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Brief paper

State and unknown input observers for nonlinear systems with delayed measurements☆

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ABSTRACT

The use of connected devices or humans-in-the-loop for measuring outputs of dynamical systems inevitably produces time-varying measurement delays. These delays can lead to instability or severe degradation of system performance. In this paper, linear matrix inequality-based sufficient conditions are proposed for the design of state and unknown input observers based on delayed measurements for a class of nonlinear systems, where the nonlinearities are characterized by incremental multiplier matrices. The proposed observer is guaranteed to perform at specified operational levels in the presence of unknown exogenous inputs acting on the states and measurement outputs. Sufficient conditions are also provided for the estimation of these unknown inputs to a specified degree of accuracy. The potential of the proposed approach is illustrated via estimation of enzyme kinetics.

1. Introduction

Delays in states and measurements are inevitable for large-scale networked systems such as communication networks, tele-operation systems or systems with humans-in-the-loop like patient monitoring systems. For linear systems, observers with delayed measurements have been proposed in, for example, Ahmed-Ali, Van Assche, Massieu, and Dorleans (2013), Boutayeb (2001) and Chakrabarty, Ayoub, Żak, and Sundaram (2017). Although these observers have demonstrated excellent performance, the introduction of nonlinearities into the system architecture results in a more challenging scenario. A method is provided in Trinh, Aldeen, and Nahavandi (2004) for de-constructing the delays into matched and unmatched components and a reduced-order functional observer design is proposed. State observer design from an explicit resolution of a time-invariant Lyapunov algebraic equation is explored in Farza, Sboui, Cherrier, and M'Saad (2010) for nonlinear systems with constant delays. In Germani, Manes, and Pepe (2002), a chain of observers for a class of uniformly observable nonlinear systems is proposed, and conditions for asymptotic stability of the observer chain in spite of constant measurement delays are formulated. These conditions are relaxed in Kazantzis and Wright (2005) using a partial differential equation based transformation of the coordinate system. Further relaxations for triangular nonlinear systems are proposed in Ahmed-Ali, Karafyllis, and Lamnabhi-Lagarrigue (2013), Ibrir (2009) and Van Assche, Ahmed-Ali, Hann, and Lamnabhi-Lagarrigue (2011) using high-gain or adaptive observers for time-varying measurement delays. Such time-varying delays are also investigated in Cacace, Germani, and Manes (2014) using Lyapunov–Razumikhin methods. For linear systems with time-varying measurement delays and exogenous disturbance inputs acting on the system, input-to-state stability and $\mathcal{H}_\infty$ based state estimators are proposed in Fridman and Shaked (2001) and Fridman, Shaked, and Xie (2003) leveraging Lyapunov–Krasovskii functionals. Recently, a formalism for estimating the state in the presence of unknown delays was proposed in Cacace, Conte, Germani, and Palombo (2017) by appending a delay identifier to a state estimator.

It may become crucial in certain applications to identify the unknown exogenous inputs acting on the dynamical system. We refer to such observers as *unknown input observers*. To the best of our knowledge, Han, Fridman, and Spurgeon (2014) is the first attempt at the design of such an unknown input observer for linear sampled-measurement systems exploiting sliding mode techniques: however, disturbances simultaneously in the actuator

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and sensor are not considered. For nonlinear systems with delays in the state, delay-dependent conditions and a descriptor system approach introduced in Fridman and Shaked (2001) was exploited in Hassan, Zemouche, and Boutayeb (2013) to estimate the state and unknown input, based on $\mathcal{H}_\infty$ and Sobolev norm-based definitions of stability.

