NEAR-OPTIMAL H^∞ CONTROL OF LINEAR SINGULARLY PERTURBED SYSTEMS

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Abstract. We consider the singularly perturbed H^∞ control problem under perfect state measurements, for both finite and infinite horizons. We suggest a construction of high-order approximations to a controller that guarantees a desired performance level on the basis of the exact decomposition of the full-order Riccati equations to the reduced-order slow and fast equations. This leads to effective asymptotic and numerical algorithms. We show that the high-order accuracy controller improves the performance.

1. Introduction

Consider the linear time-varying singularly perturbed system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u + D_1 w, \quad \varepsilon \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u + D_2 w, \quad x(0) = 0 \quad (1.1)$$

and the quadratic functional

$$J = x'(t_f)Fx(t_f) + \int_0^{t_f} [x'(t)Q(t)x(t) + u'(t)u(t)]dt, \quad (1.2)$$

where $x = \text{col}\{x_1, x_2\}$ is the state vector with $x_1(t) \in {}^{n_1}$ and $x_2(t) \in {}^{n_2}$, $u(t) \in {}^{p}$ is the control input , $w(t) \in {}^{q}$ is the disturbance. The matrices $A_{ij} = A_{ij}(t), B_i = B_i(t), D_i = D_i(t)$ $(i = 1,2\ j = 1,2)$ are infinitely differentiable functions of $t \geq 0$, and ε is a small positive parameter. The symbol $(\cdot)'$ denotes the transpose of a matrix,

$$Q = Q' = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \geq 0, \quad F = F' = \begin{pmatrix} F_{11} & \varepsilon F_{12} \\ \varepsilon F_{21} & \varepsilon F_{22} \end{pmatrix} \geq 0.$$

Denote by $|\cdot|$ the Euclidean norm of a vector and by $\|\cdot\|$ the norm in $L_2[0, t_f]$. Let $S_{ij} = B_i B_j' - \gamma^{-2} D_i D_j',\ i = 1,2,\ j = 1,2,\ B_\varepsilon = col\{B_1, \varepsilon^{-1}B_2\},\ D_\varepsilon = col\{D_1, \varepsilon^{-1}D_2\}$,

$$A_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix}, \quad S_\varepsilon = \begin{pmatrix} S_{11} & \varepsilon^{-1}S_{12} \\ \varepsilon^{-1}S_{21} & \varepsilon^{-2}S_{22} \end{pmatrix}.$$

With (1.1), (1.2) we associate the Riccati differential equation (RDE)

$$\dot{Z} + A'_\varepsilon Z + ZA_\varepsilon - ZS_\varepsilon Z + Q = 0; \quad Z(t_f) = F \tag{1.3}$$

for the matrix function

$$Z = Z' = Z(t,\varepsilon) = \begin{pmatrix} Z_{11}(t,\varepsilon) & \varepsilon Z_{12}(t,\varepsilon) \\ \varepsilon Z_{21}(t,\varepsilon) & \varepsilon Z_{22}(t,\varepsilon) \end{pmatrix} \qquad (1.4).$$

For each ε, a controller that guarantees the disturbance attenuation level γ exists (and solves the H^∞ control problem) iff (1.3) has a bounded solution on $[0, t_f]$ [1], [9]. Such a controller is determined by the formula

$$u(t) = -[B'_1;\ \varepsilon^{-1}B'_2]Zx(t)\ , \quad t \in [0; t_f]\ . \tag{1.5}$$

In the infinite horizon case we take $A_\varepsilon, B_\varepsilon, D_\varepsilon$ and $Q = C'C$ to be time-invariant , $F = 0$ and assume:

A1. The pair $(A_\varepsilon, B_\varepsilon)$ is controllable and (A_ε, C) is observable for $\varepsilon \in (0, \varepsilon_0]$ $(\varepsilon_0 > 0)$.

A controller that guarantees the performance level γ exists iff the full-order algebraic Riccati equation (ARE) of the form (1.3), where $\dot{Z} = 0$, has a solution $Z \geq 0$ such that the matrix $A_\varepsilon - S_\varepsilon Z$ is Hurwitz. Such a controller is determined by (1.5) [1], [2], [9].

Computation of the controller (1.5) for small $\varepsilon > 0$ presents serious difficulties due to high dimension and numerical stiffness. In [9] a composite controller has been designed on the basis of the reduced-order slow and fast subproblems. This controller is $O(\varepsilon)$-close to those of (1.5) and achieves the performance γ for the full-order system for small enough ε. However, for values of ε that are not too small, higher order approximations based on the reduced-order equations are needed to guarantee the desired performance.

The main results of the note are :

(A) Construction of a high-order accuracy controller on the basis of the exact decomposition of the singularly perturbed Riccati equations to the reduced-order pure-slow and pure-fast equations. New algorithms (in comparison with [8,12]) for asymptotic solutions of the Riccati equations.

(B) The fact that an $O(\varepsilon^k)$ accuracy controller achieves the performance $\gamma + O(\varepsilon^k)$.

