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Brief paper

Network-based H_∞ filtering of parabolic systems[☆]

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ABSTRACT

We design a network-based H_∞ filter for a parabolic system governed by a vector semilinear N-D diffusion equation over a rectangular domain Ω under distributed in space measurements. The sampled in time measurements are sent to the observer over a communication network according to Round-Robin scheduling protocol (one after another in a periodic manner). The objective is to enlarge the sampling time intervals and, thus, to reduce the amount of communications, while preserving a satisfactory error system performance. We suggest to divide Ω into a finite number of rectangular sub-domains N_s, where stationary or mobile sensing devices provide spatially averaged state measurements to be transmitted through communication network. Sufficient conditions in terms of Linear Matrix Inequalities (LMIs) for the internal exponential stability and L_2-gain analysis of the estimation error are derived via the time-delay approach to networked control systems. By solving these LMIs, the filter gain along with the upper bounds on the sampling time intervals, on the network induced time-delays, and on the diameters of the sub-domains can be found that preserve the internal stability of the error system and achieve a given L_2-gain. Numerical examples illustrate the efficiency of the method.

1. Introduction

Networked Control Systems (NCSs), where the plant is controlled via communication network, is a hot topic. The introduction of communication network media brings great advantages, such as low cost, reduced weight, simple installation/maintenance and long distance control. Long distance estimation/control of chemical reactors or air polluted areas (that can be modeled by diffusion PDEs Koda & Seinfeld, 1978 and Smagina & Sheintuch, 2006) is potentially of great interest. It is important to provide a stability and performance certificate that takes into account the network imperfections: variable sampling intervals and communication delays, scheduling protocols and quantization (Heemels, Teel, van de Wouw, & Nesic, 2010). Three main approaches have been used to the NCSs: the discrete-time, the hybrid system and the time-delay approaches (Donkers, Heemels, van de Wouw, & Hetel, 2011; Gao, Chen, & Lam, 2008; Heemels et al., 2010; Liu, Fridman, & Hetel, 2012).

While there exists an extensive literature on network-based control of finite dimensional systems, there are only a few works on network-based control of PDEs. For linear parabolic systems, mobile collocated sensors and actuators were considered in Demetriou (2010). The discrete-time approach to sampled-data control of linear time-invariant distributed parameter systems was developed in Logemann (2013), Logemann, Rebarber, and Townley (2005) and Tan, Trelat, Chitour, and Nesic (2009). A model-reduction-based approach to network-based control of semilinear distributed parameter systems was introduced in Ghantasala and El-Farra (2012) and Yao and El-Farra (2012), where a finite-dimensional controller was designed on the basis of a finite-dimensional system that captures the dominant (slow) dynamics of the infinite-dimensional system. The latter approach has difficulties in the case of spatially-dependent diffusion coefficients. The above methods are not applicable to the performance (exponential decay rate or L_2-gain) analysis of the closed-loop infinite-dimensional systems.

Finding constructive LMI conditions for the performance analysis in terms of sampling intervals, delays and scheduling protocols for networked estimation of diffusion PDEs is of theoretical and practical importance. In the recent papers Fridman and Bar Am (2013) and Fridman and Blighovsky (2012) sampled-data control of 1-D diffusion PDEs under the spatially averaged and the point-wise measurements respectively was studied. The results of Fridman and Bar Am (2013) and Fridman and Blighovsky (2012) were

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limited to the scalar 1-D case, whereas communication constraints (scheduling protocols) were not considered.

In the present paper we study, for the first time, a network-based H_∞ filtering of distributed parameter systems. We consider a vector N-D semilinear diffusion PDE over a rectangular domain Ω. Similar to El-Farra and Christofides (2004), we assume that a large number of "pointwise" spatial output measurements (e.g. temperature of the rod throughout the reactor) are available so that the averaged measurements over the spatial domain Ω or over its closed sub-domains are known with sufficient accuracy. The measurements are sent over communication network to the observer in the discrete-time instances. Due to communication constraints, only one measurement can be sent per transmission. The measurements are sent according to the Round-Robin protocol in a periodic manner. The objective is to enlarge the sampling time intervals and, thus, to reduce the amount of communications, while still retaining a satisfactory estimation error performance.

We suggest to divide the spatial domain into N_s rectangular sub-domains, where sensing devices provide spatially averaged measurements. Such measurements can be done either by stationary or by mobile sensors that move to the sub-domain with the measurements to be transmitted. A larger N_s allows to send a more accurate approximation of the "pointwise" measurements that may improve the performance. However, due to communication constraints, an increase in N_s enlarges the delays and, thus, worsens the performance. Sufficient conditions for the internal exponential stability and L_2-gain analysis of the error system are derived in the framework of the time-delay approach to NCSs, where the variable in time sampling intervals, network-induced delays and a Round-Robin scheduling protocol are taken into account. We show that given N_s, the division that minimizes the maximum diameter of the resulting sub-domains enlargers the sampling intervals.

1.1. Notation and preliminaries

The superscript 'T' stands for matrix transposition, $\mathbb{R}^N$ denotes the N-dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}^{N\times M}$ is the space of $N\times M$ real matrices, and the notation $P>0$ with $P\in\mathbb{R}^{N\times N}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. If $A\in\mathbb{R}^{n\times m}$ and $B\in\mathbb{R}^{p\times q}$, then the Kronecker product $A\otimes B$ is the $np\times mq$ block matrix:

$$A\otimes B=\begin{bmatrix} a_{11}B & \cdots & a_{1n}B\\ \vdots & \ddots & \vdots\\ a_{n1}B & \cdots & a_{nn}B\end{bmatrix}.$$

Continuous functions (continuously differentiable) in all arguments, are referred to as of class C (of class C^1). $L_2(\Omega)$ is the Hilbert space of square integrable $f:\Omega\to\mathbb{R}^q$, where $\Omega\subset\mathbb{R}^N$, with the norm $\|f\|_{L_2}=\sqrt{\int_\Omega |f(x)|^2dx}$. Let $\partial\Omega$ be the boundary of Ω. $L_2(0,\infty;L_2(\Omega))$ is the Hilbert space of square integrable functions $w:(0,\infty)\to L_2(\Omega)$ with the norm

$$\|\omega\|^2_{L_2(0,\infty;L_2(\Omega))}=\int_0^\infty\int_{x\in\Omega}|\omega(x,t)|^2dxdt<\infty.$$

For $z(x)=[z^1(x),\dots,z^M(x)]^T\in\mathbb{R}^M$ with $z^m:\Omega\to\mathbb{R}$ ($m=1,\dots,M$) denote $z_x^m=[\frac{\partial z^m}{\partial x_1},\dots,\frac{\partial z^m}{\partial x_N}]$, $\nabla_x z^m=(z_x^m)^T$ and $\nabla_x z\triangleq \mathrm{col}\{\nabla_x z^1,\dots,\nabla_x z^M\}\in\mathbb{R}^{NM}$.

$\mathcal{H}^1(\Omega)$ is the Sobolev space of absolutely continuous functions $z:\Omega\to\mathbb{R}^M$ with the square integrable $\nabla_x z$.

