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On delay-derivative-dependent stability of systems with fast-varying delays☆

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Abstract

Stability of linear systems with uncertain bounded time-varying delays is studied under the assumption that the nominal delay values are not equal to zero. An input-output approach to stability of such systems is known to be based on the bound of the L_2-norm of a certain integral operator. There exists a bound on this operator norm in two cases: in the case where the delay derivative is not greater than 1 and in the case without any constraints on the delay derivative. In the present note we fill the gap between the two cases by deriving a tight operator bound which is an increasing and continuous function of the delay derivative upper bound $d \geqslant 1$. For $d \to \infty$ the new bound corresponds to the second case and improves the existing bound. As a result, for the first time, delay-derivative-dependent frequency domain and time domain stability criteria are derived for systems with the delay derivative greater than 1.

Keywords: Time-varying delay; Stability; Input–output approach; L_2-norm

1. Introduction

The uncertain time-varying delay has been divided into two types in the existing literature on the stability of time-delay systems: the *slowly-varying* delay (with the delay derivative less than $d<1$) and the *fast-varying delay* (without any constraints on the delay derivative)(see e.g. Gu, Kharitonov, & Chen, 2003; Kolmanovskii & Myshkis, 1999; Niculescu, 2001 and the references therein). Recently a third type of *moderately varying delay* has been revealed in Fridman and Shaked (2006), where the delay derivative is not greater than 1 (almost for all t). This has been obtained by applying the input–output approach to stability. It is known (Gu et al., 2003; Kao & Lincoln, 2004; Quet et al., 2002) that the latter approach to systems with time-varying bounded delays is based on the bound of the L_2-norm of a certain integral operator.

In the present paper we fill the gap between the case of the delay derivative not greater than 1 and the fast-varying delay by deriving a new integral operator bound. This bound is an increasing and continuous function of the delay derivative bound $d \geqslant 1$. In the limit case (where $d \to \infty$) which corresponds to the fast-varying delay, the new bound improves the existing one. As a result, improved frequency domain and time domain stability criteria are derived for systems with the delay derivative bound greater than 1.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathscr{R}^n$ denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathscr{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathscr{R}^{n\times n}$ means that P is symmetric and positive definite. L_2 is the space of square integrable functions $v:[0,\infty)\to C^n$ with the norm $\|v\|_{L_2}=[\int_0^\infty \|v(t)\|^2\,\mathrm{d}t]^{1/2}$, $\|A\|$ denotes the Euclidean norm of a $n\times n$ (real or complex) matrix A, which is equal to the maximum singular value of A. For a transfer function matrix of a stable system $G(s)$, $s\in C$, $\|G\|_\infty=\sup_{-\infty<w<\infty}\|G(iw)\|$, $i=\sqrt{-1}$.

2. Problem formulation

We consider the following linear system with uncertain time-varying delay $\tau(t)$:

$$\dot{x}(t)=A_0x(t)+A_1x(t-\tau(t)), \tag{1}$$

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where $x(t) \in \mathscr{R}^n$ is the system state, $A_i,\ i=0,1$ are constant matrices. The uncertain delay $\tau(t)$ has a form

$$\tau(t) = h + \eta(t), \quad |\eta(t)| \leqslant \mu \leqslant h, \tag{2}$$

where h is a known nominal delay value and μ is a known upper bound on the delay uncertainty. In the existing literature (Gu et al., 2003; Kolmanovskii & Myshkis, 1999; Niculescu, 2001) the following types of uncertain time-varying delays are usually considered:

Case A (Slowly varying delay): $\tau(t)$ is a differentiable almost everywhere function, satisfying

$$\dot{\tau}(t) = \dot{\eta}(t) \leqslant d = 1 + p, \tag{3}$$

where $-1 \leqslant p < 0$.

Case B (Fast-varying delay): $\tau(t)$ is a measurable (e.g. piecewise-continuous) function.

Recently a *moderately varying* delay with $\dot{\tau}(t) \leqslant d = 1$ was introduced in Fridman and Shaked (2006). In the present note we enlarge the latter class of delays as follows:

Case C (Moderately varying delay): $\tau(t)$ is a differentiable almost everywhere function, satisfying (3) with $p \geqslant 0$.

Simple examples of fast and of moderately varying delays are $\tau(t) = 2 - \sin t^2$ (with unbounded $\dot{\tau}$) and $\tau(t) = 2 - \sin 10t$ (with $\dot{\tau}(t) \leqslant d = 10$) correspondingly. Time-varying delays appear e.g. in flow control of data communication networks (Quet et al., 2002). The delay of cases C and B corresponds to the network, where a packet sent out from the source node at a particular time may reach the bottleneck node before another packet, which was sent at an earlier time.

