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Exact slow–fast decomposition of a class of non-linear singularly perturbed optimal control problems via invariant manifolds

E. FRIDMAN*

We study a Hamilton–Jacobi partial differential equation, arising in an optimal control problem for an affine non-linear singularly perturbed system. This equation is solvable iff there exists a special invariant manifold of the corresponding Hamiltonian system. We obtain exact slow–fast decomposition of the Hamiltonian system and of the special invariant manifold into slow and fast components. We get sufficient conditions for the solvability of the Hamiltonian–Jacobi equation in terms of the reduced-order slow submanifold, or, in the hyperbolic case, in terms of a reduced-order slow Riccati equation. On the basis of this decomposition we construct asymptotic expansions of the optimal state-feedback, optimal trajectory and optimal open-loop control in powers of a small parameter.

1. Introduction

The non-linear optimal control problem for singularly perturbed systems leads to a high-dimensional Hamilton–Jacobi (HJ) partial differential equation of two time-scales for an optimal controller evaluation (Chow and Kokotovic 1978, 1981). To alleviate the difficulties caused by the high dimensionality and stiffness that result from the interaction of slow and fast dynamical modes, a composite controller was designed (Chow and Kokotovic 1978, 1981). This ϵ-independent controller was based on the reduced-order slow and fast subproblems. Also, a series expansion method was developed for approximate solution of the HJ equation. It was shown that the truncated expansion satisfies this equation to $O(\epsilon^k)$-accuracy. Bensoussan (1987) showed that the exact solution converges to the leading term of this approximation as $\epsilon \rightarrow 0$, Gaitsgory (1996) studied the limit as $\epsilon \rightarrow 0$ of the optimal value function for the finite horizon problem with an arbitrary non-linear dynamics and functional. However, near-optimality of the high-order approximation to the solution of the HJ equation (in the sense of its closeness to the exact solution) has not been studied yet.

In the linear case, high-order numerical approximations were constructed by Su *et al.* (1992) on the basis of the exact decomposition of the full-order Riccati equation into reduced-order Riccati and linear algebraic equations. Exact decomposition of the singularly perturbed H_∞ Riccati equation was obtained in Fridman (1995); asymptotic approximation to its solution was constructed in Fridman (1996). Moreover, it has been proved that the full-order Riccati equation has a stabilizing solution iff the reduced-order slow Riccati equation has such a solution.

In the present paper we obtain the non-linear counterpart of Su *et al.* (1992) and Fridman (1995, 1996). We apply the geometric approach which relates HJ equations to special invariant manifolds of Hamiltonian systems (see e.g. Lukes 1969, Isidori and Astolfi 1992, Van der Schaf 1991). We obtain the exact decomposition of the special slow–fast manifold into the reduced-order slow submanifold of the Hamiltonian system and the fast manifold of an auxiliary system. This decomposition is based on the slow–fast decomposition of the Hamiltonian system. Unlike the linear case, the fast manifold depends also on the slow variables, and there is no immediate order reduction. Still, the fast manifold can be found in the form of asymptotic expansions with terms evaluated by algebraic operations. The special manifold exists iff the Hamiltonian system possesses the slow submanifold. Thus, we get reduced-order sufficient conditions for the solvability of the HJ equation in terms of the slow submanifold or, in the hyperbolic case, in terms of a slow Riccati equation. We construct a higher-order approximation to the optimal closed-loop and open-loop controls and optimal trajectory in the form of expansion in the powers of ϵ. In the hyperbolic case we show that the high-order accuracy controller improves the performance.

The present paper is organized as follows. In the next section we formulate the non-linear singularly perturbed optimal control problem and the known sufficient conditions for its solvability in terms of the special invariant manifold of the Hamiltonian system. In § 3 we express this manifold via slow and fast components. In § 4 we study the hyperbolic case. In § 5 we construct asymptotic expansions for the optimal controller, optimal trajectory and open-loop control, and consider an illustrative example. The paper ends with an Appendix containing proofs of theorems.

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* Department of Electrical Engineering Systems, Tel Aviv University, Tel Aviv 69978, Israel. e-mail: emilia@eng.tau.ac.il

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2. Problem formulation

Consider the optimal control problem for the system

$$\dot{x}_1 = a_1(x_1) + A_1(x_1)x_2 + B_1(x_1)u \qquad (2.1a)$$

$$\epsilon\dot{x}_2 = a_2(x_1) + A_2(x_1)x_2 + B_2(x_1)u \qquad (2.1b)$$

with respect to the functional

$$J = \int_0^\infty \left[r(x_1) + s'(x_1)x_2 + x_2'Q(x_1)x_2 + u'R(x_1)u\right]\mathrm{d}t \qquad (2.1c)$$

where $x_1(t) \in \mathbf{R}^{n_1}$ and $x_2(t) \in \mathbf{R}^{n_2}$ are the state vectors, $x = \mathrm{col}\{x_1, x_2\}$, $u(t) \in \mathbf{R}^m$ is the control input. The prime denotes the transposition of a matrix. The functions a_i, A_i, B_i, p, s and Q are differentiable with respect to x a sufficient number of times. We assume also that $a_i(0) = 0$, $r(0) = 0$ and $s(0) = 0$. Assume that $Q = Q' \geq 0$, $R = R' > 0$ and $r + s'x_2 + x_2'Qx_2 \geq 0$.

The system (2.1) has a standard singularly perturbed form in the sense that it is non-linear only on the slow variable x_1 (see e.g. Chow and Kokotovic 1978, 1981, Kokotovic *et al.* 1986). However, A_{22} is allowed to be singular.

Denote by $|\cdot|$ the Euclidean norm of a vector. Consider the Hamiltonian function

$$\begin{aligned}\mathcal{H}(x_1, x_2, p_1, p_2) = {} & r + s'x_2 + x_2'Qx_2 + p_1'(a_1 + A_2x_2) \\ & + p_2'(a_2 + A_2x_2) - \tfrac{1}{4}(p_1'B_1 + p_2'B_2) \\ & \times R^{-1}(B_1'p_1 + B_2'p_2) \end{aligned} \qquad (2.2)$$

where p_1 and ϵp_2 play the role of the costate variables. Denote $S_{ij} = \frac{1}{2}B_iR^{-1}B_j'$. The corresponding Hamiltonian system has the form

$$\dot{x}_1 = f_1(x_1, p_1, x_2, p_2) \qquad (2.3a)$$

$$\dot{p}_1 = f_2(x_1, p_1, x_2, p_2) \qquad (2.3b)$$

$$\epsilon\dot{x}_2 = A_2x_2 - S_{22}p_2 + f_3(x_1, p_1) \qquad (2.3c)$$

$$\epsilon\dot{p}_2 = -Qx_2 - A_2'p_2 + f_4(x_1, p_1) \qquad (2.3d)$$

where

$$\left.\begin{aligned} f_1 &= a_1 + A_2x_2 - S_{11}p_1 - S_{12}p_2 \\ f_2 &= -\nabla_{x_1}\mathcal{H} \\ f_3 &= a_2 - S_{21}p_1 \\ f_4 &= -A_1'p_1 - s \end{aligned}\right\} \qquad (2.4)$$