MATI is the Maximum Allowable Transmission Interval, MAD is the Maximum Allowable (network-induced) Delay. $\mathcal{N}$ denotes the set $\{0,1,2,\dots\}$.

We present below some useful inequalities.

Lemma 1. *Let $\Omega=[0,l_1]\times\cdots\times[0,l_N]$. Assume $z:\Omega\to\mathbb{R}$ and $z\in\mathcal{H}^1(\Omega)$.*

(i) *(Poincare's inequality) If $\int_\Omega z(x)dx=0$, then according to Payne and Weinberger (1960)*

$$\|z\|^2_{L_2}\le\mathcal{P}^2\|\nabla_x z\|^2_{L_2},\qquad \mathcal{P}=\frac{\delta}{\pi}=\frac{\sqrt{l_1^2+\cdots+l_N^2}}{\pi}. \tag{1}$$

Here δ is the diameter of Ω, $\mathcal{P}$ is Poincare's constant.

(ii) *(Wirtinger's inequality) If $z_{|\partial\Omega}=0$, then the following inequality holds (Hardy, Littlewood, & Polya, 1988):*

$$\mathcal{W}^2\|z\|^2_{L_2}\le\|\nabla_x z\|^2_{L_2},\qquad \mathcal{W}^2=\frac{\pi^2}{l_1^2}+\cdots+\frac{\pi^2}{l_N^2}. \tag{2}$$

2. Problem formulation

Denote by Ω the N-dimensional rectangle

$$\Omega=\{x=(x_1,x_2,\dots,x_N)^T|x_k\in[0,l_k],\ l_k>0,\ k=1,2,\dots,N\},$$

with the boundary

$$\partial\Omega=\{x=(x_1,x_2,\dots,x_N)^T|\exists k\in 1,2,\dots,N \text{ s.t. } x_k=0 \text{ or } x_k=l_k\}.$$

Consider the following semilinear diffusion PDE

$$z_t(x,t)=\Delta_D z(x,t)-\beta\nabla_x z(x,t)+Az(x,t)+\phi(z(x,t),x,t)+B_1w(x,t),\quad t\ge 0,\ x\in\Omega, \tag{3}$$

where $z(x,t)=[z^1(x,t),\dots,z^M(x,t)]^T\in\mathbb{R}^M$ is the vector state, $w(x,t)\in L_2(0,\infty;L_2(\Omega))$ is the disturbance, A and B_1 are constant matrices, and $\beta\in\mathbb{R}^{M\times NM}$ is the convection coefficients matrix.

The diffusion term is given by

$$\Delta_D z(x,t)\triangleq\mathrm{col}\{\Delta_D^1 z^1(x,t),\dots,\Delta_D^M z^M(x,t)\},$$

$$\Delta_D^m z^m(x,t)=\sum_{k=1}^N\frac{\partial}{\partial x_k}\left(\sum_{j=1}^N D_{kj}^m(x)\frac{\partial z^m(x,t)}{\partial x_j}\right),\quad m=1,\dots,M,$$

where $D^m=(D^m)^T:\Omega\to\mathbb{R}^{N\times N}$ is of class C^1. It is assumed that

$$0<d_0^m I_N\le D^m(x)\in\mathbb{R}^{N\times N},\quad m=1,\dots,M. \tag{4}$$

Then

$$D_0\triangleq\mathrm{diag}\{d_0^1,\dots,d_0^M\}>0. \tag{5}$$

The function ϕ is supposed to be of class C^1 with a uniformly bounded ϕ_z, satisfying

$$\phi_z^T(z,x,t)\phi_z(z,x,t)\le Q\quad\forall z,x,t \tag{6}$$

for some constant and positive $M\times M$-matrix Q.

Consider (3) under the Dirichlet

$$z(x,t)|_{x\in\partial\Omega}=0 \tag{7}$$

or under the Neumann

$$z_x(x,t)\cdot\hat{n}|_{x\in\partial\Omega}=0 \tag{8}$$

boundary conditions, where $\hat{n}$ is a unit vector normal to the edge.

The disturbance $w(x,t)\in L_2(0,\infty;L_2(\Omega))$ is said to be *admissible* if system (3) possesses a unique strong solution being initialized with $z(\cdot,0)\in\mathcal{H}^1(\Omega)$, satisfying the boundary conditions, and if this solution is globally continuable to the right. If w is C^1 and is

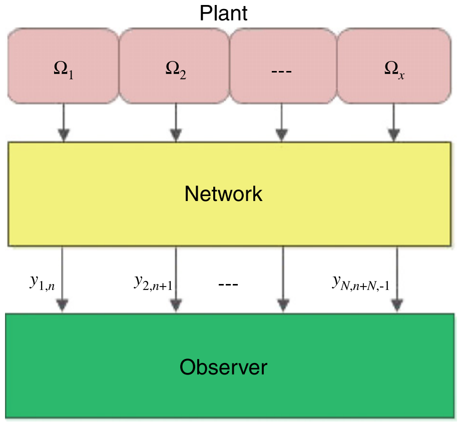

Fig. 1. Round-Robin scheduling: $n = 0, N_s, 2N_s, \ldots$.

uniformly bounded, then by the arguments of Fridman and Bar Am (2013), the strong solutions of (3) under the boundary conditions (7) or (8) initialized with $z(\cdot, 0) \in \mathcal{H}^1(\Omega)$ that satisfy the corresponding boundary conditions exist, and they are continuable for $t \geq 0$.

The measurements are sent over communication network to the observer in the discrete-time instances. In order to construct network-based H_∞ filter we suggest to divide the spatial domain into N_s rectangular sub-domains Ω_i covering the whole region $\bigcup_{i=1}^{N_s} \Omega_i = \Omega$, that intersect only on the boundaries

$$\Omega_i = \{x = (x_1, x_2, \ldots, x_N)^T \in \Omega | x_k \in [x_k^{\min}(i), x_k^{\max}(i)],$$
$$k = 1, 2, \ldots, N, \ i = 1, 2, \ldots, N_s\}.$$

We have $N_s = n^1 \times \cdots \times n^N$, where n^k is the number of subintervals of the side $[0, l_k]$ corresponding to N_s. Denote

$$\Delta_i = \int_{x_1^{\min}(i)}^{x_1^{\max}(i)} \cdots \int_{x_N^{\min}(i)}^{x_N^{\max}(i)} d\xi = \int_{\Omega_i} d\xi,$$

$$\delta_i^2 = \sum_{k=1}^{N} [x_k^{\max}(i) - x_k^{\min}(i)]^2, \quad \delta_i \leq \delta,$$

where Δ_i and δ_i are the volume and the diameter of Ω_i respectively, that can be variable. Here δ is the diameters' upper bound.

