\top R \\ 0_{3n\times n} \\ 0_{n\times n} \\ \hline 0_{(2n+\bar{N})\times n} \\ \hline \sqrt{\epsilon^\star T}B_l^\top R \end{array} & \begin{array}{c} \sqrt{\epsilon^\star T}A_i^\top \bar{H} \\ 0_{3n\times n} \\ \sqrt{\epsilon^\star T}A_d^\top \bar{H} \\ \hline 0_{n\times n} \\ \sqrt{\epsilon^\star T}\bar{H} \\ 0_{\bar{N}\times n} \\ \hline \sqrt{\epsilon^\star T}B_l^\top \bar{H} \end{array} & \begin{array}{c} \sqrt{h_M}A_i^\top W \\ 0_{3n\times n} \\ \sqrt{h_M}A_d^\top W \\ \hline 0_{n\times n} \\ \sqrt{h_M} W \\ 0_{\bar{N}\times n} \\ \hline \sqrt{h_M}B_l^\top W \end{array} \\ \hline \star & -R & 0_{n\times n} & 0_{n\times n} \\ & \star & -\bar{H} & 0_{n\times n} \\ & \star & \star & -W \end{array}\right] < 0 \quad i = 1, \dots, N,\ l = 1, \dots, \bar{N}, \tag{22}$$

where $\Omega \in \mathbb{R}^{\nu\times\nu}$ with $\nu = 7n + (\bar{N}+1)n_w$ is the symmetric block matrix whose elements are

$$\begin{aligned}
&\Omega_{11} = PA_{av} + A_{av}^\top P + 2\alpha P + S + \lambda\kappa^2 I_n - \frac{1}{h_M}\rho_H W \in \mathbb{R}^{n\times n}, \quad \Omega_{12} = -A_{av}^\top P - 2\alpha P \in \mathbb{R}^{n\times n},\\
&\Omega_{13} = \Omega_{24} = \Omega_{27} = -P \in \mathbb{R}^{n\times n},\\
&\Omega_{14} = \Omega_{17} = \Omega_{23} = P \in \mathbb{R}^{n\times n}, \quad \Omega_{15} = PA_d + \frac{1}{h_M}\rho_H (W - U) \in \mathbb{R}^{n\times n}, \quad \Omega_{16} = \frac{1}{h_M}\rho_H U \in \mathbb{R}^{n\times n},\\
&\Omega_{25} = -PA_d \in \mathbb{R}^{n\times n}, \ \Omega_{44} = -\lambda I_n \in \mathbb{R}^{n\times n},\\
&\Omega_{18} = P\begin{bmatrix} B_1 & B_2 & \dots & B_{\bar{N}} & 0_{n\times n_w}\end{bmatrix} \in \mathbb{R}^{n\times(\bar{N}+1)n_w}, \quad \Omega_{22} = -\frac{4}{\epsilon^\star T}\rho_\epsilon R + 2\alpha P \in \mathbb{R}^{n\times n}, \quad \Omega_{28} = -\Omega_{18},\\
&\Omega_{33} = -\frac{2}{\epsilon^\star T}\rho_\epsilon H \in \mathbb{R}^{n\times n},\\
&\Omega_{55} = -\frac{1}{h_M}\rho_H(2W - U - U^\top) + \lambda_d\kappa_d^2 I_n \in \mathbb{R}^{n\times n}, \quad \Omega_{56} = -\frac{1}{h_M}\rho_H (W - U) \in \mathbb{R}^{n\times n},\\
&\Omega_{66} = -\rho_H\left(S + \frac{1}{h_M}W\right) \in \mathbb{R}^{n\times n},\\
&\Omega_{77} = -\lambda_d I_n \in \mathbb{R}^{n\times n}, \quad \rho_\epsilon = e^{-2\alpha\epsilon^\star T}, \Omega_{88} = -diag\{b_1 I_{n_w}, \ \dots, \ b_{\bar{N}} I_{n_w}, \ b_0 I_{n_w}\} \in \mathbb{R}^{(\bar{N}+1)n_w\times(\bar{N}+1)n_w}, \quad \rho_H = e^{-2\alpha h_M}.
\end{aligned} \tag{23}$$

Then, for all $\epsilon \in (0, \epsilon^\star]$ there exists a positive constant ν such that the solutions of the delayed system (4) initialized by $\phi \in W[-h_M, 0]$ satisfy

$$|x(t)|^2 \leq \nu e^{-2\alpha(t-\epsilon^\star T - h_M)}\|\phi\|_W^2 + \left[\nu e^{-2\alpha(t-\epsilon^\star T-h_M)} + \frac{\sum_{l=0}^{\bar{N}} b_l}{2\alpha}\right]\|w(t)\|_\infty^2, \quad \forall t \geq 0 \tag{24}$$

for all locally essentially bounded $w(t)$ and $\phi \in W[-h_M, 0]$, meaning that (4) is ISS for all $\epsilon \in (0, \epsilon^\star]$ and $h(t) \in [0, h_M]$. Moreover, given $\Delta > 0$, the ball

$$\mathcal{X} = \left\{x \in \mathbb{R}^n : |x|^2 \leq \frac{b_0 + \dots + b_{\bar{N}}}{2\alpha}\Delta^2\right\} \tag{25}$$

is exponentially attractive with a decay rate α for (4).

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t) = V_P(t) + V_R(t) + V_H(t) + V_S(t) + V_W(t), \tag{26}$$

with

$$V_P(t) = z^\top(t)Pz(t), \tag{27}$$

$$V_R(t) = \frac{1}{\epsilon T}\int_{t-\epsilon T}^{t} e^{-2\alpha(t-s)}(s-t+\epsilon T)^2 g^\top(s)Rg(s)\,ds, \tag{28}$$

$$V_H(t) = \frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}\int_{s}^{t} e^{-2\alpha(t-\theta)}(s-t+\epsilon T)\dot{x}^\top(\theta)A^\top(\frac{s}{\epsilon})HA(\frac{s}{\epsilon})\dot{x}(\theta)\,d\theta\,ds, \tag{29}$$

$$V_S(t) = \int_{t-h_M}^{t} e^{-2\alpha(t-s)}x^\top(s)Sx(s)\,ds, \tag{30}$$

$$V_W(t) = \int_{t-h_M}^{t} (s-t+h_M)e^{-2\alpha(t-s)}\dot{x}^\top(s)W\dot{x}(s)\,ds. \tag{31}$$

Note that $V_R(t)$ and $V_H(t)$ in (28)–(29) are to compensate $G(t)$ and $Y(t)$ in (18), while $V_S(t)$ and $V_W(t)$ in (30)–(31) are standard terms for delay-dependent stability to compensate delay $x(t-h(t))$.

Differentiating $V_P(t)$ and $V_R(t)$ along the trajectories of (18) we have:

$$\begin{aligned}\dot{V}_P(t) + 2\alpha V_P(t) =& 2\left[x(t) - G(t)\right]^\top P[(A_{av} + \Delta A(\tfrac{t}{\epsilon}))x(t) - Y(t) + (A_d + \Delta A_d(t))x(t-h(t)) \\ &+ \sum_{l=1}^{\bar{N}} B_l w_l(t)] + 2\alpha\left[x(t) - G(t)\right]^\top P\left[x(t) - G(t)\right],\end{aligned} \tag{32}$$

$$\dot{V}_R(t) + 2\alpha V_R(t) = (\epsilon T)g^\top(t)Rg(t) - \frac{2}{\epsilon T}\int_{t-\epsilon T}^{t} e^{-2\alpha(t-s)}(s-t+\epsilon T)g^\top(s)Rg(s)\,ds. \tag{33}$$

For the integral term in (33), Jensen's inequality in (2) of Lemma 1 ensures that

$$2G^\top(t)RG(t) \le \int_{t-\epsilon T}^{t}(s-t+\epsilon T)g^\top(s)Rg(s)\,ds. \tag{34}$$

