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Input/output delay approach to robust sampled-data H_∞ control☆

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Abstract

Sampled-data output-feedback H_∞ control of linear systems is considered. The only restriction on the sampling and hold is that the distances between the sequel sampling times and holding times are not greater than given bounds. A new approach, which was recently introduced to sampled-data state-feedback stabilization, is developed to the H_∞ control. The system is modelled as a continuous-time one, where the control input and the measurement output have piecewise-continuous delays. Sufficient linear matrix inequalities (LMIs) conditions for H_∞ control of such systems are derived via Lyapunov–Krasovskii functionals and descriptor approach to time-delay systems. For the first time the new approach allows to develop different robust control methods for the case of sampled-data H_∞ control.

Keywords: Sampled-data control; H_∞ control; LMI; Delay model; Lyapunov–Krasovskii functionals

1. Introduction

Sampled-data H_∞ control of systems has been studied in a number of papers (see e.g. [2–4,11,15,16], and the references therein). Two main approaches have been used. The first one is based on the lifting technique [2,18] in which the problem is transformed to equivalent finite-dimensional discrete H_∞ control problem. The second, more direct, approach is based on the representation of the system in the form of hybrid discrete/continuous model and the solution is obtained in terms of differential Riccati equations with jumps. These approaches give necessary and sufficient conditions and lead to equivalent solutions.

To the best of our knowledge, the only LMI solution to sampled-data output-feedback H_∞ control was derived by Lall and Dullerod [11] for the lifted discrete system when the sampling and the hold operators are periodic and their rates are commeasurable. This solution is computationally complicated because it includes the evaluation of the matrices of the lifted system.

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The hybrid system approach has been applied recently to robust sampled-data H_2 control for the case of equidistant sampling [9]. To overcome difficulties of solving differential inequalities with jumps, a piecewise linear in time Lyapunov function has been suggested. As a result in [9] LMIs have been derived, which do not depend on the sampling interval, and thus are very conservative. The sampling interval independent LMI conditions have been also derived recently for the case of robust H_∞ filtering under sampled-data measurements [17].

Modelling of continuous-time systems with digital control as continuous systems with delayed control input was introduced by Mikheev et al., Astrom and Wittenmark [12,1]. The digital control law may be represented as delayed control as follows:

$$\begin{aligned} u(t) &= u_{\mathrm{d}}(t_k) = u_{\mathrm{d}}(t-(t-t_k)) \\ &= u_{\mathrm{d}}(t-\tau(t)), \quad t_k \leqslant t < t_{k+1}, \quad \tau(t) = t - t_k, \end{aligned} \tag{1}$$

where u_{d} is a discrete-time control signal and the time-varying delay $\tau(t) = t - t_k$ is piecewise linear with derivative $\dot{\tau}(t) = 1$ for $t \neq t_k$. Moreover, $\tau \leqslant t_{k+1} - t_k$. Recently, this input delay approach was applied to robust sampled-data stabilization via Lyapunov–Krasovskii technique in [6]. It is the purpose of the present paper to develop this approach to the case of sampled-data H_∞ control.

Bounded real lemmas (BRLs) for systems with time-varying delays were derived for the cases where the derivative of the delay is less than one via Lyapunov–Krasovskii functionals (see e.g. [15]). For the case of time-varying delay without any restrictions on the delay of the derivative a BRL was obtained in [7]. This became possible due to a new descriptor model representation of the delay system introduced in [5].

In the present paper we consider the output-feedback sampled data H_∞ control problem by finding solution for the continuous-time H_∞ control problem for systems with uncertain but bounded (by the maximum sampling and holding intervals) delays in the control input and in the measurement output. We apply the BRL of [7] and derive solution in terms of LMIs. The solution which we obtain is robust with respect to different sampling and holding with the only requirement that the maximum sampling interval and maximum holding interval are not greater than given bounds. The LMI conditions are sufficient only, but they are comparatively simple. For the first time the new approach allows to develop different robust control methods for the case of sampled-data H_∞ control. We give a solution to H_∞ control of systems with norm-bounded uncertainties.

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathscr{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$, $\mathscr{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathscr{R}^{n\times n}$ means that P is symmetric and positive definite. Let $C_n[a,b]$ denotes the space of continuous functions ϕ: $[a,b] \to \mathscr{R}^n$ with the supremum norm $|\cdot|_c$ and $L_2[0,\infty)$ be the space of the square integrable functions with the norm $\|\cdot\|_{L_2}$. We also denote $x_t(\theta) = x(t+\theta)$ $(\theta \in [-h, 0])$.

2. Main results

2.1. Problem formulation

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= C_1 x(t) + D_{12} u(t), \end{aligned} \tag{2}$$

where $x(t) \in \mathscr{R}^n$ is the state vector, $w(t) \in \mathscr{R}^{n_w}$ is the disturbance, $u(t) \in \mathscr{R}^{n_u}$ is the control input and $z(t) \in \mathscr{R}^{n_z}$ is the controlled output.

The control signal is assumed to be generated by a zero-order hold function with a sequence of hold times $0 < t_1 < \cdots < t_k < \cdots$

$$u(t) = u_{\mathrm{d}}(t_k), \quad t_k \leqslant t < t_{k+1}, \tag{3}$$

where $\lim_{k\to\infty} t_k = \infty$, and u_{d} is a discrete-time control signal.

