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Decoupling Transformation of Singularly Perturbed Systems with Small Delays and Its Applications

For linear singularly perturbed system of functional differential equations with small time delays we find a change of variables that decomposes this system into a purely-slow system of ordinary differential equations and a purely-fast functional equation. This decomposition is a generalization of Chang's decoupling transformation of singularly perturbed systems in the time-delay case. It is obtained by virtue of invariant manifolds and it can be found approximately in the form of asymptotic expansion. Using this transformation we get the reduced-order approximate models, stability and stabilizability criteria. This transformation can be further used in different control problems.

1. Decoupling transformation and its asymptotic approximation

Let $\mathbb{R}^m$ be Euclidean space and $C[a,b]$ be the space of continuous functions $\phi : [a,b] \to \mathbb{R}^m$ with the supremum norm $|\cdot|$. Consider the system

$$\dot{x}(t) = A_{11}x(t) + A_{12}z_t + B_1u(t), \quad x(0) = x_0, \tag{1a}$$

$$\epsilon\dot{z}(t) = A_{21}x(t) + A_{22}z_t + B_2u(t), \quad z(\zeta) = z_0 \quad (-\epsilon \le \zeta \le 0), \tag{1b}$$

where $x(t) \in \mathbb{R}^n,\ z(t) \in \mathbb{R}^m$ are the state vectors, $u(t) \in \mathbb{R}^q$ is the control input, $A_{i1}, B_i\ (i = 1,2)$ are the matrices of the appropriate dimensions, $z_t = z(t+\zeta)\ (\zeta \in [-\epsilon h_k; 0])$ and ϵ is a small positive parameter. Assume that $z_0 \in C[-\epsilon h_k, 0], u \in L_1^{loc}[0,\infty)$ and that the linear mappings $A_{i2},\ i = 1,2$ are given by

$$A_{i2}z_t = \sum_{j=0}^{k} C_{ji}z(t-\epsilon h_j) + \int_{-h_k}^{0} D_i(\theta)z(t+\epsilon\theta)d\theta, \quad 0 = h_0 < h_1 < ... < h_k,$$

where D_i is integrable on $[-h_k, 0]$. The important tool in studying different control problems for (1) (e.g. construction of approximate models, controllability, filtering etc. [6]) in the case of ordinary differential equations is Chang's transformation [2], decoupling (1) into slow and fast subsystems. In the present paper we construct the similar transformation in the case of retarded type equations. The singularly perturbed systems with small delays were considered e.g. in [7;8], where the existence of slow manifolds and of exponential dichotomy were established. In [3] the decoupling transformation has been constructed in the case of the homogeneous system with one discrete delay and advance.

For each $\epsilon > 0$ denote by $S(t,\epsilon) : C[-\epsilon h_k, 0] \to C[-\epsilon h_k, 0]$ the semigroup of the shift operators, corresponding to the fast linear equation

$$\epsilon\dot{z}(t) = A_{22}z_t, \tag{2}$$

and $X(t+\zeta,\epsilon) = S(t,\epsilon)X_0(\zeta \in [-\epsilon, 0])$ is the fundamental matrix of (2), $X_0(0) = 0, X_0(\zeta) = 0(\zeta < 0)$. Applying the variation of constants formula [5] to (1b) , we get the equivalent to (1) system of differential and integral equations

$$\dot{x}(t) = A_{11}x(t) + A_{12}z_t + B_1u(t), \tag{3a),}$$

$$z_t = S(t,\epsilon)z_0 + \frac{1}{\epsilon}\int_0^t S(t-s,\epsilon)[X_0A_{21}x(s) + B_2u(s)]ds, \tag{3b}$$

where integral in (3b) is understood as the integral in $\mathbb{R}^n$.

Our main assumption is:

A1. The roots of the characteristic equation

$$det[\lambda I - \sum_{j=0}^{k} C_{j2}e^{-\lambda h_j} - \int_{-h_k}^{0} D_2(\theta)e^{\lambda\theta}d\theta] = 0 \tag{4}$$

have negative real parts.