Then, inequality (33) can be re-written as

$$\dot{V}_R(t) + 2\alpha V_R(t) \le (\epsilon T)g^\top(t)Rg(t) - \frac{4}{\epsilon T}e^{-2\alpha\epsilon T}G^\top(t)RG(t), \tag{35}$$

with $g(t)$ presented as

$$g(t) = \sum_{i=1}^{N}\rho_i(\tfrac{t}{\epsilon})A_i x(t) + \sum_{l=1}^{\bar{N}} f_l(\tfrac{t}{\epsilon})B_l w(t), \tag{36}$$

where Assumptions 2 and 3 have been exploited. By differentiating $V_H(t)$ in (29) along (18), we find:

$$\dot{V}_H(t) + 2\alpha V_H(t) \le \dot{x}^\top(t)\Big(\frac{1}{\epsilon T}\int_{t-\epsilon T}^{t}(s-t+\epsilon T)A^\top(\frac{s}{\epsilon})HA(\frac{s}{\epsilon})\,ds\Big)\dot{x}(t) - \frac{1}{\epsilon T}e^{-2\alpha\epsilon T}\int_{t-\epsilon T}^{t}\int_{s}^{t}\dot{x}^\top(\theta)A^\top(\frac{s}{\epsilon})HA(\frac{s}{\epsilon})\dot{x}(\theta)\,d\theta\,ds. \tag{37}$$

For the first integral term in (37), as in [6], we apply the change of variable $s = \epsilon\zeta$ and employ inequality (21), i.e.,

$$\frac{1}{\epsilon^2 T^2}\int_{t-\epsilon T}^{t}(s-t+\epsilon T)A^\top(\frac{s}{\epsilon})HA(\frac{s}{\epsilon})\,ds = \frac{1}{T^2}\int_{\frac{t}{\epsilon}-T}^{\frac{t}{\epsilon}}(\zeta - \tfrac{t}{\epsilon} + T)A^\top(\zeta)HA(\zeta)\,d\zeta \le \bar{H}. \tag{38}$$

For the second integral term of (37) we leverage the extended Jensen's inequality in (3) of Lemma 1, i.e.,

$$2Y^\top(t)HY(t) \le \int_{t-\epsilon T}^{t}\int_{s}^{t}\dot{x}^\top(\theta)A^\top(\frac{s}{\epsilon})HA(\frac{s}{\epsilon})\dot{x}(\theta)\,d\theta\,ds, \tag{39}$$

thus obtaining the following inequality

$$\dot{V}_H(t) + 2\alpha V_H(t) \le (\epsilon T)\dot{x}^\top(t)\bar{H}\dot{x}(t) - \frac{2}{\epsilon T}e^{-2\alpha\epsilon T}Y^\top(t)HY(t). \tag{40}$$

Moreover, to compensate delayed terms, we differentiate $V_S(t)$ in (30) and $V_W(t)$ in (31), thus obtaining:

$$\begin{aligned}\dot V_S(t)+2\alpha V_S(t) =& x^\top(t)Sx(t)-e^{-2\alpha h_M}x^\top(t-h_M)Sx(t-h_M),\\ \dot V_W(t)+2\alpha V_W(t) =& h_M\dot x^\top(t)W\dot x(t)-\int_{t-h_M}^{t}e^{-2\alpha(t-s)}\dot x^\top(s)W\dot x(s)\,ds = h_M\dot x^\top(t)W\dot x(t)-\int_{t-h(t)}^{t}e^{-2\alpha(t-s)}\dot x^\top(s)W\dot x(s)\,ds\\ &-\int_{t-h_M}^{t-h(t)}e^{-2\alpha(t-s)}\dot x^\top(s)W\dot x(s)\,ds.\end{aligned}\tag{41}$$

By applying Jensen's inequality to the last two integral terms of (41) together with Park inequality (see Lemma 3.4 in [29,31]) it yields:

$$\dot V_W(t)+2\alpha V_W(t)\le h_M\dot x^\top(t)W\dot x(t)-\frac{e^{-2\alpha h_M}}{h_M}\begin{bmatrix} x(t)-x(t-h(t))\\ x(t-h(t))-x(t-h_M)\end{bmatrix}^\top\begin{bmatrix} W & U\\ \star & W\end{bmatrix}\begin{bmatrix} x(t)-x(t-h(t))\\ x(t-h(t))-x(t-h_M)\end{bmatrix},\tag{42}$$

with matrix $U\in\mathbb{R}^{n\times n}$ such that inequality (20) holds. Summing-up (32)–(35)–(40)–(41)–(42) and by applying S-procedure to compensate the terms $\Delta A(\frac{t}{\epsilon})x(t)$ and $\Delta A_d(t)x(t-h(t))$ in (32), along (18), for $t\ge\epsilon T+h_M$ it yields:

$$\dot V(t)+2\alpha V(t)\le \dot V(t)+2\alpha V(t)+\lambda[\kappa^2|x(t)|^2-|\Delta A(\tfrac{t}{\epsilon})x(t)|^2]+\lambda_d[\kappa_d^2|x(t-h(t))|^2-|\Delta A_d(t)x(t-h(t))|^2],\tag{43}$$

with some constant $\lambda>0$ and $\lambda_d>0$. Moreover, under Assumption 2 and Assumption 3, the derivative $\dot x(t)$ in (40)–(41) can be presented as

$$\dot x(t)=\sum_{i=1}^{N}\rho_i(\tfrac{t}{\epsilon})A_ix(t)+(A_d+\Delta A_d(t))x(t-h(t))+\sum_{l=1}^{\bar N}f_l(\tfrac{t}{\epsilon})B_lw(t).\tag{44}$$

Then, by introducing the following vectors

$$\begin{aligned}&\zeta_1(t)=\mathrm{col}\{x(t),\ G(t),\ Y(t),\ \Delta A(\tfrac{t}{\epsilon})x(t),\ x(t-h(t)),\ x(t-h_M),\ \Delta A_d(t)x(t-h(t))\}\ \in\mathbb{R}^{7n},\\ &\bar w(t)=\mathrm{col}\{w_1(t),\ w_2(t),\ \dots,\ w_{\bar N}(t),\ w(t)\}\ \in\mathbb{R}^{(\bar N+1)n_w},\quad \xi(t)=\mathrm{col}\{\zeta_1(t),\ \bar w(t)\}\ \in\mathbb{R}^{\nu},\end{aligned}\tag{45}$$

with $\nu=7n+(\bar N+1)n_w$, inequality (43) can be recast as

$$\dot V(t)+2\alpha V(t)-b_0|w(t)|^2-\sum_{l=1}^{\bar N}b_l|w_l(t)|^2\le\xi^\top(t)\Omega\xi(t)+\dot x^\top(t)\left[(\epsilon^\star T)\bar H+h_MW\right]\dot x(t)+(\epsilon^\star T)g^\top(t)Rg(t),\quad \forall t\ge\epsilon T,\tag{46}$$

where $\Omega\in\mathbb{R}^{\nu\times\nu}$ is the symmetric block matrix whose elements are detailed in (23).

Furthermore, by Schur complement, if

$$\begin{bmatrix}\Omega & \Xi_{12}\\ \star & \Xi_{22}\end{bmatrix}<0,\tag{47}$$

with $\Xi_{12}\in\mathbb{R}^{\nu\times 3n}$ and $\Xi_{22}\in\mathbb{R}^{3n\times 3n}$ given as

$$\Xi_{12}=\begin{bmatrix}\sqrt{\epsilon^\star T}\sum_{i=1}^{N}\rho_i(\frac{t}{\epsilon})A_i^\top R & \sqrt{\epsilon^\star T}\sum_{i=1}^{N}\rho_i(\frac{t}{\epsilon})A_i^\top\bar H & \sqrt{h_M}\sum_{i=1}^{N}\rho_i(\frac{t}{\epsilon})A_i^\top W\\ 0_{3n\times n} & 0_{3n\times n} & 0_{3n\times n}\\ 0_{n\times n} & \sqrt{\epsilon^\star T}A_d^\top\bar H & \sqrt{h_M}A_d^\top W\\ & 0_{n\times n} & 0_{n\times n}\\ 0_{(2n+\bar N)\times n} & \sqrt{\epsilon^\star T}\bar H & \sqrt{h_M}W\\ & 0_{\bar N\times n} & 0_{\bar N\times n}\\ \sqrt{\epsilon^\star T}\sum_{l=1}^{\bar N}f_l(\frac{t}{\epsilon})B_l^\top R & \sqrt{\epsilon^\star T}\sum_{l=1}^{\bar N}f_l(\frac{t}{\epsilon})B_l^\top\bar H & \sqrt{h_M}\sum_{l=1}^{\bar N}f_l(\frac{t}{\epsilon})B_l^\top W\end{bmatrix},\quad \Xi_{22}=-diag\{R,\bar H,W\},\tag{48}$$

for $t\ge\epsilon T+h_M$ we have:

$$\dot V(t)+2\alpha V(t)-b_0|w(t)|^2-\sum_{l=1}^{\bar N}b_l|w_l(t)|^2\le 0.\tag{49}$$

Note that, (22) implies 1 (and thus (49)) since 1 is affine in $\sum_{i=1}^{N}\rho_i(\frac{t}{\epsilon})A_i$ and $\sum_{l=1}^{\bar N}f_l(\frac{t}{\epsilon})B_l$.