By stability in the present paper we understand the uniform (with respect to initial time) asymptotic stability of the system (see Kolmanovskii & Myshkis, 1999, p. 200). Our objective is to improve the stability results in cases B and C by applying input–output approach and by deriving new inequalities. The results are easily generalized to the case of any finite number of the delays.

We represent (1) in the form:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) - A_1 \int_{-h-\eta}^{-h} \dot{x}(t+s)\,\mathrm{d}s. \tag{4}$$

Following Fridman and Shaked (2006) we introduce an auxiliary system:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + \sqrt{\mu} A_1 X^{-1} u(t), \\ y(t) &= \sqrt{\mu \mathscr{F}(p)} X \dot{x}(t), \end{aligned} \tag{5}$$

with the feedback

$$u(t) = -\frac{1}{\mu \cdot \sqrt{\mathscr{F}(p)}} \int_{-h-\eta}^{-h} y(t+s)\,\mathrm{d}s, \tag{6}$$

where $\mathscr{F} : [-1, \infty] \to R^+$ is a scalar function which will be shortly defined, p is given by (3) and X is a scaling nonsingular matrix. The results for the delay of case B correspond to $p = \infty$, i.e. to $\mathscr{F}(\infty)$ in the input–output model (5) and (6). Substitution of (6) in (5) leads to (4).

We are looking for $\mathscr{F}(p)$ which satisfies the following inequality:

$$\|u\|^2_{L_2} \leqslant \|y\|^2_{L_2}, \quad \forall y \in L_2[0,\infty), \quad y|_{[-\infty,0]} \equiv 0, \tag{7}$$

where u is given by (6). This is equivalent to the fact that $\mu\sqrt{\mathscr{F}(p)}$ is an upper bound on the L_2-norm of the integral operator $\Delta : L_2[0,\infty) \to L_2[0,\infty)$

$$z(t) = \Delta y(t) = \int_{-h-\eta}^{-h} y(t+s)\,\mathrm{d}s, \quad y|_{[-\infty,0]} \equiv 0, \tag{8}$$

i.e. that

$$\|z\|^2_{L_2} \leqslant \mu^2 \mathscr{F}(p) \|y\|^2_{L_2}, \quad \forall y \in L_2[0,\infty), \quad y|_{[-\infty,0]} \equiv 0. \tag{9}$$

Our objective is to find $\mathscr{F}(p)$ (as small as possible) such that (7) (or equivalently (9)) holds.

For $-1 \leqslant p < 0$ (case A) it was established in Gu et al. (2003) that $\mathscr{F}(p)$ can be chosen to be 1. For $p \geqslant 0$ the following was found in Fridman and Shaked (2006): $\mathscr{F}(0) = 1$ and $\mathscr{F}(p) \equiv 2$ for $p \in (0, \infty]$.

Remark 1. The value 1 of $\mathscr{F}(p)$ for $-1 \leqslant p \leqslant 0$ cannot be improved (i.e. chosen to be less than 1). Indeed, taking constant delay $\eta(t) \equiv \mu$, which satisfies the condition of case A for any $-1 \leqslant p \leqslant 0$, we consider the functions $y_\theta(t) = 1$ as $0 \leqslant t \leqslant \theta$, and $y_\theta(t) = 0$ as $t > \theta$, where $\theta > \mu$. Using formula (6) with $\mathscr{F}(p) = 1$ we immediately obtain $\|y_\theta\|^2_{L_2} = \theta$,

$$\begin{aligned} -\mu u(t) &= \int_{t-h-\mu}^{t-h} y_\theta(r)\,\mathrm{d}r \\ &= \begin{cases} t-h, & \text{if } h \leqslant t \leqslant h+\mu, \\ \mu, & \text{if } h+\mu < t < \theta + h, \\ \theta + h + \mu - t, & \text{if } \theta + h \leqslant t \leqslant \theta + h + \mu, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Hence $\|u\|^2_{L_2} = \theta - \frac{1}{3}\mu$, and $\lim_{\theta\to\infty} \|u\|_{L_2} / \|y_\theta\|_{L_2} = 1$.

In the present paper we will improve the values of $\mathscr{F}(p)$ for $p > 0$ by showing that $\mathscr{F}(p)$ can be chosen as a *continuous increasing function of* $p \geqslant 0$ satisfying $\mathscr{F}(0) = 1$ (as in Fridman & Shaked, 2006), but $\mathscr{F}(p) < \mathscr{F}(\infty) = 1.75$ for $p > 0$. The improved values of $\mathscr{F}(p)$ will readily lead to improved stability criteria.

3. Main results

3.1. New bounds

Proofs of the lemmas of this section are given in Appendix A.