For each $\epsilon > 0$, if $V(x)$ is a C^2 solution of the HJ equation

$$\begin{aligned}&\left[r + s'x_1 + x_2'Qx_2\right] + V_{x_1}(a_1 + A_1x_2) \\ &+ \frac{1}{\epsilon}V_{x_2}(a_2 + A_2x_2) - \tfrac{1}{4}\left(V_{x_1}B_1 + \tfrac{1}{\epsilon}V_{x_2}B_2\right)R^{-1} \\ &\times\left(B_1'V_{x_1}' + \frac{1}{\epsilon}B_2'V_{x_2}'\right) = 0, \quad V(0) = 0 \end{aligned} \qquad (2.5)$$

where (V_{x_1}, V_{x_2}) denotes the Jacobian matrix of V, such that the system

$$\dot{x}_1 = a_1 + A_2x_2 - S_{11}V_{x_1}' - S_{12}V_{x_2}'$$

$$\epsilon\dot{x}_2 = a_2 + A_2x_2 - S_{21}V_{x_1}' - S_{22}V_{x_2}'$$

is asymptotically stable, then $V \geq 0$ (which implies also $V_x(0) = 0$) and the controller given by

$$u = -\tfrac{1}{2}R^{-1}\left[B_1', \epsilon^{-1}B_2'\right]V_x' = -\tfrac{1}{2}R^{-1}B_1'Z_1 - \tfrac{1}{2}R^{-1}B_2'Z_2 \qquad (2.6)$$

is the minimizing one. The latter is equivalent (see e.g. Isidori and Astolfi 1992) to the existence of the invariant manifold of (2.3)

$$p_1 = Z_1(x_1, x_2), \quad p_2 = Z_2(x_1, x_2) \qquad (2.7)$$

where

$$V_{x_1} = Z_1', \quad V_{x_2} = \epsilon Z_2' \qquad (2.8)$$

and $V(0) = 0$, with asymptotically stable flow

$$\left.\begin{aligned}\dot{x}_1 &= a_1 + A_2x_2 - S_{11}Z_1 - S_{12}Z_2, \\ \epsilon\dot{x}_2 &= a_2 + A_2x_2 - S_{21}Z_1 - S_{22}Z_2\end{aligned}\right\} \qquad (2.9)$$

Note that the manifold (2.7) is not necessarily the stable manifold of the Hamiltonian system (2.3) because (2.9) does not need to be exponentially stable (Isidori and Astolfi 1992).

We shall reduce the analysis of the $(2n_1+2n_2)$-dimensional Hamiltonian system (2.3) to the slow $2n_1$-dimensional subsystem that corresponds to the restriction of (2.3) to its slow (centre) manifold. Namely, we shall show that the existence of (2.7) is equivalent to the existence of the reduced-order invariant manifold of the slow subsystem. Moreover, we shall find the functions Z_1 and Z_2 from algebraic equations by means of the latter manifold and a fast manifold of an auxiliary system.

3. Decomposition of the slow–fast manifold

For each $x_1 \in \mathbf{R}^{n_1}$ consider the fast linear subproblem

$$\left.\begin{aligned}\dot{x}_2 &= A_2(x_1)x_2 + B_2(x_1)u \\ J &= \int_0^\infty \left[x_2'Q(x_1)x_2 + u'R(x_1)u\right]\mathrm{d}t\end{aligned}\right\} \qquad (3.1)$$

and the corresponding algebraic Riccati equation

$$A_2'M + MA_2 + Q - MS_{22}M = 0 \tag{3.2}$$

We assume further

Assumption 1: *For each* $x_1 \in \mathbf{R}^{n_1}$ *the triple* $\{A_2(x_1), B_2(x_1), Q(x_1)\}$ *is controllable-observable.*

Under this assumption, for each $x_1 \in \mathbf{R}^{n_1}$ (3.2) has a positive definite symmetry solution $M(x_1)$, such that the matrix $\Lambda = A_2 - S_{22}M$ is Hurwitz. This solution is smooth on x_1 since A_2, Q and S_{22} are smooth on x_1.

Consider the Hamiltonian matrix

$$P(x_1) = \begin{pmatrix} A_2 & -S_{22} \\ -Q & -A_2' \end{pmatrix} = \begin{pmatrix} I & 0 \\ M & I \end{pmatrix} \begin{pmatrix} \Lambda & -S_{22} \\ 0 & -\Lambda' \end{pmatrix} \begin{pmatrix} I & 0 \\ -M & I \end{pmatrix} \tag{3.3}$$

Under Assumption 1, for any $m > 0$ P possesses the following property: it has n_2 stable eigenvalues λ, $\mathrm{Re}\,\lambda < -\alpha < 0$, and n_2 unstable ones λ, $\mathrm{Re}\,\lambda > \alpha$ for all $|x_1| \le m$. Then for any $m > 0$ there exists $\epsilon_m > 0$, such that for all $\epsilon \in (0, \epsilon_m]$ and $|x_1| + |p_1| < m$ the system (2.3) has the slow manifold (Kokotovic *et al.* 1986, Sobolev 1984)

$$\begin{pmatrix} x_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} L_3^*(x_1, p_1, \epsilon) \\ L_4^*(x_1, p_1, \epsilon) \end{pmatrix} = L^*(x_1, p_1, \epsilon) \tag{3.4}$$

The subscripts of L^* correspond to the third and fourth variables in the system of (2.3). To avoid cumbersome notation we shall omit the ϵ argument in the functions below.

Setting (3.4) into (2.3 a, b) and substituting v_1 and w_1 for x_1 and p_1 respectively, we get the $2n_1$-dimensional system for the flow on the slow manifold

$$\dot v_1 = f_1[v_1, w_1, L_3^*(v_1, w_1), L_4^*(v_1, w_1)] \tag{3.5a}$$

$$\dot w_1 = f_2(v_1, w_1, L_3^*(v_1, w_1), L_4^*(v_1, w_1)] \tag{3.5b}$$

The function L^* can be found in the form of an expansion

$$L^*(x_1, p_1, \epsilon) = \sum_{j=0}^{q} \epsilon^j l_j^*(x_1, p_1) + O(\epsilon^{q+1}) \tag{3.6}$$

The terms of (3.6) can be determined from the equation

$$\epsilon \frac{\partial L^*}{\partial x_1} f_1 + \epsilon \frac{\partial L^*}{\partial p_1} f_2 = \begin{pmatrix} A_2 L_3^* - S_{22} L_4^* + f_3(x_1, p_1) \\ -QL_3^* - A_2' L_4^* + f_4(x_1, p_1) \end{pmatrix} \tag{3.7}$$

where $f_i = f_i(x_1, p_1, L_3^*, L_4^*)$, $i = 1, 2$, by algebraic operations. Thus, $l_0^* = -P^{-1} f_0$, where $f_0 = \mathrm{col}\,\{f_3, f_4\}$. Note that (3.7) can be derived by differentiating on t of (3.4), where $x_1 = v_1(t)$, $p_1 = w_1(t)$, $x_2 = x_2(t)$, $p_2 = p_2(t)$, and by substituting for $\dot v_1$ and $\dot w_1$ the right-hand sides of (3.5).