Assume that sensors provide N_s averaged measurements

$$y_i(t) = \frac{\int_{\Omega_i} Cz(\xi, t) d\xi}{\Delta_i} + v^i(t), \quad i = 1, 2, \ldots, N_s, \tag{9}$$

where C is a constant $l \times M$-matrix ($l \leq M$), $v^i(t) \in \mathbb{R}^l$ is the measurement noise. The latter measurements are transmitted via a communication network to the observer. Let s_k denote the unbounded monotonously increasing sequence of sampling instances, i.e.

$$0 = s_0 < s_1 < \cdots < s_k < \cdots, \quad k \in \mathcal{N}, \quad \lim_{k\to\infty} s_k = \infty.$$

At each sampling instant s_k, one of the measurements $y_i(t)$ is transmitted via the network. The choice of the $y_i(t)$ to be sampled is ruled by a Round-Robin scheduling protocol: $y_i(t)$ are sampled one after another in the periodic manner, i.e. $y_i(t)$ is transmitted only at the sampling instants $t = s_{N_s p+i-1}, p \in \mathcal{N}$. After each reception, the values of $y_i(t)$ are updated with the newly received ones, while the other values of $y_j(t)$ for $j \neq i$ remain the same, as no additional information is received. This leads to the constrained data exchange expressed as

$$y_{i,k} = \frac{\int_{\Omega_i} Cz(\xi, s_k) d\xi}{\Delta_i} + v_{i,k},$$
$$i = 1, 2, .., N_s, \ k = N_s p + i - 1, \ p \in \mathcal{N}, \tag{10}$$

where $v_{i,k} = v^i(t_k)$ is an additive measurement disturbance (see Fig. 1). Denote

$$v(x, t) = v_{i,k}, \quad x \in \Omega_i, \ t \in [t_k, t_{k+N_s}),$$
$$i = 1, \ldots, N_s, \ k = N_s p + i - 1, \ p \in \mathcal{N}. \tag{11}$$

We suppose that the transmission of the information over the network is subject to a variable and bounded communication delay $h_k \leq$ MAD. Then $t_k = s_k + h_k$ is the updating instant time of the observer. A time-delay approach to finite-dimensional network-based control under the Round-Robin scheduling was introduced in Liu et al. (2012). As in Liu et al. (2012) we do not restrict the network-induced delay to be small with $t_k = s_k + h_k < s_{k+1}$, i.e. $h_k < s_{k+1} - s_k$. We assume the following:

A1. The order of the measurements y_i ($i = 1, 2, \ldots, N_s$) is not changed over the network

$$t_{k+N_s} = s_{k+N_s} + h_{k+N_s} \geq s_k + h_k = t_k,$$
$$k = N_s p + i - 1, \ p \in \mathcal{N}, \tag{12}$$

whereas the time span between the most recent updating and the oldest sampling instant is bounded

$$t_{k+N_s} - t_k + h_k = s_{k+N_s} - s_k + h_{k+N_s}$$
$$\leq \text{MATI} \cdot N_s + \text{MAD} \triangleq \tau_M. \tag{13}$$

A2. The measurements are sent with the time-stamps.

The assumption (12) makes the scheduling reasonable. The assumption A2 means that $s_k = t_k - h_k$ is known on the observer side. The latter allows to use the Luenberger type observer of the form

$$\hat{z}_t(x, t) = \Delta_D \hat{z}(x, t) - \beta \nabla_x \hat{z}(x, t) + A\hat{z}(x, t) + \phi(\hat{z}, x, t),$$
$$t \in [0, t_{i-1}),$$
$$\hat{z}_t(x, t) = \Delta_D \hat{z}(x, t) - \beta \nabla_x \hat{z}(x, t) + A\hat{z}(x, t) + \phi(\hat{z}, x, t)$$
$$+ K_o \left[y_{i,k} - \frac{\int_{\Omega_i} C\hat{z}(\xi, s_k) d\xi}{\Delta_i} \right], \quad t \in [t_k, t_{k+N_s}), \tag{14}$$
$$x \in \Omega_i, \quad i = 1, \ldots, N_s, \ k = N_s p + i - 1, \ p \in \mathcal{N}$$

with $\hat{z}(x, t) \in R^M$ and a constant observer gain K_o. The observer dynamics is subject to the same boundary conditions as the state dynamics: $\hat{z}(x, t)|_{x\in\partial\Omega} = 0$ for (3), (7) or $\hat{z}_x(x, t) \cdot \hat{n}|_{x\in\partial\Omega} = 0$ for (3), (8). By using the step method (i.e. considering $t \in [0, t_0)$, $t \in [t_0, t_1), \ldots$) and applying the arguments of Fridman and Bar Am (2013), the strong solutions of (14) under the corresponding boundary conditions initialized with $\hat{z}(\cdot, 0) \in \mathcal{H}^1(\Omega)$ that satisfy the boundary conditions exist. Moreover, these solutions are continuable for $t \geq 0$.

Let $e(x, t) = z(x, t) - \hat{z}(x, t)$ be the estimation error. Then the error dynamics is governed by

$$e_t(x, t) = \Delta_D e_x(x, t) - \beta \nabla_x e(x, t) + Ae(x, t)$$
$$+ \phi' e(x, t) + B_1 w(x, t), \quad t \in [0, t_{i-1}),$$
$$e_t(x, t) = \Delta_D e_x(x, t) - \beta \nabla_x e(x, t) + Ae(x, t)$$
$$+ \phi' e(x, t) - K_o \left[\frac{\int_{\Omega_i} Ce(\xi, t_k - h_k) d\xi}{\Delta_i} \right] \tag{15}$$
$$- K_o v(x, t) + B_1 w(x, t), \quad t \in [t_k, t_{k+N_s}),$$
$$x \in \Omega_i, \ i = 1, 2, \ldots, N_s, \ k = N_s p + i - 1, \ p \in \mathcal{N},$$

where $\phi' \triangleq \int_0^1 \phi_z(\hat{z}(x, t) + \alpha e(x, t), x, t) d\alpha$. Due to (6), by applying Jensen's inequality we obtain

$$\phi'^T \phi' \leq \int_0^1 \phi_z^T(\hat{z} + \alpha e, x, t) \phi_z(\hat{z} + \alpha e, x, t) d\alpha \leq Q \tag{16}$$

for all $\hat{z}, e, x, t$. The boundary conditions for the error dynamics are

$$e(x,t)|_{x\in\partial\Omega} = 0 \tag{17}$$

for (3) under the Dirichlet boundary conditions (7) and

$$e_x(x,t)\cdot\hat{n}|_{x\in\partial\Omega} = 0 \tag{18}$$

for (3) under the Neumann boundary conditions (8).