The measurement output $y_k \in \mathscr{R}^{n_y}$ is assumed to be available at discrete sampling instants $0 < \sigma_1 < \cdots < \sigma_k < \cdots$, $\lim_{k\to\infty} \sigma_k = \infty$, and it may be corrupted by $w_k = w(\sigma_k)$:

$$y_k = C_2 x(\sigma_k) + D_{21} w_k, \quad k = 0, 1, 2, \ldots . \tag{4}$$

We consider an output-feedback sampled-data H_∞ control. Assume that

A1. $C_1^{\mathrm{T}} D_{12} = 0$.
A2. $t_{k+1} - t_k \leqslant h \ \forall k \geqslant 0$.
A3. $\sigma_{k+1} - \sigma_k \leqslant g \ \forall k \geqslant 0$.

We define the following performance index for a prescribed scalar $\gamma > 0$:

$$J_{\mathrm{c}}(w) = \int_0^\infty (z^{\mathrm{T}}(s) z(s) - \gamma^2 w^{\mathrm{T}}(s) w(s))\, \mathrm{d}s. \tag{5}$$

Our objective is to find a dynamic output-feedback control law of the form

$$\begin{aligned} u(t) &= C_{\mathrm{c}} x_{\mathrm{c}}(t_k) + D_{\mathrm{c}} y_k, \ \ t_k \leqslant t < t_{k+1}, \\ \dot{x}_{\mathrm{c}}(t) &= A_{\mathrm{c}} x_{\mathrm{c}}(t) + B_{\mathrm{c}} y_k, \end{aligned} \tag{6}$$

which for all sampling and hold times satisfying A2 and A3 internally stabilizes the system and leads to $J_{\mathrm{c}} < 0$ for $x(0) = 0$, and for all non-zero $w \in L_2[0, \infty)$.

2.2. The input and the output delay model

We consider the following piecewise-constant measurement:

$$\begin{aligned} y(t - \eta(t)) &= C_2 x(t - \eta(t)) + D_{21} w(t - \eta(t)), \\ \eta(t) &= t - \sigma_k, \ \ t \leqslant \sigma_k < \sigma_{k+1}. \end{aligned} \tag{7}$$

From A3 it follows that $0 \leqslant \eta(t) \leqslant g$.

The output-feedback controller law is described by

$$\begin{aligned} \dot{x}_{\mathrm{c}}(t) &= A_{\mathrm{c}} x_{\mathrm{c}}(t) + B_{\mathrm{c}} y(t - \eta(t)), \\ x_{\mathrm{c}}(t) &= 0, \quad t \in [-h, 0], \\ \bar{u}(t) &= C_{\mathrm{c}} x_{\mathrm{c}}(t) + D_{\mathrm{c}} y(t - \eta(t)), \end{aligned} \tag{8a–c}$$

where the sampled version $u(t) = \bar{u}(t - \tau(t))$ is applied to (2). We represent z in the form

$$z(t) = [\, C_1 \quad D_{12} C_{\mathrm{c}} \,] \begin{bmatrix} x(t) \\ x_{\mathrm{c}}(t) \end{bmatrix}. \tag{9}$$

In order to restore the transference property of the sample and hold component, namely to recover the filtering property of the sample and hold which filters out the high-frequency part of the sampled signal, and in order to conveniently describe the sampling of $y(t)$ and $u(t)$, we introduce prescribed LTI components that are connected in series to $y(t)$ of (7) and $\bar{u}(t)$ of (8c) and produce the sampled version of $y(t)$ and $u(t)$. These components will be later added to the controller of (8). We thus consider the following two components:

$$\dot{\xi}(t) = -\rho I_{n_y}\xi(t) + \rho y(t),$$
$$\dot{\xi}_1(t) = -\rho I_{n_u}\xi_1(t) + C_{\mathrm{c}}x_{\mathrm{c}}(t) + D_{\mathrm{c}}\xi(t), \tag{10a,b}$$

where $\xi \in \mathscr{R}^{n_y}$, $\xi_1 \in \mathscr{R}^{n_u}$, and $\rho < 1$ is a positive scalar.

Denoting $\zeta(t) = \mathrm{col}\{x(t),\ \xi(t),\ \xi_1,\ x_{\mathrm{c}}(t)\}$ the following closed-loop system is obtained:

$$\dot{\zeta}(t) = \mathscr{A}_0\zeta(t) + \mathscr{A}_1\zeta(t-\eta(t)) + \mathscr{A}_2\zeta(t-\tau(t)) + \mathscr{B}w(t), \tag{11a}$$

where

$$\mathscr{A}_0 \triangleq \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & -\rho I_{n_y} & 0 & 0 \\ 0 & D_{\mathrm{c}} & -\rho I_{n_u} & C_{\mathrm{c}} \\ 0 & B_{\mathrm{c}} & 0 & A_{\mathrm{c}} \end{bmatrix},$$
$$\mathscr{A}_1 \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ \rho C_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\mathscr{A}_2 \triangleq \begin{bmatrix} 0 & 0 & \rho B_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\mathscr{B} \triangleq \begin{bmatrix} B_1 & 0 \\ 0 & \rho D_{21} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{11b–e}$$

The corresponding $z(t)$ is given by

$$z(t) = \mathscr{C}\zeta(t), \quad \text{where} \quad \mathscr{C} = [\,C_1 \quad 0 \quad D_{12} \quad 0\,]. \tag{12}$$

2.3. BRL for linear systems with time-varying delays

Consider an auxiliary system

$$\dot{x}(t) = A_0x(t) + A_1x(t-\eta(t)) + A_2x(t-\tau(t)) + B_1w(t),$$
$$z(t) = Cx(t), \tag{13a,b}$$

with the performance index (5), where $x(t) \in \mathscr{R}^n$, $w(t) \in R^{n_w}$, $z(t) \in \mathscr{R}^{n_z}$, $\tau(t)$ and $\eta(t)$ are piecewise-continuous delays satisfying $\tau(t) \leqslant h$, $\eta(t) \leqslant g$, and A_i, $i = 0, 1, 2$, B_1 and C are constant matrices.

