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State-feedback H_∞ control of nonlinear singularly perturbed systems

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SUMMARY

We study the H_∞ control problem for an affine singularly perturbed system, which is nonlinear in the state variables. Under suitable assumptions on the linearized problem, we construct ε-independent composite and linear controllers that solve the local H_∞ control problem for the full-order system for all small enough ε. These controllers solve also the corresponding problem for the descriptor system. The 'central' nonlinear controller can be approximated in the form of expansions in the powers of ε. An illustrative example shows that the higher-order approximate controller achieves the better performance, while the composite (zero-order approximate) controller leads to the better performance than the linear one. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: H_∞ control; nonlinear systems; singular perturbations; descriptor systems

1. INTRODUCTION

H_∞-control of a class of singularly perturbed system being nonlinear only on the slow variable have been studied in References [1–3]. In Reference [1] a composite ε-independent controller has been designed. In Reference [2] it has been shown that any ε-independent H_∞ controller of the linearized singularly perturbed system is a local H_∞ controller for nonlinear problem. In Reference [3] a high-order approximate controller has been constructed in the form of expansions in the powers of ε.

In the present note, we consider a general singularly perturbed system, being affine in the control and nonlinear in both, the slow and the fast variables. We generalize results of References [4, 5] obtained for the case of optimal control problem of such systems to the H_∞ control: we construct an ε-independent composite controller and a high-order approximate one by expanding in the powers of ε the 'central' controller, that solves the problem for each ε. Assuming that the

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corresponding linearized H_∞ control problem is solvable we show that

(i) the composite controller is $O(\varepsilon)$-close to the 'central' one and solves the local H_∞ control problem for the singularly perturbed system for all small enough $\varepsilon > 0$;
(ii) the composite controller solves the local H_∞ control problem for the corresponding descriptor system;
(iii) the truncated expansion approximates the 'central' controller and solves the local H_∞ control problem.

Proofs of the theorems are given in the appendix.

2. PROBLEM FORMULATION

Consider the system

$$\dot{x}_1 = F_1(x_1, x_2) + B_1(x_1, x_2)u + D_1(x_1, x_2)w \tag{1a}$$

$$\varepsilon\dot{x}_2 = F_2(x_1, x_2) + B_2(x_1, x_2)u + D_2(x_1, x_2)w \tag{1b}$$

$$z = \mathrm{col}\{k(x_1, x_2), u\} \tag{1c}$$

where $x_1(t) \in \mathbf{R}^{n_1}$ and $x_2(t) \in \mathbf{R}^{n_2}$ are the state vectors, $x = \mathrm{col}\{x_1, x_2\}$, $u(t) \in \mathbf{R}^m$ is the control input, $w \in \mathbf{R}^q$ is the disturbance and $z \in \mathbf{R}^s$ is the output to be controlled. The functions F_i, B_i, D_i and k are differentiable with respect to x a sufficient number of times. We assume also that $F_i(0, 0) = 0$ and $k(0, 0) = 0$.

Unlike Reference [1] we consider a non-standard singularly perturbed problem in the sense that we do not require the solvability with respect to x_2 of the algebraic equation

$$F_2(x_1, x_2) + B_2(x_1, x_2)u + D_2(x_1, x_2)w = 0$$

For more information on non-standard singularly perturbed system we refer to Reference [5] and example therein, where the above algebraic equation has the form $x_1 + u = 0$.

Denote by $|\cdot|$ the Euclidean norm of a vector. Let γ be a fixed positive constant. Then, the nonlinear H_∞ control problem (for performance level γ) is to find a nonlinear state-feedback

$$u = \beta(x), \quad \beta(0) = 0 \tag{2}$$

such that the closed-loop system of (1) and (2) has a L_2-gain less than or equal to γ (see Reference [6]). It means that the following inequality holds:

$$\int_0^\tau |z(t)|^2 \, \mathrm{d}t \leqslant \gamma^2 \int_0^\tau |w(t)|^2 \, \mathrm{d}t \tag{3}$$

for all $w \in L_2[0, \tau]$ and all $\tau \geqslant 0$, where z denotes the response of the closed-loop system of (1) and (2) for $w \in L_2[0, \tau]$ and the initial condition $x(0) = 0$ (see References [6, 7]). The H_∞ control problem is solvable on $\Omega \subset \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ containing 0 as an interior point if (3) holds for every $\tau \geqslant 0$ and for every $w \in L_2[0, \tau]$ for which the state trajectory of the closed-loop system (1) and (2) starting from 0 remains in Ω for all $t \in [0, \tau]$.