With the aim of proving ISS, it is worth noting that for all $\epsilon \in (0, \epsilon^\star]$, $V(t)$ is positive-definite since, by Jensen's inequality, we have (19). The comparison principle applied to (49) leads to

$$|x(t)|^2 \le V(t) \le e^{-2\alpha(t-\epsilon T-h_M)}V(\epsilon T) + \frac{\sum_{l=0}^{\bar N} b_l}{2\alpha}\|w[0,t]\|_\infty^2 \quad t \ge \epsilon T, \epsilon \in (0, \epsilon^\star]. \tag{50}$$

In addition, by definition of (26), for some positive ϵ-independent ν_1, the following holds:

$$V(\epsilon T) \le \nu_1\left[\|x_{\epsilon T}\|_W^2 + \int_{-h_M}^{\epsilon T} |\dot x(s)|^2\,ds\right]. \tag{51}$$

By denoting $x_t(\theta) = x(t+\theta)$ with $\theta \in [-h_M, 0]$, from (4), it follows:

$$x_t(\theta) = \begin{cases} \phi(t+\theta), & t+\theta < 0 \\ \phi(0) + \int_0^{t+\theta}[A(\frac{s}{\epsilon})x(s) + (A_d + \Delta A_d(t))x(s-h(s)) + B(\frac{s}{\epsilon})w(s)]\,ds, & t+\theta \ge 0 \end{cases} \tag{52}$$

Due to Assumption 3, there exists $b > 0$ such that $\|B(\tau)\| \ge b$. Then, from (52), the following holds:

$$\|x_t\|_W \le \|\phi\|_W + \int_{-\theta}^{0} \nu_2\|\phi(s)\|_W\,ds + b(\epsilon T + h_M)\|w[0,t]\|_\infty, \quad t \ge 0, \tag{53}$$

for some ϵ-independent $\nu_2 > 0$. By Gronwall's inequality, (53) implies

$$\|x_t\|_W \le e^{\nu_2 h_M}\|\phi\|_W + b(\epsilon T + h_M)\|w[0,t]\|_\infty \quad t \in [0, \epsilon T + h_M]. \tag{54}$$

From this latter, it follows that

$$\|x_t\|_W^2 \le e^{2\nu_2 h_M}\|\phi\|_W + b^2(\epsilon T + h_M)^2\|w[0,t]\|_\infty^2 \quad t \in [0, \epsilon T + h_M]. \tag{55}$$

Similarly, from (4), we have:

$$|\dot x(t)|^2 \le \nu_3\|\phi\|_W^2 + b^2\|w[0,t]\|_\infty^2 \qquad t \in [0, \epsilon T + h_M], \tag{56}$$

for some ϵ-independent $\nu_3 > 0$. Substitution of (55)–(56) into (51) leads to

$$V(\epsilon T) \le \nu_1\left[e^{2\nu_2 h_M}\|\phi\|_W^2 + b^2(\epsilon T + h_M)^2\|w[0,t]\|_\infty^2\right]. \tag{57}$$

Hence, taking into account (50) and (57), it is easy to verify that inequality (24) holds (and, thus, (25)) for some ϵ-independent $\nu > 0$. □

To minimize ellipsoid radius in Eq. (25), while guaranteeing the fulfillment of Theorem 1, the following constrained optimization problem can be solved:

Let Assumptions 1–3 hold. Given the set of tuning parameters $\{\kappa, \kappa_d, \alpha, \epsilon^\star, T, h_M\}$ and matrices $\{A_{av}, A_i, A_d, B_l\}$ of Theorem 1, find positive definite $n \times n$ matrices $P,\ R,\ H,\ S,\ W,\ \bar H$, matrix U, scalars $\lambda > 0,\ \lambda_d > 0$ and $b_l,\ \forall l = 0, \dots, \bar N$ such that

$$\begin{aligned} &\min_{b_1, b_2, \dots, b_{\bar N}} \sum_{l=0}^{\bar N} b_l\Big(\{\kappa, \kappa_d, \alpha, \epsilon^\star, T, h_M\}, \{A_{av}, A_i, A_d, B_l\}\Big) \\ &\qquad \text{subject to (19)–(22)} \end{aligned} \tag{58}$$

To solve (58), several optimization software tools are available, e.g. MOSEK [32], which has a MATLAB API accessible via the YALMIP parser [33].

Remark 2. Note that in [4] A_d was supposed to be time-varying with the average $A_{d_{av}}$ and with Hurwitz $A_{av} + A_{d_{av}}$, whereas $G(t)$ was defined by (10) with $g(t)$ changed by $\dot x$. The latter led to conditions that were feasible for $h_M \sim \mathcal{O}(\epsilon)$. Extension of averaging to non-small delays is important due to non-small delays that appear in feedback controllers or internal dynamics latencies, that require proper compensation in the stability analysis (see [5]). Compared to [4], the change of $g(t)$ (and, thus, of $G(t)$) leads to a novel neutral type transformation in (18), where $z(t) = x(t) - G(t)$. Moreover, a novel Lyapunov-Krasovskii candidate in (26) with additional terms (30)–(31) has been exploited to compensate non-small delays.

Remark 3. Assume $A_{av} + A_d$ to be Hurwitz. Given $h_M > 0$, let there exist positive constants $\alpha,\ \kappa$ and κ_d such that the following standard delay-dependent condition

$$\begin{bmatrix} \Omega_0+S & P & PA_d+\frac{1}{h_M}\rho_H(W-U) & \frac{1}{h_M}\rho_H U & P & \sqrt{h_M}A_{av}^\top W \\ \star & -\lambda I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ \star & \star & -\frac{1}{h_M}\rho_H(2W-U-U^\top)+\lambda_d\kappa_d^2 I_n & -\frac{1}{h_M}\rho_H(W-U) & 0_{n\times n} & \sqrt{h_M}A_d^\top W \\ \star & \star & 0_{n\times n} & -\rho_H(S+\frac{1}{h_M}W) & 0_{n\times n} & 0_{n\times n} \\ \star & \star & \star & \star & -\lambda_d I_n & \sqrt{h_M}W \\ \star & \star & \star & \star & \star & -W \end{bmatrix} < 0 \tag{59}$$

and (20) hold with $\Omega_0 = PA_{av}+A_{av}^\top P+2\alpha P+\lambda\kappa^2 I_n$. Then, the averaged system (6) is exponentially stable with a decay rate $\alpha > 0$ for all $h(t) \le h_M$ and for small enough $\kappa > 0$ and $\kappa_d > 0$ [29]. Therefore, given non-small ϵ-independent $h_M > 0$ satisfying (20) and (59), LMIs of Theorem 1 are always feasible for small enough $\epsilon^\star > 0$ with the same $\alpha > 0,\ \kappa > 0$ and $\kappa_d > 0$ as in (59) since, by Schur complements, (22) is $\mathcal{O}(\epsilon)$-perturbation of (59).