Following the time-delay approach to sampled-data control (Fridman, Seuret, & Richard, 2004), denote

$$\begin{aligned}&\tau_i(t) = t - t_k + h_k, \quad t\in[t_k, t_{k+N_s}),\\ &\quad i = 1,2,\ldots,N_s,\ k = N_s p + i - 1,\ p\in\mathcal{N}.\end{aligned} \tag{19}$$

Then $t_k - h_k = t - \tau_i(t)$ and, due to (13),

$$\tau_i(t) \le t_{k+N_s} - t_k + h_k \le \tau_M, \quad i = 1,2,\ldots,N_s.$$

Similar to Fridman and Bar Am (2013), we shall use the elementary relation

$$\frac{\int_{\Omega_i} e(\xi, t-\tau_i(t))d\xi}{\Delta_i} = e(x,t) - f_i(x,t) - \rho_i, \quad x\in\Omega_i,$$

where $\rho_i = \rho_i(t)$ and where

$$f_i(x,t) \triangleq \frac{\int_{\Omega_i}[e(x,t) - e(\xi,t)]d\xi}{\Delta_i},$$

$$\rho_i \triangleq \frac{\int_{\Omega_i}\int_{t-\tau_i(t)}^{t} e_s(\xi,s)dsd\xi}{\Delta_i}.$$

Hence, the error system can be presented as

$$\begin{aligned}e_t(x,t) &= \Delta_D e(x,t) - \beta\nabla_x e(x,t) + Ae(x,t)\\ &\quad + (\phi' - K_oC)e(x,t) + K_oCf_i(x,t)\\ &\quad + K_oC\rho_i + B_1 w(x,t) - K_o v(x,t),\\ &\quad t\ge t_{N_s-1},\ x\in\Omega_i,\ i = 1,\ldots,N_s.\end{aligned} \tag{20}$$

The initial condition $e(x,t)$ $(t\in[0, t_{N_s-1}])$ for (20) is defined as a strong solution of (15), where $e(\cdot,0)\in\mathcal{H}^1(\Omega)$ satisfies the boundary conditions.

Since $f_{ix}(x,t) = e_x(x,t)$ and $\int_{\Omega_i} f_i(x,t)dx = 0$, Poincare's inequality (1) implies

$$\int_{\Omega_i}\left[|\nabla_x e^m(x,t)|^2 - \frac{\pi^2}{\delta_i^2}|f_i^m(x,t)|^2\right]dx \ge 0, \quad t\ge t_{N_s-1}. \tag{21}$$

Note that the Lyapunov-based analysis of (20) under the corresponding boundary conditions in the case of scalar z and x with $s_k = t_k$ and $h_k = 0$ was considered in Fridman and Bar Am (2013), where ρ_i was multiplied by $\frac{1}{t-t_k}$.

For the system (20), we choose the Lyapunov–Krasovskii functional of the form

$$\begin{aligned}&V(t) = V_P + V_S + V_R,\\ &V_P = \int_\Omega e^T(x,t)P_1 e(x,t)dx\\ &\qquad + \sum_{m=1}^{M}\int_\Omega p_3^m e_x^m(x,t)D^m(x)\nabla_x e^m(x,t)dx,\\ &V_S = \int_\Omega\int_{t-\tau_M}^{t} e^{2\alpha(s-t)}e^T(x,s)Se(x,s)dsdx,\\ &V_R = \tau_M\int_\Omega\int_{-\tau_M}^{0}\int_{t+\theta}^{t} e^{2\alpha(s-t)}e_s^T(x,s)Re_s(x,s)dsd\theta dx,\end{aligned} \tag{22}$$

with positive matrices $P_1, S, R\in\mathbb{R}^{M\times M}$ and positive diagonal $M\times M$-matrix $P_3 = \mathrm{diag}\{p_3^1,\ldots,p_3^M\}$. This V extends the construction of Fridman and Blighovsky (2012) to N-D case. Following Liu and

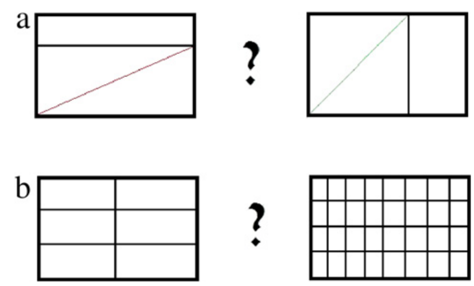

Fig. 2. Spatial design dilemmas for 2-D problem.

Fridman (2014), in order to guarantee that $V(t)$ is defined for all $t\ge t_{N_s-1}$ we set $e(x,t)\equiv e(x,0)$ for $t<0$.

Our objective is to find a constant gain K_o that internally exponentially stabilizes the error system, i.e. exponentially stabilizes the disturbance-free system in the sense that the following holds:

$$V(t) \le e^{-2\alpha(t-t_{N_s-1})}V(t_{N_s-1}), \quad t\ge t_{N_s-1},$$

where $\alpha > 0$ is the decay rate. While internally stabilizing the parabolic error process, the influence of the admissible external disturbances on the controlled output

$$\zeta(x,t) = C_1 e(x,t) \tag{23}$$

with a constant matrix $C_1\in R^{q\times M}$ is to be attenuated.

The following H_∞ filtering problem is thus under study. Given $\gamma > 0$, it is required to find an observer (14) that internally exponentially stabilizes the estimation error dynamics (20) and leads to a negative performance index

$$\begin{aligned}J(T) &= \int_{t_{N_s-1}}^{T}\int_\Omega\Big[|\zeta(x,t)|^2 - \gamma^2[|w(x,t)|^2 + |v(x,t)|^2]\Big]dxdt\\ &< V(t_{N_s-1}) \quad \forall T > t_{N_s-1}\end{aligned} \tag{24}$$

for all admissible disturbances such that

$$\int_\Omega [|w(x,t)|^2 + |v(x,t)|^2]dx > 0, \quad t > t_{N_s-1}. \tag{25}$$

Then it is said that the error dynamics (20) has an L_2-gain less than γ.

In the design process, the geometrical properties of the sensors (leading to a certain choice of convenient sub-domains Ω_i) should be taken into account. We will show that given N_s, the division with the minimum diameters' bound δ enlarges τ_M leading to a larger maximum sampling interval MATI $= (\tau_M - \mathrm{MAD})/N_s$. In the 2-D case as shown in Fig. 2(a), where $N_s = 2$, the choice of the right rectangular with the smallest maximum diagonal leads to a larger MATI. Increasing N_s (the accuracy of the measurements to be transmitted) enlarges τ_M as well as the amount of communications. Our objective is to maximize MATI. By verifying the feasibility of the LMIs of Theorem 1 below, we will optimize the choice of N_s. For example, in the case of Fig. 2(b), for MAD $\to 0$ the left division may be preferable, but for MAD $\to \tau_M$ it may happen that only the right one guarantees the desired performance of the error system.

3. Main results: L_2-gain analysis and design

In order to solve the problem we will derive sufficient conditions for the following dissipative inequality

$$\begin{aligned}W(t) &\triangleq \dot{V}(t) + 2\alpha V(t) + \sum_{i=1}^{N_s}\int_{\Omega_i}\{|\zeta(x,t)|^2 - \gamma^2[|w(x,t)|^2\\ &\quad + |v(x,t)|^2]\}dx < 0, \quad \alpha > 0,\ t\ge t_{N_s-1}\end{aligned} \tag{26}$$

to hold along the trajectories of (20) with the corresponding boundary conditions provided (25) is valid. The integration of (26) in t from t_{N_s-1} to T would yield (24) since $V \geq 0$. For the unperturbed system (20), (26) implies $\dot{V}(t) + 2\alpha V(t) \leq 0$ and, thus, the exponential stability of (20) with the decay rate α.