Consider the Hamiltonian function

$$\mathscr{H}(x_1, x_2, p_1, p_2) = p_1' F_1(x_1, x_2) + p_2' F_2(x_1, x_2) - \frac{1}{2}(p_1' p_2') \begin{pmatrix} S_{11}(x) & S_{12}(x) \\ S_{21}(x) & S_{22}(x) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \frac{1}{2} k'(x_1, x_2) k(x_1, x_2) \quad (4)$$

where prime denotes the transposition of a matrix, p_1 and εp_2 play the role of the co-state variables and $S_{ij} = B_i B_j' - 1/\gamma^2 D_i D_j'$. The corresponding Hamiltonian system has the form

$$\dot{x}_1 = f_1(x_1, p_1, x_2, p_2) \quad (5a)$$

$$\dot{p}_1 = f_2(x_1, p_1, x_2, p_2) \quad (5b)$$

$$\varepsilon \dot{x}_2 = f_3(x_1, p_1, x_2, p_2) \quad (5c)$$

$$\varepsilon \dot{p}_2 = f_4(x_1, p_1, x_2, p_2) \quad (5d)$$

where $f_1 = (\partial \mathscr{H}/\partial p_1)'$, $f_2 = -(\partial \mathscr{H}/\partial x_1)'$, $f_3 = (\partial \mathscr{H}/\partial p_2)'$, $f_4 = -(\partial \mathscr{H}/\partial x_2)'$.

For each $\varepsilon > 0$ the problem is solvable on $\Omega \subset \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ if there exists a C^2 non-negative solution $V : \Omega \to \mathbf{R}$ to the Hamilton–Jacobi (HJ) partial differential equation

$$V_{x_1} F_1(x_1, x_2) + \varepsilon^{-1} V_{x_2} F_2(x_1, x_2) - \frac{1}{2}(V_{x_1}\, \varepsilon^{-1} V_{x_2}) \begin{pmatrix} S_{11}(x) & S_{12}(x) \\ S_{21}(x) & S_{22}(x) \end{pmatrix} \begin{pmatrix} V_{x_1}' \\ \varepsilon^{-1} V_{x_2}' \end{pmatrix} + \frac{1}{2} k'(x_1, x_2) k(x_1, x_2), \quad V(0) = 0 \quad (6)$$

with the property that the system

$$\dot{x}_1 = f_1(x_1, V_{x_1}', x_2, \varepsilon^{-1} V_{x_2}'), \quad \varepsilon \dot{x}_2 = f_3(x_1, V_{x_1}', x_2, \varepsilon^{-1} V_{x_2}') \quad (7)$$

is asymptotically stable (see References [6, 7]), where $V_x = (V_{x_1}, V_{x_2})$ denotes the Jacobian matrix of V. The latter is equivalent to the existence of the invariant manifold of (5) $p_1 = V_{x_1}'$, $p_2 = \varepsilon^{-1} V_{x_2}'$, with asymptotically stable flow, such that $V \geqslant 0$, $V(0) = 0$ (that implies $V_x(0) = 0$). The controller that solves the problem is then given by

$$u = -[B_1', \varepsilon^{-1} B_2']\, V_x' \quad (8)$$

Similarly to the linear case, the latter will be denoted as the 'central' controller.

Note that the 'central' controller of (8) is found by solving high-order ε-dependent HJ partial differential equation (6). We shall construct H_∞ controllers by solving the simplified ε-independent reduced-order partial differential and algebraic equations.