Lemma 1. *Consider case C. For all $y(t) \in L_2[0,\infty)$ and such that $y(t) = 0\ \forall t \leqslant 0$ and for $u(t)$ given by (6) inequality (7) holds with $\mathscr{F}$ given by*

$$\mathscr{F}(p) = \begin{cases} \dfrac{2p+1}{p+1} & \text{if } 0 \leqslant p < 1, \\ \dfrac{7p-1}{4p} & \text{if } p \geqslant 1. \end{cases} \tag{10}$$

As it was mentioned above, $\mathscr{F}$ is increasing continuous function satisfying for $p>0$ the following inequality: $1=\mathscr{F}(0)<\mathscr{F}(p)<\lim_{p\to\infty}\mathscr{F}(p)=\frac{7}{4}$.

Lemma 2. *Consider case B. For all $y(t)\in L_2[0,\infty)$ and such that $y(t)=0\ \forall t\leqslant 0$ and for $u(t)$ given by* (6) *inequality* (7) *holds with $\mathscr{F}(\infty):=7/4$.*

Remark 2. The value $\frac{7}{4}=1.75$ for $\mathscr{F}(\infty)$ in Lemma 2 is not far from an optimal one. The following example shows that it cannot be less than 1.5. Namely, define

$$y(t)=\begin{cases} t & \text{if } 0\leqslant t\leqslant\mu,\\ 2\mu-t & \text{if } \mu\leqslant t\leqslant 2\mu,\\ 0 & \text{if } t(2\mu-t)<0,\end{cases}$$

$$\eta(t)=\begin{cases} -\mu & \text{if } t\leqslant\mu,\\ \mu & \text{if } t>\mu.\end{cases}$$

Setting in (6) $\mathscr{F}(\infty)=\frac{3}{2}$ we have $u(t)=-1/\mu\sqrt{3/2}z(t)$, where

$$z(t+h)=\int_{t-\eta(t)}^{t} y(s)\,\mathrm{d}s$$
$$=\begin{cases} -(t+\mu)^2/2 & \text{if } -\mu\leqslant t+h\leqslant 0,\\ -(\mu^2+2\mu t-2t^2)/2 & \text{if } 0<t+h\leqslant\mu,\\ (6\mu t-3\mu^2-2t^2)/2 & \text{if } \mu<t+h\leqslant 2\mu,\\ (t-3\mu)^2/2 & \text{if } 2\mu<t+h\leqslant 3\mu,\\ 0 & \text{otherwise.}\end{cases}$$

We achieve equality in (7) since

$$\|y\|_{L_2}^2=\frac{2}{3}\mu^3,\quad \|u\|_{L_2}^2=\frac{2}{3\mu^2}\|z\|_{L_2}^2=\frac{2}{3\mu^2}\cdot\mu^5=\frac{2}{3}\mu^3.$$

3.2. An improved frequency domain stability criterion

We assume

A1 Given the nominal value of the delay $h>0$, the nominal system

$$\dot{x}(t)=A_0x(t)+A_1x(t-h), \tag{11}$$

is asymptotically stable.

The auxiliary system (5) can be written as $y=Gu$ with the transfer matrix

$$G(s)=\mu\sqrt{\mathscr{F}(p)}sX(sI-A_0-A_1\mathrm{e}^{-hs})^{-1}A_1X^{-1}. \tag{12}$$

By the small gain theorem (see e.g. Gu et al., 2003) system (1) is input–output stable (and thus uniformly asymptotically stable, since the nominal system is time-invariant) if

$$\|G\|_\infty<1. \tag{13}$$

Theorem 1. *Consider* (1) *with delay given by* (2). *Under* A1 *the system is uniformly asymptotically stable if there exists a nonsingular matrix X such that* (13) *is satisfied, where G is given by* (12) *with $\mathscr{F}(p)$ of* (10) *and where $p\in[0,\infty)$ corresponds to case C, while $\mathscr{F}(\infty)=\frac{7}{4}$ corresponds to case B.*

Remark 3. From Theorem 1 it follows that under A1 (1) is stable if $\mu<k/\sqrt{\mathscr{F}(p)}$,

$$k=\|sX(sI-A_0-A_1\mathrm{e}^{-hs})^{-1}A_1X^{-1}\|_\infty^{-1}.$$

By Fridman and Shaked (2006) $\mathscr{F}(p)=2$, $p>0$ and thus (1) (with $\dot{\tau}(t)\leqslant 1+p$, $p>0$ or with $\tau(t)$ of case B) is asymptotically stable for $\tau(t)\in[h-\mu,h+\mu]$, where $\mu<0.7071k$. By the new bounds of Lemmas 2 and 1 we obtain wider stability intervals:

$$\begin{array}{llll} p=0.1, & \dot{\tau}(t)\leqslant 1.1, & \mathscr{F}(p)=1.0909, & \mu<0.9574k,\\ p=1, & \dot{\tau}(t)\leqslant 2, & \mathscr{F}(p)=1.5, & \mu<0.8165k,\\ p=\infty, & \text{case B}, & \mathscr{F}(p)=1.75, & \mu<0.7559k.\end{array}$$