Applying to (13) and (5) the Lyapunov–Krasovskii functional of the form

$$V(\bar{x}(t), \dot{x}_t) = V_1(\bar{x}(t)) + V_2(\dot{x}_t), \tag{14}$$

where

$$\bar{x}(t) = col\{x(t),\ \dot{x}(t)\}, \quad E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix},$$
$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 = P_1^{\mathrm{T}} > 0 \tag{15a–d}$$

and

$$V_1(\bar{x}(t)) = \bar{x}^{\mathrm{T}}(t) E P \bar{x}(t),$$
$$V_2(\dot{x}_t) = \int_{-g}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) R_1 \dot{x}(s)\,\mathrm{d}s + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) R_2 \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta, \tag{15e,f}$$

and finding the conditions that $(\mathrm{d}/\mathrm{d}t)V(\bar{x}(t), \dot{x}_t) + z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) < 0$, we obtain similarly to [7,8] the following BRL:

Lemma 1. *Consider* (13). *For a prescribed* $\gamma > 0$, *the cost function* (5) *achieves* $J_{\mathrm{c}}(w) < 0$ *for all non-zero* $w \in \mathscr{L}^q[0,\ \infty)$ *and for all piecewise-continuous delays* $\tau(\cdot), \eta(\cdot)$, *satisfying inequalities* $\tau(t) \leqslant h, \eta(t) \leqslant g$, *if there exist* $n \times n$*-matrices* $P_1 > 0, P_2, P_3$ *and* $R_i = R_i^{\mathrm{T}}$ *that satisfy the following LMIs*:

$$\begin{bmatrix} \Psi_1 & P^{\mathrm{T}}\begin{bmatrix} 0 \\ B_1 \end{bmatrix} & \begin{bmatrix} C^{\mathrm{T}} \\ 0 \end{bmatrix} \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0,$$

and

$$\begin{bmatrix} R_i & [0\ A_i^{\mathrm{T}}]P \\ * & Z_i \end{bmatrix} \geqslant 0, \ \ i = 1, 2, \tag{16a,b}$$

where P is given by (15c) *and*

$$\Psi_1 = \Psi_0 + gZ_1 + hZ_2 + \begin{bmatrix} 0 & 0 \\ 0 & gR_1 + hR_2 \end{bmatrix},$$
$$\Psi_0 = P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i & -I \end{bmatrix} + \begin{bmatrix} 0 & \sum_{i=0}^{2} A_i^{\mathrm{T}} \\ I & -I \end{bmatrix} P.$$

2.4. Output-feedback H_∞ control

Consider the closed-loop system (11), (12) and the performance index J_{c}.

We apply Lemma 1 to system (11), (12) directly, where we replace R_i by diag$\{R_i,\ 0\}$ with $R_i \in \mathscr{R}^{n_{\mathrm{c}} \times n_{\mathrm{c}}}$, we denote $n_{\mathrm{c}} = n + n_y + n_u$. Pre- and post-multiplying the resulting LMI that is equivalent to (16a) by diag$\{Q^{\mathrm{T}},\ I,\ I\}$ and diag$\{Q,\ I,\ I\}$, respectively, and the one equivalent to (16b) by diag$\{I,\ Q^{\mathrm{T}}\}$ and diag$\{I, Q\}$, respectively, where $P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}$. The following is obtained for $\hat{Z}_i = Q^{\mathrm{T}} Z_i Q$.

Lemma 2. *Consider* (11), (12). *For a prescribed* $\gamma > 0$, *the cost function* (5) *achieves* $J_{\mathrm{c}}(w) < 0$ *for all non-zero* $w \in \mathscr{L}^q[0, \infty)$ *and for all piecewise-continuous delays* $\tau(\cdot), \eta(\cdot)$, *satisfying inequalities* $\tau(t) \leqslant h, \eta(t) \leqslant g$, *if there exist* $2n_{\mathrm{c}} \times 2n_{\mathrm{c}}$*matrices* $Q_1 > 0, Q_2, Q_3$ *and* $n_{\mathrm{c}} \times n_{\mathrm{c}}$ *matrices* $R_i = R_i^{\mathrm{T}}$ *that satisfy the following* *inequalities*:

$$\begin{bmatrix} \Phi & \begin{bmatrix} 0 \\ \mathscr{B} \end{bmatrix} & \begin{bmatrix} Q_1\mathscr{C}^{\mathrm{T}} \\ 0 \end{bmatrix} & g\begin{bmatrix} Q_2^{\mathrm{T}} \\ Q_3^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_{n_c} \\ 0 \end{bmatrix} & h\begin{bmatrix} Q_2^{\mathrm{T}} \\ Q_3^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_{n_c} \\ 0 \end{bmatrix} \\ * & -\gamma^2 I_{n_w} & 0 & 0 & 0 \\ * & * & -I_{n_z} & 0 & 0 \\ * & * & * & -gR_1^{-1} & 0 \\ * & * & * & * & -hR_2^{-1} \end{bmatrix} < 0,$$