2.3. Example: [5] Stabilization of the inverted pendulum in the presence of feedback delays and disturbances

Consider the system consisting of a cart and a planar pendulum apparatus in a reference frame subjected to a periodic amplitude and frequency disturbances along the horizontal axis. To stabilize the inverted pendulum, a delayed proportional controller for pendulum position has been introduced. Following the approach of [5] in coordinate changing and by linearizing the model at the upper equilibrium position, i.e. $x_1 = \pi$ and $x_2 = 0$, we consider:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \cos^2\frac{t}{\epsilon}-1 & -(c+\Delta c) \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ K_p & 0 \end{bmatrix} x(t-h(t)) + \begin{bmatrix} \cos^2\frac{t}{\epsilon} \\ \sin\frac{t}{\epsilon} \end{bmatrix} w(t) \tag{60}$$

with parameter $c > 0$ and periodic $B(\frac{t}{\epsilon})$ due to attempting control on an unsteady platform. Furthermore, it is reasonable to add uncertainties Δc on this latter arising from the presence of the damping coefficient of the planar joint and such that $|\Delta c| \le c_1$, with $c_1 > 0$. From $A(\frac{t}{\epsilon})$ in (60), Assumptions 1–3 hold with $T = 2\pi$ and

$$A_{av} = \begin{bmatrix} 0 & 1 \\ -0.5 & -c \end{bmatrix},\quad \Delta A = \begin{bmatrix} 0 & 0 \\ 0 & -\Delta c \end{bmatrix}. \tag{61}$$

Note that, to compute matrices in (61) it has been used that $\cos\frac{t}{\epsilon} \in [-1, 1]$ and its average is zero, while $\cos^2\frac{t}{\epsilon} \in [0, 1]$ and its average is 0.5. Moreover, $c > 0$ guarantees that $A_{av}+A_d$ is Hurwitz. We choose $c = 0.05$. Under above assumptions, $A(\frac{t}{\epsilon})$ and $B(\frac{t}{\epsilon})$ can be expressed as the following convex combination of $N = 4$ and $\bar{N} = 2$ constant matrices, respectively:

$$A_i = \begin{bmatrix} 0 & 1 \\ -0.5\pm 0.5 & -0.05+c_1 \end{bmatrix},\ i=1,2,\quad A_i = \begin{bmatrix} 0 & 1 \\ -0.5\pm 0.5 & -0.05-c_1 \end{bmatrix},\ i=3,4,\quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{62}$$

In order to prove the feasibility of (58), firstly we have to verify that condition in (21) holds. Following the approach of [6], we assume that $H = hI_2$, with $h > 0$. Then, from (21) we have:

$$\begin{aligned} &\frac{1}{T^2}\int_{\frac{t}{\epsilon}-T}^{\frac{t}{\epsilon}} (\zeta - \tfrac{t}{\epsilon} + T)A^\top(\zeta)HA(\zeta)\,d\zeta \le \frac{h}{T^2}\int_{\frac{t}{\epsilon}-T}^{\frac{t}{\epsilon}} (\zeta - \tfrac{t}{\epsilon} + T)A^\top(\zeta)A(\zeta)\,d\zeta \le \frac{h}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &+ \frac{h}{T}\int_{\frac{t}{\epsilon}-T}^{\frac{t}{\epsilon}} \begin{bmatrix} (\cos^2\zeta - 1)^2 & (0.05+c_1)(1-\cos^2\zeta) \\ \star & (1+c_1)^2 \end{bmatrix} d\zeta = h\Lambda = \bar{H}, \end{aligned} \tag{63}$$

where Λ has the following form:

$$\Lambda = \begin{bmatrix} 0.3750 & 0.5(0.05+c_1) \\ \star & (0.05+c_1)^2+0.5 \end{bmatrix}. \tag{64}$$

We consider two cases: *i* nominal case, where $c_1 = 0$, meaning that no uncertainties on damping coefficient are imposed, and *ii* uncertain case, where $c_1 \ne 0$, with $c_1 = 0.01$. Clearly, in the nominal case, $\kappa = 0$ and the number of vertices in (62) are $N = 2$ and $\bar{N} = 2$, respectively, while $\bar{H} = h\Lambda$ with $c_1 = 0$. In the uncertain case, $c_1 = 0.01$ leads to $\kappa = c_1$ and a number of vertices $N = 4$ and $\bar{N} = 2$, while $\bar{H} = h\Lambda$. Both for nominal and uncertain scenarios two different values for the decay rate are considered in order to disclose the impact of the convergence rate on $\epsilon^\star$ and h_M.

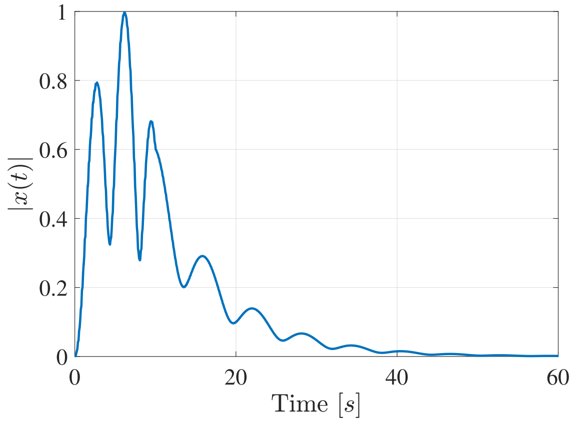

Fig. 1. Time history of state trajectories of system (60) with $\epsilon^\star = 0.054$, $h_M = 0.8$ and $w(t) = \sin(t)$ if $t \le 10$, $w(t) = 0$ otherwise.

Table 1
Example 2.3. Upper bound h_M for each set of tuning parameters $\{\sigma_i, \alpha_j, \epsilon^\star, T\}$, with $i, j = 1, 2$ and $\epsilon^\star = 0.038$.

	α	h_M
$\epsilon^\star = 0.038,\ \ c_1 = 0$	$\frac{1}{10\pi}$	0.946
	0.005	0.970
$\epsilon^\star = 0.038,\ \ c_1 = 0.01$	$\frac{1}{10\pi}$	0.913
	0.005	0.948

Maximum delay bound h_M: Firstly, our aim is to find the upper bound h_M for the time-varying delay $h(t)$ that preserves the ISS by satisfying (58) for each set of tuning parameters $\{\kappa_i, \alpha_j, \epsilon^\star, T\}$, $i, j = 1, 2$. This means that the following four sets of tuning parameters are considered: $\mathcal{S}_{1,j} = \{\kappa_1, \alpha_j, \epsilon^\star, T\}$ and $\mathcal{S}_{2,j} = \{\kappa_2, \alpha_j, \epsilon^\star, T\}$, $j = 1, 2$, with $\kappa_1 = 0$, $\kappa_2 = 0.01$, $\alpha_1 = \frac{1}{10\pi}$ and $\alpha_2 = 0.005$. Given $\mathcal{S}_{i,j}$, we solve (58) by verifying the feasibility of the LMIs of Theorem 1. Specifically, we fix $\epsilon^\star = 0.038$ and for each set of tuning parameters we iteratively increase the value of h_M in order to find its maximum value till LMIs of Theorem 1 still holds. Results are shown in Table 1, where it is possible to observe that for all $\epsilon \in (0, 0.038]$ and for $\alpha = \frac{1}{10\pi}$ the fulfillment of Theorem 1 is guaranteed for $h_M = 0.946$, while in uncertain scenario a lower value is found, i.e. $h_M = 0.913$. A smaller convergence rate (i.e. $\alpha = 0.005$) leads to larger upper bounds for time-varying delays, both in nominal and uncertain scenarios, i.e. $h_M = 0.970$ and $h_M = 0.948$, respectively. The above results confirm that ISS is preserved in the presence of larger delays, whose values are essentially larger than ϵ.

Maximum ϵ bound $\epsilon^\star$: Starting from results in Table 1, now we fix the value of h_M and iteratively increase the value of ϵ in order to find its upper bound $\epsilon^\star$ that preserves the ISS of system (60), for all $\epsilon \in (0, \epsilon^\star]$ and $h(t) \in [0, h_M]$. For the four sets of tuning parameters $\mathcal{S}^h_{i,j} = \{\kappa_i, \alpha_j, h_M, T\}$, $i, j = 1, 2$, we choose $h_M = 0.8$, while κ_i and α_j are selected as previously.

Table 2 shows the results of this latter analysis. In particular, it has been found that for the couple ($h_M = 0.8, \alpha = \frac{1}{10\pi}$) ISS is preserved for all $\epsilon \in (0, 0.057]$ in nominal scenario, while this range is restricted in the uncertain scenario, where $\epsilon^\star = 0.051$ due to polytopic uncertainties. Similar results have been obtained for the couple ($h_M = 0.8, \alpha = 0.005$), even though with larger values for $\epsilon^\star$ due to an improved convergence rate ($\alpha = 0.005$), i.e. $\epsilon^\star = 0.061$ in nominal scenario and $\epsilon^\star = 0.054$ in uncertain scenario, respectively.

Compared with [5], where both the values of $\epsilon = 0.1$ and $h_M = 0.5$ have been fixed by numerical simulations, firstly our approach leads to larger maximum admissible delays, and also it provides for the first time stability conditions expressed as LMIs whose solution allows quantifying theoretical upper bounds $\epsilon^\star$ and h_M, even if these latter could result smaller w.r.t. the ones found by simulations. Moreover, differently from [4] (see Example 5.1 in [4]), Tables 1 and 2 confirm the feasibility of LMIs in (19)–(22) for non-small delays whose upper bound h_M is essentially larger than ϵ.