3.3. On improved time domain stability criteria

Theorem 2. *System* (1) *is uniformly asymptotically stable for all delays of* (2), *if there exist $n\times n$ matrices $0<P_1$, P_2, P_3, $S>0$, Y_1, Y_2, R, R_a such that the following Linear Matrix Inequality* (*LMI*)

$$\Gamma=\left[\begin{array}{c|cc} & \mu P_2^{\mathrm{T}}A_1 & 0\\ \Gamma_n & \mu P_3^{\mathrm{T}}A_1 & \mu\mathscr{F}(p)R_a\\ & 0 & 0\\ \hline * & -\mu R_a & 0\\ * & * & -\mu\mathscr{F}(p)R_a\end{array}\right]<0 \tag{14}$$

is feasible, where $*$ denotes the symmetric elements and

$$\Gamma_n=\begin{bmatrix}\Psi_n & P_2^{\mathrm{T}}A_1-Y_1^{\mathrm{T}} & hY_1^{\mathrm{T}}\\ & P_3^{\mathrm{T}}A_1-Y_2^{\mathrm{T}} & hY_2^{\mathrm{T}}\\ * & -S & 0\\ * & * & -hR\end{bmatrix},\quad P=\begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix},$$

$$\Psi_n=P^{\mathrm{T}}\begin{bmatrix}0 & I\\ A_0 & -I\end{bmatrix}+\begin{bmatrix}0 & A_0^{\mathrm{T}}\\ I & -I\end{bmatrix}P+\begin{bmatrix}S+Y_1+Y_1^{\mathrm{T}} & Y_2\\ Y_2^{\mathrm{T}} & hR\end{bmatrix}.$$

Here, $\mathscr{F}(p)$ is given by (10) *in case C and $\mathscr{F}(\infty)=\frac{7}{4}$ in case B.*

Proof of Theorem 2 is similar to the time domain results of Fridman and Shaked (2006), derived via the simple descriptor Lyapunov functional

$$V_n=x^{\mathrm{T}}(t)P_1x(t)+\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta$$
$$+\int_{t-h}^{t}x^{\mathrm{T}}(s)Sx(s)\,\mathrm{d}s.$$

The idea of the proof is the following: A1 and (13) (i.e. the conditions of Theorem 1) are satisfied if along the trajectories of (5) $W:=\dot{V}_n+y^{\mathrm{T}}(t)y(t)-u^{\mathrm{T}}(t)u(t)<0$. The standard calculations and application of the Schur complements to $y^{\mathrm{T}}(t)y(t)$ lead to $W\leqslant\zeta^{\mathrm{T}}\Gamma\zeta$, where $R_a=X^{\mathrm{T}}X$ and $\zeta=col\{x(t),\dot{x}(t),x(t-h),\frac{1}{h}\int_{-h}^{0}\dot{x}(s)\,\mathrm{d}s,u(t),\dot{x}(t)\}$. Hence, (14) implies the stability of (1).

LMI (14) is convex in $\mathscr{F}(p)$ and thus smaller values of $\mathscr{F}(p)$ lead to a less restrictive conditions. Moreover, $\Gamma_n<0$ corresponds to the stability of the nominal system (11) and

Table 1

d	1	1.1	1.5	2	∞ (fast delay)
μ	0.383	0.367	0.331	0.313	0.289

may be replaced by any other appropriate matrix, that guarantees the stability of the nominal system. Thus, stability conditions via descriptor discretized Lyapunov functional have the following form: LMI (11) of Fridman (2006b) and the above LMI (14), where Γ_n in (14) should be substituted by the left-hand side of (17) in Fridman (2006b).

Example 1 (*(Kharitonov & Niculescu, 2003)*). Consider the system

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}x(t)+\begin{bmatrix}0 & 0\\ -1 & 1\end{bmatrix}x(t-\tau(t)),$$

where $\tau(t)=1+\eta(t),\ |\eta(t)|\leqslant\mu,\ \dot{\tau}(t)\leqslant d$. In Fridman (2006a) the maximum value of μ, for which the system is asymptotically stable, was found to be $\mu=0.271$ for all $d\geqslant 1$. The latter result was less restrictive than the one by Kharitonov and Niculescu (2003). By the time domain criterion of Fridman and Shaked (2006) for $d=1$ the resulting μ is greater ($\mu=0.383$), while for $d>1$ the result is the same ($\mu=0.271$). Theorem 2 of the present paper leads to a wider stability interval for $d>1$ (see Table 1).