and

$$\begin{bmatrix} R_i & [I_{n_c} \quad 0][0 \ \mathscr{A}_i^{\mathrm{T}}] \\ * & \hat{Z}_i \end{bmatrix} \geqslant 0, \ \ i = 1, 2, \tag{17a,b}$$

where

$$\Phi = \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} \mathscr{A}_i & -I \end{bmatrix} Q + Q^{\mathrm{T}} \begin{bmatrix} 0 & \sum_{i=0}^{2} \mathscr{A}_i^{\mathrm{T}} \\ I & -I \end{bmatrix} + h\hat{Z}_2 + g\hat{Z}_1.$$

We further denote

$$Q_1 = \begin{bmatrix} X & M^{\mathrm{T}} \\ M & U \end{bmatrix}, \ \ Q_1^{-1} = \begin{bmatrix} Y & N^{\mathrm{T}} \\ N & V \end{bmatrix},$$

$$J = \begin{bmatrix} I & Y \\ 0 & N \end{bmatrix}, \ \text{and} \ \bar{J} = \mathrm{diag}\{J, \ J\} \tag{18}$$

Multiplying (17a) by $\mathrm{diag}\{\bar{J}^{\mathrm{T}}, I\}$ from the left and by $\mathrm{diag}\{\bar{J}, I\}$ from the right, respectively, and (17b) by $\mathrm{diag}\{I, \ \bar{J}^{\mathrm{T}}\}$ from the left and by $\mathrm{diag}\{I, \ \bar{J}\}$ from the right, respectively, we obtain

$$\begin{bmatrix} \begin{bmatrix} \bar{Q}_2 + \bar{Q}_2^{\mathrm{T}} & \bar{Q}_3 - \bar{Q}_2^{\mathrm{T}} + J^{\mathrm{T}} Q_1 (\sum_{i=0}^{2} \mathscr{A}_i^{\mathrm{T}}) J \\ * & -\bar{Q}_3 - \bar{Q}_3^{\mathrm{T}} \end{bmatrix} + h\bar{Z}_2 + g\bar{Z}_1 & \begin{bmatrix} 0 \\ J^{\mathrm{T}}\mathscr{B} \end{bmatrix} & \begin{bmatrix} J^{\mathrm{T}} Q_1 \mathscr{C}^{\mathrm{T}} \\ 0 \end{bmatrix} & g\begin{bmatrix} \bar{Q}_2^{\mathrm{T}} \\ \bar{Q}_3^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_{n_c} \\ 0 \end{bmatrix} & h\begin{bmatrix} \bar{Q}_2^{\mathrm{T}} \\ \bar{Q}_3^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_{n_c} \\ 0 \end{bmatrix} \\ * & -\gamma^2 I_{n_w} & 0 & 0 & 0 \\ * & * & -I_{n_z} & 0 & 0 \\ * & * & * & -gR_1^{-1} & 0 \\ * & * & * & * & -hR_2^{-1} \end{bmatrix} < 0,$$

and

$$\begin{bmatrix} R_i & [I_{n_c} \quad 0][0 \ \mathscr{A}_i^{\mathrm{T}}]\bar{J} \\ * & \bar{Z}_i \end{bmatrix} \geqslant 0, \ \ i = 1, 2, \tag{19a,b}$$

where $\bar{Q}_{i+1} = J^{\mathrm{T}} Q_{i+1} J$ and $\bar{Z}_i = J^{\mathrm{T}} \hat{Z}_i J$, $i = 1, 2$ and where it is required that

$$J^{\mathrm{T}} Q_1 J = \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{20}$$

It is readily found that

$$\sum_{i=0}^{2} \mathscr{A}_i = \mathrm{diag}\{\bar{A}, \ 0\} + \bar{B}_2 \Theta \bar{C}_2, \tag{21a}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 & \rho B_2 \\ \rho C_2 & -\rho I_{n_y} & 0 \\ 0 & 0 & -\rho I_{n_u} \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 & \hat{B}_2 \\ I_{n_c} & 0 \end{bmatrix},$$
$$\bar{C}_2 = \begin{bmatrix} 0 & I_{n_c} \\ \hat{C}_2 & 0 \end{bmatrix}, \text{ and } \Theta = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \tag{21b–d}$$

with

$$\hat{B}_2 = \begin{bmatrix} 0 \\ 0 \\ I_{n_u} \end{bmatrix} \text{ and } \hat{C}_2 = [0 \quad I_{n_y} \quad 0].$$