Note that (4) corresponds to (2) written in the fast time $\tau = t/\epsilon$. Set $t = \epsilon\tau$ in (2) to obtain $S(t,\epsilon)z_0 = S(\tau,1)\bar{z}_0,\ z_0(\epsilon\theta) = \bar{z}_0(\theta,\epsilon)(\theta \in [-h_k, 0])$. Then under A1 the following inequality holds:

$$|S(t,\epsilon)z_0| = |S(\tau,1)\bar{z}_0| \le Ke^{-\frac{\alpha}{\epsilon}t}|z_0|, \quad \alpha > 0,\ K > 1.$$

By standard argument for existence of invariant manifolds (see e.g. [5;7]), for all small enough ϵ the homogeneous system (1) (with $B_i = 0$) has a center manifold $z_t = -Lx$, where $L = L(\epsilon) : \mathbb{R}^n \to C_[-\epsilon h_k, 0]$ is the linear bounded operator (see e.g.[3]).

For continuously differentiable function $\phi \in C[-\epsilon h_k, 0]$ denote

$$\mathcal{A}(\epsilon)\phi = \begin{cases} \dot{\phi}, & \text{if } \zeta \in [-\epsilon h_k, 0) \\ \frac{1}{\epsilon}\sum_{j=0}^{k} C_{j2}z(t-\epsilon h_j) + \frac{1}{\epsilon}\int_{-h_k}^{0} D_2(s)z(t+\epsilon s)ds, & \text{if } \zeta = 0. \end{cases}$$

Similarly to [4] the following Lemma can be proved:

Lemma 1. *Under A1 for all small enough $\epsilon > 0$*

1. *the continuously differentiable $m \times n$-matrix function $L(\zeta)$ determines the center manifold $z_t = -Lx(t)$ of (3) iff for every $\zeta \in [-\epsilon h_k, 0]$ it satisfies the equation*

$$\epsilon L(A_{11} - A_{12}L) = \epsilon\mathcal{A}(\epsilon)L - X_0A_{21}; \tag{5}$$

2. *the following approximation holds uniformly on $[-h_k, 0]$:*

$$L(\epsilon\theta) = L_0(\theta) + \epsilon L_1(\theta) + ... + \epsilon^p L_p + O(\epsilon^{p+1}). \tag{6}$$

Substitute (6) into (5) and equate the coefficients of equal powers of ϵ. Since $\mathcal{A}(\epsilon)L = \epsilon^{-1}A(1)\sum\epsilon^i L_i$, we get $\mathcal{A}(1)L_0 - X_0A_{21} = 0$ that implies the following solution:

$$L_0 = (A_{22}I)^{-1}A_{21}, \quad A_{22}I = \sum_{j=0}^{k} C_{j2} + \int_{-h_k}^{0} D_2(\theta)d\theta,$$

where under A1 the inverse matrix exists since $\lambda = 0$ does not satisfy (4). The next term L_1 satisfies the equation $\mathcal{A}(1)L_1 = L_0(A_{11} - A_{12}L_0)$. Solving the latter equation we get

$$L_1 = (\theta I + M)L_0(A_{11} - A_{12}L_0), \quad M = (A_{22}I)^{-1}[I + \sum_{j=0}^{k} C_{j2}h_j - \int_{-h_k}^{0} D_2(\theta)\theta d\theta].$$

Similarly the higher order terms $L_i (i \ge 1)$ can be found.

Changing variables in (3) $\xi_t = z_t + Lx(t)$ and using the formula [4]:

$$S(t,\epsilon)Lx_0 - Lx(t) = \int_0^t S(t-s,\epsilon)A(\epsilon)Lx(s)ds - \int_0^t S(t-s,\epsilon)L\dot{x}(s)ds,$$

we get the following system

$$\dot{x}(t) = (A_{11} - A_{12}L)x + A_{12}\xi_t + B_1u(t), \tag{7a}$$

$$\xi_t = S(t,\epsilon)\xi_0 + \frac{1}{\epsilon}\int_0^t S(t-s,\epsilon)[\epsilon LA_{12}\xi_s + (X_0B_2 + \epsilon LB_1)u(s)]ds. \tag{7b}$$

For small enough ϵ homogeneous system (7) has the stable manifold $x = \epsilon H\xi$, where $H : C[-\epsilon h_k, 0] \to \mathbb{R}^n$. Similarly to [3,4] the following Lemma can be proved:

Lemma 2. *Under A1 for all small enough $\epsilon > 0$*

1. *the linear bounded operator $H : C[-\epsilon h_k, 0] \to \mathbb{R}^n$ determines the stable manifold $x(t) = Hz_t$ of (3) iff for every continuously differentiable $\xi \in C[-\epsilon h_k, 0]$ it satisfies the following equation:*

$$\epsilon H[\mathcal{A}(\epsilon)\xi + LA_{12}\xi] = [(A_{11} - A_{12}L)\epsilon H + A_{12}]\xi; \tag{8}$$

2. the following approximation holds :

$$H\xi = H_0\bar{\xi} + \epsilon H_1\bar{\xi} + ... + \epsilon^p H_p\bar{\xi} + O(\epsilon^{p+1}|\xi|),$$

where $\xi \in C[-\epsilon h_k, 0]$, $\bar{\xi}(\theta, \epsilon) = \xi(\epsilon\theta)$ $(\theta \in [-h_k, 0])$, $\bar{\xi} \in C[-h_k, 0]$.

Substitution of $\sum \epsilon^i H_i$ and $\epsilon^{-1}\mathcal{A}(1)\bar{\xi}$ for H and $\mathcal{A}(\epsilon)\xi$ into (8) and equating the like coefficients of ϵ leads to the equations for H_i determination. Thus, for H_0 we get the equation $H_0\mathcal{A}(1)\bar{\xi} = A_{12}\bar{\xi}$. The latter equation by Lemma 2 has a solution, determining the stable manifold $x = H_0\bar{\xi}$ of the system

$$\dot{x}(\tau) = A_{12}\bar{\xi}_\tau, \quad \dot{\bar{\xi}}(\tau) = A_{22}\bar{\xi}_\tau, \quad A_{i2}\bar{\xi}_\tau = \sum_{j=0}^{k} C_{ji}\bar{\xi}(\tau - h_j) + \int_{-h_k}^{0} D_i(\theta)\bar{\xi}(\tau+\theta)d\theta \ (i = 1, 2).$$

This solution is given by

$$H_0\bar{\xi} = A_{12}I \cdot (A_{22}I)^{-1}[\bar{\xi}(0) + \sum_{i=0}^{k} C_{i2}\int_{-h_i}^{0} \bar{\xi}(s)ds + \int_{-h_k}^{0}\int_{-h_k}^{s} D_2(r)dr\bar{\xi}(s)ds]$$

$$-\sum_{i=0}^{k} C_{i1}\int_{-h_i}^{0} \bar{\xi}(s)ds - \int_{-h_k}^{0}\int_{-h_k}^{s} D_1(r)dr\bar{\xi}(s)ds],$$

where $A_{12}I = \sum_{i=0}^{k} C_{i1} + \int_{-h_k}^{0} D_1(s)ds$. Really substituting the latter expression for H_0 into the equation for H_0 and further integrating by parts, we get an identity. Similarly the higher order terms H_i $(i \geq 1)$ can be found.

After the next change of variables $\eta(t) = x(t) - \epsilon H\xi_t$ we obtain the decoupled system

$$\dot{\eta}(t) = (A_{11} - A_{12}L)\eta(t) + [B_1 - \epsilon HLB_1 - HX_0B_2]u(t), \tag{9a}$$

$$\xi_t = S(t,\epsilon)\xi_0 + \frac{1}{\epsilon}\int_0^t S(t-s,\epsilon)[\epsilon LA_{12}\xi_s + (X_0B_2 + \epsilon LB_1)u(s)]ds. \tag{9b}$$

Expressing x and z in terms of η and ξ from the formulas for the variables changes we get

Theorem 1. *Under A1 for all small enough* ϵ *there exists a nonsingular transformation* $T : R^n \times C[-\epsilon h_k, 0] \to R^n \times C[-\epsilon h_k, 0]$ *given by*

$$\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} I & \epsilon H \\ -L & I - \epsilon LH \end{pmatrix}\begin{pmatrix} \eta \\ \xi \end{pmatrix} = T\begin{pmatrix} \eta \\ \xi \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{pmatrix}, \tag{10}$$

that transforms (3) to the purely-slow system (9a) and the purely-fast system (9b). The following approximation is valid:

$$T = T_0 + \epsilon T_1 + ... + \epsilon^p T_p + O(\epsilon^{p+1}). \tag{11}$$

The expansion (11) can be easily obtained from (10) by using expansions of L and H.