3. MAIN RESULTS

3.1. H_∞ composite controller

Composite controller will be constructed similar to the one in the optimal control case (see Reference [5]). Consider the linearization of (1) at $x = 0$:

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_{10} u + D_{10} w, \quad \varepsilon \dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_{20} u + D_{20} w$$
$$z = \mathrm{col}\{C_1 x_1 + C_2 x_2, u\} \quad (9)$$

where $A_{ij} = (\partial f_i/\partial x_j)\ (0,0)$, $B_{i0} = B_i(0,0)$, $D_{i0} = D_i(0,0)$, $C_i = (\partial k/\partial x_i)(0,0)$, $i = 1, 2;\ j = 1, 2$. Hamiltonian system that corresponds to (9) can be written in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \text{Ham} \begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \tag{10a}$$

$$\text{Ham} = \begin{bmatrix} T_{11} & T_{12} \\ \varepsilon^{-1} T_{21} & \varepsilon^{-1} T_{22} \end{bmatrix} \tag{10b}$$

$$T_{ij} = \begin{bmatrix} A_{ij} & -S_{ij}(0) \\ -C_i' C_j & -A_{ji}' \end{bmatrix} \tag{10c}$$

To guarantee that for all small ε this linear H_∞ control problem is solvable we assume:

A1

For a given γ a fast Riccati equation

$$A_{22}' X_f + X_f A_{22} + C_2' C_2 - X_f S_{22}(0) X_f = 0 \tag{11}$$

has a solution $X_f = X_f' \geqslant 0$, such that the matrix $\Lambda_f = A_{22} - S_{22} X_f$ is Hurwitz.

A2

For a given γ a slow Riccati equation

$$X_0 A_0 + A_0' X_0 - X_0 S_0 X_0 + Q_0 = 0 \tag{12}$$

where

$$\begin{bmatrix} A_0 & -S_0 \\ -Q_0 & -A_0' \end{bmatrix} = T_{11} - T_{12} T_{22}^{-1} T_{21} \tag{13}$$

has a solution $X_0 = X_0' \geqslant 0$ such that the matrix $\Lambda_s = A_0 - S_0 X_0$ in Hurwitz.

Note that under assumptions of stabilizability–detectability and absence of invariant zeros on the imaginary axis of Reference [8] assumptions A1 and A2 are necessary and sufficient for solvability of the linear H_∞ control problem for all small enough ε. It is known (see Reference [8]) that under A1 and A2 for all small enough ε the linear controller

$$u_l = -B_{10}' X_0 x_1 - B_{20}'(X_c x_1 + X_f x_2), \quad X_c = [X_f, -I] T_{22}^{-1} T_{21} \begin{bmatrix} I \\ X_0 \end{bmatrix} \tag{14}$$

solves the linear H_∞ control problem. Therefore, for each small ε, the H_∞ control problem is solvable on a small enough neighbourhood of $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ containing 0 (see Reference [6]). We shall show that there exists an ε-independent neighbourhood that is appropriate for all small enough ε.

Under A1 the Hamiltonian matrix T_{22} has n_2 eigenvalues with negative real parts and n_2 with positive ones. Under A1 and A2 the Hamiltonian matrix $T_0 = T_{11} - T_{12} T_{22}^{-1} T_{21}$, has n_1 eigenvalues with negative real parts and n_1 with positive ones. Then by implicit function theorem in a small enough neighbourhood of $\mathbf{R}^{n_2} \times \mathbf{R}^{n_2}$ containing 0 the system of equations

$$f_3(x_1, p_1, x_2, p_2) = 0, \quad f_4(x_1, p_1, x_2, p_2) = 0$$

has an isolated solution

$$x_2 = \phi(x_1, p_1), \quad p_2 = \psi(x_1, p_1) \tag{15}$$

Consider the *reduced* Hamiltonian system

$$\dot{x}_1 = f_1(x_1, p_1, \phi(x_1, p_1), \psi(x_1, p_1)) \tag{16a}$$

$$\dot{p}_1 = f_2(x_1, p_1, \phi(x_1, p_1), \psi(x_1, p_1)) \tag{16b}$$

This system results after substituting (15) into (5a) and (5b). From A2 and the theory of nonlinear differential equations (see e.g. Reference [9]) it follows that this system has a stable manifold