Theorem 1. *Given positive scalars $N_s, \delta, \gamma, \alpha, \tau_M$ and a matrix $K_o \in \mathbb{R}^{M\times l}$, let there exist a matrix $G \in \mathbb{R}^{M\times M}$, positive $M \times M$-matrices P_1, S and R, diagonal $M \times M$-matrices $P_3 > 0$ and P_2, and scalars $\lambda_Q > 0, \lambda_j > 0$ $(j = 1, 2)$ and $\lambda_0 > 0$ for the Dirichlet or $\lambda_0 = 0$ for the Neumann boundary conditions such that the following LMIs are feasible:*

$$\Xi \triangleq \begin{bmatrix} R & G \\ & R \end{bmatrix} > 0, \qquad \Phi_\gamma < 0, \tag{27}$$

where

$$\Phi_\gamma = \left[\begin{array}{c|cc} & P_2B_1 & P_2K_o \\ \Psi_s & P_3B_1 & P_3K_o \\ & 0 & 0 \\ \hline & -\gamma^2 I_q & 0 \\ & * & -\gamma^2 I_M \end{array}\right] \tag{28}$$

and

$$\Psi_s = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & -P_2\beta & \Phi_{16} & \Phi_{17} & P_2 \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & 0 & 0 & \Phi_{28} \\ * & * & -\frac{\lambda_1\pi^2}{\delta^2}I_M & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\lambda_2 I_M & 0 & 0 & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & \Phi_{67} & 0 \\ * & * & * & * & * & * & \Phi_{77} & 0 \\ * & * & * & * & * & * & * & -\lambda_Q I_M \end{bmatrix}. \tag{29}$$

Here

$$\begin{aligned} \Phi_{11} &= 2\alpha P_1 + P_2(A - K_oC) + (A - K_oC)^T P_2 + S \\ &\quad - Re^{-2\alpha\tau_M} + \lambda_2 I_M - \lambda_0 \sum_{k=1}^{M} \frac{\pi^2}{l_k^2} I_M + \lambda_Q Q + C_1^T C_1, \end{aligned}$$

$$\Phi_{12} = P_1 - P_2 + P_3(A - K_oC),$$
$$\Phi_{13} = \Phi_{14} = P_2K_oC,$$
$$\Phi_{16} = (R - G)e^{-2\alpha\tau_M} - \lambda_2 I_M, \qquad \Phi_{17} = Ge^{-2\alpha\tau_M},$$
$$\Phi_{22} = R\tau_M^2 - 2P_3, \qquad \Phi_{23} = \Phi_{24} = P_3K_oC,$$
$$\Phi_{25} = -P_3\beta, \qquad \Phi_{28} = P_3,$$
$$\Phi_{55} = -2D_0(P_2 - \alpha P_3) \otimes I_N + [\lambda_0 + \lambda_1]I_{MN},$$
$$\Phi_{66} = (G + G^T - 2R)e^{-2\alpha\tau_M} + \lambda_2 I_M,$$
$$\Phi_{67} = (R - G)e^{-2\alpha\tau_M}, \qquad \Phi_{77} = -Re^{-2\alpha\tau_M} - Se^{-2\alpha\tau_M}.$$

Then the error system (20) under the Dirichlet (17) or under the Neumann (18) boundary conditions is internally exponentially stable with the decay rate α and has L_2-gain less than γ. Moreover, if the above conditions are feasible with $\alpha = 0$, then (20) is internally exponentially stable with a small enough decay rate and has L_2-gain less than γ.

See Appendix for the proof.

Remark 1. Note that δ appears only in $\Phi_{33} = -\frac{\lambda_1\pi^2}{\delta^2}I_M$ of Φ_γ, meaning that a smaller δ enlarges τ_M that preserves the H_∞ performance. Therefore, given N_s one has to choose such a division of Ω that minimizes the maximum diameter of the resulting subdomains. This choice enlarges MATI $= \frac{\tau_M - \text{MAD}}{N_s}$.

Remark 2. Compared to Fridman and Bar Am (2013) and Fridman and Blighovsky (2012), an improved technique (based on S-procedure) is presented (see Appendix) leading to less restrictive LMIs under the Dirichlet than under the Neumann boundary conditions. The LMIs are less restrictive due to the negative term $-\lambda_0 \sum_{k=1}^{M} \frac{\pi^2}{l_k^2} I_M$ in Φ_{11} of (28).

If K_o is unknown then the matrix inequalities of Theorem 1 are nonlinear. In order to linearize these inequalities we follow the method of Suplin, Fridman, and Shaked (2007): assume $P_3 = \varepsilon P_2$ and denote $Y = P_2K_o$. We obtain the following result for the H_∞ filter design:

Corollary 2. *Given positive scalars $N_s, \delta, \gamma, \alpha, \tau_M$ and a tuning parameter ε, let there exist matrices $G \in \mathbb{R}^{M\times M}$ and $Y \in \mathbb{R}^{M\times l}$, positive $M\times M$-matrices P_1, S and R, diagonal $M\times M$-matrices $P_3 > 0$ and P_2, and scalars $\lambda_Q > 0, \lambda_j > 0$ $(j = 1, 2)$ and $\lambda_0 > 0$ for the Dirichlet or $\lambda_0 = 0$ for the Neumann boundary conditions such that the LMIs (27) are feasible, where Φ_γ is given by (28) with P_3B_1, P_2K_0 and P_3K_0 replaced by εP_2B_1, Y and εY respectively, and where Ψ_s is given by (29) with*

$$\begin{aligned} \Phi_{11} &= 2\alpha P_1 + P_2A + A^T P_2 - YC - C^T Y^T + S \\ &\quad - Re^{-2\alpha\tau_M} + \lambda_2 I_M - \lambda_0 \sum_{k=1}^{N} \frac{\pi^2}{l_k^2} I_M + \lambda_Q Q + C_1^T C_1, \end{aligned}$$

$$\Phi_{12} = P_1 - P_2 + \varepsilon P_2 A - \varepsilon YC,$$
$$\Phi_{13} = \Phi_{14} = P_2A - YC,$$
$$\Phi_{16} = (R - G)e^{-2\alpha\tau_M} - \lambda_2 I_M, \qquad \Phi_{17} = Ge^{-2\alpha\tau_M},$$
$$\Phi_{22} = R\tau_M^2 - 2P_3, \qquad \Phi_{23} = \Phi_{24} = \varepsilon(P_2A - YC),$$
$$\Phi_{25} = -\varepsilon P_2\beta, \qquad \Phi_{28} = \varepsilon P_2,$$
$$\Phi_{55} = -2D_0(P_2 - \alpha\varepsilon P_2) \otimes I_N + [\lambda_0 + \lambda_1]I_{MN},$$
$$\Phi_{66} = (G + G^T - 2R)e^{-2\alpha\tau_M} + \lambda_2 I_M,$$
$$\Phi_{67} = (R - G)e^{-2\alpha\tau_M}, \qquad \Phi_{77} = -Re^{-2\alpha\tau_M} - Se^{-2\alpha\tau_M}.$$

Then the error system (20) with the observer gain $K_o = P_2^{-1}Y$ under the Dirichlet (17) or under the Neumann (18) boundary conditions is internally exponentially stable with the decay rate α and has L_2-gain less than γ. Moreover, if the above conditions are feasible with $\alpha = 0$, then (20) is internally exponentially stable with a small enough decay rate and has L_2-gain less than γ.