Finally, state trajectories of system (60) can be seen in Fig. 1, where the external disturbance $w(t)$ has been selected as $w(t) = \sin(t)$ for $t \in [0, 10]$ $[s]$ and $w(t) = 0$ otherwise, thus confirming theoretical derivation.

3 Robust stabilization of affine systems by time-dependent switching

In this section we will apply the averaging via the time-delay approach to switched affine systems with non-small delays

$$\dot{x}(t) = \tilde{A}_{\sigma(t)}x(t) + (A_d + \Delta A_d(t))\, x(t - h(t)) + \tilde{B}_{\sigma(t)}, \tag{65}$$

Table 2
Example 2.3. Upper bound $\epsilon^\star$ for each set of tuning parameters $\{\sigma_i, \alpha_j, \epsilon^\star, T\}$, with $i, j = 1, 2$ and $h_M = 0.8$.

	α	$\epsilon^\star$
$h_M = 0.8, \quad c_1 = 0$	$\frac{1}{10\pi}$	0.057
	0.005	0.061
$h_M = 0.8, \quad c_1 = 0.01$	$\frac{1}{10\pi}$	0.051
	0.005	0.054

where $x(t) \in \mathbb{R}^n$, $\tilde{A}_{\sigma(t)} = A_{\sigma(t)} + \Delta A_{\sigma(t)}(t)$, $\tilde{B}_{\sigma(t)} = B_{\sigma(t)} + \Delta B_{\sigma(t)}(t)$, $\sigma : \mathbb{R} \to \mathcal{I} = \{1, 2, \ldots, N\}$ is a switching law, $\epsilon > 0$ is a small enough positive constant, $A_d \in \mathbb{R}^{n\times n}$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^n$ $(i \in \mathcal{I})$ are the nominal matrices, while $\Delta A_d(t) \in \mathbb{R}^{n\times n}$, $\Delta A_i(t) \in \mathbb{R}^{n\times n}$, $\Delta B_i(t) \in \mathbb{R}^n$ $(i \in \mathcal{I})$ are the perturbations with respect to the nominal values satisfying

$$\|\Delta A_d\| \le \kappa_d, \qquad \|\Delta A_i\| \le \kappa, \qquad |\Delta B_i| \le \kappa_b, \; i \in \mathcal{I}.$$

Here κ_d, κ and κ_b are some small enough positive constants. For this class of systems, given the simplex $\Lambda = \Big\{\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]^{\mathrm{T}} \in \mathbf{R}^N,\; \lambda_i \ge 0,\; \sum_{i=1}^N \lambda_i = 1\Big\}$, generating the convex combinations

$$\tilde{A}(\lambda) = \sum_{i=1}^N \lambda_i A_i + A_d \triangleq A(\lambda) + A_d, \qquad B(\lambda) = \sum_{i=1}^N \lambda_i B_i,\; \lambda \in \Lambda,$$

we assume that there exists the subset $\Lambda_H \subseteq \Lambda$ such that $\Lambda_H = \{\lambda \in \Lambda : \tilde{A}(\lambda)$ is Hurwitz$\}$. In the absence of uncertainties and time-delay, the set of equilibrium points for (65) is given by $\mathcal{S}_e = \Big\{x_e : x_e = -\tilde{A}^{-1}(\lambda)B(\lambda),\; \lambda \in \Lambda_H\Big\}$. Moreover, any $x_e = -\tilde{A}^{-1}(\lambda)B(\lambda)$ is also an admissible equilibrium point for delayed system (65) without uncertainties, since $x(t)$ approaches it for $t \to \infty$ together with $x(t - h(t))$ (see e.g. [23]).

Given an equilibrium point $x_e \ne 0 \in \mathcal{S}_e$ and denote the error $e(t) = x(t) - x_e$, where $x(t)$ is solution of (65). It follows that system (65) can be presented as

$$\dot{e}(t) = \tilde{A}_{\sigma(t)}e(t) + (A_d + \Delta A_d)e(t - h(t)) + \bar{B}_{\sigma(t)} + \Delta\bar{B}_{\sigma(t)}, \tag{66}$$

with $\bar{B}_{\sigma(t)} = B_{\sigma(t)} + (A_{\sigma(t)} + A_d)x_e$ and $\Delta\bar{B}_{\sigma(t)} = \Delta B_\sigma + (\Delta A_{\sigma(t)} + \Delta A_d)x_e$. As $x_e = -\tilde{A}^{-1}(\lambda)B(\lambda)$, $\lambda \in \Lambda_H$, then

$$\sum_{i=1}^N \lambda_i \bar{B}_{\sigma(t)} = B(\lambda) + \tilde{A}(\lambda)x_e = 0.$$

Thus, without loss of generality, we can assume that $x_e = 0$ is the equilibrium point of affine system (65). Hence, we leverage the following assumption.

Assumption 4. There exists $\lambda \in \Lambda_H$ such that $\tilde{A}(\lambda)$ is Hurwitz and $B(\lambda) = 0$.

Hence, we can design the time-dependent periodic switching law $\sigma(t)$ as

$$\sigma(t) = i,\; t \in \left[\left(k + \sum_{j=0}^{i-1}\lambda_j\right)\epsilon, \left(k + \sum_{j=0}^{i}\lambda_j\right)\epsilon\right),\; i \in \mathcal{I} \tag{67}$$

with $\lambda \in \Lambda_H$, $\lambda_0 = 0$ and $k = 0, 1, 2, \ldots$. For each time interval in (67), we introduce the indicator function $\chi_i(\tau) = \chi_{[(k+\sum_{j=1}^{i-1}\lambda_j)\epsilon,(k+\sum_{j=1}^{i}\lambda_j)\epsilon)}$, with $\tau = \frac{t}{\epsilon} \in [k, k+1]$ and $\sum_{i=1}^N \chi_i(\tau) = 1$. Hence, system (65) can be presented as

$$\dot{x}(t) = \sum_{i=1}^N \chi_i(\tau)(A_i + \Delta A_i(\tau))x(t) + (A_d + \Delta A_d)x(t - h(t)) + \sum_{i=1}^N \chi_i(\tau)(B_i + \Delta B_i(\tau)), \quad \forall i \in \mathcal{I}, \tag{68}$$

with $\lambda \in \Lambda_H,\; k = 0, 1, 2\ldots,\; \tau = \frac{t}{\epsilon} \in [k, k+1]$.

Using notations (9)–(10) and (14) and integrating (68) $[t - \epsilon, t]$ for $t \ge \epsilon + h_M$, we finally obtain

$$\dot{z}(t) = [A_{av} + \Delta A_{\sigma(t)}]x(t) - Y(t) + (A_d + \Delta A_d)\, x(t - h(t)) + \Delta B_{\sigma(t)}, \qquad t \ge \epsilon + h_M. \tag{69}$$

Here $z(t) = x(t) - G(t)$, $A_{av} = A(\lambda)$, $x(t)$ satisfies (65), $g(s) = A_{\sigma(s)}x(s) + B_{\sigma(s)}$ and $Y(t)$ as in (14) with $A(\frac{s}{\epsilon})$ replaced by $A_{\sigma(s)}$.

Therefore, system (65) is practically stable if the time-delay system (69) is practically stable . By using arguments of Theorem 1, the following result is obtained for delayed switched affine systems:

Theorem 2. *Consider the switched affine system with time-varying delays (65) and let Assumption 4 hold. Given matrices A_{av}, $A_i (i = 1, \dots, N)$, A_d, ΔA_d, $B_i (i = 1, \dots, N)$ and positive constants κ, κ_d, κ_b, α, $\epsilon^\star$, $T = 1$ and h_M, let there exist positive-definite matrices P, R, H, W, S and $\bar{H} \in \mathbb{R}^{n\times n}$, a matrix $U \in \mathbb{R}^{n\times n}$ and scalars $\lambda > 0$, λ_d, b_0 and $b > 0$ that satisfy (19), (20), (21) and the following LMIs*

$$\begin{bmatrix} \Upsilon & \begin{bmatrix} \sqrt{2}A_i^\top R & \sqrt{2}A_i^\top(\epsilon^\star\bar{H}+h_M W) \\ 0_{2n\times n} & 0_{2n\times n} \\ 0_{n\times n} & \sqrt{2}(\epsilon^\star\bar{H}+h_M W) \\ 0_{n\times n} & \sqrt{2}A_d^\top(\epsilon^\star\bar{H}+h_M W) \\ 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \sqrt{2}(\epsilon^\star\bar{H}+h_M W) \\ 0_{n\times n} & \sqrt{2}(\epsilon^\star\bar{H}+h_M W) \end{bmatrix} \\ \star & \begin{bmatrix} -\frac{1}{\epsilon^\star}R & 0_{n\times n} \\ \star & -(\epsilon^\star\bar{H}+h_M W) \end{bmatrix} \end{bmatrix} < 0, \quad i = 1, \dots, N, \tag{70}$$

$$\begin{bmatrix} b_0(\epsilon^\star+h_M) & B_i^\top[2\epsilon^\star(R+\bar{H})+2h_M W] \\ \star & 2\epsilon^\star(R+\bar{H})+2h_M W \end{bmatrix} > 0, \tag{71}$$

with

$$\Upsilon = \begin{bmatrix} \bar{\Upsilon} & \begin{bmatrix} P \\ -P \\ 0_{5n\times 5n} \end{bmatrix} \\ \star & -b \end{bmatrix}, \tag{72}$$