4. Conclusions

Linear systems with bounded time-varying delays are analyzed under the assumption that the nominal delay values are not equal to zero. Two cases of delay are considered: case B (without any constraints on the delay derivative) and case C (where the delay derivative is not greater than $d\geqslant 1$). An input–output approach to stability of such systems is known to be based on the bound of the L_2-norm of a certain integral operator. In the present paper for the first time a tight *d-dependent* bound is derived. The existing bound in case B is also improved. The new bounds lead to improved stability criteria and give tools for further improvements.

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Appendix A.

Proof of Lemma 2. Our argument will prove the asserted statement for the functions in the space $L_2(\mathbb{R})$, which is a stronger statement, since the space $L_2[0,\infty)$ can be included into $L_2(\mathbb{R})$ without change of the norm by extending the functions of $L_2[0,\infty)$ as identically zero for the negative values of the variable. Our problem is to estimate the norm $\|\Delta\|_{L_2}$ of the operator Δ defined by (8) in the space $L_2(\mathbb{R})$.

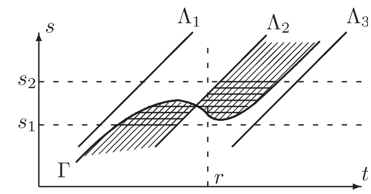

Fig. 1. Domains D and $D(t_0)$: $\Lambda_1: s{=}t{-}h{+}\mu$; $\Lambda_2: s{=}t{-}h$; $\Lambda_3: s{=}t{-}h{-}\mu$; $\Gamma: s=t-h-\eta(t)$; $s_1:=t_0-h-\eta(t_0)$; $s_2:=t_0-h$.

Denote by $\varphi:\mathbb{R}^2\to\{0,1\}$ the characteristic function of the domain D in the plane $\mathbb{R}^2=\{(t,s):t,s\in\mathbb{R}\}$, bounded by the line $s=t-h$ and by the graph of the function $s=t-h-\eta(t)$, i.e. for arbitrary $t,s\in\mathbb{R}$,

$$\varphi(t,s)=\begin{cases}1 & \text{if } (t-h-s)(t-h-\eta(t)-s)\leqslant 0,\\ 0 & \text{if } (t-h-s)(t-h-\eta(t)-s)>0\end{cases}$$

(shaded region in Fig. 1). Since $|\eta(t)|\leqslant\mu$, the domain D entirely lies in the strip between the lines $\Lambda_1=\{s=t-h+\mu\}$ and $\Lambda_3=\{s=t-h-\mu\}$. Then the operator Δ introduced by (8), can be expressed as an integral linear operator

$$z(t)=\operatorname{sign}\eta(t)\cdot\int_{-\infty}^{\infty}\varphi(t,s)y(s)\,\mathrm{d}s.$$

It is well-known that $\|\Delta\|_{L_2}=\|\Delta^*\|_{L_2}$, where Δ^* is the adjoint operator, given by the formula

$$\widetilde{z}(s)=\Delta^*(y)(s)=\int_{-\infty}^{\infty}\operatorname{sign}\eta(t)\cdot\varphi(t,s)y(t)\,\mathrm{d}t$$

(see e.g. Jörgens, 1982, Theorem 3.17 in Section 3.8 and Eq. (8.56) in Section 8.6). Then we derive that

$$\begin{aligned}\|\widetilde{z}\|_{L_2}^2&=\int_{-\infty}^{\infty}\widetilde{z}^{\mathrm{T}}(s)\widetilde{z}(s)\,\mathrm{d}s\\&=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\operatorname{sign}\eta(t_0)\cdot\varphi(t_0,s)y(t_0)\,\mathrm{d}t_0\right)^{\mathrm{T}}\\&\quad\times\left(\int_{-\infty}^{\infty}\operatorname{sign}\eta(t)\cdot\varphi(t,s)y(t)\,\mathrm{d}t\right)\mathrm{d}s\\&\leqslant\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t_0,s)\varphi(t,s)\,\mathrm{d}s\,|y^{\mathrm{T}}(t_0)y(t)|\,\mathrm{d}t_0\,\mathrm{d}t.\end{aligned}$$

Bounding in the latter triple integral $|y^{\mathrm{T}}(t_0)y(t)|\leqslant\frac{1}{2}(\|y(t_0)\|^2+\|y(t)\|^2)$ and denoting

$$K(t_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t_0,s)\varphi(t,s)\,\mathrm{d}s\,\mathrm{d}t, \tag{15}$$

we continue to estimate $\|\widetilde{z}\|_{L_2}^2$ as

$$\begin{aligned}\|\widetilde{z}\|_{L_2}^2&\leqslant\frac{1}{2}\int_{-\infty}^{\infty}K(t_0)\|y(t_0)\|^2\,\mathrm{d}t_0\\&\quad+\frac{1}{2}\int_{-\infty}^{\infty}K(t)\|y(t)\|^2\,\mathrm{d}t\leqslant\sup_{t_0\in\mathbb{R}}K(t_0)\|y\|_{L_2}^2.\end{aligned}$$

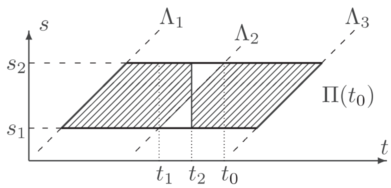

Fig. 2. Upper bound for the area of $D(t_0)$: $t_1 = t_0 - \eta(t_0)$; $t_2 = t_0 - \eta(t_0)/2$.