Consider the slow system (3.5). Denote by $\Omega_{m_i} = \{x_1 \in \mathbf{R}^{n_i} : |x_i| < m_i\}$, $i = 1, 2$. Our next assumption is

Assumption 2: *There exist* $m_1 > 0$ *and* $\epsilon_1 > 0$ *such that for all* $\epsilon \in (0, \epsilon_1]$ *and* $v_1 \in \Omega_{2m_1}$ *the system* (3.5) *possesses the invariant manifold*

$$w_1 = N(v_1) \tag{3.8}$$

where the function $N = N(v_1, \epsilon)$ *is continuous on both arguments and uniformly bounded together with its first derivative on* v_1, *and* $N(0) = 0$.

The restriction of (3.5) to (3.8) is governed by the n_1-dimensional system

$$\dot v_1 = F_1(v_1) \tag{3.9}$$

where

$$F_i(v_1) = f_i[v_1, N(v_1), L_3^*(v_1, N(v_1)), L_4^*(v_1, N(v_1))] \tag{3.10}$$

and $i = 1$. Additionally, we assume

Assumption 3: *For all* $\epsilon \in (0, \epsilon_1]$, *equaiton* (3.9) *is asymptoptically stable.*

The theorem below states that Assumptions (A2 and A3) are necessary conditions for the existence of the invariant manifold (2.7) with asymptotically stable flow (2.9).

Theorem 1: *Let* A1 *hold, and for all small enough* ϵ *there exist* $\bar m_1$ *and* $\bar m_2$ *such that the* $(2n_1 + 2n_2)$*-dimensional Hamiltonian system* (2.3) *has an invariant on* $\Omega_{\bar m_1} \times \Omega_{\bar m_2}$ *manifold* (2.7) *with* (2.9) *asymptotically stable, where* V *has continuous and uniformly bounded derivatives on* $(x_1, x_2) \in \Omega_{\bar m_1} \times \Omega_{\bar m_2}$ *up to the second order, then* A2 *and* A3 *are valid.*

Note that the stable solutions of (2.3) are exponentially approaching the solutions of the slow manifold (3.4) (Pliss 1977, Sobolev 1984). Under A1–A3 we shall construct the invariant manifold (2.7) with the stable flow by means of the slow submanifold (3.8) and a fast manifold of an auxiliary system. To get the latter system let us introduce the following change of variables:

$$\left.\begin{aligned} \begin{pmatrix} v_2 \\ \bar p_2 \end{pmatrix} &= \begin{pmatrix} x_2 \\ p_2 \end{pmatrix} - L^*(x_1, p_1) \\ \begin{pmatrix} \bar x_1 \\ \bar p_1 \end{pmatrix} &= \begin{pmatrix} x_1 \\ p_1 \end{pmatrix} - \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \end{aligned}\right\} \tag{3.11}$$

where v_1 and w_1 satisfy (3.5). For the new variables we obtain the system

$$
\left.
\begin{aligned}
\dot{\bar{x}}_1 &= g_1(v_1, w_1, \bar{x}_1, \bar{p}_1, v_2, \bar{p}_2)\\
\dot{\bar{p}}_1 &= g_2(v_1, w_1, \bar{x}_1, \bar{p}_1, v_2, \bar{p}_2)\\
\epsilon \dot{v}_2 &= A_2(\bar{x}_1 + v_1)v_2 - S_{22}(\bar{x}_1 + v_1)\bar{p}_2\\
&\quad + g_3(v_1, w_1, \bar{x}_1, \bar{p}_1, v_2, \bar{p}_2)\\
\epsilon \dot{\bar{p}}_2 &= -Q(\bar{x}_1 + v_1)v_2 - A_2'(\bar{x}_1 + v_1)\bar{p}_2\\
&\quad + g_4(v_1, w_1, \bar{x}_1, \bar{p}_1, v_2, \bar{p}_2)
\end{aligned}
\right\} \tag{3.12}
$$

where for $i = 1, 2$

$$
\begin{aligned}
g_i = f_i\Big[&\bar{x}_1 + v_1, \bar{p}_1 + w_1, v_2 + L_3^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)\\
&\bar{p}_2 + L_4^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)\Big]\\
&- f_i\Big[v_1, w_1, L_3^*(v_1, w_1), L_4^*(v_1, w_1)\Big]
\end{aligned}
$$

and for $i = 3, 4$

$$
\begin{aligned}
g_i = &-\epsilon \frac{\partial L_i^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)}{\partial x_1}\Delta f_1\\
&-\epsilon \frac{\partial L_i^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)}{\partial p_1}\Delta f_2
\end{aligned}
$$

$$
\begin{aligned}
\Delta f_j = f_j\Big[&\bar{x}_1 + v_1, \bar{p}_1 + w_1, v_2 + L_3^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)\\
&\bar{p}_2 + L_4^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)\Big]\\
&- f_j\Big[\bar{x}_1 + v_1, \bar{p}_1 + w_1, L_3^*(v_1 + \bar{x}_1, w_1 + \bar{p}_1)\\
&L_4^*(v_1 + \bar{x}_1, w_1 + \bar{p}_1)\Big], \quad j = 1, 2
\end{aligned}
$$

Let $m_2 > 0$ be any positive. We choose m' such that

$$
\left|x_2 - L_3^*(x_1, N(x_1))\right| \leq m'/2, \ (x_1, x_2) \in \Omega_{2m_1} \times \Omega_{m_2}
$$

Then under A1 there exists ϵ' such that for all $\epsilon \in (0, \epsilon']$ the system of (3.5) and (3.12) has the fast (stable) manifold for $|v_2| < m'$ (Sobolev 1984, Henry 1982)

$$
\left.
\begin{aligned}
\bar{x}_1 &= \epsilon L_1^+(v_1, w_1, v_2)\\
\bar{p}_1 &= \epsilon L_2^+(v_1, w_1, v_2)\\
\bar{p}_2 &= L_4^+(v_1, w_1, v_2)
\end{aligned}
\right\} \tag{3.13}
$$

where $L_4^+ = M(v_1)v_2 + O(\epsilon)$. The functions L_i^+ $(i = 1, 2, 4)$ satisfy the inequalities

$$
\left.
\begin{aligned}
&\left|L_i^+(v_1, w_1, v_2)\right| \leq c|v_2|\\
&\left|L_i^+(v_1, w_1, v_2) - L_i^+(v_1, w_1, \tilde{v}_2)\right| \leq c|v_2 - \tilde{v}_2|\\
&\left|L_i^+(v_1, w_1, v_2) - L_i^+(\tilde{v}_1, \tilde{w}_1, v_2)\right| \leq c|v_2|\big(|v_1 - \tilde{v}_1|\\
&\qquad + |w_1 - \tilde{w}_1|\big)
\end{aligned}
\right\} \tag{3.14}
$$