2. Exact decomposition of the full-order Riccati equations

We will develop the method of exact decomposition of the singularly perturbed Riccati equations initiated with [3], [11], to H^∞ control problem. We begin with $t_f < \infty$. Consider the Hamiltonian system corresponding to (1.3) with the adjoint variables $y_1, \varepsilon y_2$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \varepsilon \dot{x}_2 \\ \varepsilon \dot{y}_2 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}, \quad R_{ij} = \begin{pmatrix} A_{ij} & -S_{ij} \\ -Q_{ij} & -A'_{ji} \end{pmatrix}, \tag{2.1}$$

$$x_1(t_f) = x_1^0, \quad y_1(t_f) = F_{11}x_1^0 + \varepsilon F_{12}x_2^0, \quad x_2(t_f) = x_2^0, \quad y_2(t_f) = F_{21}x_1^0 + F_{22}x_2^0. \tag{2.2}$$

Proposition [4]. *For each $\varepsilon > 0$, (1.3) has a bounded on $[0, t_f]$ solution iff there exists the matrix function of the form (1.4) such that for all $x_1^{(0)} \in {}^{n_1}$, $x_2^{(0)} \in {}^{n_2}$ a solution of (2.1), (2.2) can be represented as follows:*

$$col\{y_1, \varepsilon y_2\} = Zx, \quad t \in [0, t_f]. \tag{2.3}$$

Let $C_2'C_2 = Q_{22}$. Consider the following ARE

$$A'_{22}M^{(0)} + M^{(0)}A_{22} + Q_{22} - M^{(0)}S_{22}M^{(0)} = 0 \ , \quad t \in [0, t_f] \ , \tag{2.4}$$

which corresponds, for each $t \in [0, t_f]$, to the fast infinite horizon subproblem. Assume

A2. The pair $\{A_{22}, B_2\}$ is controllable and $\{A_{22}, C_2\}$ is observable for all $t \in [0, t_f]$.

Let $\gamma_f^t = \inf\{\gamma' \mid$ ARE (2.4) has a solution $M^{(0)} > 0$ such that $\Lambda_0 = A_{22} - S_{22}M^{(0)}$ is Hurwitz$\}$. Under A2 $\gamma_f = \sup_{t \in [0, t_f]} \gamma_f^t < \infty$ [9]. We assume

A3. The performance level $\gamma > \gamma_f$.

From [2, Lemma 4] and from the continuous dependence of R_{22} on $t \in [0, t_f]$ it follows that for all $t \in [0, t_f]$ the matrix R_{22} has n_2 stable eigenvalues $\lambda, \mathrm{Re}\,\lambda < -\alpha < 0$ and n_2 unstable ones, $Re\lambda > \alpha$. This implies for small ε the existence of the matrix functions $H = -R_{22}^{-1}R_{21} + O(\varepsilon)$, $P = R_{12}R_{22}^{-1} + O(\varepsilon), M = M^{(0)} + O(\varepsilon)$ and $L = L^{(0)} + O(\varepsilon)$ that for all $t \in [0, t_f]$ satisfy the equations [10], [7, p.210-212]:

$$\varepsilon \dot{H} + \varepsilon H(R_{11} + R_{12}H) = R_{21} + R_{22}H, \tag{2.5a}$$

$$\varepsilon \dot{P} + P(R_{22} - \varepsilon H R_{12}) = \varepsilon(R_{11} + R_{12}H)P + R_{12}. \tag{2.5b}$$

$$\varepsilon \dot{M} + M[A_{22} + \varepsilon K_1 + (\varepsilon K_2 - S_{22})M] = -Q_{22} + \varepsilon K_3 + (-A'_{22} + \varepsilon K_4)M, \tag{2.5c}$$

$$\varepsilon \dot{L} - L[A'_{22} - \varepsilon K_4 + M(\varepsilon K_2 - S_{22})] = [A_{22} + \varepsilon K_1 + (\varepsilon K_2 - S_{22})M]L + \varepsilon K_2 - S_{22}, \tag{2.5d}$$

where

$$\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} = -HR_{12}, \quad H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}, \quad P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}.$$

The matrix $M^{(0)}$ is a solution of (2.4) and $L^{(0)}$ satisfies the Lyapunov equation, that results from (2.5d) by setting $\varepsilon = 0$. Note that we do not specify initial conditions in (2.5), following [7], [10]. The functions H, P, M and L can be easily found in the form of asymptotic expansions [10]. For all small enough ε the nonsingular transformation [10]

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} I & 0 & \varepsilon G_1 & \varepsilon G_2 \\ 0 & I & \varepsilon G_3 & \varepsilon G_4 \\ H_1 & H_2 & E_1 & E_2 \\ H_3 & H_4 & E_3 & E_4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}, \tag{2.6}$$

where

$$\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} = (I + \varepsilon HP) \begin{pmatrix} I & L \\ M & I + ML \end{pmatrix}, \quad \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = P \begin{pmatrix} I & L \\ M & I + ML \end{pmatrix},$$

decomposes (2.1) into the slow system for $u_1 \in {}^{n_1}$ and $v_1 \in {}^{n_1}$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{v}_1 \end{pmatrix} = W \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} = R_{11} + R_{12} H, \tag{2.7a}$$

and the two fast decoupled equations for $u_2 \in {}^{n_2}$ and $v_2 \in {}^{n_2}$

$$\varepsilon \dot{u}_2 = (A_{22} + \varepsilon K_1 + (-S_{22} + \varepsilon K_2) M) u_2, \quad \varepsilon \dot{v}_2 = (-A'_{22} + \varepsilon K_4 + M(S_{22} - \varepsilon K_2)) v_2. \tag{2.7b}$$