Hence

$$J^{\mathrm{T}} Q_1 \left(\sum_{i=0}^{2} \mathscr{A}_i^{\mathrm{T}} \right) J$$
$$= \begin{bmatrix} X \\ I_{n_c} \end{bmatrix} \bar{A}^{\mathrm{T}} [\, I_{n_c} \quad Y\,] + \begin{bmatrix} M^{\mathrm{T}} & X\hat{C}_2^{\mathrm{T}} \\ 0 & \hat{C}_2^{\mathrm{T}} \end{bmatrix} \Theta^{\mathrm{T}} \begin{bmatrix} 0 & N \\ B_2^{\mathrm{T}} & \hat{B}_2 Y \end{bmatrix}$$
$$= \begin{bmatrix} X\bar{A}^{\mathrm{T}} & 0 \\ \bar{A}^{\mathrm{T}} & \bar{A}^{\mathrm{T}} Y \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \hat{C}_2^{\mathrm{T}} \end{bmatrix} K^{\mathrm{T}} \begin{bmatrix} \hat{B}_2^{\mathrm{T}} & 0 \\ 0 & I \end{bmatrix} \triangleq \Xi, \tag{22}$$

where

$$K = \begin{bmatrix} 0 & I_{n_u} \\ N^{\mathrm{T}} & Y\hat{B}_2 \end{bmatrix} \Theta \begin{bmatrix} M & 0 \\ \hat{C}_2 X & I_{n_y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Y\bar{A}X & 0 \end{bmatrix}. \tag{23}$$

Substituting the above result in (19a,b) and choosing $R_1 = \mathrm{diag}\{\varepsilon_1 I_n,\ \bar{\varepsilon} I_{n_y},\ \bar{\varepsilon} I_{n_u}\}$ and $R_2 = \mathrm{diag}\{\bar{\varepsilon} I_{n+ny},\ \varepsilon_2 I_{n_u}\}$, where $0 < \bar{\varepsilon}$ tends to zero, and ε_1 and ε_2 are positive tuning parameters, we obtain the following.

Theorem 1. *Consider system* (2). *For prescribed scalars* $\gamma > 0$ *and* $0 < \rho$, $J_c(w) < 0\ \forall\ w(t) \in L_2[0\ \infty)$ *under the sampled-data output-feedback controller of* (8) *and* (10) *for all holding and sampling times satisfying* A2 *and* A3, *respectively, if for some tuning scalar parameters* ε_1 *and* ε_2 *there exist* $4n_c \times 4n_c$ *matrices* $\bar{Z}_1,\ \bar{Z}_2$, $2n_c \times 2n_c$ *matrices* $\bar{Q}_2,\ \bar{Q}_2$, $n_c \times n_c$ *matrices* $X,\ Y$, *and a* $(n_c + n_u) \times (n_c + n_y)$*-matrix K that satisfy* (20) *and the following three LMIs*:

$$\Gamma + h\bar{Z}_2 + g\bar{Z}_1 < 0,$$
$$\begin{bmatrix} \varepsilon_1 I_n & [0 \quad [0 \quad \rho C_2^{\mathrm{T}} \quad 0\,][\, I_{n_c} \quad Y\,]] \\ * & \bar{Z}_1 \end{bmatrix} \geqslant 0,$$
$$\begin{bmatrix} \varepsilon_2 I_{n_u} & [0 \quad \rho B_2^{\mathrm{T}} [\, I_n \quad 0 \quad 0\,][\, I_{n_c} \quad Y\,]] \\ * & \bar{Z}_2 \end{bmatrix} \geqslant 0, \tag{24a–c}$$

where

$$\Gamma \triangleq \begin{bmatrix} \begin{bmatrix} \bar{Q}_2 + \bar{Q}_2^{\mathrm{T}} & \bar{Q}_3 - \bar{Q}_2^{\mathrm{T}} + \Xi \\ * & -\bar{Q}_3 - \bar{Q}_3^{\mathrm{T}} \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ Y \end{bmatrix} \hat{B}_1 & \begin{bmatrix} X \\ I \\ 0 \end{bmatrix} \hat{C}_1^{\mathrm{T}} & g \begin{bmatrix} \bar{Q}_2^{\mathrm{T}} \\ \bar{Q}_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0_{n,n_y} \\ I_{n_y} \\ 0 \end{bmatrix} & h \begin{bmatrix} \bar{Q}_2^{\mathrm{T}} \\ \bar{Q}_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I_{n_u} \end{bmatrix} \\ * & -\gamma^2 I_{n_w+n_v} & 0 & 0 & 0 \\ * & * & -I_{n_z} & 0 & 0 \\ * & * & * & -g\varepsilon_1^{-1} I_{n_y} & 0 \\ * & * & * & * & -h\varepsilon_2^{-1} I_{n_u} \end{bmatrix}, \tag{25}$$

$$\hat{B}_1 = \begin{bmatrix} B_1 & 0 \\ 0 & \rho D_{21} \\ 0 & 0 \end{bmatrix}, \quad and \quad \hat{C}_1 = [\, C_1 \quad 0 \quad D_{12} \,].$$

If a solution to the above exists then the output-feedback controller that achieves the required performance is described by the series connection of the pre- and the post-controllers (10a) *and* (10b), *respectively, with the system described in* (8), *where the matrices of the latter are obtained by solving* (23) *for* Θ *and taking* $N = I_{n_c}$, $M = I_{n_c} - YX$.

Remark 1. In (23) the matrices N and M can be obtained by factorizing $I_{n_c} - YX = N^{\mathrm{T}} M$. One of these factorizations yields $N = I_{n_c}$. The transfer function matrix of the obtained controller will not depend on the specific factors chosen.