Remark 3. Practical implementation of the above H_∞ filter involves a synchronization method for the sensors and observer nodes that may lead to errors in s_k. For simplicity, in the present paper we ignore the latter errors. See e.g. Seuret and Richard (2008) for the corresponding stability analysis (not including H_∞ performance and scheduling protocols) in the case of finite-dimensional systems.

Remark 4. Different from the distributed estimation considered e.g. in Zhang, Feng, and Yu (2012), where each node has a local estimate of the state by employing his measurement and estimates from his neighbors, we construct a global (centralized) observer that is based on all (but delayed) sensor measurements. Thus we have no problem of disagreements among the various local estimates.

3.1. Discussions: the case of "pointwise" measurements

For large-scale plants (e.g. air polluted areas), a more realistic situation arises when there are some subregions in Ω without measurements. In the H_∞ filtering framework, our results can be extended to this case provided the subregions without measurements are non-disturbed. Consider (3). Denote by Ω_0 the non-disturbed sub-domain of Ω (where $w \equiv 0$), and by Ω_i ($i = 1, \dots, N_s$) the disturbed small rectangular sub-domains with diagonals δ_i. Assume that $\bigcup_{i=0}^{N_s} \Omega_i = \Omega$ and that Ω_i ($i = 0, \dots, N_s$) intersect only on the boundaries. Suppose that N_s sensors provide "pointwise" measurements defined by (9).

Under the same assumptions on s_k, η_k as above and under the Round Robin scheduling protocol, let the measurements be defined by (10). Consider the observer given by (14) and by

$$\hat{z}_t(x,t) = \Delta_D \hat{z}(x,t) - \beta \nabla_x \hat{z}(x,t) + A\hat{z}(x,t) + \phi(\hat{z}, x, t), \quad t \geq 0, \ x \in \Omega_0. \tag{30}$$

Then the error system can be presented as (20) and

$$e_t(x,t) = \Delta_D e(x,t) - \beta \nabla_x e(x,t) + Ae(x,t) + \phi' e(x,t) \quad t \geq 0, \ x \in \Omega_0. \tag{31}$$

Consider $J(T)$ defined by (24). We are looking for conditions that guarantee $J(T) < 0$ for all $T > t_{N_s-1}$ and for all admissible $w \in L_2(0, \infty; L_2(\Omega))$: $w_{|x \in \Omega_0} \equiv 0$ subject to (25).

By modifying the proof of Theorem 1, we arrive at the following conditions that guarantee L_2-gain less than γ for the error system: (27) and $\Phi_{s|K_0=0} < 0$. Note that the latter LMI guarantees the exponential stability of the unperturbed error dynamics with $K_0 = 0$. Therefore, the observer with $K_0 = 0$ (that does not use the measurements) leads to a finite γ of the error system, but $K_0 \neq 0$ may significantly enhance the performance (see Example 1 below).

Remark 5. In the case of non-collocated disturbances and sensors, an extension of the presented results to the case of "pointwise" measurements is not clear. Thus, in 1-D case and the "pointwise" measurements, the Lyapunov analysis may be based on a combination of the Lyapunov–Krasovskii method with Halanay's inequality (Fridman & Blighovsky, 2012). However, Halanay's inequality is not applicable to L_2-gain analysis. The presented results can be extended to networked Luenberger type observer of the unperturbed N-D diffusion PDEs under the "pointwise" measurements. Such an extension may be a topic for the future research.

4. Examples

Example 1. Consider H_∞ filtering of the scalar 2-D PDE (3) with the controlled output (23) and the measurements (10), where $d_0 = 10^{-4}$, $A = \beta = \phi = 0$, $\Omega = [0, 0.1] \times [0, 0.06]$ and $C_1 = B_1 = C = 1$. This PDE with $B_1 = 0$ was considered in Demetriou (2010). We take N_s as in the Table 1 and choose such a division of Ω into N_s equal rectangles that corresponds to the minimum diameter δ (see Table 1). Thus, for $N_s = 2$ the division of the side [0, 0.1] into two subintervals (i.e. $N_s = 2 \times 1$) leads to a smaller $\delta^2 = 0.05^2 + 0.06^2 = 0.0061$ than the division of the side [0, 0.06] into two subintervals ($N_s = 1 \times 2$) with $\delta^2 = 0.1^2 + 0.03^2 = 0.019$.

Consider the Neumann boundary conditions. Using Theorem 1 with $\lambda_0 = 0$, $K_0 = 0.2$, $\alpha = 0$ and $\gamma = 10.1$ we find the maximum values of τ_M that guarantee $J(T) < 0$ (see Table 1). For MAD = 0.4, the strategy of dividing the whole region Ω into two parts results in the maximum sampling interval bound MATI. For large MAD = 3.5, only the division into $N_s = 6$ subdomains preserves the error performance provided the sampling is fast enough with MATI = 0.04.

Note that under the Dirichlet boundary conditions, where the LMIs of Theorem 1 are used with $\lambda_0 \neq 0$, the resulting values of γ (for the same values of τ_M and δ) are essentially smaller: e.g. for $N_s = 2 \times 1$ and $\tau_M = 2.5$ the resulting $\gamma = 5.3$ (instead of 10.1 in Table 1). Simulation of $J(T)$ under the Dirichlet boundary conditions corresponding to the error system with the zero initial conditions and $N_s = 2 \times 1$, $\gamma = 5.3$, $c \equiv 1$, $h_k \equiv h = 0.4$, $s_{k+1} - s_k = 1.05$, where $v_{i,k} = 50\cos(t_k - h)e^{-0.01(t_k - h)}$, $w(x,t) = e^{-0.01t}$, and $V(t_k) = 0$, $k < N_s$, confirms the theoretical result (see Fig. 3). Moreover, simulations show that for the above choice of N_s, h, MATI and the disturbances, $J(T)$ becomes non-negative for some $T > 0$ if $\gamma = 5.1$ (to be compared with $\gamma = 5.3$ in Table 1), i.e. the results of Theorem 1 are not conservative.

Table 1
Example 1: $\gamma = 10.1$ and the Neumann boundary conditions.