$$p_1 = N_0(x_1), \quad N_0(x_1) = X_0 x_1 + O(|x_1|^2) \tag{17}$$

with asymptotically stable flow

$$\dot{x}_1 = f_1(x_1, N_0(x_1), \phi(x_1, N_0(x_1)), \psi(x_1, N_0(x_1)) \tag{18}$$

for x_1 from small enough neighbourhood of 0. Function $N_0 = N_0(x_1)$ satisfies the *slow* ε-independent partial differential equation (PDE)

$$\frac{\partial N_0}{\partial x_1} f_1(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)) = f_2(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)) \tag{19}$$

Approximate solution of the slow PDE may be obtained by the power-series method as given in Reference [10] (see also Reference [6] and references therein). Due to this method N_0 can be approximately found in the form of expansion in the powers of x_1:

$$N_0(x_1) = X_0 x_1 + N^{(2)}(x_1) + N^{(3)}(x_1) + \cdots \tag{20}$$

with $N^{(i)}$ denoting the ith-order terms of $N_0(x_1)$. Substitution of (20) into (19) and equating of terms of the same order lead to algebraic equations with respect to $N^{(i)}$. From the latter equations $N^{(i)}$ can be computed inductively via $X_0, \ldots, N^{(i-1)}$.

For each x_1 such that (17) exists consider the 'fast' system

$$\dot{x}_2 = \bar{f}_3(x_1, x_2, p_2), \quad \dot{p}_2 = \bar{f}_4(x_1, x_2, p_2) \tag{21}$$

where

$$\bar{f}_i = f_i(x_1, N_0, x_2 + \phi(x_1, N_0), p_2 + \psi(x_1, N_0)), \quad i = 3, 4$$

For each x_i (21) has a stable manifold $p_2 = M_0(x_1, x_2)$ with asymptotically stable flow

$$\dot{x}_2 = \bar{f}_3(x_1, x_2, M_0(x_1, x_2))$$

where x_1 and x_2 from small enough neighbourhood Ω of $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ containing 0. Function

$$M_0 = M_0(x_1, x_2) = X_f x_2 + O((|x_1| + |x_2|)|x_2|) \tag{22}$$

satisfies the *fast* ε-independent PDE

$$\frac{\partial M_0}{\partial x_2}\bar{f}_3(x_1, x_2, M_0) = \bar{f}_4(x_1, x_2, M_0) \tag{23}$$

Similarly to N_0, function M_0 can be found in the form of expansion in the powers of x_2:

$$M_0(x_1, x_2) = X_f x_2 + M^{(2)}(x_1, x_2) + M^{(3)}(x_1, x_2) + \cdots$$

with $M^{(i)}(x_1, x_2)$ denoting the ith-order terms with respect to x_2 and depending on the parameter x_1.

Define the *composite* controller as follows:

$$u_0(x) = -B_1' N_0(x_1) - B_2'[\psi(x_1, N_0(x_1)) + M_0[x_1, x_2 - \phi(x_1, N_0(x_1))]] \tag{24}$$

From (14), (17) and (22) it follows that

$$u_0 = u_l + O(|x_1|^2 + |x_2|^2) \tag{25}$$

Both ε-independent controllers, the composite (24) and the linear (14), solve H_∞ control problem on some neighbourhood for all small enough ε:

Theorem 3.1

Under A1 and A2 there exist $m_1 > 0, m_2 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the following holds:

(i) There exists a C^2 function $V: \Omega_{m_1} \times \Omega_{m_2} \to [0, \infty)$, satisfying the HJ equation (6). The solution to HJ equation and the 'central' controller have the following approximations:

$$V(x_1, x_2) = V_0(x_1) + O(\varepsilon), \quad u(x) = u_0(x) + O(\varepsilon) \tag{26}$$

where u_0 is given by (24) and $\partial V_0/\partial x_1 = N_0'(x_1)$.