The LMIs in Theorem 2 apply variables that are in fact redundant. Inequalities (24b,c) set lower bounds on $\bar{Z}_1$ and $\bar{Z}_2$. The latter matrices can thus be replaced in (24a) by their respective bounds. We thus obtain the following:

Corollary 1. *Consider system* (2). $J_c(w) < 0 \ \forall \ w(t) \in L_2[0 \ \infty)$ *for prescribed scalars* $\gamma > 0$ *and* $0 < \rho$, *under the sampled-data output-feedback controller of* (8) *and* (10) *for all holding and sampling times satisfying* A2 *and* A3, *respectively, if for some tuning scalar parameters* ε_1 *and* ε_2 *there exists a* $2n_c \times 2n_c$ *matrices* $\bar{Q}_2, \ \bar{Q}_3$, $n_c \times n_c$ *matrices* $X, \ Y$, *and a* $(n_c + n_u) \times (n_c + n_y)$*-matrix K that satisfy* (20) *and the following inequality*:

$$\begin{bmatrix} \Gamma & \begin{bmatrix} g \begin{bmatrix} 0 \\ I \\ Y \end{bmatrix} \begin{bmatrix} 0 \\ \rho C_2 \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} & \begin{bmatrix} h \begin{bmatrix} 0 \\ I \\ Y \end{bmatrix} \begin{bmatrix} \rho B_2 \\ 0 \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} \\ * & -g\varepsilon_1 I_n & 0 \\ * & * & -h\varepsilon_2 I_{n_u} \end{bmatrix} < 0. \tag{26}$$

2.5. The case of systems with norm-bounded uncertainties

The results of the previous section can be easily generalized to the case of systems with norm-bounded uncertainties.

Theorem 2. *Consider system* (2) *where* A, B_1, B_2, C_2, *and* D_{21} *are replaced by* $A + \Delta A$, $B_1 + \Delta B_1$, $B_2 + \Delta B_2$, $C_2 + \Delta C_2$ *and* $D_{21} + \Delta D_{21}$, *respectively, and where*

$$[\, \Delta A \quad \Delta B_1 \quad \Delta B_2 \,] = H\Delta(t)\, [\, E \quad E_1 \quad E_2 \,], \quad i = 0, 1, 2 \quad and$$

$$[\, \Delta C_2 \quad \Delta D_{21} \,] = H_c \bar{\Delta}(t)\, [\, E_c \quad E_d \,] \tag{27a,b}$$

with Δ *and* $\bar{\Delta}$ *satisfying*

$$\Delta(t)^{\mathrm{T}} \Delta(t) \leqslant I \quad and \quad \bar{\Delta}(t)^{\mathrm{T}} \bar{\Delta}(t) \leqslant I. \tag{28a,b}$$

For prescribed scalars $\gamma > 0$ *and* $0 < \rho$, $J_c(w) < 0 \ \forall \ w(t) \in L_2[0 \ \infty)$ *under the sampled-data output-feedback controller of* (8) *and* (10) *for all holding and sampling times satisfying* A2 *and* A3, *respectively, if for some tuning scalar parameters* ε_1 *and* ε_2, r_1 *and* r_2 *there exist* $4n_c \times 4n_c$ *matrices* $\bar{Z}_1, \ \bar{Z}_2$, $2n_c \times 2n_c$ *matrices* $\bar{Q}_2, \ \bar{Q}_2$, $n_c \times n_c$

matrices $X,\ Y$, *and a* $(n_c+n_u)\times(n_c+n_y)$*-matrix* K *that satisfy* (20) *and the following inequalities*:

$$\begin{bmatrix} \Gamma + h\hat{Z}_2 + g\hat{Z}_1 & \begin{bmatrix} 0 \\ r_1 \begin{bmatrix} I \\ Y \end{bmatrix}\begin{bmatrix} H \\ 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} 0 \\ r_2 \begin{bmatrix} I \\ Y \end{bmatrix}\begin{bmatrix} 0 \\ H_c \\ 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} X \\ I \end{bmatrix}\begin{bmatrix} E^T \\ 0 \\ \rho E_2^T \\ 0 \end{bmatrix} \\ \begin{bmatrix} E_1^T \\ 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} X \\ I \end{bmatrix}\begin{bmatrix} \rho E_c^T \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ \rho E_d^T \\ 0 \end{bmatrix} \end{bmatrix} \\ * & -r_1 I & 0 & 0 & 0 \\ * & * & -r_2 I & 0 & 0 \\ * & * & * & -r_1 I & 0 \\ * & * & * & * & -r_2 I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \varepsilon_1 I_n & [0 \quad [0 \quad \rho C_2^T \quad 0][I_{n_c} \quad Y]] & \rho E_c^T & 0 \\ * & \bar{Z}_1 & 0 & \begin{bmatrix} 0 \\ r_2 H_c \\ 0 \end{bmatrix} \\ * & * & r_2 I & 0 \\ * & * & * & r_2 I \end{bmatrix} \geqslant 0,$$

$$\begin{bmatrix} \varepsilon_2 I_{n_u} & [0 \quad [\rho B_2^T \quad 0 \quad 0][I_{n_c} \quad Y]] & \rho E_2^T & 0 \\ * & \bar{Z}_2 & 0 & \begin{bmatrix} 0 \\ r_1 H \\ 0 \end{bmatrix} \\ * & * & r_1 I & 0 \\ * & * & * & r_1 I \end{bmatrix} \geqslant 0, \qquad \text{(29a–c)}$$

where Γ *is defined in* (25).