We shall show that $K(t_0) \leqslant \frac{7}{4}\mu^2$ for all $t_0 \in \mathbb{R}$. Fix some number $t_0 \in \mathbb{R}$. We can assume that $\eta(t_0) \geqslant 0$, since the treatment of the case $\eta(t_0) \leqslant 0$ is completely similar. Geometrically, $K(t_0)$ *is the area of the part* $D(t_0)$ of the domain D cut out by the strip $t_0 - h \geqslant s \geqslant t_0 - h - \eta(t_0)$ (double shaded region in Fig. 1). Indeed, $\varphi(t_0, s) = 1$ as far as $t_0 - h \geqslant s \geqslant t_0 - h - \eta(t_0)$, and $\varphi(t, s) = 1$ as far as $(t, s) \in D$, and their product vanishes outside these limits, that is formula (15) gives the area of $D(t_0)$.

Thus, $D(t_0)$ lies inside the parallelogram

$$\Pi(t_0) := \{(t, s) \in \mathbb{R}^2 : t_0 - h - \eta(t_0) \leqslant s \leqslant t_0 - h,\\ t - h - \mu \leqslant s \leqslant t - h + \mu\}$$

(see Fig. 2). The vertical lines $t = t_0$ and $t = t_1 := t_0 - \eta(t_0)$, passing through the intersection points of the line Λ_2 with the sides of $\Pi(t_0)$ (see Fig. 2), divide $\Pi(t_0)$ into two trapezes of total area $\mu^2 - (\mu - \eta(t_0))^2$, and the square $S = \{t_1 \leqslant t \leqslant t_0,\ s_1 \leqslant s \leqslant s_2\}$. Our objective is to estimate the area of $D(t_0) \cap S$.

Consider the intersection of the vertical line $t = r$, where $t_1 \leqslant r \leqslant t_0$, with $D(t_0)$ (see Fig. 1). The intersection of $\{t = r\}$ with $\Pi(t_0)$ is divided by the line Λ_2 into two segments: $s_1 \leqslant s \leqslant r - h$ and $r - h \leqslant s \leqslant s_2$. The intersection of $D(t_0)$ with $\{t = r\}$ is entirely contained in one of these segments: in the upper one if $\eta(r) \leqslant 0$ and in the lower one if $\eta(r) \geqslant 0$. Hence, the area of $D(t_0) \cap S$ does not exceed

$$\int_{t_0-\eta(t_0)}^{t_0} \max\{r - h - s_1, s_2 - (r - h)\}\, dr\\ = \int_{t_0-\eta(t_0)}^{t_0} \max\{r - (t_0 - \eta(t_0)), t_0 - r\}\, dr = \frac{3}{4}\eta(t_0)^2.$$

So the total area of $D(t_0)$ does not exceed $\mu^2 - (\mu - \eta(t_0))^2 + \frac{3}{4}\eta(t_0)^2 \leqslant \frac{7}{4}\mu^2$, which implies the required bound (9). □

Remark 4. The bound $\frac{7}{4}\mu^2$ can be interpreted geometrically as the maximum (attained at $\eta(t_0) = \mu$) of the area of the shaded region in Fig. 2. The bound $\|\Delta\|^2_{L_2} \leqslant 2\mu^2$ of Fridman and Shaked (2006) corresponds to the maximal area of the whole parallelogram $\Pi(t_0)$.

Proof of Lemma 1. As in the proof of Lemma 2, we come to estimation from above of $K(t_0)$, or, equivalently, of the area of the domain $D(t_0)$ for any given real t_0. Without loss of generality we can assume that the zero locus of η is locally finite. Indeed, such functions form a dense (in the sense of both sup-norm and L_1-norm) subset in the space of almost everywhere differentiable functions, and hence, our restriction does not affect the estimation of $K(t_0)$. Again we also assume