Substituting (2.6) into the terminal conditions (2.2) and further eliminating x_1^0 and x_2^0, we obtain the following terminal conditions for u_1, v_1, u_2, v_2:

$$u_1(t_f) = u_1^0, \quad u_2(t_f) = u_2^0, \quad v_1(t_f) = U_{11} u_1^0 + \varepsilon U_{12} u_2^0, \quad v_2(t_f) = U_{21} u_1^0 + U_{22} u_2^0. \tag{2.8}$$

where

$$\begin{pmatrix} U_{11} & \varepsilon U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} Y_2 \\ Y_4 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix}^{-1} \Bigg|_{t=t_f}, \quad \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \begin{pmatrix} \Phi_1 & \Phi_2 & -\varepsilon P_1 & -\varepsilon P_2 \\ \Phi_3 & \Phi_4 & -\varepsilon P_3 & -\varepsilon P_4 \\ \Psi_1 & \Psi_2 & \Xi_1 & \Xi_2 \\ \Psi_3 & \Psi_4 & \Xi_3 & \Xi_4 \end{pmatrix} \begin{pmatrix} I & 0 \\ F_{11} & \varepsilon F_{12} \\ 0 & I \\ F_{21} & F_{22} \end{pmatrix} \tag{2.9}$$

$$\begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix} = I + \varepsilon PH, \begin{pmatrix} \Xi_1 & \Xi_2 \\ \Xi_3 & \Xi_4 \end{pmatrix} = \begin{pmatrix} I + LM & -L \\ -M & I \end{pmatrix}, \begin{pmatrix} \Psi_1 & \Psi_2 \\ \Psi_3 & \Psi_4 \end{pmatrix} = -\begin{pmatrix} \Xi_1 & \Xi_2 \\ \Xi_3 & \Xi_4 \end{pmatrix} H.$$

By straightforward computations we get

$$\begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \dots & I + L^{(0)}(M^{(0)} - F_{22}) \end{pmatrix} + O(\varepsilon).$$

To assure the existence of the inverse matrix in (2.9) we assume

A4. The matrix $I + L^{(0)}(M^{(0)} - F_{22})$ is invertible at $t = t_f$.

Consider the pure-slow RDE for the $n_1 \times n_1$-matrix function $N = N(t, \varepsilon)$

$$\dot{N} + N(W_1 + W_2 N) = W_3 + W_4 N, \quad N(t_f) = U_{11}, \tag{2.10}$$

and the pure-fast linear equations for the $n_i \times n_j$-matrix functions $N_{ij} = N_{ij}(t, \varepsilon)$:

$$\varepsilon \dot{N}_{12} = -N_{12}(\Lambda + \varepsilon(K_1 + K_2 M + W_2)) + \varepsilon W_4 N_{12}, \quad N_{12}(t_f) = U_{12}, \tag{2.11}$$

$$\varepsilon \dot{N}_{21} = -(\Lambda' - \varepsilon(K_4 - MK_2))N_{21} - \varepsilon N_{21}(W_1 + W_2 N), \quad N_{21}(t_f) = U_{21}, \tag{2.12}$$

$$\varepsilon \dot{N}_{22} = -N_{22}(\Lambda + \varepsilon(K_1 + K_2 M)) - (\Lambda' - \varepsilon(K_4 - MK_2))N_{22}, \quad N_{22}(t_f) = U_{22}, \tag{2.13}$$

where $\Lambda = A_{22} - S_{22}M$, and matrices K_i and U_{ij} can be obtained by solving (2.5). Similarly to Proposition, equations (2.10)-(2.13) have bounded solutions on $[0, t_f]$ iff for every $u_1^0 \in {}^{n_1}, u_2^0 \in {}^{n_2}$, a solution of (2.7) can be represented in the form $v_1 = Nu_1 + \varepsilon N_{12} u_2$, $v_2 = N_{21} u_1 + N_{22} u_2$, $t \in [0, t_f]$. Finally, substituting the latter relations and (2.6) into (2.3) and equating separately terms with u_1 and u_2, we get

$$Z \begin{pmatrix} I + \varepsilon G_2 N_{21} & \varepsilon G_1 + \varepsilon G_2 N_{22} \\ H_1 + H_2 N + E_2 N_{21} & E_1 + E_2 N_{22} + \varepsilon H_2 N_{12} \end{pmatrix} = \begin{pmatrix} N + \varepsilon G_4 N_{21} & \varepsilon N_{12} + \varepsilon G_3 + \varepsilon G_4 N_{22} \\ \varepsilon(H_3 + H_4 N + E_4 N_{21}) & \varepsilon E_3 + \varepsilon E_4 N_{22} + \varepsilon^2 H_4 N_{12} \end{pmatrix}. \tag{2.14}$$

If for small ε RDE (2.10) has a uniformly bounded solution on $[0, t_f]$ then the linear equations (2.11)-(2.13) have solutions, satisfying the inequality $|N_{ij}(t, \varepsilon)| \leq K e^{\alpha(t - t_f)/\varepsilon}$, $t \in [0, t_f]$, $K > 0$, and algebraic equation (2.14) has a unique solution [4]. Thus, after solving (2.5) and (2.10)-(2.13) we can obtain Z_{ij} from (2.14).