Ξ_{22} defined in (48) as $\Xi_{22} = -diag\{R, \bar{H}, W\}$ and $\bar{\Upsilon}$ obtained from Ω in (23) by removing the last block-column and block-row. Then, the delayed switched affine system (65) is practically exponentially stable with a decay rate $\alpha > 0$ for all $\epsilon \in (0, \epsilon^\star]$ and $h(t) \in [0, h_M]$, meaning that there exists a positive constant $\tilde{\nu}$ such that the solutions of the delayed system (65) initialized by $\phi \in W[-h_M, 0]$ satisfy

$$|x(t)|^2 \leq \tilde{\nu} e^{-2\alpha(t-\epsilon^\star-h_M)}\|\phi\|_W^2 + \left[\tilde{\nu} e^{-2\alpha(t-\epsilon^\star-h_M)} + \frac{b_0(\epsilon^\star+h_M)+b\kappa_b^2}{2\alpha}\right], \qquad \forall t \geq 0. \tag{73}$$

Moreover, the ball

$$\mathcal{X}_{\epsilon^\star} = \left\{x \in \mathbb{R}^n : |x|^2 \leq \frac{b_0(\epsilon^\star+h_M)+b\kappa_b^2}{2\alpha}\right\}, \tag{74}$$

is exponentially attractive with decay rate $\alpha > 0$ for (65) for all $\phi \in W[-h_M, 0]$.

Proof. Choose the Lyapunov–Krasovskii functional $V(t)$ as in (26), with $A(\frac{s}{\epsilon}) = A_{\sigma(s)}$. Then, following arguments of Theorem 1, for $t \geq \epsilon + h_M$, we have:

$$\begin{aligned} \dot{V}(t) + 2\alpha V(t) - b|\Delta B_{\sigma(t)}|^2 - b_0(\epsilon^\star + h_M) \leq& \xi_1(t)\Upsilon\xi_1(t) + \epsilon^\star(A_{\sigma(t)}x(t) + B_{\sigma(t)})^\top R(A_{\sigma(t)}x(t) + B_{\sigma(t)}) \\ &+ \dot{x}^\top(t)(\epsilon^\star\bar{H} + h_M W)\dot{x}(t) - b_0(\epsilon^\star + h_M), \end{aligned} \tag{75}$$

with matrix $\bar{H}$ as in (21), $\xi_1^\top(t) = [\zeta_1^\top(t), \ \ \Delta B_{\sigma(t)}^\top]$ and Υ as in (72). Substituting (65) into (75) and applying Young's inequality, we obtain:

$$\begin{aligned} \dot{V}(t)+&2\alpha V(t) - b|\Delta B_{\sigma(t)}|^2 - b_0(\epsilon^\star + h_M) \leq \xi_1^\top(t)\Upsilon\xi_1(t) + 2\epsilon^\star x^\top(t)A_{\sigma(t)}^\top RA_{\sigma(t)}x(t) + 2\epsilon^\star B_{\sigma(t)}^\top RB_{\sigma(t)} \\ &- b_0(\epsilon^\star + h_M) + 2B_{\sigma(t)}^\top(\epsilon^\star\bar{H} + h_M W)B_{\sigma(t)} + 2\Big[(A_{\sigma(t)} + \Delta A_{\sigma(t)})x(t) + (A_d + \Delta A_d)x(t-h(t)) + \Delta B_{\sigma(t)}\Big]^\top \\ &\times (\epsilon^\star\bar{H} + h_M W)\Big[(A_{\sigma(t)} + \Delta A_{\sigma(t)})x(t) + (A_d + \Delta A_d)x(t-h(t)) + \Delta B_{\sigma(t)}\Big] \\ &= \xi_1^\top(t)[\Upsilon + \Sigma]\xi_1(t) + 2\epsilon^\star B_{\sigma(t)}^\top RB_{\sigma(t)} - b_0(\epsilon^\star + h_M) + 2B_{\sigma(t)}^\top(\epsilon^\star\bar{H} + h_M W)B_{\sigma(t)}, \end{aligned} \tag{76}$$

where Σ is the symmetric block matrix whose elements are

$$\begin{aligned} &\Sigma_{11} = 2\sum_{i=1}^N \chi_i(\tau)A_i^\top(\epsilon^\star\bar{H} + h_M W)\sum_{i=1}^N \chi_i(\tau)A_i + 2\epsilon^\star\sum_{i=1}^N \chi_i(\tau)A_i^\top R\sum_{i=1}^N \chi_i(\tau)A_i, \\ &\Sigma_{14} = \Sigma_{17} = \Sigma_{18} = 2\sum_{i=1}^N \chi_i(\tau)A_i^\top(\epsilon^\star\bar{H} + h_M W), \quad \Sigma_{15} = 2\sum_{i=1}^N \chi_i(\tau)A_i^\top(\epsilon^\star\bar{H} + h_M W)A_d, \\ &\Sigma_{44} = \Sigma_{47} = \Sigma_{48} = \Sigma_{77} = \Sigma_{78} = \Sigma_{88} = 2(\epsilon^\star\bar{H} + h_M W), \quad \Sigma_{45} = 2(\epsilon^\star\bar{H} + h_M W)A_d, \\ &\Sigma_{55} = 2A_d^\top(\epsilon^\star\bar{H} + h_M W)A_d, \quad \Sigma_{57} = \Sigma_{58} = 2A_d^\top(\epsilon^\star\bar{H} + h_M W), \end{aligned} \tag{77}$$

and other blocks are zero matrices. By applying Schur complement to (76) and taking into account that Σ is affine in $\sum_{i=1}^{N} \chi_i(\tau)A_i$, if (70)–(71) hold, then we have

$$\dot{V}(t) + 2\alpha V(t) - b|\Delta B_{\sigma(t)}|^2 - b_0(\epsilon^\star + h_M) \leq 0, \quad t \geq \epsilon. \tag{78}$$

The rest of the proof is similar to that in Theorem 1. □

Remark 4. Note that system (65) can be presented as (4) with $A(\tau) = \sum_{i=1}^{N} \chi_i(\tau)(A_i + \Delta A_i(\tau))$, $B(\tau) = \sum_{i=1}^{N} \chi_i(\tau)(B_i + \Delta B_i(\tau))$, $\tau = \frac{t}{\epsilon}$. For $\Delta A_i = \Delta B_i = 0$, both $A(\tau)$ and $B(\tau)$ are $T = 1$-periodic. Then, we have:

$$\Delta A(\tau) = \sum_{i=1}^{N} \int_{\lambda_{i-1}}^{\lambda_i} \Delta A_i(\tau - \theta) \mathrm{d}\theta, \qquad \Delta B(\tau) = \sum_{i=1}^{N} \int_{\lambda_{i-1}}^{\lambda_i} \Delta B_i(\tau - \theta) \mathrm{d}\theta,$$

with $\lambda_0 = 0$ and

$$\|\Delta A(\tau)\| \leq sum_{i=1}^{N} \int_{\lambda_{i-1}}^{\lambda_i} \|\Delta A_i(\tau - \theta)\| \, \mathrm{d}\theta \leq \kappa, \qquad \|\Delta B(\tau)\| \leq \sum_{i=1}^{N} \int_{\lambda_{i-1}}^{\lambda_i} \|\Delta B_i(\tau - \theta)\| \, \mathrm{d}\theta \leq \kappa_b.$$

However, different from previous section, the term $g(s, \epsilon)$ includes only the nominal part that leads to simpler LMIs.