N_s	δ	τ_M	MATI (MAD = 0.4)
$1 = 1 \times 1$	0.1166	–	–
$2 = 2 \times 1$	0.0781	2.5	1.05
$4 = 2 \times 2$	0.0583	3.37	0.7425
$6 = 3 \times 2$	0.0448	3.75	0.5583

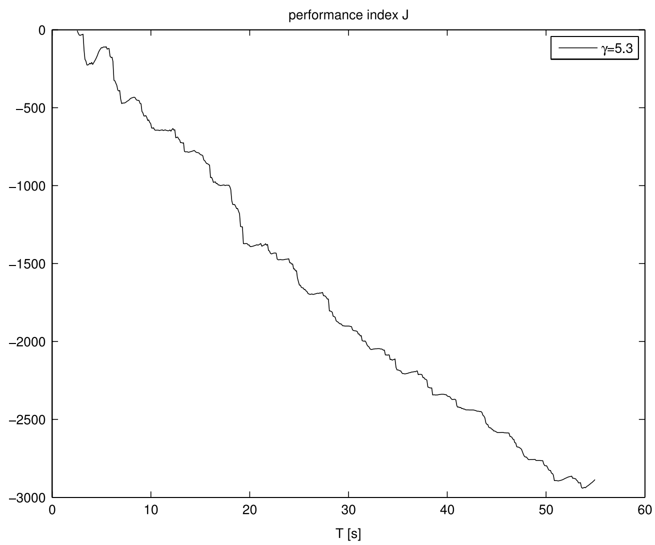

Fig. 3. Ex. 1: $N_s = 2$, $h_k \equiv 0.4$, $s_{k+1} - s_k \equiv 1.05$, $\gamma = 5.3$.

Consider next the case of "pointwise" collocated disturbances and sensors under the Dirichlet boundary conditions. Note that for $K_o = 0$ the conditions of Theorem 1 guarantee that $J(T) < 0$ for $\gamma = 2.7$. For $K_0 = 0.5$ and $\delta = 0.001$ Theorem 1 guarantees a smaller $\gamma = 1.5$ for $\tau_M = 0.6$. Thus, in the case of $N_s = 5$ "pointwise" disturbances and sensors inside of rectangles with the diagonals not greater than 0.001, MAD = 0.1, MATI = 0.1 and under Round Robin scheduling protocol, $K_0 = 0.5$ leads to error dynamics with a reduced L_2-gain $\gamma = 1.5$.

Example 2. Consider the chemical reactor model from Smagina and Sheintuch (2006) governed by (3) under the Neumann boundary conditions with the controlled output (23) and the measurements (10), where $M = 2$, $N = 1$, $\Omega = [0, 10]$, $D_0 = \mathrm{diag}\{0.01, 0.005\}$, $\beta = \mathrm{diag}\{0.011, 1.1\}$,

$$A = \begin{bmatrix} 0 & 0.01 \\ -0.45 & -0.2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$C_1 = [1\ 0], \qquad C = I_2.$$

The nonlinearity $\phi = \mathrm{col}\{\phi_1(z^1), 0\}$ is assumed to satisfy $\frac{d\phi_1}{dz^1} \in [0, 0.01]$. This model accounts for an activator temperature z^1, that undergoes reaction (expressed as $0.01z^2 + \phi$), advection and diffusion, and for a fast inhibitor concentration z^2, which may be advected by the flow. The elements of β are convective velocities.

By dividing the interval $\Omega = [0, 10]$ into $N_s = 20$ equal subintervals of the length δ and applying Corollary 2 with $\tau_M = \epsilon = 0.25$ and $\gamma = 10$ we arrive at $K_o = \begin{bmatrix} 0.5777 & -0.0542 \\ -0.4080 & 0.1216 \end{bmatrix}$. Applying

Table 2
Example 2: LMI results for $\gamma = 10$.

N_s	$\delta = \frac{10}{N_s}$	τ_M	MATI (MAD = 0.1)
20	0.5	0.68	0.029
30	0.333	1.26	0.0387
50	0.2	1.47	0.0274

further Theorem 1 to the resulting error dynamics with the latter K_o and $\gamma = 10$ we find the maximum values of τ_M that guarantee $J(T) < 0$ (see Table 2). As seen from Table 2, the strategy of dividing the whole interval [0, 10] into $N_s = 30$ equal parts results in the maximum sampling interval MATI $= 0.0387$. Moreover, for $N_s = 20$ the resulting $\tau_M = 0.68$ is essentially larger than $\tau_M = 0.25$ achieved by Corollary 2. This mirrors the conservatism of Corollary 2. Also in this example simulations confirm the theoretical results.

5. Conclusion

H_∞ filter has been designed for convection–diffusion PDEs over rectangular N-D domain Ω in the situation, where distributed in space measurements are sent to observer through communication network. The objective is to enlarge the sampling time intervals, while preserving a satisfactory error system performance in spite of variable sampling, network-induced delays and Round Robin scheduling of communication protocols. We have suggested to divide Ω into a finite number N_s of rectangular sub-domains, where stationary or mobile sensing devices provide spatially averaged measurements. Given N_s, we have found that the division that minimizes the maximum diameter of the resulting sub-domains is advantageous leading to larger sampling intervals.

Appendix. Proof of Theorem 1

We will derive LMI conditions for $W(t) < 0$ $(t \ge t_{N_s-1})$ via the descriptor method (Fridman, 2001), where

$$\begin{aligned}0 \equiv\ & 2\int_\Omega [e^T(x,t)P_2 + e_t^T(x,t)P_3][-e_t(x,t)\\ &+ \Delta_D e(x,t)Ae(x,t) - \beta\nabla_x e(x,t) + (\phi' - K_oC)e(x,t)\\ &+ B_1 w(x,t)]dx + 2\sum_{i=1}^{N_s}\int_{\Omega_i}[e^T(x,t)P_2\\ &+ e_t^T(x,t)P_3]K_0C[f_i(x,t) + \rho_i - v(x,t)]dx \end{aligned} \tag{A.1}$$

with some free diagonal matrix $P_2 = \mathrm{diag}\{p_2^1, \dots, p_2^M\}$ is added to $\dot V + 2\alpha V$. Taking into account the boundary conditions, by Green's formula we obtain

$$\begin{aligned}&2\int_\Omega e_t^T(x,t)P_3\Delta_D e(x,t)dx = 2\sum_{m=1}^M p_3^m\int_\Omega e_t^m(x,t)\Delta_D^m e^m(x,t)dx\\ &\quad = -2\sum_{m=1}^M p_3^m \int_\Omega e_{xt}^m(x,t)D^m(x)\nabla_x e^m(x,t)dx,\\ &2\int_\Omega e^T(x,t)P_2\Delta_D e(x,t)dx\\ &\quad = -2\sum_{m=1}^M p_2^m\int_\Omega e_x^m(x,t)D^m(x)\nabla_x e^m(x,t)dx.\end{aligned} \tag{A.2}$$

We have

$$\begin{aligned}\dot V_P(t) + 2\alpha V_P(t) =\ & 2\int_\Omega e^T(x,t)P_1 e_t(x,t)dx\\ &+ 2\sum_{m=1}^M\int_\Omega p_3^m e_{xt}^m(x,t)D^m(x)\nabla_x e^m(x,t)dx\\ &+ 2\alpha\int_\Omega\Bigg[e^T(x,t)P_1e^T(x,t)dx + \sum_{m=1}^M\int_\Omega p_3^m e_x^m\\ &\times (x,t)D^m(x)\nabla_x e^m(x,t)dx\Bigg]dx.\end{aligned} \tag{A.3}$$