(ii) H_∞ control problem is solvable on $\Omega_{m_1} \times \Omega_{m_2}$ by the composite controller of (24).

(iii) H_∞ control problem is solvable on $\Omega_{m_1} \times \Omega_{m_2}$ by the linear controller of (14).

Remark 3.1

Example in Reference [3] and the one below show that the composite controller leads to the better performance than the linear controller.

3.2. H_∞ *control of nonlinear descriptor system*

In this subsection, we show that similarly to the linear case (see Reference [8]), the composite controller solves H_∞ control problem for the corresponding descriptor system

$$E\dot{x} = F(x) + B(x)u + D(x)w, \quad z = \mathrm{col}\{k(x), u\} \tag{27}$$

where $k(x) = k(x_1, x_2)$ and

$$E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix}, \quad B = \begin{bmatrix} B_1(x_1, x_2) \\ B_2(x_1, x_2) \end{bmatrix}, \quad D = \begin{bmatrix} D_1(x_1, x_2) \\ D_2(x_1, x_2) \end{bmatrix}$$

Unlike the non-descriptor case, the initial condition for (27) is defined only for the slow variable x_1. We restrict ourselves to *admissible* controllers of form (8) which guarantee a unique solution to the closed-loop system (27) and (8) for any small enough initial condition $Ex(0)$ and for any square integrable function $w(\cdot)$ from small enough neighbourhood of 0 in R^q. An admissible controller solves local H_∞ control problem for (27) on $\Omega \subset R^{n_1} \times R^{n_2}$ containing 0 as an interior point if (3) holds for every $\tau \geqslant 0$ and for every square integrable $w \in L_2[0, \tau]$ for which the state trajectory of the closed-loop system (27) and (8) starting from $Ex(0) = 0$ is uniquely defined and remains in Ω for all $t \in [0, \tau]$.

Theorem 3.2

Under A1 and A2 the following holds:

(i) Let there exists a twice continuously differentiable function $V_d : \Omega_{m_1} \times \{0\} \to R$ such that $V_d(Ex) \geqslant 0$,

$$\frac{\partial V_d(Ex)}{\partial x} = W(x)E \tag{28}$$

$$2W(x)F(x) - W(x)S(x)W'(x) + k'(x)k(x) = 0, \quad S = BB' - \gamma^{-2}DD' \tag{29}$$

with the property that $\tilde{f}_{x_2}(0)$ and $\tilde{F}_{x_2}(0)$ are non-singular, where $\tilde{f} = F_2 - B_2B_2'W'$, $\tilde{F} = F_2 - SW'$, and that the systems

$$E\dot{x} = F(x) - B(x)B'(x)W'(x), \quad E\dot{x} = F(x) - S(x)W'(x) \tag{30}$$

are asymptotically stable. Then the controller

$$u_d(x) = -B'(x)W'(x) \tag{31}$$

solves the local H_∞ control problem for the descriptor system (27).

(ii) The composite controller (24) solves the local H_∞ control problem for (27).

(iii) The linear controller (9) solves the local H_∞ control problem for (27).

3.3. High-order approximate controller

An asymptotic approximation to the controller (8)

$$u(x) = u_q(x) + O(\varepsilon^{q+1}), \quad u_q(x) = \sum_{i=0}^{q} \varepsilon^i u^{(i)}(x) \tag{32}$$

can be found similarly to Reference [4] by expanding the invariant manifolds into the powers of ε. The controller u_q solves local H_∞ control problem:

Theorem 3.3

Given $\gamma > 0$, under A1 and A2 there exist m_1, m_2, ε_1 such that for all $\varepsilon \in (0, \varepsilon_1]$ the following holds on $\Omega_{m_1} \times \Omega_{m_2}$:

(i) The 'central' controller exists and can be approximated by (32), where all functions are continuously differentiable in x and approximation is uniform on x and ε.

(ii) H_∞ control problem is solvable on $\Omega_{m_1} \times \Omega_{m_2}$ by the controller u_q.

Examples in Reference [3] and the one below show that the higher-order terms in (32) lead to improved performance.