The flow on this manifold is governed by the decoupled system of the slow equations (3.5) and the fast equation

$$
\epsilon \dot{v}_2 = A_2 v_2 - S_{22}L_4^+ + g_3(v_1, w_1, \epsilon L_1^+, \epsilon L_2^+, v_2, L_4^+) \tag{3.15}
$$

where $L_i^+ = L_i^+(v_1, w_1, v_2)$ $(i = 1, 2, 4)$, $A_2 = A_2(v_1 + \epsilon L_1^+)$ and $S_{22} = S_{22}(v_1 + \epsilon L_1^+)$. The solution of (3.15) with the initial value $v_2(0) = v_2^0$ satisfies the inequality

$$
|v_2(t)| \leq K \exp\Big(-\frac{\alpha}{\epsilon}t\Big) \cdot |v_2(0)|, \quad K > 0, t > 0 \tag{3.16}
$$

Hence, due to the first inequality of (3.14), the solutions of (3.12) lying on the fast manifold of (3.13) are rapidly exponentially decaying as t increases. Substituting (3.8) and (3.13) into (3.11), we get the algebraic equations for determining Z_1 and Z_2:

$$
x_1 = v_1 + \epsilon L_1^+\big[v_1, N(v_1), v_2\big] \tag{3.17a}
$$

$$
x_2 = v_2 + L_3^*\big[x_1, N(v_1) + \epsilon L_2^+(v_1, N(v_1), v_2)\big] \tag{3.17b}
$$

and

$$
p_1 = N(v_1) + \epsilon L_2^+\big[v_1, N(v_1), v_2\big] \tag{3.18a}
$$

$$
\begin{aligned}
p_2 = L_4^*&\big[x_1, N(v_1) + \epsilon L_2^+(v_1, N(v_1), v_2)\big]\\
&+ L_4^+(v_1, N(v_1), v_2)
\end{aligned} \tag{3.18b}
$$

Consider (3.17) as the system with respect to v_1 and v_2. Using the contraction principle argument, one can prove that there exists ϵ_2 such that, for $\epsilon \in (0, \epsilon_2]$, the system (3.17) has a unique solution on $\Omega_{m_1} \times \Omega_{m_2}$

$$
v_1 = U_1(x_1, x_2) = x_1 + \epsilon \bar{U}_1(x_1, x_2) \tag{3.19a}
$$

$$
\begin{aligned}
v_2 &= U_2(x_1, x_2)\\
&= x_2 - L_3^*(x_1, N(x_1)) + \epsilon \bar{U}_2(x_1, x_2)
\end{aligned} \tag{3.19b}
$$

where the functions $\bar{U}_1$ and $\bar{U}_2$ are Lipschitzian on x_1 and x_2, they vanish at $(x_1, x_2) = 0$, and satisfy the inequalities $\epsilon_2|\bar{U}_1| \leq m_1$, $\epsilon_2|\bar{U}_2| \leq m'/2$. Further, applying the implicit function theorem, one can show that U_1 and U_2 are continuously differentiable on x_1 and x_2. Substituting (3.19) into (3.18a) and (3.18b) we get (2.7), where

$$
Z_1 = N(U_1) + \epsilon L_2^+\big[U_1, N(U_1), U_2\big] \tag{3.20a}
$$

$$
\begin{aligned}
Z_2 = L_4^*&\big[x_1, N(U_1) + \epsilon L_2^+(U_1, N(U_1), U_2)\big]\\
&+ L_4^+(U_1, N(U_1), U_2)
\end{aligned} \tag{3.20b}
$$

Theorem 2: *Under* A1–A3 *for any* $m_2 > 0$ *there exists* $\epsilon_2 > 0$ *such that, for all* $\epsilon \in (0, \epsilon_2]$,

(i) *the* $(2n_1 + 2n_2)$*-dimensional Hamiltonian system* (2.3) *has the invariant on the* $\Omega_{m_1} \times \Omega_{m_2}$ *manifold* (2.7) *with* (2.9) *asymptotically stable, where continuously differentiable on* x_1 *and* x_2 *functions* Z_1 *and* Z_2 *are defined by formula* (3·20) *from the algebraic systems of* (3·17) *and* (3·18);

(ii) *there exists a* C^2 *function* $V : \Omega_{m_1} \times \Omega_{m_2} \to \mathbf{R}$, *satisfying the HJ equation* (2·5) *and relations* (2·8) (*therefore the optimal control problem is solvable by the controller of* (2·6)).

4. Hyperbolic case

Assumption A2 is not easily verifiable. In this section we will consider a particular case when A2 holds. For simplicity we assume that $\partial r(0)/\partial x_1 = 0$. Consider the linearization of (2.1) at 0

$$\left.\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_{10}u \\ \epsilon\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_{20}u \end{aligned}\right\} \quad (4{\cdot}1\,a)$$

$$J = \int_0^\infty \left[\sum_{i=1}^{2}\sum_{j=1}^{2} x_i' Q_{ij} x_j + u' R(0) u\right] \mathrm{d}t \quad (4{\cdot}1\,b)$$

where $A_{i1} = (\partial a_i/\partial x_1)(0)$, $A_{i2} = A_{i2}(0)$, $B_{i0} = D_i(0)$, $Q_{11} = \partial^2 r(0)/\partial x_1^2$, $Q_{12} = \frac{1}{2}(\partial s'(0)/\partial x_1)$ and $Q_{22} = Q(0)$. The Hamiltonian matrix corresponding to (4.1) is similar to the matrix Ham_γ that corresponds to the linearization at 0 of the Hamiltonian system (2.3). It is easy to see that

$$Ham_\gamma = \begin{pmatrix} P_{11} & P_{12} \\ \epsilon^{-1}P_{21} & \epsilon^{-1}P_{22} \end{pmatrix} \quad (4{\cdot}2)$$

where for $i = 1, 2$, $j = 1, 2$

$$P_{ij} = \begin{pmatrix} A_{ij} & -S_{ij} \\ -Q_{ij} & -A'_{ji} \end{pmatrix}, \quad S_{ij} = \tfrac{1}{2} B_{i0} R^{-1}(0) B'_{j0}$$