Lemma 1 [4]**.** *Under A2-A4 for a prechosen γ and all small enough $\varepsilon > 0$*

(i) the full-order RDE (1.3) has a bounded solution on $[0, t_f]$ iff the slow RDE (2.10) has a bounded solution on $[0, t_f]$;

(ii) if (1.3) has a bounded solution on $[0, t_f]$, then this solution can be uniquely defined from the equations (2.5), the decoupled pure-slow and pure-fast differential equations (2.10)-(2.13) and the linear algebraic equation (2.14).

In the infinite-horizon case we take $A_\varepsilon, B_\varepsilon, D_\varepsilon, Q$ to be time-invariant and $F = 0$. In this case (2.5) are algebraic equations and H, P, M and L are time-invariant.

Lemma 2 [4]. *Under A1-A3 for a prechosen γ and all small enough $\varepsilon > 0$*

(i) the full-order ARE of (1.3) has a unique solution Z, such that the matrix $A_\varepsilon - S_\varepsilon Z$ is Hurwitz, iff the slow ARE of (2.10) , where $\dot{N} = 0$, has a unique solution such that $W_1 + W_2 N$ is Hurwitz;

(ii) the solution of the full-order ARE can be uniquely defined from the equations (2.5), the slow ARE (2.10) and the linear algebraic equation (2.14), where $N_{ij} = 0$.

3. High-order approximations in H^∞-control

The exact decomposition can be used for the development of high-order accuracy methods for singularly perturbed H^∞ control problems. Solutions to the pure-slow and pure-fast equations (2.10)-(2.13) can be found without difficulty by standard asymptotic and numerical methods. This would lead to effective reduced-order algorithms. In the case of optimal control problem numerical reduced-order algorithms were developed in [5],[11].

In the present note we shall construct asymptotic solutions. We start with the finite horizon case. It is easy to see that at $\varepsilon = 0$ system (2.10) has the form:

$$\dot{N}^{(0)} + N^{(0)}(W_1^{(0)} + W_2^{(0)} N^{(0)}) = W_3^{(0)} + W_4^{(0)} N^{(0)}, \quad N^{(0)}(t_f) = F_{11}, \tag{3.1}$$

where $W^{(0)} = W\Big|_{\varepsilon=0}, W_i^{(0)} = W_i\Big|_{\varepsilon=0}$, $i = 1, ..., 4$. From Lemma 1 and (2.14) it follows that, for small ε, (2.10) has a bounded on $[0, t_f]$ solution of the form $N = N^{(0)}(t) + O(\varepsilon)$ iff (1.3) has a bounded on $[0, t_f]$ solution of the form $Z = diag\{Z_{11}^{(0)}(t), 0\} + O(\varepsilon)$, and $Z_{11}^{(0)} = N^{(0)}$. Let $\gamma_s = \inf\{\gamma' > 0 \mid \forall \gamma > \gamma' \ \ (3.1)$ has a bounded solution on $[0, t_f]\}$ and $\overline{\gamma} = \max\{\gamma_s, \gamma_f\}$. To guarantee $\overline{\gamma} < \infty$ we assume, following [8] and [9]:

A5. The matrices A_{22} and Q_{22} are invertible for all $t \in [0, t_f]$.

A6. The matrix $M^{(0)}(t_f) - F_{22}$ is either positive definite or zero for $\gamma > \overline{\gamma}$.

For each $\gamma > \overline{\gamma}$ and small enough ε we find asymptotic expansions of H, P, M and L as described in [10]. Substituting these expansions into (2.9) we easily obtain $U_{ij} = U_{ij}^{(0)} + \varepsilon U_{ij}^{(1)} + ...$. Thus, $U_{11}^{(0)} = F_{11}$, $U_{22}^{(0)} = [F_{22} - M^{(0)}][I + L^{(0)}(M^{(0)} - F_{22})]^{-1}\Big|_{t=t_f}$.

From the regularly perturbed RDE (2.10) and from the stable (as $t \to -\infty$) linear equations (2.12), (2.13) written in the fast time $\tau = \varepsilon^{-1}(t - t_f)$, $\tau \leq 0$ we further find

$$N = N^{(0)}(t) + \varepsilon N^{(1)}(t) + ... + O(\varepsilon^{m+1}), \quad N_{ij} = N_{ij}^{(0)}(\tau) + \varepsilon N_{ij}^{(1)}(\tau) + ... + O(\varepsilon^{m+1}), \tag{3.2}$$

where $i = 2, j = 1, 2$. For $N^{(0)}$ we get (3.1), while for the other terms of these expansions we get linear terminal value problems by successively equating coefficients of equal powers of ε. Thus, $N_{22}^{(0)} = e^{-\Lambda'_f \tau} U_{22}^{(0)} e^{-\Lambda_f \tau}$, where $\Lambda_f = A_{22} - S_{22} M^{(0)}\Big|_{t=t_f}$. By the standard asymptotic methods argument (see e.g. [6, Chapter 7]), the approximations in (3.2) are uniform in $t \in [0, t_f], \tau \leq 0$.