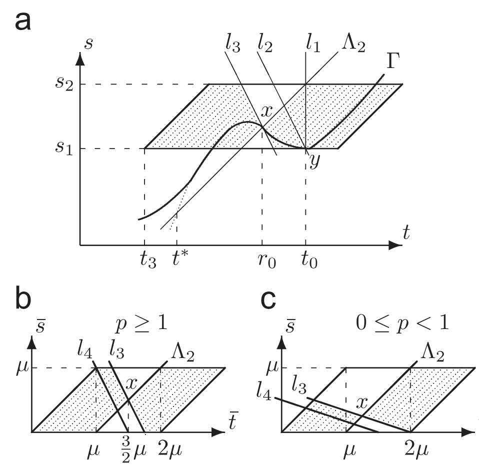

Fig. 3. The case $N = 1$.

that $\eta(t_0) > 0$, which, in particular, means that the vertical line $t = t_0$ crosses the domain $D(t_0)$ along the segment $s_1 \leqslant s \leqslant s_2$ (cf. Fig. 3(a)). Introduce the number N of the zeroes of the function $\eta(t)$ in the interval $t_3 := t_0 - \eta(t_0) - \mu < t < t_0$, where it changes its sign (see the interval in Fig. 3(a)). Consider a few possibilities.

Step 1: Suppose that $N = 0$. Then the graph Γ does not cross the segment of the line Λ_2 inside the parallelogram Π_0, and hence the domain $D(t_0)$ is entirely contained in one of the halves of $\Pi(t_0)$, cut out by the line Λ_2. Its area does not exceed $\frac{1}{2}\mathrm{Area}(\Pi(t_0)) = \eta(t_0)\mu \leqslant \mu^2 \leqslant \mathscr{F}(p)\mu^2$.

Step 2: Suppose that $N = 1$, and let $r_0 \in (t_3, t_0)$ be the (unique) zero of $\eta(t)$, where it changes its sign (see Fig. 3(a)). Geometrically, it is the abscissa of the intersection point x of the line $\Lambda_2 = \{s = t - h\}$ and the graph $\Gamma = \{s = t - h - \eta(t)\}$. Since the intersection point y of the line $l_1 := \{t = t_0\}$ with the lower base of the parallelogram $\Pi(t_0)$ belongs to Γ (it has coordinates $(t_0, t - h - \eta(t_0))$), and since

$$\frac{d}{dt}(t - h - \eta(t)) \geqslant -p, \tag{16}$$

we decide that for $t \geqslant r_0$,

(i) the graph Γ lies below the line Λ_2 and above the line l_3, passing through x with the slope $-p$, and
(ii) the point x must lie below the line l_2, passing through the point y with the slope $-p$.

Similarly, for $t \leqslant r_0$, the graph Γ must lie above the line Λ_2 and below the line l_3. So, the domain $D(t_0)$, bounded by Λ_2 and Γ, is entirely contained in the polygonal region R in $\Pi(t_0)$ shown shaded in Fig. 3(a).

The area of R depends only on the value of $\eta(t_0)$ and on the position of the point x on the line Λ_2. First, we observe that, increasing $\eta(t_0)$ up to μ by pulling the lower base of $\Pi(t_0)$ down when the other sides remain fixed, we enlarge the region R and respectively its area.

Assuming now that $\eta(t_0)=\mu$, we vary the point x along Λ_2 below the line l_2. If $p \geqslant 1$, then the maximal area is achieved as x becomes the middle point of the segment of Λ_2 cut out by $\Pi(t_0)$ (see Fig. 3(b), where $\Pi(t_0)$ is shown in a convenient coordinate system $\bar{t}, \bar{s}$). Indeed, if we move the point x and, respectively, the line l_3 away from the middle position (the new position of l_3 is shown as l_4 in Fig. 3(b)), then in the region R we replace the shaded trapeze between the lines l_3, l_4 by clear trapeze, the latter one having shorter bases than the former one, and thus the area of R would decrease.

In the case $0 \leqslant p < 1$, we cannot move the point x and line l_3 above the position shown in Fig. 3(c), since the point y, which after equating $\eta(t_0)=\mu$ become the right lower vertex of $\Pi(t_0)$, cannot lie below the line l_3 (see restriction (ii) in the beginning of Step 2). We claim that this highest position of l_3 gives the maximal area of the region R. Indeed, if we move the line l_3 down (its new position is shown as l_4 in Fig. 3(c)), then we replace a shaded parallelogram between the lines l_3, l_4 by a clear trapeze, whose bases are shorter than those for the parallelogram, and the area of R would decrease.

It remains to compute the shaded areas shown in Fig. 3(b,c). In Fig. 3(b) we remove from $\Pi(t_0)$ two symmetric triangles with the height $\mu/2$ and the base $\mu(1+1/p)/2$, and hence the area of R is $2\mu^2-\mu^2(1+1/p)/4=(7p-1)/4p\mu^2$.