Under A1 the matrix P_{22} has no purely imaginary eigenvalues since $P_{22} = P(0)$. The matrix (4.2) has one group of $2n_1$ small eigenvalues $O(\epsilon)$ close to those of $P_0 = P_{11} - P_{12}P_{22}^{-1}P_{21}$ and another group of $2n_2$ large eigenvalues $O(1)$ close to those of $\epsilon^{-1}P_{22}$ (Kokotovic *et al.* 1986). To guarantee that, for all small enough ϵ, the matrix (4.2) has no purely imaginary eigenvalues, i.e. the vector field defined by (2.3) is hyperbolic, we suppose

Assumption 0: *The matrix* $P_0 = P_{11} - P_{12}P_{22}^{-1}P_{21}$ *has no purely imaginary eigenvalues.*

It is known that in the hyperbolic case, for each ϵ, the optimal control problem is solvable on a small enough neighbourhood of $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ containing 0 if the linearized problem is solvable. The latter is equivalent to the existence of a non-negative definite stabilizing solution to a corresponding $(n_1 + n_2) \times (n_1 + n_2)$-algebraic Riccati equation (ARE). We will get the reduced-order (in terms of $n_1 \times n_1$-ARE) sufficient conditions for the solvability of the optimal control problems on the domains containing large values of x_2 for all sufficiently small ϵ.

Under A0 the matrix P_0 has n_1 eigenvalues with negative real parts and n_1 with positive ones. This fact follows from the symmetry of the eigenvalues of Ham_γ and of P_{22}. Note that P_0 coincides with the linearization of the slow subsystem (3.5) on v_1, w_1 at $(v_1, w_1, \epsilon) = 0$. Hence under A0 the invariant manifold (3.8) of stable solutions of (3.5) (if it exists) is a stable manifold of (3.5). To guarantee its existence we consider system (3.5), linearized on v_1 and w_1 at $(v_1, w_1, \epsilon) = 0$:

$$\left.\begin{aligned} \dot{v}_1 &= T_1 v_1 + T_2 w_1 \\ \dot{w}_1 &= T_3 v_1 + T_4 w_1 \\ \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} &= P_0 \end{aligned}\right\} \quad (4{\cdot}3)$$

Suppose that the stable manifold (i.e. the stable eigenspace) of (4.3) can be parametrized by v_1-coordinates in the form $w_1 = N^{(0)}v_1$. Then $N^{(0)}$ satisfies the following $n_1 \times n_1$-ARE:

$$N^{(0)}(T_1 + T_2N^{(0)}) = T_3 + T_4N^{(0)} \quad (4{\cdot}4)$$

and the matrix $T_1 + T_2N^{(0)}$ is Hurwitz. We suppose further

Assumption 4: *ARE* (4·4) *has a solution* $N^{(0)}$ *such that the matrix* $T_1 + T_2N^{(0)}$ *is Hurwitz.*

Note that A4 implies that A0 since the stable eigenvalues of P_0 coincide with the eigenvalues of $T_1 + T_2N^{(0)}$. From the theory of non-linear differential equation it is known that under A4 the hypotheses A2 and A3 are valid.

Thus we obtain

Corollary 1: *Under* A1 *and* A4 *for any* $m_2 > 0$ *there exist* $m_1 > 0$ *and* $\epsilon_3 > 0$ *such that for all* $\epsilon \in (0, \epsilon_3]$ *items* (i) *and* (ii) *of Theorem 2 are valid.*

5. Asymptotic expansion of the optimal control and optimal trajectory

5.1. *Asymptotic approximation of the invariant manifolds*

We assume further

Assumption 5: *For small enough* ϵ *and* $|v_1| \le 2m_1$ *the function N can be represented in the form*

$$N(v_1,\epsilon) = \sum_{j=0}^{q} \epsilon^j N_j(v_1) + O(\epsilon^{q+1}) \qquad (5.1)$$

where N_j are smooth functions.

Note that A4 implies A5. The terms of (5.1) can be found from the partial differential equation

$$\frac{\partial N}{\partial v_1} F_1(v_1) = F_2(v_1) \qquad (5.2)$$

where F_i are defined by (3.10).

Analogously to (3.6), we construct the expansion of the function $L^+ = \mathrm{col}\{L_1^+, L_2^+, L_4^+\}$

$$L^+ = \sum_{j=0}^{q} \epsilon^j l_j^+ + O(\epsilon^{q+1}) \qquad (5.3)$$

from the equation (Sobolev 1984)

$$\epsilon \frac{\partial L^+}{\partial v_1} f_1 + \epsilon \frac{\partial L^+}{\partial w_1} f_2 + \frac{\partial L^+}{\partial v_2}\left[A_2 v_2 - S_{22} L_4^+ + g_3\right] = \mathrm{col}\left\{g_1, g_2, -Q v_2 - A_2' L_4^+ + g_4\right\} \qquad (5.4)$$

where A_2, S_{22} and Q depend on $v_1 + \epsilon L_1^+$, and $f_i = f_i(v_1, w_1, L_3^*, L_4^*)$, $i = 1, 2$, $g_k = g_k(v_1, w_1, \epsilon L_1^+, \epsilon L_2^+, v_2, L_4^+)$, $k = 1, \dots, 4$, $L^+ = L^+(v_1, w_1, v_2)$.

For the terms of (5.3) we get successively equations of the form

$$\frac{\partial l_{4j}^+}{\partial v_2}\left[A_2(v_1) - S_{22}(v_1) M(v_1)\right] v_2 = -A_2' l_{4j}^+ + G_{4j}(v_1, w_1, v_2) \qquad (5.5)$$

where $l_j^+ = \mathrm{col}\{l_{1j}^+, l_{2j}^+, l_{4j}^+\}$ and G_{4j} is a known function such that $G_{4j}[v_1, w_1, 0] = 0$. Equation (5.5) depends on v_1 and w_1, as on the parameters and its solution is given by

$$l_{4j}^+ = \int_0^\infty e^{A_2' t} G_{4j}\left[v_1, w_1, e^{[A_2(v_1) - S_{22}(v_1)M(v_1)]t} v_2\right] dt$$

Then the equations for l_{41}^+ and l_{42}^+ have the form

$$\frac{\partial l_{ij}^+}{\partial v_2}\left[A_2(v_1) - S_{22}(v_1) M(v_1)\right] v_2 = G_{ij}(v_1, w_1, v_2) \qquad i = 1, 2 \qquad (5.6)$$

where G_{ij} are known functions such that $G_{ij}[v_1, w_1, 0] = 0$. Solutions to the latter equations are given by

$$l_{ij}^+ = \int_0^\infty G_{ij}\left[v_1, w_1, e^{[A_2(v_1) - S_{22}(v_1)M(v_1)]t} v_2\right] dt$$

In the case of the system of (2.1), l_j^+ is a $(j+1)$-order polynomial with respect to v_2. The coefficients of this polynomial can be found from (5.5) and (5.6) by algebraic operations.