3.4. Example

Consider the systems

$$\dot{x}_1 = -\tan x_2 + 2u + w, \quad \varepsilon\dot{x}_2 = \tan x_2 - u, \quad z = (x_1\ u)' \tag{33}$$

Here (11) has a stabilizing solution $X_f = 2$. We obtain the following Hamiltonian function:

$$\mathscr{H} = -p_1 \tan x_2 + p_2 \tan x_2 + 1/2(1/\gamma^2 - 4)p_1^2 + 2p_1p_2 - 1/2p_2^2 + 1/2x_1^2$$

and the corresponding Hamiltonian system

$$\dot{x}_1 = -\tan x_2 + (1/\gamma^2 - 4)p_1 + 2p_2, \quad \dot{p}_1 = -x_1$$

$$\varepsilon\dot{x}_2 = \tan x_2 + 2p_1 - p_2, \quad \varepsilon\dot{p}_2 = \cos^{-2} x_2(p_1 - p_2)$$

We find

$$\phi = -\arctan p_1, \quad \psi = p_1, \quad N_0 = Kx_1, \quad M_0 = 2\tan[x_2 - \arctan(Kx_1)] + 2Kx_1$$

where $K = (1 - 1/\gamma^2)^{-0.5}$, and A2 is satisfied for $\gamma > 1$. The linear (14) and the composite (24) controllers have the following form:

$$u_l = Kx_1 + 2x_2, \quad u_0 = Kx_1 + 2\tan x_2 \tag{34}$$

By algorithm of Reference [4] we find the first-order approximate controller u_1:

$$u_1 = u_0 + 3\varepsilon K(x_2 + \arctan(Kx_1)) + \frac{\varepsilon x_1}{1 + K^2x_1^2} - 4\varepsilon\frac{x_1 \sec^2 x_2}{(1 + K^2x_1^2)^2} + \varepsilon l, \quad l = O(|x_2 + \arctan(Kx_1)|^2)$$

We made some simulations of the behaviour of (33) under u_l, u_0 and u_1 (with $l = 0$) and constant and sinusoidal disturbance functions. Choosing $\gamma = 1.1 > 1$ we found $K = 2.4004$. Thus, for $w(t) = 0.5 \sin 3t$ the values of the functional cost

$$J = \int_0^t [x_1^2(s) + u^2(s) - \gamma^2 w^2(s)]\, \mathrm{d}s$$

under all these controllers are negative for the following values of $\varepsilon < 0.08$: $\varepsilon = 0.07$, 0.001 and 0. Therefore, under this particular disturbance input all the controllers achieve the performance bound of $\gamma = 1.1$.

We include in Figure 1 plots of the functional cost for $w(t) = 0.5 \sin 3t$, $\varepsilon = 0.08$ and $\varepsilon = 0.09$, where '.' is the plot under $u = u_l$, '- -' is the plot under $u = u_0$ and '–' is the plot under $u = u_1$. For $\varepsilon = 0.08$ functional costs under u_0 and u_1 are negative, while for u_l it becomes positive for $t > 15$. Hence, under this particular disturbance input the composite and the first-order controllers achieve the performance bound of $\gamma = 1.1$, whereas, the linear controller does not achieve this bound. For $\varepsilon = 0.9$ only the first-order controller u_1 leads to negative values of cost and, hence achieves the performance bound $\gamma = 1.1$. Therefore, the linear controller (14) is robust for $0 \leqslant \varepsilon \leqslant 0.07$, the composite controller (34) is robust for $0 \leqslant \varepsilon \leqslant 0.08$ in the sense that they achieve $\gamma = 1.1$ for $w = 0.5 \sin 3t$. For $\varepsilon = 0.09$ only u_1 achieves $\gamma = 1.1$ for $w = 0.5 \sin 3t$. Thus the $O(\varepsilon^2)$-approximation to the 'central' controller (8) achieves the better performance bound than its