The relatively unexplored problem of designing unknown input observers for nonlinear systems with delayed measurements serves as the motivation for this paper. We design observers with guaranteed performance in the presence of unknown exogenous inputs acting on the states and measurement outputs. Sufficient conditions are provided for the estimation of these unknown inputs to a specified degree of accuracy. A preliminary version of this paper was presented in Chakrabarty, Buzzard, Fridman, and Żak (2016). Herein, we present the key results of the preliminary paper with proofs, and add new results that arise in special cases such as constant delays and linear error dynamics to enable the construction of simpler LMIs. For this latter scenario, we show that detectability is sufficient to provide finite state and unknown input estimation error bounds for any bounded unknown input.

2. Notation

We denote by $\mathbb{R}$ the set of real numbers, and $\mathbb{R}^{n\times m}$ the set of real $n \times m$ matrices. For any matrix P, we denote $P^\top$ its transpose; $\overline{\lambda}(P)$ is the largest eigenvalue of P, $\underline{\lambda}(P)$ is the smallest eigenvalue of P, and $\|P\|$ as the maximum singular value of P. For a vector v and symmetric matrix P, the quadratic form $v^\top P v$ is written as $\|v\|_P^2$ for brevity. For a symmetric matrix $M = M^\top$, we use the star notation to avoid rewriting symmetric terms, that is, $\begin{bmatrix} M_a & \star \\ M_b^\top & M_c \end{bmatrix} \equiv \begin{bmatrix} M_a & M_b \\ \star & M_c \end{bmatrix} \equiv \begin{bmatrix} M_a & M_b \\ M_b^\top & M_c \end{bmatrix}$.For any vector $v \in \mathbb{R}^n$, we consider the norm $\|v\| = \sqrt{v^\top v}$. For a bounded function $v(\cdot) : \mathbb{R} \to \mathbb{R}^n$, we consider the norm $\|v(\cdot)\|_\infty = \sup_t \|v(t)\|$, and the space of functions which satisfy $\|v(\cdot)\|_\infty < \infty$ is denoted $\mathcal{L}_\infty$. The space of square integrable functions is denoted $\mathcal{L}_2$, with a corresponding norm $\|\cdot\|_{\mathcal{L}_2}$. We also denote $\mathcal{D}f$ as the derivative of a differentiable function f. The space of continuously differentiable functions $\phi : [a, b] \to \mathbb{R}^n$ is denoted $\mathcal{C}[a, b]$ and a norm $\|\phi\|_{\mathcal{C}} \triangleq \sup_{[a,b]} |\phi(\cdot)|$ is defined. The Banach space of absolutely continuous functions $\psi : [a, b] \to \mathbb{R}^n$ with $\mathcal{D}\psi \in \mathcal{L}_2(a, b)$ is denoted as $\mathcal{W}[a, b]$ possessing the norm $\|\psi(\cdot)\|_{\mathcal{W}} = \|\psi(\cdot)\|_{\mathcal{C}} + \|\mathcal{D}\psi\|_{\mathcal{L}_2}$. The notation $\langle\cdot,\cdot\rangle$ denotes an inner product.

3. Problem statement

In this section, we present the class of systems considered and provide an observer architecture for the estimation of system states to a specified degree of accuracy in spite of corruptive exogenous inputs. We consider a class of nonlinear systems with measurement delays modeled by

$$\dot{x} = Ax + B_f f(t, q) + Bw + g(t, u, y) \tag{1a}$$
$$y = Cx_\tau + Dw \tag{1b}$$
$$q = C_q x + D_q w \tag{1c}$$
$$x(t) = \phi_0(t), \quad \forall t \in [-h, 0], \tag{1d}$$