5.2. *Asymptotic approximation of optimal feedback*

Next we obtain from (3.17)

$$v_i = U_i(x_1, x_2, \epsilon) = \sum_{j=0}^{q} \epsilon^j U_{ij}(x_1, x_2) + O(\epsilon^{q+1}), \quad i = 1, 2 \qquad (5.7)$$

where U_{ij} are sufficiently smooth with respect to x since L^*, N and L^+ are smooth with respect to v_1, w_1 and v_2. Thus, $U_{10} = x_1$, $U_{20} = x_2 - L_{30}[v_1, N_0(v_1)]$. We substitute the expansions (5.7), (5.1), (5.3) and (3.6) into (3.20*a*) and (3.20*b*):

$$Z_1 = \sum_{i=0}^{q} \epsilon^i N_i\left(\sum_{j=0}^{q} \epsilon^j U_{1j}\right) + \sum_{i=0}^{q} \epsilon^{i+1} \times l_{2i}^+\left[\sum_{j=0}^{q} \epsilon^j U_{1j}, \sum_{i=0}^{q} \epsilon^i N_i\left(\sum_{j=0}^{q} \epsilon^j U_{1j}\right), \sum \epsilon^j U_{2j}\right] + O(\epsilon^{q+1})$$

$$Z_2 = \sum_{i=0}^{q} \epsilon^i l_{4i}^*\left[x_1, Z_1\right] + \sum_{i=0}^{q} \epsilon^i \times l_{4i}^+\left[\sum_{j=0}^{q} \epsilon^j U_{1j}, \sum_{i=0}^{q} \epsilon^i N_i\left(\sum_{j=0}^{q} \epsilon^j U_{1j}\right), \sum \epsilon^j U_{2j}\right] + O(\epsilon^{q+1})$$

Expanding the right-hand sides of the latter equations in the powers of ϵ we get asymptotic approximations to Z_1 and Z_2

$$Z_i(x_1, x_2, \epsilon) = \sum_{j=0}^{q} \epsilon^j Z_{ij}(x_1, x_2) + O(\epsilon^{q+1}), \quad i = 1, 2 \qquad (5.8)$$

where Z_{ij} are sufficiently smooth. Note that

$$\left.\begin{aligned} Z_{10} &= N_0(x_1) \\ Z_{20} &= l_{40}^*\left[x_1, N_0(x_1)\right] + M(x_1)\left[x_2 - l_{30}^*(x_1, N_0(x_1))\right] \end{aligned}\right\} \qquad (5.9)$$

Substituting (5.8) into (2.6) we get the following $O(\epsilon^{q+1})$-approximation to the optimal controller

$$\left.\begin{aligned} u &= u_q + O(\epsilon^{q+1}) \\ u_q &= -\sum_{k=1}^{2}\sum_{j=0}^{q} \epsilon^j B_k' Z_{kj}(x_1, x_2) \end{aligned}\right\} \qquad (5.10)$$

5.3. *Asymptotic expansion of optimal trajectory and open-loop control*

These operations on the finite segment $[0, T]$ (for every $T > 0$) can be found from (3.17), (3.18), (3.9), (3.15) and the relation

$$u(t) = -\tfrac{1}{2} R^{-1} B_1' p_1(t) - \tfrac{1}{2} R^{-1} B_2' p_2(t) \qquad (5.11)$$

Applying standard asymptotic methods (see e.g. O'Malley 1974) to the decoupled equations (3.9) and (3.15), where $w_1 = N(v_1)$, we obtain correspondingly

$$v_1(t) = \sum_{i=0}^{q} \epsilon^i v_1^{(i)}(t) + \epsilon^{q+1} r_{1q}(t, \epsilon) \qquad (5{\cdot}12a)$$

$$v_2(t) = \sum_{i=0}^{q} \epsilon^i v_2^{(i)}(\tau) + \epsilon^{q+1} r_{2q}(\tau, \epsilon) \qquad (5{\cdot}12b)$$

where $\tau = t/\epsilon$ and r_1 and r_2 are uniformly bounded for $t \in [0, T]$ and $\tau \geq 0$. Substituting (5·12*a*) into (3.9) and equating coefficients of equal powers of ϵ, we find differential equations for $v_1^{(i)}$ with initial values defined by (3·19*a*), where $t = 0$. Similarly, from (3.15), (3.8), (5·12*b*) and (3·19*b*), we obtain initial value problems for $v_2^{(i)}$.

Finally, substituting expansions of v_1, v_2, L^+ and L^* into (3.17), (3.18) and (5.11) and expanding right-hand sides of the resulting equations in the powers of ϵ, we find the following approximations

$$x(t) = \sum_{i=0}^{q} \epsilon^i x^{(i)}(t) + \sum_{i=0}^{q} \epsilon^i \Pi_1^{(i)}(\tau) + \epsilon^{q+1} R_{1q}(t, \epsilon) \quad (5{\cdot}13a)$$

$$u(t) = \sum_{i=0}^{q} \epsilon^i u^{(i)}(t) + \sum_{i=0}^{q} \epsilon^i \Pi_2^{(i)}(\tau) + \epsilon^{q+1} R_{2q}(t, \epsilon) \quad (5{\cdot}13b)$$

where Π_1 and Π_2 are boundary layer terms exponentially decaying when $\tau \to \infty$, remainders R_{1q} and R_{2q} are uniformly bounded for $t \in [0, T]$. Note that (5·13*b*) can be found also by substitution of (5·13*a*) into the expansion of the optimal feedback (5.10).

The higher-order terms in the approximation (5.10) lead to improved performance:

Theorem 3: (i) *Under* A1–A3 *and* A5 (*or* A1 *and* A4) *for small enough* ϵ *and* $|x_1| \leq m_1$, $|x_2| \leq m_2$, *the invariant manifold* (2·7) *exists and can be represented in the form* (5·8)*; the optimal controller exists and can be approximated by* (5·10)*; the optimal trajectory and optimal open-loop control can be approximated by* (5·13), *where approximation is uniform on every finite segment* $[0, T]$.
(ii) *Under* A1 *and* A4 *for any* $m_2 > 0$ *there exist* m_1' *and* ϵ_1' *such that, for all* $\epsilon \in (0, \epsilon_1']$ *and initial conditions from* $\Omega_{m_1} \times \Omega_{m_2}$, *the controller* u_q *achieves the performance* $O(\epsilon^{q+1})$*-close to the optimal one.*

Remark 1: Chow and Kokotovic (1978) showed that the generating function V can be found from HJ equation (2.5) in the form of nested expansion. They proved that the truncated series of this expansion satisfies the HJ equation with a high order of accuracy. From Theorem 3 above the stronger result follows: under assumptions of (ii) the truncated series (5.10) lead to near-optimal controller and, in the hyperbolic case, to near-optimal performance.