If a solution to the above exists then the output-feedback controller that achieves the required performance is described by the series connection of the pre- and the post-controllers (10a) *and* (10b), *respectively, with the system described in* (8), *where the matrices of the latter are obtained by solving* (23) *for* Θ *and taking* $N = I_{n_c}$, $M = I_{n_c} - YX$.

Proof. The result follows by replacing in (24) A, B_1, B_2, C_2, and D_{21} by $A+\Delta A$, $B_1+\Delta B_1$, $B_2+\Delta B_2$, $C_2+\Delta C_2$, and $D_{21}+\Delta D_{21}$ respectively, and by using inequalities of the type

$$\begin{aligned} &-rHH^T - r^{-1}E^TE \\ &\quad \leqslant H\Delta E + E^T\Delta^T H^T \\ &\quad \leqslant rHH^T + r^{-1}E^TE \quad \forall \ \text{scalars } 0<r \end{aligned}$$

and applying Schur complements formula. □

Similarly to Corollary 1, the three LMIs of Theorem 2 can be combined to one LMI.

2.6. On state-feedback H_∞ control

Assume A2 and

A1′. $D_{12} = 0$.

Our objective is to find a state-feedback controller of the form

$$u(t) = Kx(t_k), \quad t_k \leqslant t < t_{k+1}, \qquad (30)$$

which for all hold times satisfying A2 internally stabilizes the system and leads to $J_c < 0$, for $x(0) = 0$ and for all non-zero $w \in L_2[0, \infty)$. Under A1′ a simple solution to state-feedback H_∞ control will be derived.

We represent a piecewise constant control law as continuous time control with time-varying piecewise-continuous delay $\tau = t - t_k$, as given in (1). We will thus look for a state-feedback controller of the form

$$u(t) = Kx(t - \tau(t)). \tag{31}$$

Substituting (31) into (2), we obtain the following closed-loop system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2 K x(t - \tau(t)) + B_1 w(t), \\ z(t) &= C_1 x(t). \end{aligned} \tag{32}$$

From A2 it follows that $0 \leqslant \tau(t) \leqslant h$ since $\tau \leqslant t_{k+1} - t_k$. We will further consider (32) as the system with uncertain and bounded delay, and we require that $J_c < 0$ holds for all non-zero $w \in L_2[0, \infty)$ and for zero initial condition $x(t) = 0,\ t \leqslant 0$.

In order to apply Lemma 1 to (32), we denote $P_3 = \varepsilon P_2$, where ε is a scalar, $\bar{P} = P_2^{-1}$, $\bar{P}_1 = \bar{P}^{\mathrm{T}} P_1 \bar{P}$, $\bar{R} = \bar{P}^{\mathrm{T}} R \bar{P}$, $\bar{Z}_i = \bar{P}^{\mathrm{T}} Z_i \bar{P},\ i = 1, 2, 3$ and $\bar{Y} = K\bar{P}$. Multiplying (16a) by diag$\{\bar{P}^{\mathrm{T}},\ \bar{P}^{\mathrm{T}},\ I,\ I\}$ and diag$\{\bar{P},\ \bar{P},\ I,\ I\}$ and (16b) by diag$\{\bar{P}^{\mathrm{T}},\ \bar{P}^{\mathrm{T}},\ \bar{P}^{\mathrm{T}}\}$ and diag $\{\bar{P},\ \bar{P},\ \bar{P}\}$, from the right and the left, respectively, the following is obtained.

Theorem 3. *Assume* A1′. *Consider* (2). *For a prescribed scalar* $\gamma > 0$, $J_c(w) < 0\ \forall\ w(t) \in L_2[0\ \infty)$ *under sampled-data state-feedback controller for all holding times satisfying* A2, *if for some tuning scalar parameter* ε *there exist* $n \times n$ *matrices* $0 < \bar{P}_1,\ \bar{P},\ \bar{Z}_1,\ \bar{Z}_2, \bar{Z}_3,\ \bar{R}$, *and* $n_u \times n$- *matrix* $\bar{Y}$ *that satisfy*

$$\begin{bmatrix} \bar{P}^{\mathrm{T}} A^{\mathrm{T}} + A\bar{P} + B_2\bar{Y} + \bar{Y}^{\mathrm{T}} B_2 + h\bar{Z}_1 & \bar{P}_1 - \bar{P} + \varepsilon\bar{P}^{\mathrm{T}} A^{\mathrm{T}} + \varepsilon\bar{Y}^{\mathrm{T}} B_2^{\mathrm{T}} + h\bar{Z}_2 & \bar{P}^{\mathrm{T}} C_1^{\mathrm{T}} & B_1 \\ * & -\varepsilon\bar{P}^{\mathrm{T}} - \varepsilon\bar{P} + h(\bar{R} + \bar{Z}_3) & 0 & \varepsilon B_1 \\ * & * & -I & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \bar{R} & \bar{Y}^{\mathrm{T}} B_2^{\mathrm{T}} & \varepsilon\bar{Y}^{\mathrm{T}} B_2^{\mathrm{T}} \\ * & \bar{Z}_1 & \bar{Z}_2 \\ * & * & \bar{Z}_3 \end{bmatrix} \geqslant 0. \tag{33a,b}$$