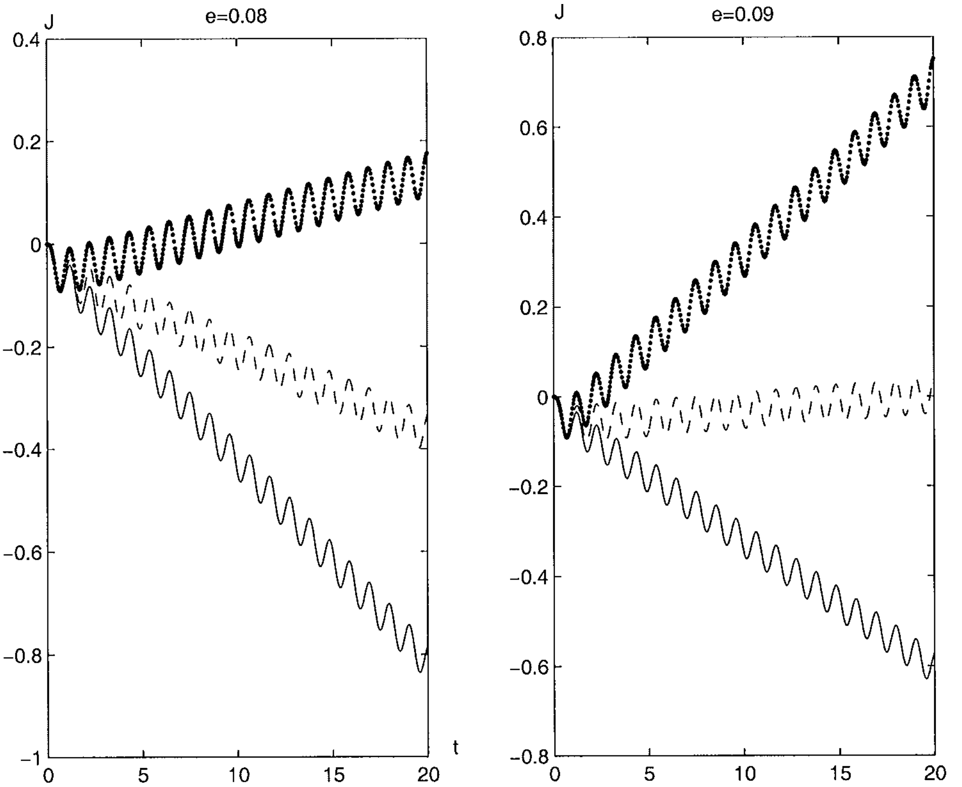

Figure 1. Functional costs for $\gamma = 1.1$ and $w = 0.5 \sin 3t$: $(\cdots\cdots)$ $u = u_l$; (- - -) $u = u_0$; (—) $u = u_1$.

$O(\varepsilon)$-approximation, whereas, the composite controller leads to the better performance than the linear one.

4. CONCLUSIONS

We have designed robust ε-independent H_∞ controllers for singularly perturbed systems that are nonlinear in both, the slow and the fast state variables. We have shown that these controllers achieve γ performance for all small enough ε. Moreover, they solve a local H_∞-control problem for the corresponding descriptor system. We have shown that a higher-order accuracy controller improves the performance.

APPENDIX

Proof of Theorem 3.1. (i) The existence of C^2 solution to (6) and (26) follows from Reference [4]. We shall show that $V \geqslant 0$ by proving that the closed-loop system of (1) and (8) with $w = 0$ is asymptotically stable (see References [6, 7]). To prove the latter consider linearization of the closed-loop system in the 0:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{22}\\ \varepsilon^{-1}\tilde{A}_{21} & \varepsilon^{-1}\tilde{A}_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}A_{11} & A_{22}\\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}B_{10}\\ \varepsilon^{-1}B_{20}\end{bmatrix}u_l \tag{A1}$$

Under A1 and A2 the fast matrix $\tilde{A}_{22}$ and the reduced one $\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ are Hurwitz (see proof of Lemma 3.4 from Reference [11]). The system of (1) and (8) with $w=0$ is then asymptotically stable since its fast and reduced systems are stable (see e.g. Reference [12]).