Setting $Z(t) = \bar{Z}(t) + \Pi(\tau)$ in (2.14), where $|\Pi(\tau)| \leq K e^{\alpha\tau}$, and equating separately the slow and the fast (exponentially decaying) terms, we get the outer solution $\bar{Z}$

$$\bar{Z} = \begin{pmatrix} N & \varepsilon G_3 \\ \varepsilon(H_3 + H_4 N) & \varepsilon E_3 \end{pmatrix} \begin{pmatrix} I & \varepsilon G_1 \\ H_1 + H_2 N & E_1 \end{pmatrix}^{-1}, \quad \bar{Z} = \begin{pmatrix} \bar{Z}_{11} & \varepsilon \bar{Z}_{12} \\ \varepsilon \bar{Z}_{21} & \varepsilon \bar{Z}_{22} \end{pmatrix}, \tag{3.3}$$

and the algebraic equation for the boundary layer correction Π :

$$\Pi \begin{pmatrix} I + \varepsilon G_2 N_{21} & \varepsilon G_1 + \varepsilon G_2 N_{22} \\ H_1 + H_2 N + E_2 N_{21} & E_1 + E_2 N_{22} + \varepsilon H_2 N_{12} \end{pmatrix} + \bar{Z} \begin{pmatrix} \varepsilon G_2 N_{21} & \varepsilon G_2 N_{22} \\ E_2 N_{21} & E_2 N_{22} + \varepsilon H_2 N_{12} \end{pmatrix} =$$
$$\begin{pmatrix} \varepsilon G_4 N_{21} & \varepsilon N_{12} + \varepsilon G_4 N_{22} \\ \varepsilon E_4 N_{21} & \varepsilon E_4 N_{22} + \varepsilon^2 H_4 N_{12} \end{pmatrix}, \quad \Pi = \begin{pmatrix} \Pi_{11} & \varepsilon \Pi_{12} \\ \varepsilon \Pi_{21} & \varepsilon \Pi_{22} \end{pmatrix}. \tag{3.4}$$

For small enough ε the inverse matrix in (3.3) exists, since $E_1 = I + O(\varepsilon)$. Substituting for the functions in the right sides of (3.3) their expansions we get the outer expansion

$$\bar{Z}_{ij}(t) = \sum_{k=0}^{m} \varepsilon^k Z_{ij}^{(k)}(t) + O(\varepsilon^{m+1}), \tag{3.5}$$

where the approximation is uniform in $t \in [0, t_f]$. Thus, $Z_{11}^{(0)} = N^{(0)}(t)$, $Z_{22}^{(0)} = M^{(0)}(t)$, $Z_{12}^{(0)} = Z_{21}^{(0)\prime} = G_3^{(0)}(t) - N^{(0)}(t) G_1^{(0)}(t)$, where $G_i^{(0)} = G_i\Big|_{\varepsilon=0}$.

To get the boundary layer corrections $\Pi_{11}, \Pi_{12} = \Pi'_{21}, \Pi_{22}$ in the form

$$\Pi_{ij}(\tau) = \sum_{k=0}^{m} \varepsilon^k \Pi_{ij}^{(k)}(\tau) + O(\varepsilon^{m+1}), \tag{3.6}$$

and the asymptotic expansion of N_{12} in the form of (3.2), we put $t = t_f + \varepsilon\tau$ in (3.4), expand all the functions of (3.4) into the powers of ε and equate the coefficients of equal powers of ε. The resulting linear algebraic equations for the terms of the latter approximations have unique solution, since under A6 the matrix $E_1 + E_2 N_{22}\Big|_{\varepsilon=0} = I + L^{(0)}(t_f) N_{22}^{(0)}(\tau)$ is nonsingular for all $\tau \le 0$ (see [8], invertibility of (2.19)). Thus, $\Pi_{11}^{(0)} = 0$ and

$$\Pi_{22}^{(0)} = [E_4^{(0)}(t_f) - Z_{22}^{(0)}(t_f)L^{(0)}(t_f)]N_{22}^{(0)}(\tau)[I + L^{(0)}(t_f)N_{22}^{(0)}(\tau)]^{-1}, \quad E_4^{(0)} = I + M^{(0)}L^{(0)},$$

$$\Pi_{21}^{(0)} = \Pi_{12}^{(0)\prime} = \{E_4^{(0)}(t_f) - [\Pi_{22}^{(0)}(\tau) + Z_{22}^{(0)}(t_f)]L^{(0)}(t_f)\}N_{21}^{(0)}(\tau) -$$
$$\Pi_{22}^{(0)}(\tau)[H_1^{(0)}(t_f) + H_2^{(0)}(t_f)F_{11}].$$