Remark 5. From (73) and (74), it is clear that for $t \to \infty$, the trajectories of switched affine system (65) exponentially approach the attractive ball $|x|^2 \leq \frac{b_0(\epsilon^\star + h_M) + b\kappa_b^2}{2\alpha}$. To obtain a smaller ball, firstly it is possible to minimize b_0 and b. However, this minimization leads to weak performances in terms of convergence rate, which can be improved by increasing the value of decision variables b_0 and b. Moreover, due to (75), b_0 is of the order of $\mathcal{O}(h_M + \epsilon^\star)$. Hence, larger values of h_M and ϵ increase the ball radius. Therefore, a good trade-off between non-small delays, frequency switching, convergence rate and attractive ball size has to be also reached.

Remark 6. It is worth noting that many recent results on the stabilization of switched affine systems suggest state/output-dependent switching laws (see, e.g. [20,34–36]). Although state/output-dependent switching laws may have advantages in robustness to disturbances with respect to time-dependent switching, time-dependent switching law is simpler for implementation due to no need of measurements and on-line calculation of the switching law. Moreover, it can be useful to switch from a state-dependent to a time-dependent switching law in some practical applications [37], e.g., when sensor-faults occur.

3.1 Examples: Stabilization of switched affine systems.

3.1.1 Example 1 [21]

Consider the delayed version of the switched affine system in [21]:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + A_d x(t - h(t)) + B_1, & t \in [k\epsilon, (k + \lambda_1)\epsilon], \\ A_2 x(t) + A_d x(t - h(t)) + B_2, & t \in [(k + \lambda_1)\epsilon, (k + \lambda_1 + \lambda_2)\epsilon], \\ A_3 x(t) + A_d x(t - h(t)) + B_3, & t \in [(k + \lambda_1 + \lambda_2)\epsilon, (k + 1)\epsilon], \end{cases} \tag{79}$$

with $\epsilon > 0$, $k = 0, 1, \ldots$ and $\lambda \in (0, 1)$. Matrices in (79) are given as follows:

$$A_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.1 & 0 \\ -1 & -1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.045 & 0.13 \\ -0.5 & -0.8 \end{bmatrix}.$$

Then (79) can be presented as (4) with $\Delta A = \Delta A_d = \Delta B = 0$ and

$$A(\tau) = \sum_{i=1}^{N} \chi_i(\tau) A_i, \qquad B(\tau) = \sum_{i=1}^{N} \chi_i(\tau) B_i,$$

with $\chi_i(\tau)$ the indicator function. As in [21], we choose $\lambda = [0.4, \; 0.47, \; 0.13] \in \Lambda_H$ leading to the desired operating point $x_e = [0.1 \;\; 0.2]^\top$ and to matrix

$$A_{av} = A(\lambda) + A_d = \sum_{i=1}^{3} \lambda_i A_i + A_d,$$

Table 3
Example 3.1.1. Solution of (58) for different values of h_M and the corresponding ultimate bound.

	h_M	$b_0^\star$	$UB = \sqrt{\frac{b^\star \kappa_b^2 + b_0^\star(\epsilon^\star + h_M)}{2\alpha}}$
$\epsilon^\star = 0.2, \quad \alpha_1 = 0.005$	0.1	0.38258	3.3878
	0.3	0.39848	4.4637
	0.5	0.64144	6.7008
	0.7	0.80207	8.4962
	0.9	0.86383	9.7479
$\epsilon^\star = 0.2, \quad \alpha_2 = 0.5$	0.1	0.3915	0.3427
	0.3	0.41319	0.4545
	0.5	0.68916	0.6946
	0.7	1.0503	0.9723

Table 4
Example 3.1.1. Solution of (58) for different values of $\epsilon \in (0, \epsilon^\star]$ and corresponding ultimate bound.

	ϵ	$b_0^\star$	$UB = \sqrt{\frac{b^\star \kappa_b^2 + b_0^\star(\epsilon + h_M)}{2\alpha}}$
$h_M = 0.5, \quad \alpha_1 = 0.005$	0.2	0.64144	6.7008
	0.3	0.7935	7.9674
	0.4	0.79296	8.4479
	0.53	1.2663	11.4207
$h_M = 0.5, \quad \alpha_2 = 0.5$	0.2	0.68916	0.6946
	0.3	0.85376	0.8264
	0.4	0.85747	0.8785
	0.5	1.2455	1.1160
	0.55	1.3293	1.1814

Moreover, both $A(\tau)$ and $B(\tau)$ are 1-periodic. Hence, for $\tau \geq T = 1$, inequality (21) can be easily computed as

$$\begin{aligned}\int_{\tau-1}^{\tau} (\zeta - \tau + 1) A^\top(\zeta) H A(\zeta)\, d\zeta \leq & \int_{\tau-\lambda_1}^{\tau} (\zeta - \tau + 1)\, d\zeta A_1^\top H A_1 \\ & + \int_{\tau-(\lambda_1+\lambda_2)}^{\tau} (\zeta - \tau + 1)\, d\zeta A_2^\top H A_2 + \int_{\tau-(1-\lambda_1-\lambda_2)}^{\tau} (\zeta - \tau + 1)\, d\zeta A_3^\top H A_3 \\ & = \frac{1-(1-\lambda_1)^2}{2} A_1^\top H A_1 + \frac{1-\lambda_3^2}{2} A_2^\top H A_2 + \frac{1-(\lambda_1+\lambda_2)^2}{2} A_3^\top H A_3 = \bar{H}.\end{aligned}$$

Firstly, we analyze two different sets $\{\alpha_i, \ \epsilon^\star, \ T\}, \ i = 1, 2$ of tuning parameters, which involve different values of decay rate, i.e., $\alpha_1 = 0.005$ and $\alpha_2 = 0.5$ in order to show the impact of this latter on the feasibility of the LMIs of Theorem 2 and the corresponding UB. It is clear that for both sets of tuning parameters, $\kappa = \kappa_d = \kappa_b = 0$ since no system uncertainties occur, i.e. $\Delta A(\tau) = \Delta A_d = \Delta B(\tau) = 0$.

Maximum delay bound h_M: For each set, here we fix $\epsilon^\star = 0.2$ and iteratively increase the value of h_M in order to find its upper bound that guarantees the existence of a solution for (58) for all $\epsilon \in (0, \epsilon^\star], \ h(t) \in [0, h_M]$. Results are reported in Table 3, which shows that for all $\epsilon \in (0, 0.2]$ and $\alpha_1 = 0.005$, (58) is feasible until $h_M = 0.9$, whereas $UB = 9.7476$. For $\alpha_2 = 0.5$, it is found that for all $\epsilon \in (0, 0.2]$, problem (58) holds for $h_M = 0.7$, which leads to an ultimate bound $UB = 0.9723$. As expected, comparing the above results of Table 3, for a fixed values of $\epsilon^\star$ and h_M, smaller values of the decay rate lead to a larger attractive ball, thus deteriorating the performances. Hence, a good trade-off between UB size and convergence rate has to be found to satisfy specific control requirements.

Maximum ϵ bound $\epsilon^\star$: Here we fix h_M, while ϵ has been increased in order to find its upper bound $\epsilon^\star$, whose value preserves the feasibility of (58) (and, thus, Theorem 2). Note that, the value of h_M has been fixed as $h_M = 0.5$ according to the results of Table 3. The results of the optimization procedure for $\alpha_i, \ i = 1, 2$ can be seen in Table 4, where it is possible to observe the values of the UB for different values of ϵ. In particular, for $\alpha_1 = 0.005$, practical stability can be guaranteed for all $\epsilon \in (0, \epsilon^\star]$ with $\epsilon^\star = 0.53$, which leads to $UB = 11.4207$. On the other hand, for $\alpha_2 = 0.5$, the LMIs of Theorem 2 are still feasible for all $\epsilon \in (0, 0.55]$, with $UB = 1.1814$. Also in this case, given the value h_M, decay rate α_1 leads to a larger attractive ball w.r.t. the one obtained with α_2, thus deteriorating performances in terms of ellipsoid radius.