The state-feedback gain is given by

$$K = \bar{Y}\bar{P}^{-1}. \tag{34}$$

Consider now (2) under the continuous state-feedback $u(t) = Kx(t)$ and the index (5). It is well-known that in the continuous case $J_c < 0$ iff there exist $n \times n$-matrix $0 < P_1$, and $n_u \times n$-matrix Y that satisfy

$$\begin{bmatrix} P_1 A^{\mathrm{T}} + AP_1 + B_2 Y + Y^{\mathrm{T}} B_2 & P_1 C_1^{\mathrm{T}} & B_1 \\ * & -I & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{35}$$

where $K = YP_1^{-1}$. If (35) is feasible than for small enough $h > 0$ (33) is feasible too (take e.g. the same P_1, $Y = \bar{Y}$, and $\bar{P} = P_1, \varepsilon \to 0$, while R and Z are any matrices satisfying (33b)). We, therefore, obtain the following result (similar to the one of [13] which was proved by using the lifting technique):

Corollary 2. *Consider* (2) *under* A1′, A2. *If the continuous-time state-feedback* $u = Kx(t)$ *achieves* $J_c(w) < 0$, *then there exists* $h^* > 0$ *such that for all* $h \in (0, h^*]$ *the sampled-data state-feedback* (30), *with the same gain* K, *achieves* $J_c(w) < 0$.

The LMIs of Theorem 3 are affine in the system matrices and thus the solution for the system with polytopic type uncertainty readily follows. The results of this subsection may be easily adapted to the case of systems with norm-bounded uncertainties.

2.7. Examples

Example 1. We consider an example of [14]. The system is

$$\begin{aligned}
&\dot{x}(t) = -0.8x(t) + w(t) + 2u(t),\\
&z(t) = \begin{bmatrix}1\\0\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t),\\
&y(k) = x(\sigma_k) + 0.3v_k, \quad k = 0, 1, \ldots \;.
\end{aligned} \tag{36}$$

We choose $\rho = 0.01$ and apply Theorem 1, for $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 0.4$ we obtain $\gamma_{\min} = 1.25$. This result is higher than the minimum attenuation level of $\gamma_{\min} = 0.816$ reported in [14] for the case of equidistant sampling with the period $g = \pi/4$ and holding with the period $h = 1$.

Consider next the state-feedback sampled-data H_∞ control of (36a) with $z(t) = \begin{bmatrix}1\\0\end{bmatrix}x(t)$.

Applying Theorem 3 for holding bound of $h = 1$ a minimum attenuation level of $\gamma = 0.8866$ is obtained for $\varepsilon = 0.74$. The corresponding feedback gain is $K = -0.2678$. The result obtained for holding that tends to zero is $\gamma = 1.872 \times 10^{-6}$.

Example 2. We consider here another example of [14]. Given the system:

$$\begin{aligned}
&\dot{x}(t) = \begin{bmatrix}0 & 1\\-16 & -4.8\end{bmatrix}x(t) + \begin{bmatrix}0\\16\end{bmatrix}w(t) + \begin{bmatrix}0\\16\end{bmatrix}u(t),\\
&z(t) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}x(t) + \begin{bmatrix}0\\0.1\end{bmatrix}u(t),\\
&y_k = [1\ 0]x(\sigma_k) + 0.1v_k, \;\; k = 0, 1, \ldots \;.
\end{aligned} \tag{37}$$

For the output-feedback control with $h=1$ and $g=\pi/4$ we choose $\rho=0.01$ and apply Theorem 1. For $\varepsilon_1=0.4\times 10^{-4}$ and $\varepsilon_2 = 1.2 \times 10^{-3}$ we obtain $\gamma_{\min} = 1.125$. This result is close to the minimum attenuation level of $\gamma_{\min} = 1.02$ found in [14] for the case of equidistant holding with the period 1 and equidistant sampling with the period $\pi/4$. It is noted that the two poles introduced by the components of (8) and (10) at -0.01 are canceled by zeros of the controller so that the overall feedback controller possesses four poles and two zeros.

We assume next that the model of (37) encounters norm bounded uncertainty of type (27) with $H = [0\ \ 1]^{\mathrm{T}}$, $E=[3\ \ 0]$, and with zero E_1, E_2 and H_c. Applying Theorem 2, for the above value of $\rho=0.01$, we obtain a minimum value of $\gamma_{\min} = 1.432$ for $r_1 = 39$, $\varepsilon_1 = 0.4 \times 10^{-4}$, and $\varepsilon_2 = 1.2 \times 10^{-3}$.

3. Conclusions

A new approach, which was recently introduced to sampled-data state-feedback stabilization, is developed to the sampled-data H_∞ control. The system is modelled as a continuous-time one, where the control input and the measurement output have piecewise-continuous delays. It is assumed that the maximum holding interval is not greater than $h > 0$ and the maximum sampling interval is not greater than $g > 0$. The h and g-dependent sufficient LMI conditions are derived for output-feedback H_∞ control of such systems via Lyapunov–Krasovskii functionals and descriptor approach to time-delay systems. The results are generalized to the case of systems with norm-bounded uncertainties.

The solution of the output-feedback control problem is based on introducing simple filters that precede the sampling of the measurement and the control input. The steady-state gain of these filters is one, and they filter out signal components of frequencies equal to or larger than the corresponding Nyquist frequencies [10]. Although the poles of these components are cancelled by the zeros of the controller that is obtained by solving the LMIs of Theorems 1 and 2, the fact that this controller should possess zeros at prespecified locations is somewhat restrictive.

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