where $t \geq t_0 \in \mathbb{R}$ is the time variable, $\tau(t) : \mathbb{R} \mapsto [0, h]$ is a known, time-varying, bounded delay function, $x \triangleq x(t) \in \mathbb{R}^{n_x}$ is the state-vector, $x_\tau \triangleq x(t - \tau(t))$ is the delayed state vector, and $w \triangleq w(t) \in \mathbb{R}^{n_w}$ models unknown state and sensor attack vectors, which we will equivalently refer to as the exogenous input vector. If the unknown inputs in the state are denoted $w_s(t)$ and in the outputs $w_o(t - \tau(t))$, then $w(t)^\top := \begin{bmatrix} w_s(t)^\top & w_o(t-\tau(t))^\top \end{bmatrix}$.The variable $y \triangleq y(t) \in \mathbb{R}^{n_y}$ denotes a delayed measured output, and the function $\phi_0 \in \mathcal{C}([-h, 0])$ is a continuous map of initial conditions. The known nonlinearity $f : \mathbb{R} \times \mathbb{R}^{n_q} \to \mathbb{R}^{n_f}$ contains the argument $q \triangleq q(t) \in \mathbb{R}^{n_q}$. Since q is a function of the state x, it is not known for all time. Therefore, q has to be estimated. The nonlinearity $g : \mathbb{R} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ models the completely known information, such as the control input u and the measured output y. The matrices A, B_f, B, C, D, C_q and D_q are of appropriate dimensions. In this paper, we characterize nonlinearities via their incremental multiplier matrices (Chakrabarty, Corless, Buzzard, Żak, & Rundell, 2017).

Definition 1 (*Incremental Multiplier Matrices*)**.** A symmetric matrix $M \in \mathbb{R}^{(n_q+n_f)\times(n_q+n_f)}$ is an **incremental multiplier matrix** (δMM) for f if it satisfies the following **incremental quadratic constraint** (δQC) for all $t \in \mathbb{R}$ and $q_1, q_2 \in \mathbb{R}^{n_q}$:

$$\begin{bmatrix} \Delta q \\ \Delta f \end{bmatrix}^\top M \begin{bmatrix} \Delta q \\ \Delta f \end{bmatrix} \geq 0, \tag{2}$$

where $\Delta q \triangleq q_1 - q_2$ and $\Delta f \triangleq f(t, q_1) - f(t, q_2)$.

Example 1. Consider the nonlinearity $f(q) = q^{\frac{1}{3}}$: note that this is not globally Lipschitz. However, it satisfies the inequality $(q_1 - q_2)(f(q_1) - f(q_2)) \geq 0$. Hence, an δMM for f is given by $M = \kappa \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, with $\kappa > 0$.

Characterizing nonlinearities using incremental multipliers allows us to generalize our observer design methodology to a broad class of nonlinear systems. A library of δMMs for many nonlinearities are provided in Açıkmeşe and Corless (2011) and Chakrabarty, Corless et al. (2017).

In order to estimate the states of the plant (1), we propose an observer of the form

$$\dot{\hat{x}} = A\hat{x} + B_f f\left(t, \hat{q}\right) + L_1(\hat{y} - y) + g(t, u, y) \tag{3a}$$
$$\hat{y} = C\hat{x}_\tau \tag{3b}$$
$$\hat{q} = C_q \hat{x}, \tag{3c}$$

with initial condition $\hat{\phi}_0$. Here, $L_1 \in \mathbb{R}^{n_x \times n_y}$ is the observer gain, $\hat{x}$ is the estimate of the plant state, and $\hat{q}$ is the estimate of the argument of the nonlinearity f.

Let the state estimation error be denoted $\epsilon \triangleq \hat{x} - x$. Then the observer error dynamics are given by

$$\dot{\epsilon} = A\epsilon + L_1 C\epsilon_\tau + B_f \Delta f - (B + L_1 D)w, \tag{4}$$

where $\epsilon_\tau = \epsilon(t - \tau(t))$,

$$\Delta f \triangleq f(t, \hat{q}) - f(t, q). \tag{5}$$

For convenience, we write

$$\Delta q \triangleq \hat{q} - q = C_q \epsilon - D_q w. \tag{6}$$

We also define a performance output

$$z(t) = H\epsilon(t) \tag{7}$$

in case all components of the error states are not equally important from a disturbance attenuation perspective. Before we formally state our objective, we need the following definition. This is a modification of $\mathcal{L}_\infty$-stability for continuous and discrete time systems, discussed in Chakrabarty, Corless et al. (2017) and Chakrabarty, Żak, and Sundaram (2016) to time-delay systems.