It can be shown (by applying the contraction principle argument to the equations for the remainders in the expansions (3.6) and (3.2)), that for small ε the weakly nonlinear algebraic system of (3.4), where $\Pi_{12} = \Pi_{21}'$, has a unique solution $\Pi_{11}, \Pi_{21}, \Pi_{22}, N_{12}$, represented in the form (3.6), (3.2). The approximations in (3.6) and (3.2) are uniform in $\tau \le 0$, and $|\Pi_{ij}^{(k)}(\tau)| \le K e^{\alpha_1 \tau}$ $(\alpha_1 > 0)$. From (3.5) and (3.6) the following uniform on $t \in [0, t_f]$ approximation of the solution to the full-order RDE (1.3) follows:

$$Z_{ij}(t) = \bar{Z}_{ij}(t) + \Pi_{ij}(\tau) = \sum_{k=0}^{m} \varepsilon^k [Z_{ij}^{(k)}(t) + \Pi_{ij}^{(k)}(\tau)] + O(\varepsilon^{m+1}), \quad \tau = \varepsilon^{-1}(t - t_f). \quad (3.7)$$

For analogous result, in the case of optimal-control problem, obtained by boundary layer method see [12], [8]. In the case of $m = 0$, (3.7) has been obtained in [9]. As compared with the boundary layer method, applied to the full-order RDE (1.3), our method allows the evaluation of the outer expansion (i.e. the asymptotic solution of (1.3) away from $t = t_f$) independently of the boundary layer correction. Moreover, we invert matrices, whereas by the boundary layer algorithm one has to integrate on the infinite interval (see [8], [12], the computation of $\Pi_{11}^{(k)}(\tau), Z_{11}^{(k+1)}(t_f), k \ge 0$).

Put $Z_{12}^{(-1)} + \Pi_{12}^{(-1)} = 0$. It follows from (3.7) that

$$u = \left[V_m + O(\varepsilon^{m+1})\right]x, \quad V_m = -\sum_{j=1}^{2} B_j' \sum_{k=0}^{m} \varepsilon^k [Z_{j1}^{(k)} + \Pi_{j1}^{(k)}; Z_{j2}^{(k+j-2)} + \Pi_{j2}^{(k+j-2)}]. \quad (3.8)$$

For $t_f = \infty$ we take $\gamma_s = \inf\{\gamma' > 0 | \ \forall \gamma > \gamma'$ ARE (3.1), where $\dot{N}^{(0)} = 0$, has a solution such that $\Delta_1 = W_1^{(0)} + W_2^{(0)} N^{(0)}$ is Hurwitz$\}$. The matrix Ham of the Hamiltonian system of (2.1) has $2n_2$ eigenvalues tending to infinity as $\varepsilon \to 0$ and $2n_1$ eigenvalues tending to those of $R_{11} - R_{12}R_{22}^{-1}R_{21} = W^{(0)}$ [7, §2.3]. From the symmetry of the eigenvalues of Ham it follows that if λ is an eigenvalue of $W^{(0)}$, then so is $-\lambda$. For $\gamma > \overline{\gamma}$ the matrix $W^{(0)}$ can be represented in the form

$$W^{(0)} = \begin{pmatrix} I & 0 \\ N^{(0)} & I \end{pmatrix} \begin{pmatrix} \Delta_1 & W_2^{(0)} \\ 0 & \Delta_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ -N^{(0)} & I \end{pmatrix}, \quad \Delta_2 = W_4^{(0)} - N^{(0)} W_2^{(0)}.$$

Then the matrix $W^{(0)}$ has n_1 stable eigenvalues λ, corresponding to Δ_1, and n_1 unstable ones $-\lambda$, corresponding to Δ_2. The solution of ARE of (2.10) determines the stable manifold $v_1 = Nu_1$ of (2.7a), and N can be found in the form (3.2) by standard argument for asymptotic expansions of invariant manifolds (see e.g. [7], [10]). The terms $N^{(i)} (i \geq 1)$ satisfy the linear algebraic equation of the form $N^{(i)}\Delta_1 - \Delta_2 N^{(i)} = C^{(i)}$, where $C^{(i)}$ is known, having a unique solution, since the matrices Δ_1 and $-\Delta_2$ are Hurwitz. Then from (2.14) and (1.5), Z and u can be approximated by (3.7) and (3.8), where $\Pi_{ij}^{(k)} = 0$.

To assure the internal stability of (1.1), (1.5), i.e. the matrix $\widetilde{A}_\varepsilon = A_\varepsilon - B_\varepsilon B_\varepsilon' Z$ to be Hurwitz (and thus $Z \geq 0$ [2]), consider the matrix $\bar{A}_\varepsilon = A_\varepsilon + B_\varepsilon V_0$, where

$$\bar{A}_\varepsilon = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \varepsilon^{-1}\bar{A}_{21} & \varepsilon^{-1}\bar{A}_{22} \end{pmatrix}, \quad \widetilde{A}_\varepsilon = \begin{pmatrix} \widetilde{A}_{11}(\varepsilon) & \widetilde{A}_{12}(\varepsilon) \\ \varepsilon^{-1}\widetilde{A}_{21}(\varepsilon) & \varepsilon^{-1}\widetilde{A}_{22}(\varepsilon) \end{pmatrix}. \tag{3.9}$$

Due to (3.8) and (1.5) $\widetilde{A}_{ij}(\varepsilon) = \bar{A}_{ij} + O(\varepsilon), i = 1,2, j = 1,2$. The matrix $\bar{A}_{22} = A_{22} - B_2 B_2' M^{(0)}$ is Hurwitz [1]. Then for small $\varepsilon > 0$ the matrix $\widetilde{A}_\varepsilon$ is Hurwitz under the following assumption (see e.g. [6, Theorem 8.3]):

A7. The matrix $A_0 = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$ is Hurwitz.