Remark 2: All the results of the present paper are also valid for the systems containing non-linear on x_2 terms of the order of $O(\epsilon)$. In this case V cannot be found in the form of the expansion of Chow and Kokotovic (1978), having a more complicated structure.

Example. We consider an example from Chow and Kokotovic (1978):

$$\dot{x}_1 = x_1 x_2, \quad \epsilon \dot{x}_2 = -x_2 + u, \quad J = \int_0^\infty (x_1^4 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}u^2)\, dt$$

This is a non-hyperbolic case since $P_0 = 0$. We obtain the following Hamiltonian function

$$\mathcal{H} = x_1^4 + 1/2x_2^2 + p_1 x_1 x_2 - x_2 p_2 - 1/2 p_2^2$$

and Hamiltonian system

$$\left.\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{p}_1 &= -4x_1^3 - p_1 x_2 \\ \epsilon \dot{x}_2 &= -x_2 - p_2 \\ \epsilon \dot{p}_2 &= -x_2 - p_1 x_1 + p_2 \end{aligned}\right\} \qquad (5{\cdot}14)$$

Further, we neglect terms of the order $O(\epsilon^2)$. Then from (3.7), (3.9) and (5.4) we find

$$L^* = \left[-1/2 p_1 x_1, 1/2 p_1 x_1 - 2\epsilon x_1^4\right]', \quad N = 2x_1$$

$$L^+ = \left[-\epsilon\sqrt{2}/2 v_1 v_2, \epsilon\sqrt{2}/2 w_1 v_2, (-1 + \sqrt{2}) v_2\right]'$$

Equations (3.16) and (3.18) have the form

$$x_1 = v_1 - \epsilon\sqrt{2}/2 v_1 v_2 \qquad (5{\cdot}15a)$$

$$x_2 = v_2 - x_1(v_1 + \epsilon\sqrt{2}/2 v_1 v_2) \qquad (5{\cdot}15b)$$

$$p_1 = 2v_1 + \sqrt{2}\epsilon v_1 v_2 \qquad (5{\cdot}16a)$$

$$p_2 = x_1(v_1 + \sqrt{2}/\epsilon v_1 v_2) - 2\epsilon x_1^4 + (\sqrt{2} - 1) v_2 \qquad (5{\cdot}16b)$$

From (5.15) we obtain

$$\left.\begin{aligned} v_1 &= x_1 + \epsilon\sqrt{2}/2 x_1 (x_2 + x_1^2) \\ v_2 &= x_2 + x_1^2 + \epsilon\sqrt{2} x_1^2 (x_2 + x_1^2) \end{aligned}\right\} \qquad (5{\cdot}17)$$

Substituting (5.17) into (5.16), we find

$$p_1 = Z_1(x_1, x_2) = 2x_1 + 2\sqrt{2}\epsilon x_1 (x_2 + x_1^2)$$

$$p_2 = Z_2(x_1, x_2) = \sqrt{2} x_1^2 + (\sqrt{2} - 1) x_2 + 2\epsilon x_1^2 x_2$$

By (2.6), $u = -Z_2(x_1, x_2)$ and thus we obtain the same near-optimal feedback as in Chow and Kokotovic (1978)

$$u_1 = -\sqrt{2}x_1^2 - (\sqrt{2} - 1)x_2 - 2\epsilon x_1^2 x_2 \quad (5.18)$$

Our results show that this feedback is $O(\epsilon^2)$-close to the optimal one if (3.9) is asymptotically stable.

Equation (3.9) has the form

$$\dot{v}_1 = v_1 L_3^*(v_1, N(v_1)) = -v_1(v_1^2 + O(\epsilon v_1^2)) \quad (5.19)$$

since L_3^*, $\partial L_3^*/\partial x_1$ and $\partial L_3^*/\partial p_1$ vanish in the origin as well as the right-hand sides of (5.14) together with their partial derivatives on p_1 and x_1. Therefore (5.19), for small enough ϵ, is asymptotically stable (to check this, one can take $V_1 = v_1^2$ as a Lyapunov function). Hence optimal control exists and is $O(\epsilon^2)$-close to (5.18).

Neglectig terms of the order $O(\epsilon^2)$, we obtain (3.9) and (3.15)

$$\dot{v}_1 = -v_1^3$$

$$\epsilon \dot{v}_2 = -\sqrt{2}v_2$$

Then v_1 and v_2 are given by

$$\left.\begin{aligned} v_1(t) &= \frac{\operatorname{sgn} v_1(0)}{\sqrt{2t + v_1(0)^{-2}}} \\ v_2(t) &= e^{-\frac{\sqrt{2}t}{\epsilon}} v_2(0) \end{aligned}\right\} \quad (5.20)$$

Substituting (5.20) with the initial values given by (5.17), where $t = 0$, into (5.15) and (5.16), we find asymptotic approximation to optimal trajectory and open-loop control $u = -p_2$.

6. Conclusions

We have developed a geometric approach to a singularly perturbed optimal control problem, non-linear on the slow state variables. We have obtained the exact decomposition of the slow–fast invariant manifold of the Hamiltonian system into the reduced-order slow manifold and a fast manifold. As a result, sufficient conditions for the solvability of the optimal control problem in terms of the slow manifold have been obtained. Also, an asymptotic expansion of the optimal controller has been constructed by solving partial differential equations, depending only on the slow variables. At the same time we have obtained decomposition of the Hamiltonian system to the slow and fast subsystems. This leads to asymptotic approximation to optimal trajectory and open-loop control. We have shown that a higher-order accuracy controller improves performance. The results are valid on the domains containing large values of the fast variables.

Appendix

Proof of Theorem 1: Under A1 the system (2.3) has a centre-stable manifold (for analogous derivations see Fridman 1992, Kelley 1967 and Pliss 1977)

$$p_2 = L^{*+}(x_1, p_1, x_2) \quad (A.1)$$

such that all the stable solutions of (2.3) belong to it. Let Z_1 and Z_2 be defined by (2.8). Then (2.7) determines an invariant on $\Omega_{\bar{m}_1} \times \Omega_{\bar{m}_2}$ manifold of stable solutions to (2.3), i.e. (2.7) is the submanifold of (A.1). Therefore p_1 and p_2, defined by (2.7), satisfy also (A.1), which implies the following relation

$$Z_2(x_1, x_2) = L^{*+}\left[x_1, Z_1(x_1, x_2), x_2\right], \quad (x_1, x_2) \in \Omega_{\bar{m}_1} \times \Omega_{\bar{m}_2} \quad (A.2)$$