If $0 \leqslant p < 1$, using the equation $l_3=\{\bar{s}=-p(\bar{t}-2\mu)$, we compute the coordinates of $x=((2p+1)/(p+1)\mu, p/(p+1)\mu)$ and of the intersection of l_3 with the left side of $\Pi(t_0)$, $(2p/(p+1)\mu, 2p/(p+1)\mu)$. The clear triangle and trapeze in Fig. 3(c) have the height $\mu/\sqrt{2}$ with the bases $\sqrt{2}p/(p+1)\mu$ and $\sqrt{2}/(p+1)\mu$, $(\sqrt{2}(1-p))/(p+1)\mu$, respectively. The area of R is equal to

$$2\mu^2-\frac{\mu}{2\sqrt{2}}\sqrt{2}\mu\frac{p+1+1-p}{p+1}=\frac{2p+1}{p+1}\mu^2.$$

Step 3: Suppose that $N>1$. We can take N to be odd, i.e., $N=2n+1$, adding (if necessary) one more zero as follows: modify the graph Γ outside the parallelogram $\Pi(t_0)$ (see the dotted line of Fig. 3(a)) in such a way that Γ additionally crosses Λ_2 in some point with the abscissa $t^* > t_3$ (such a modification shown by the dotted line if Fig. 3(a)). This operation does not affect $D(t_0)$. Then, using a certain inductive procedure, we intend to show that the case $N=2n+1$ reduces to $N=1$, considered above.

Let $r_1 > r_2 > \cdots > r_{2n+1}$ be all the roots of $\eta(t)$ in the interval (t_3, t_0), where $\eta(t)$ changes its sign. These are abscissas of the respective intersection points $x_1, x_2, \ldots, x_{2n+1}$ of the graph Γ with the line Λ_2, where Γ passes from one side of Λ_2 to the other. More precisely, treating the point x_1 (the highest one) as the point x in Step 2, we see that the graph Γ passes through x_1 from the left half plane to the right one, and hence the same holds for all odd numbered points x_{2i+1}, whereas through the even numbered point x_{2i} the graph Γ passes in the opposite direction (see Fig. 4(a)). Consider now the points x_1, x_2, x_3. Draw the lines l_1', l_2' through x_1, x_3 with the slope $-p$, and the vertical line l_1'' through x_2. In view of (16), the part of the domain $D(t_0)$ in the strip $r_3 \leqslant t \leqslant r_1$ is contained in the region R in $\Pi(t_0)$, bounded by the lines $\Lambda_2, l_1', l_2', l_1''$ (shown shaded in Fig. 4(a)).

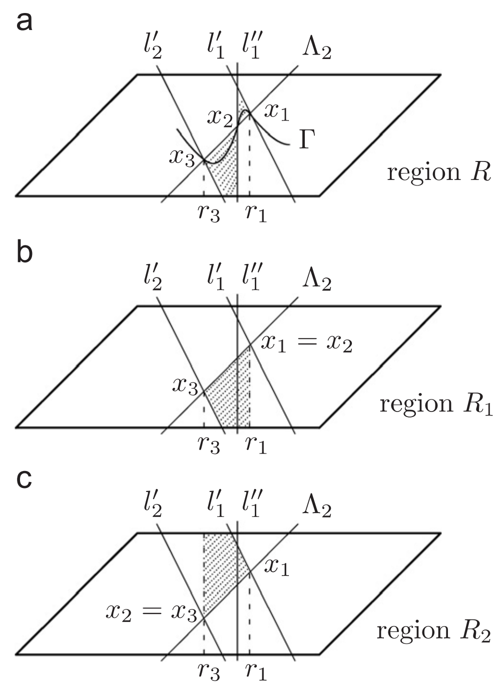

Fig. 4. The case $N>1$.

The line l_1'' contains two segments in the boundary of R, joined by the point x_2. Denote the length of the upper segment by λ_1, and of the lower one by λ_2. If $\lambda_1 \leqslant \lambda_2$, then the area of R is less than the area of the region R_1 shown shaded in Fig. 4(b) (bounded from the left by Λ_2 and by l_2'). Indeed, here we replace a triangle or a trapeze by a trapeze of the same height, but with bigger bases. Similarly, if $\lambda_1 \geqslant \lambda_2$, then the area of R is less than the area of the region R_2 shown shaded in Fig. 4(c) (bounded from the right by Λ_2 and l_1'). Now we observe that the replacement of R by R_1 or by R_2 is equivalent to the following operation: having fixed points x_1 and x_3, we move the point x_2 along Λ_2 between x_1 and x_3 so that it merges either x_1, or x_3. The meaning of both the operations is a modification of the function $\eta(t)$ so that two of its roots in (t_3, t_0) coincide and thus $\eta(t)$ does not change sign in such multiple root. Hence, the number of the roots of $\eta(t)$, where it changes its sign, decreases by 2, whereas the area of the region (which is further used for estimation of the area of $D(t_0)$) increases. Repeating the above procedure n times, we come to $N=1$. □

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