Theorem. *Under A2-A6 in the finite horizon case and under A1-A3, A7 in the infinite horizon case, for a prechosen $\gamma > \overline{\gamma}$ and all small enough $\varepsilon > 0$ the following holds:*

(i) A controller guaranteeing the performance level γ exists and can be approximated by (3.8), where in the infinite horizon case $\Pi_{ij}^{(k)} = 0$;

(ii) The approximate controller $u_m = V_m x$ guarantees the performance level $\gamma + O(\varepsilon^{m+1})$.

For proof of Theorem see Appendix.

Example . Consider the following system and functional:

$$\dot{x}_1 = u + w, \quad \varepsilon \dot{x}_2 = x_2 - u, \quad J = \int_0^\infty (x_1^2 + x_2^2 + u^2)dt.$$

Here $\gamma_f = 0$ and $\overline{\gamma} = 1.4142$. The values of the H^∞-optimum performance $\gamma^*(\varepsilon)$ are shown in Table 1. Approximations u_0 and u_1 to the controller (1.5) have the form

$$u_0 = \frac{x_1}{\sqrt{1 - 2/\gamma^2}} + (1 + \sqrt{2})x_2, \quad u_1 = u_0 + \varepsilon \frac{\sqrt{2}/2 + 1}{\sqrt{1 - 2/\gamma^2}} [\frac{x_1}{\sqrt{2 - 4/\gamma^2}} + x_2].$$

Choosing $\gamma = 2 > \gamma^*(\varepsilon)$ we get $u_0 = 1.4142x_1 + 2.4142x_2$. Applying this controller on the system we determine the corresponding performance bounds γ_0 for different values of ε. Further we find $u_1 = M_1x_1 + M_2x_2$ and the corresponding performance bounds γ_1. The values M_1, M_2, γ_0 and γ_1 are given in Table 1. We see that the controller u_1 improves the performance for $\varepsilon > 0.3$. For values of γ closer to $\gamma^*(\varepsilon)$, u_1 improves the performance for the smaller values of ε. Thus, for $\gamma = 1.5$ and $\varepsilon = 0.2$ we have: $u_0 = 3.0000x_1 + 2.4142x_2$, $\gamma_0 = 1.7876$; $u_1 = 5.1728x_1 + 3.4385x_2$, $\gamma_1 = 1.5488$.

Table 1.

ε	0.2	0.3	0.4	0.5	0.6
$\gamma^*(\varepsilon)$	1.454	1.500	1.559	1.628	1.7048
M_1	1.8977	2.1385	2.3799	2.6213	2.8627
M_2	2.8971	3.1385	3.3799	3.6213	3.8627
γ_0	1.7321	1.7321	1.9664	2.5312	3.4506
γ_1	1.7321	1.7321	1.7433	2.0363	2.6459

Conclusions

We have developed in the case of H^∞ control problem the method of exact decomposition of the full-order Riccati equation into decoupled equations of smaller order. We have found asymptotic controllers from these reduced-order equations and have shown that a higher-order accuracy controller improves the performance.

Appendix

Proof of Theorem . The proof of (i) precedes the Theorem. We give the proof of (ii) for $t_f = \infty$. For $t_f < \infty$ the proof is similar. We use $u = -B_\varepsilon' Zx$ to (1.1) and (1.2):

$$\dot{x} = \bar{A}_\varepsilon x + \varepsilon\beta_\varepsilon x + D_\varepsilon w, \quad x(0) = 0, \quad J = \int_0^\infty x'(t)\bar{Q}x(t)dt, \tag{A.1}$$

where $\bar{A}_\varepsilon = A_\varepsilon + B_\varepsilon V_0$ is given by (3.9), $\bar{Q} = Q + ZB_\varepsilon B_\varepsilon' Z, \beta_\varepsilon = col\{\beta_1, \beta_2/\varepsilon\}$, $\varepsilon\beta_i = B_i[-B_\varepsilon' Z - V_0]$, $i = 1, 2$. From (3.8) and (1.5) it follows that the functions $\beta_i = \beta_i(\varepsilon)$ are uniformly bounded for all small enough ε.

Since $\bar{A}_{22}$ is Hurwitz there is a transformation $y = T^{-1}x$ that block diagonalizes $\bar{A}_\varepsilon$ [7,p.210]: $T^{-1}\bar{A}_\varepsilon T = diag\{A_s, A_f/\varepsilon\}$, $A_s = \bar{A}_{11} - \bar{A}_{12}L$, $A_f = \bar{A}_{22} + \varepsilon L\bar{A}_{12}$, where

$$T^{-1} = \begin{pmatrix} I - \varepsilon HL & -\varepsilon H \\ L & I \end{pmatrix}, \quad T = \begin{pmatrix} I & \varepsilon H \\ -L & I - \varepsilon LH \end{pmatrix}, \quad L = L(\varepsilon), \quad H = H(\varepsilon),$$