Note that, in [21] state-dependent periodic-time and event-triggered control laws for switched affine systems are proposed, which may be restrictive when state measurements are not available. Moreover, compared with [21], where no state delays have been considered, by verifying the feasibility of Theorem 2 for $h(t) = 0, \ \alpha = 0.005$, we obtain $\epsilon^\star = 1.12$, as well as the result of our optimization procedure leads to $UB = 0.4065$. Finally, numerical simulations shown in Fig. 2 highlight the stabilization of system (79) for all $\epsilon \in (0, \epsilon^\star]$ and $h(t) \in [0, 0.5]$, thus confirming theoretical derivation.

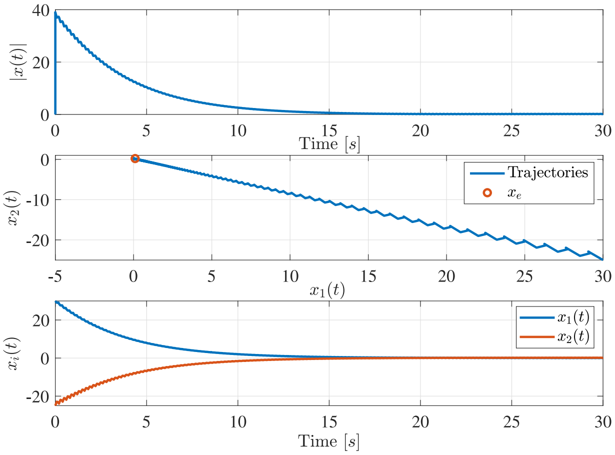

Fig. 2. Practical stabilization of switched affine systems (79) with $\epsilon^\star = 0.2$ and $h_M = 0.8$.

3.1.2 Example 2 [24]

Consider the delayed dynamics of flyback power converter from [24],[38], where the model has the form of

$$\dot{x}(t) = A_{\sigma(t)}x(t) + (A_d + \Delta A_d)x(t - h(t)) + B_{\sigma(t)}u(t) + D_{\sigma(t)}w(t) + b_{\sigma(t)}, \tag{80}$$

with $u(t) = Kx(t)$ and matrices $\Delta A_1 = \Delta A_2 = 0$,

$$\begin{aligned} &A_1 = \begin{bmatrix} -r/L_m & 0 \\ 0 & -1/R_LC \end{bmatrix},\ A_2 = \begin{bmatrix} -r/L_m & -n/L_m \\ n/C & -1/R_LC \end{bmatrix},\ A_d = \begin{bmatrix} -0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},\ b_1 = \begin{bmatrix} E_{in}/L_m \\ 0 \end{bmatrix},\ \Delta A_d = \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.2 \end{bmatrix},\ b_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ &B_1 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.3 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\ D_2 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix},\ K_1 = \begin{bmatrix} 19.7287 & -18.9826 \\ -21.1811 & -1.6378 \end{bmatrix},\ K_2 = \begin{bmatrix} 58.9652 & -14.0308 \\ -20.4310 & -1.9181 \end{bmatrix}. \end{aligned} \tag{81}$$

Note that, controller $u(t)$ has been used as in [24] in order to provide a fair comparison in terms of maximum delay bound h_M and parameter ϵ. However, $u(t)$ can be required in flyback converter due to its non-minimum-phase nature and the presence of a right-half-plane-zero in voltage transfer function in order to guarantee the indirect regulation of the output voltage [39]. Moreover, both non-minimum-phase nature and the presence of a right-half-plane-zero in voltage transfer function may lead to state delay $h(t)$ [38]. The parameters are given as $E_{in} = 6$ V, $L_m = 10$ mH, $r = 3\ \Omega$, $C = 2$ mF, $R_L = 1.5\ \Omega$ and the transformer turns ratio $n = 1$. As in [24], we choose $\lambda = 0.5 \in \Lambda_H$, leading to the desired equilibrium point $x_e = [0.7547\ 0.7925]$ and to Hurwitz matrix $A_{av} = 0.5(A_1 + B_1K_1) + 0.5(A_2 + B_2K_2) + A_d$. Moreover, in this example $\Delta A = 0$ and $\Delta A_d \neq 0$ brings to $\kappa = 0$ and $\|\Delta A_d\| \leq \kappa_d = 0.2921$ respectively, while we set $D_{\sigma(t)} = \Delta B_{\sigma(t)}$, which leads to $|\Delta\bar{B}| = |\Delta B_{\sigma(t)}| + |\Delta A_d x_e| \leq \kappa_b = 0.5001$. For all $\tau \geq T = 1$, inequality (21) becomes

$$\begin{aligned} &\int_{\tau-1}^{\tau} (\zeta - \tau + 1)(A(\zeta) + K(\zeta)B(\zeta))^\top H(A(\zeta) + K(\zeta)B(\zeta))\, d\zeta \\ &\leq \int_{\tau-\lambda}^{\tau} (\zeta - \tau + 1)\, d\zeta (A_1 + B_1K_1)^\top H(A_1 + B_1K_1) + \int_{\tau-(1-\lambda)}^{\tau} (\zeta - \tau + 1)\, d\zeta (A_2 + B_2K_2)^\top H(A_2 + B_2K_2) \\ &= \frac{1 - (1-\lambda)^2}{2}(A_1 + B_1K_1)^\top H(A_1 + B_1K_1) + \frac{1 - \lambda^2}{2}(A_2 + B_2K_2)^\top H(A_2 + B_2K_2) = \bar{H}. \end{aligned}$$

By verifying the feasibility of (19), (20), (21) and (70) with $\alpha = 0.005$, $h_M = 0.4$, we find the maximum value of $\epsilon^\star = 0.32$ that guarantees the practical stability of (80)–(81) for all $\epsilon \in (0, \epsilon^\star]$, $h(t) \in [0, h_M]$ and decay rate $\alpha = 0.005$. Moreover, by solving (58), we obtain $b_0^\star = 0.85$, $b^\star = 5.3404$ and, hence, the resulting ball radius $UB = 10.4529$. On the other hand, by solving LMIs (19), (20), (21) and (70) with $\alpha = 0.005$, $\epsilon^\star = 0.2$, the resulting maximum delay bound can be obtained as $h_M = 2.2$, which leads to $UB = 23.6443$ provided by $b^\star = 9.3171$ and $b_0^\star = 1.98$.

Therefore, compared with [24], where a state dependent switching rule along with event-triggered control protocol have been implemented with a fixed sampling period $T_{max} = 0.01$ and a delay bound $h_M \approx 0.2$, our strategy allows quantifying the bounds $\epsilon^\star$ and h_M on the small parameter and non-small delay, respectively, without the need of reliable state measurements. Finally, Fig. 3 shows the practical stabilization of switched affine flyback converter with $\epsilon^\star = 0.32$ and $h_M = 0.4$, thus confirming that the switched system (80) (81) exponentially converges to the set $\mathcal{X}_{\epsilon^\star}$ in (74).

4 Conclusion

In this paper the recent time-delay approach to averaging is extended to the class of linear systems with fast-varying coefficients and perturbations in the presence of non-small delays. An appropriate Lyapunov–Krasovskii functional is

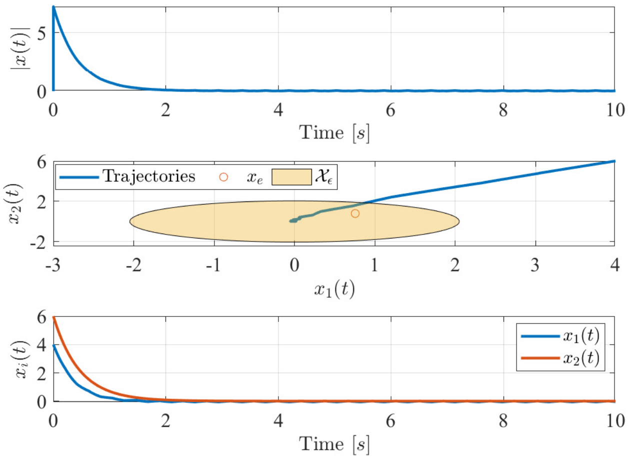

Fig. 3. Practical stabilization of switched affine system (80)–(81) with $\epsilon^\star = 0.32$ and $h_M = 0.4$.

constructed to prove the ISS of such class of time-delayed systems, thus providing ISS conditions in terms of LMIs, whose solution allow finding upper bounds on both small parameter and non-small delays. The proposed approach is extended to stabilization of uncertain delayed affine systems by periodic time-dependent switching. Numerical examples from the literature illustrate the efficiency of the method.

CRediT authorship contribution statement

Bianca Caiazzo: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data curation. **Emilia Fridman:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data curation. **Xuefei Yang:** Validation, Resources, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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