2. Some applications

Consider linear state-feedback design for (1). To alleviate difficulties caused by high-dimensionality and ill-conditioning resulting from interaction of slow and fast dynamic modes, we approximately decompose (1) into the slow and the fast subsystems. The n-th order slow system is

$$\dot{x}^s(t) = A_0x^s(t) + B_0u^s(t), \quad A_{21}x^s(t) + A_{22}I \cdot z^s(t) + B_2u^s(t) = 0, \quad x^s(0) = x^0, \tag{12}$$

where

$$A_0 = A_{11} - A_{12}I \cdot (A_{22}I)^{-1}A_{21}, \quad B_0 = B_1 - A_{12}I \cdot (A_{22}I)^{-1}B_2,$$

and x^s, z^s, u^s are the slow parts of x, z, u in (1). Note that (12) results from (1) by setting $\epsilon = 0$. The m-the order fast system in the $\tau = t/\epsilon$ is

$$\dot{z}^f(\tau) = \sum_{j=0}^{k} C_{j2}z^f(\tau - h_j) + \int_{-h_k}^{0} D_2(\theta)z^f(\tau+\theta)d\theta + B_2u^f(\tau), \quad z_0^f = (A_{22}I)^{-1}A_{21}x_0 + z_0, \tag{13}$$

where $z^f = z - z^s$ and $u^f = u - u^s$ denote fast parts of u and z. It is appropriate to consider feedback controls

$$u^s(t) = G_0 x^s(t), \quad u^f(\tau) = G_2 z^f_\tau = g_0 z^f(\tau) + \int_{-h_k}^{0} g_1(\theta) z^f(\tau + \theta) d\theta,$$

separately designed for the slow and the fast systems (12) and (13). Note that the above form of u^f arises e.g. in stabilization and optimal control problems (see e.g. [1]). We take a composite control for the full system (1) as

$$u(t) = u^s(t) + u^f(\tau) = G_0 x^s(t) + G_2 z^f_\tau = G_0 x(t) + G_2[z_\tau + (A_{22}I)^{-1}(A_{21} + B_2 G_0)x(t)]. \tag{14}$$

Similarly to Theorem 2.1 [6, p.95] (by using the decoupling transformation to the closed-loop system (1), (14)), the following Corollary can be proved:

Corollary 1. *Assume that the matrix $A_{22}I$ is nonsingular. Let G_2 be designed such that the roots of the characteristic equation*

$$det\left\{\lambda I - B_2 g_0 - \sum_{j=0}^{k} C_{j2} e^{-\lambda h_j} - \int_{-h_k}^{0} [D_2(\theta) + B_2 g_1(\theta)] e^{\lambda\theta} d\theta\right\} = 0,$$

that corresponds to the operator $A_{22} + B_2 G_2$, have negative real parts. Then for all small enough $\epsilon > 0$ if the composite control

$$u(t) = [(I + G_2 I \cdot (A_{22}I)^{-1} B_2) G_0 + G_2 I \cdot (A_{22}I)^{-1} A_{21}] x(t) + G_2 z_\tau, \tag{15}$$

is applied to (1), the state and control of the resulting closed-loop system are approximated according to

$$x(t) = x^s(t) + O(\epsilon), \quad z(t) = -(A_{22}I)^{-1} A_{21} x^s(t) + z^f(\tau) + O(\epsilon), \quad u(t) = u^s(t) + u^f(\tau) + O(\epsilon) \tag{16}$$

for all finite $t \geq 0$. If in addition G_0 is designed such that $A_0 + B_0 G_0$ is Hurwitz , then for all small enough ϵ the resulting closed-loop system is asymptotically stable and approximations (16) hold for all $t \geq 0$.

From Corollary 1 it follows

Corollary 2. *Under A1 for small enough ϵ*

1. *The homogeneous system (1) is asymptotically stable if the matrix A_0 is Hurwitz;*
2. *If the controller $u = G_0 x$ is designed such that $A_0 + B_0 G_0$ is Hurwitz, then the closed loop system (1) is asymptotically stable.*

Similarly the higher order approximations of solutions to (1) under linear state-feedback control can be achieved through the use of separate corrected slow and fast designs. Further the decoupling transformation can be applied to controllability, optimal control, filtering and other problems for (1).

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3. References

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