L and H are defined by (3.2) and (3.3) from [5,p.210] with $A_{ij} = \bar{A}_{ij}$. For y we get

$$\dot{y}_1 = A_s y_1 + \varepsilon\beta_s y + D_s w, \quad \varepsilon\dot{y}_2 = A_f y_2 + \varepsilon\beta_f y + D_f w, \quad y(0) = 0, \qquad (A.2)$$

where $col\{D_s, D_f/\varepsilon\} = T^{-1}D$, $\beta_s = [(I - \varepsilon HL)\beta_1 - H\beta_2]T$, $\beta_f = (\varepsilon L\beta_1 + \beta_2)T$. Note that D_s, D_f, β_s and β_f are uniformly bounded for small ε and $A_s = A_0 + O(\varepsilon)$, $A_f = \bar{A}_{22} + O(\varepsilon)$.

Similarly substituting u_m for u in (1.1) and (1.2) we get (A.1) with $x_m, \varepsilon\beta_m = B_\varepsilon(V_m - V_0), Q_m = Q + V_m' V_m$ and J_m substituted for $x, \varepsilon\beta, \bar{Q}$ and J. Applying the same transformation $z = T^{-1}x_m$ to the differential equation for x_m we obtain (A.2) with $z_1, z_2, \beta_{sm}, \beta_{fm}$ substituted for $y_1, y_2, \beta_s, \beta_f$, where $\beta_{sm} = [(I - \varepsilon HL)\beta_{1m} - H\beta_{2m}T$, $\beta_{fm} = (\varepsilon L\beta_{1m} + \beta_{2m})T$, $\beta_{im} = \varepsilon^{-1}B_i(V_m - V_0)$, $i = 1, 2$.

From (3.8) and (1.5) we deduce: $\beta_i - \beta_{im} = \varepsilon^{-1}B_i[-B_\varepsilon' Z - V_m] = O(\varepsilon^m)$. This relation together with formulas for $\beta_s, \beta_f, \beta_{sm}$ and β_{fm} imply that β_{sm} and β_{fm} are $O(\varepsilon^m)$ close to β_s and β_f. Similarly $Q_m - \bar{Q} = V_m' V_m - ZB_\varepsilon B_\varepsilon' Z = O(\varepsilon^{m+1})$. Then substitution of $x = Ty$ and $x_m = Tz$ into J and J_m and application of the Schwartz inequality yield:

$$\begin{aligned} |J - J_m| &\le \int_0^\infty K_1[|\Delta y(t)||y(t)| + |\Delta y(t)||z(t)| + \varepsilon^{m+1}|z(t)|^2]dt \\ &\le K_1[||\Delta y||(||y|| + ||z||) + \varepsilon^{m+1}||z||^2], \quad K_1 > 0, \end{aligned} \qquad (A.3)$$

where $\Delta y = y - z = col\{\Delta y_1, \Delta y_2\}$ vanishes at $t = 0$ and satisfies the system:

$$\dot{\Delta} y_1 = A_s \Delta y_1 + \varepsilon\beta_{sm}\Delta y + O(\varepsilon^{m+1})y, \quad \varepsilon\dot{\Delta} y_2 = A_f \Delta y_2 + \varepsilon\beta_{fm}\Delta y + O(\varepsilon^{m+1})y. \quad (A.4)$$

Applying to the second equation of (A.2) the variation of constants formula and estimating from above $|exp\{A_f t/\varepsilon\}|, |\beta_f|$ and $|D_f|$, we obtain

$$||y_2||^2 \le \int_0^\infty \int_0^t \int_0^t \frac{K}{\varepsilon^2} e^{-\frac{\alpha}{\varepsilon}(2t-r-p)}[|w(p)| + \varepsilon|y(p)|][|w(r)| + \varepsilon|y(r)|]drdpdt, \qquad (A.5)$$

where $K > 0$, $\alpha > 0$. Estimating further the product of the square brackets by $|w(p)|^2 + |w(r)|^2 + \varepsilon^2[|y(p)|^2 + |y(r)|^2]$ and reversing the order of integration we deduce

$$||y_2||^2 \le \frac{2K}{\varepsilon^2}\int_0^\infty \int_p^\infty \int_0^t e^{-\frac{\alpha}{\varepsilon}(2t-r-p)}drdt[|w(p)|^2 + \varepsilon^2|y(p)|^2]dp \le \frac{2K}{\alpha^2}[||w||^2 + \varepsilon^2||y||^2].$$

Analogously we get $||y_1||^2 \le 2K/\alpha^2[||w||^2 + \varepsilon^2||y||^2]$. Then for small ε we have $||y||^2 \le c||w||^2$, $c > 0$. Similarly one can derive $||z||^2 \le c||w||^2$, and $||\Delta y||^2 \le c\varepsilon^{2m+2}||y||^2 \le c^2\varepsilon^{2m+2}||w||^2$. The latter inequalities together with (A.3) imply $J_m = J + O(\varepsilon^{m+1})||w||^2$. By the condition $J \le \gamma^2||w||^2$. Hence, $J_m \le [\gamma^2 + O(\varepsilon^{m+1})]||w||^2 = [\gamma + O(\varepsilon^{m+1})]^2||w||^2$.

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