Then adding (A.1)–(A.3) and taking into account (A.2) we arrive at

$$\begin{aligned}&\dot V_P(t) + 2\alpha V_P(t)\\ &= 2\int_\Omega e^T(x,t)P_1e_t(x,t)dx + 2\int_\Omega\Bigg[\alpha e^T(x,t)P_1e(x,t)\\ &\quad - \sum_{m=1}^M (p_2^m - \alpha p_3^m)e_x^m(x,t)D^m(x)\nabla_x e^m(x,t)\Bigg]dx\\ &\quad \times 2\sum_{i=1}^{N_s}\int_{\Omega_i}[e^T(x,t)P_2 + e_t^T(x,t)P_3]\\ &\quad \times\Big[-e_t(x,t) - \beta\nabla_x e(x,t) + [A + \phi' - K_oC]e(x,t)\\ &\quad + B_1w(x,t) + K_0C[f_i(x,t) + \rho_i - v(x,t)]\Big]dx.\end{aligned} \tag{A.4}$$

The feasibility of $\Phi_\gamma < 0$ implies that $\Phi_{55} > 0$ and, thus, $p_2^m - \alpha p_3^m > 0$. Hence, due to (4),

$$\begin{aligned}&-2\sum_{m=1}^M[(p_2^m - \alpha p_3^m)\int_\Omega e_x^m(x,t)D^m(x)\nabla_x e^m(x,t)]dx\\ &\le -2\int_\Omega \nabla_x^T e(x,t)[D_0(P_2 - \alpha P_3)\otimes I_N]\nabla_x e(x,t)dx.\end{aligned}$$

Further we find

$$\begin{aligned}\dot V_S(t) + 2\alpha V_S\ =\ & \int_\Omega [e^T(x,t)Se(x,t)\\ &- e^{-2\alpha\tau_M}e^T(x,t-\tau_M)Se(x,t-\tau_M)]dx,\\ \dot V_R(t) + 2\alpha V_R \le\ & \tau_M^2\int_\Omega e_t^T(x,t)Re_t(x,t)dx\\ &- \tau_M e^{-2\alpha\tau_M}\sum_{i=1}^N\int_{\Omega_i}\int_{t-\tau_M}^t e_s^T(x,s)Re_s(x,s)dsdx.\end{aligned}$$

Denote $\alpha_1 = \frac{\tau_i(t)}{\tau_M}$, $\alpha_2 = \frac{\tau_M - \tau_i(t)}{\tau_M}$, $\xi^T(x,t) = [e^T(x,t) - e^T(x,t-\tau_i(t))\ e^T(x,t-\tau_i(t)) - e^T(x,t-\tau_M)]$. Applying Jensen's inequality and convex analysis of Park, Ko, and Jeong (2011), we obtain

$$\begin{aligned}&-\tau_M\left\{\int_{t-\tau_i(t)}^t e_s^T(x,s)Re_s(x,s)ds + \int_{t-\tau_M}^{t-\tau_i(t)} e_s^T(x,s)Re_s(x,s)ds\right\}\\ &\le -\xi^T(x,t)\mathrm{diag}\left\{\frac{1}{\alpha_1}R, \frac{1}{\alpha_2}R\right\}\xi(x,t) \le -\xi^T(x,t)\Xi\xi(x,t),\end{aligned}$$

where $\Xi > 0$ due to (27). Then

$$\begin{aligned}\dot V_R(t) + 2\alpha V_R\ \le\ & \tau_M^2\int_\Omega e_t^T(x,t)Re_t(x,t)dx - e^{-2\alpha\tau_M}\\ &\times\int_\Omega \eta_0^T\begin{bmatrix}R & G-R & -G\\ * & 2R-G-G^T & G-R\\ * & * & R\end{bmatrix}\eta_0 dx,\end{aligned}$$

$$\eta_0^T = \begin{bmatrix}e^T(x,t) & e^T(x,t-\tau_i(t)) & e^T(x,t-\tau_M)\end{bmatrix}.$$

By Jensen's inequality for $i = 1, 2, .., N_s$

$$\int_{\Omega_i} |e(x,t) - e(x, t-\tau_i(t))|^2 dx \geq \frac{1}{\Delta_i} \left\{ \left| \int_{\Omega_i} [e(x,t) - e(x,t-\tau_i(t))] dx \right| \right\}^2 = \Delta_i |\rho_i|^2.$$

Therefore,

$$\int_{\Omega_i} [|e(x,t) - e(x,t-\tau_i(t))|^2 - |\rho_i|^2] dx \geq 0, \tag{A.5}$$

where $i = 1, 2, \ldots, N_s$. Summation in (21) leads to

$$\sum_{i=1}^{N_s} \int_{\Omega_i} \left[|\nabla_x e(x,t)|^2 - \frac{\pi^2}{\delta_i^2} |f_i(x,t)|^2 \right] dx \geq 0. \tag{A.6}$$

Similarly, under the Dirichlet boundary conditions, the Wirtinger inequality (2) implies

$$\sum_{i=1}^{N_s} \int_{\Omega_i} \left[|\nabla_x e(x,t)|^2 - \mathcal{W}^2 |e(x,t)|^2 \right] dx \geq 0. \tag{A.7}$$

Taking into account (A.5)–(A.7) and (16) and applying the S-procedure, we add to $\dot{V} + 2\alpha V$ the left-hand sides of

$$\begin{aligned} &\lambda_0 \left\{ \sum_{i=1}^{N_s} \int_{\Omega_i} \left[|\nabla_x e(x,t)|^2 - \mathcal{W}^2 |e(x,t)|^2 \right] dx \right\} \geq 0, \\ &\lambda_1 \left\{ \sum_{i=1}^{N_s} \int_{\Omega_i} \left[|\nabla_x e(x,t)|^2 - \frac{\pi^2}{\delta_i^2} |f_i(x,t)|^2 \right] dx \right\} \geq 0, \\ &\lambda_2 \left\{ \sum_{i=1}^{N_s} \int_{\Omega_i} \left[|e(x,t) - e(x,t-\tau_i(t))|^2 - |\rho_i|^2 \right] dx \right\} \geq 0, \\ &\lambda_Q \left\{ \sum_{i=1}^{N_s} \int_{\Omega_i} \left[e^T(x,t) Q e(x,t) - |\phi' e(x,t)|^2 \right] dx \right\} \geq 0, \end{aligned} \tag{A.8}$$

where $\lambda_j \geq 0 (j = 0, 1, 2)$ and $\lambda_Q > 0$ are some constants. Finally, from (A.4)–(A.8) it follows that (27) yields (26) provided (25) holds, where

$$W(t) \leq \sum_{i=1}^{N_s} \int_{\Omega_i} \eta_i^T \Phi_\gamma \eta_i dx < 0, \quad t \geq t_{N_s - 1},$$

$$\begin{aligned} \eta_i = \ &\text{col}\{e(x,t), e_t(x,t), f_i(x,t), \rho_i, \nabla_x e(x,t), \\ &e(x, t-\tau_i(t)), e(x, t-\tau_M), \phi' e(x,t), w(x,t), v(x,t)\}. \quad \square \end{aligned}$$

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