(ii) Consider the closed-loop system of (1) and (24). We have to prove that the corresponding to this system HJ equation has a non-negative stabilizing solution. Due to (ii) of this theorem we have to verify that assumptions A1 and A2 are satisfied for the corresponding linearized problem

$$\dot{x}_1=\tilde{A}_{11}x_1+\tilde{A}_{12}x_2+D_{10}w,\quad \varepsilon\dot{x}_2=\tilde{A}_{21}x_1+\tilde{A}_{22}x_2+D_{20}w,\quad z=[\tilde{C}_1\ \ \tilde{C}_2]\,x \tag{A2}$$

where $\tilde{C}_1=[C_1,B'_{10}X_0-B'_{20}X_c]$, $\tilde{C}_2=[C_2,\ -B'_{20}X_f]$. Note that the latter system coincides with the closed-loop system of (9) and (14). Hamiltonian system that corresponds to (A2) has the form

$$\begin{bmatrix}\dot{x}_1\\ \dot{p}_1\\ \dot{x}_2\\ \dot{p}_2\end{bmatrix}=\tilde{\mathrm{H}}\mathrm{am}\begin{bmatrix}x_1\\ p_1\\ x_2\\ p_2\end{bmatrix} \tag{A3}$$

where $\tilde{\mathrm{H}}\mathrm{am}$ is defined by (10) with all matrices taken with tilde and $\tilde{S}_{ij}=-\gamma^{-2}D_iD'_j$.

We observe that A1 and A2 means that the fast and the reduced systems corresponding to (A3) have stable manifolds. Set $\varepsilon=0$ and see that substitution of $p_1=X_0x_1$ and $p_2=X_cx_1+X_fx_2$ leads (A3) to (10) with $\varepsilon=0$. Therefore, $p_2=X_fx_2$ and $p_1=X_0x_1$ are the above-mentioned stable manifolds.

Proof of (iii) is similar to (ii).

Proof of Theorem 3.2. (i) Let $x(t)$ satisfies (27) and initial condition $Ex(0)=0$. Applying (28), (27) and (29) we find

$$2\,\frac{\mathrm{d}V_d(Ex)}{\mathrm{d}t}-\gamma^2w'w+z'z=2W(x)(f(x)+B(x)u+D(x)w)-\gamma^2w'w+k'k+u'u$$

$$=(u'+W(x)B(x))(u+B'(x)W'(x))-\gamma^2(w'-\gamma^{-2}W'(x)D'(x))(w-\gamma^{-2}D(x)W) \tag{A4}$$

Substituting in (A4) $u=u_d$ and integrating the resulting inequality on t from 0 to τ (for any $\tau>0$) we find

$$2V_d(Ex(\tau))-\gamma^2\int_0^\tau|w(t)|^2\,\mathrm{d}t+\int_0^\tau|z(t)|^2\,\mathrm{d}t\leqslant 0$$

that implies (3). Due to non-singularity of $\tilde{f}_{x_2}(0)$ and $\tilde{F}_{x_2}(0)$ systems (30) have unique solution for small enough $Ex(0)$ and the closed-loop system (27) and (31) has a unique solution for small enough $Ex(0)$ and $w(\cdot)$. Hence, controller (31) is admissible.

(ii) Choosing $V_d(Ex) = V_0(x_1)$, where V_0 satisfies (26), we see that

$$\frac{\partial V_d(Ex)}{\partial x} = W(x)E, \quad W(x) = [N_0'(x_1)\ \psi'(x_1, N_0(x_1)) + M'[x_1, x_2 - \phi(x_1, N_0(x_1))]]$$

Since by Theorem 3.1 $[V_{x_1}\ \varepsilon^{-1} V_{x_2}] = W + O(\varepsilon)$, where V is solution to HJ equation (6), then W satisfies (29). Moreover, (31) coincides with (24) that together with item (i) of this theorem completes the proof. Proof of (iii) is similar to (ii) of Theorem 3.1. □

Proof of Theorem 3.3. Item (i) follows from Reference [4]. Proof of (ii) is similar to (ii) of Theorem 3.1.

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