Let $|Z_1(x_1, x_2)| \le m_2$ for $(x_1, x_2) \in \Omega_{\bar{m}_1} \times \Omega_{\bar{m}_2}$, and $m = \bar{m}_1 + m_3$. Further, let (3.4) determine a centre manifold of (2.3) for $|x_1| + |p| \le m$. We shall prove that for any $v_1^0 \in \Omega_{2m_1}$, where m_1 will be chosen below (from the solvability of (A.4a) for w_1), there exists $w_1^0 \in R^{n_1}$ such that the solution of (2.3), lying on its centre manifold

$$x_1 = v_1, p_1 = w_1, x_2 = L_3^*(v_1, w_1), p_2 = L_4^*(v_1, w_1), \quad t \in \mathbf{R} \quad (A.3a)$$

$$x_1(0) = v_1^0, \quad p_1(0) = w_1^0 \quad (A.3b)$$

lies also on the invariant manifold (2.7), i.e. satisfies on some $t_1 < 0 < t_2$ the equations

$$w_1 = Z_1\left[v_1, L_3^*(v_1, w_1)\right] \quad (A.4a)$$

$$p_2 = Z_2\left[v_1, L_3^*(v_1, w_1)\right] \quad (A.4b)$$

Note that (A.4b) follows from (A.2)–(A.4a). Really, substituting the first and the third of the relations (A.3a) into (A.2) and applying further (A.4a), we have

$$Z_2(v_1, L_3^*(v_1, w_1)) = L^{*+}\left[v_1, Z_1(v_1, L_3^*(v_1, w_1)), L_3^*(v_1, w_1)\right] = L^{*+}\left[v_1, w_1, L_3^*(v_1, w_1)\right] \quad (A.5)$$

The expression in the right-hand side of (A.5) coincides with $L_4^*(v_1, w_1)$ since the centre manifold is an invariant submanifold of the centre-stable manifold. This, together with the last of (A.3a), implies (A.4b).

Consider (A.4a) as a system for evaluating w_1. First, we shall show that Z_1 can be represented as follows

$$Z_1(x_1, x_2) = C_1(x_1) + \epsilon C_2(x_1, x_2, \epsilon) \quad (A.6)$$

where C_1 and C_2 are Lipschitzian on $(x_1, x_2) \in \Omega_{\bar{m}_1} \times \Omega_{\bar{m}_2}$. Really, differentiating the first of the relations (2.8) on x_2, and the second on x_1, we get $Z_{1x_2} = \epsilon Z_{2x_1} = V'_{x_1x_2}$, which yields the representation (A.6). Substituting (A.6) into (A.4a) and applying to the latter equation the contraction principle, one can show that there exists $m_1 > 0$ such that (A.4a) has a solution (4.1) for $|v_1| \leq 2m_1$, where N is Lipschitzian and $N(0) = 0$. Further, applying the implicit function theorem, one can prove that N is continuously differentiable for $|v_1| < 2m_1$.

Let $|v_1| < 2m_1$ for $t \in (t_1, t_2)$. Then from (A.4a) it follows that (3.8) is valid on (t_1, t_2), i.e. (3.8) defines an invariant manifold of (3.5). The solutions of the latter invariant manifold are asymptotically stable, being at the same time the solutions of (2.7) with asymptotically stable (2.9). □

Proof of Theorem 2: (i) The relations (2.7) define the invariant on $\Omega_{m_1} \times \Omega_{m_2}$ manifold of (2.3) if, for any $(x_1^0, x_2^0) \in \Omega_{m_1} \times \Omega_{m_2}$, there exists $t_1 < 0 < t_2$ such that a solution of (2.3) with the initial values

$$\left.\begin{aligned} x_1(0) &= x_1^0, \quad x_2(0) = x_2^0, \\ p_1(0) &= Z_1(x_1^0, x_2^0), \quad p_2(0) = Z_2(x_1^0, x_2^0) \end{aligned}\right\} \quad (A.7)$$

satisfies (2.7) for $t \in (t_1, t_2)$.

Let $(x_1^0, x_2^0) \in \Omega_{m_1} \times \Omega_{m_2}$ be any prechosen. Let v_1 and v_2 be solutions of (3.9) and (3.15), where w_1 is defined by (3.8) and with the following initial conditions

$$v_1(0) = U_1(x_1^0, x_2^0), \quad v_2(0) = U_2(x_1^0, x_2^0) \quad (A.8)$$

Denote by x_1, x_2, p_1, p_2 a solution of (2.3), (A.7). Note that the relations (A.7) and (A.8) imply (3.17) and (3.18) at $t = 0$. Let $t_1 < 0 < t_2$ be such an interval that, for $t \in (t_1, t_2)$, we have $(x_1, x_2) \in \Omega_{m_1} \times \Omega_{m_2}$ and $|v_1| \leq 2m_1$. Then, due to the uniqueness of the solution of (2.3), (A.7), the relations (3.17) and (3.18) are satisfied for all $t \in (t_1, t_2)$. This yields (2.7) for all $t \in (t_1, t_2)$. Hence, the relations (2.7) define an invariant on $\Omega_{m_1} \times \Omega_{m_2}$ manifold of (2.3). The asymptotic stability of (2.7) follows from the same property of v_1, v_2 and from the relations (3.17); this completes the proof of (i).

The invariant manifold (2.7) with asymptotically stable (2.9) is Lagrangian (it can be proved as Lemma 1 of Van der Schaft 1991) and is projectable on the simply connected manifold $\Omega_{m_1} \times \Omega_{m_2}$, which implies the existence of the generating function V, satisfying (2.8) and (2.3) (Van der Schaft 1991).

Asymptotic stability of (2.1), (2.6) follows from (3.17), (3.18) and asymptotic stability of (3.9) and (3.15). □

Proof of Theorem 3: We have to prove only (iii). Substitute u given by (5.10) and u_q into (2.1) and denote by x and $\bar{x}$ the solutions to the resulting equations. Under A4 the latter equations are exponentially stable. Then, similarly to (5.13), the following approximations can be obtained for small enough $x_1(0)$

$$x(t) = \sum_{i=0}^{q} \epsilon^i x^{(i)}(t) + \sum_{i=0}^{q} \epsilon^i \Pi_1^{(i)}(\tau) + \epsilon^{q+1} R_q(t, \epsilon) \quad (A.9a)$$

$$\bar{x}(t) = \sum_{i=0}^{q} \epsilon^i x^{(i)}(t) + \sum_{i=0}^{q} \epsilon^i \Pi_1^{(i)}(\tau) + \epsilon^{q+1} \bar{R}_q(t, \epsilon) \quad (A.9b)$$

where, due to A4

$$\left|x^{(i)}(t)\right| \leq C e^{-\alpha t}, \quad \left|R_q(t)\right| \leq C e^{-\alpha t}$$
$$\left|\bar{R}_q(t)\right| \leq C e^{-\alpha t} \quad (t \geq 0, \alpha > 0) \quad (A.10)$$

Note that the terms $x^{(i)}$ and $\Pi_1^{(i)}$ in the expansions (A.9) are the same since the right-hand sides of the corresponding differential equations are $O(\epsilon^{q+1})$-close. Substitution of (A.9) and (5.10) into J leads to (iii). □

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