Definition 2. Consider a general time-delayed nonlinear error system

$$\dot{\epsilon} = F(t, \epsilon, \epsilon_\tau, w) \tag{8a}$$

with initial condition $\epsilon(t) = \phi_0$ for $t \in [-h, 0]$, and performance output

$$z = G(t, \epsilon, w), \tag{8b}$$

where $t \in \mathbb{R}$, $\epsilon, \epsilon_\tau \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$, and $z \in \mathbb{R}^{n_z}$. The time-delay system (8) is globally uniformly $\mathcal{L}_\infty$-stable with performance level $\gamma > 0$ for any delay $0 \le \tau(t) \le h$ if the following conditions are satisfied:

(P1) **Global uniform exponential stability.** The nominal system, obtained by setting $w \equiv 0$, is globally uniformly exponentially stable with respect to the origin. That is, for every initial condition $\phi_0 \in \mathcal{W}[-h, 0]$, there exist constants $K \ge 1$ and $\alpha > 0$ such that

$$\|\epsilon(t)\|^2 \le K e^{-\alpha(t-t_0)} \|\phi_0(\cdot)\|^2_{\mathcal{W}}.$$

(P2) **Global uniform boundedness of the error state.** For every initial condition map $\phi_0 \in \mathcal{W}[-h, 0]$ and every exogenous input $w \in \mathcal{L}_\infty$, there exists a bound $\beta(\|\phi_0(\cdot)\|^2_{\mathcal{W}}, \|w(\cdot)\|_\infty)$ such that

$$\|\epsilon(t)\| \le \beta(\|\phi_0(\cdot)\|^2_{\mathcal{W}}, \|w(\cdot)\|_\infty)$$

for all $t \ge t_0$.

(P3) **Output response for zero initial error state.** For a zero initial condition map $\phi_0 \equiv 0$, and every exogenous input $w \in \mathcal{L}_\infty$, we have

$$\|z(t)\| \le \gamma \|w(\cdot)\|_\infty$$

for all $t \ge t_0$.

(P4) **Ultimate output response.** For every initial condition map $\phi_0 \in \mathcal{W}[-h, 0]$, and every exogenous input $w \in \mathcal{L}_\infty$, we have

$$\limsup_{t\to\infty} \|z(t)\| \le \gamma \|w(\cdot)\|_\infty. \tag{9}$$

Our objective is to construct an observer of the form (3) for estimating the states of the plant (1) with delayed measurements while attenuating the effect of the unknown exogenous inputs. To this end, we design observer gains L_1 and L_2 using convex programming methods to ensure that the observer error dynamical system (4) is $\mathcal{L}_\infty$-stable operating at a specified performance level for any delay with known bounds.

4. State and unknown input estimation for nonlinear systems with delayed measurements

In this section, we provide delay-dependent LMI conditions for the observer design. The following lemma from Chakrabarty, Buzzard et al. (2016) will be used to arrive at our main result.

Lemma 1. *Let $\epsilon(t) \in \mathbb{R}^{n_x}$ be a solution to the error system (8a) with initial condition $\phi_0 \in \mathcal{W}[-h, 0]$, and $z(t)$ be a performance output of the form (8b). Let $\phi_t(s) \triangleq \epsilon(t+s)$ for $s \in [-h, 0]$ and $t \ge 0$. Let $V : \mathcal{W}[-h, 0] \times \mathcal{L}_2(-h, 0) \to [0, \infty)$ be a functional of the form*

$$V(\phi_t, \dot\phi_t) = \langle \mathcal{T}_1 \phi_t, \phi_t \rangle + \langle \mathcal{T}_2 \dot\phi_t, \dot\phi_t \rangle \tag{10}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are linear operators whose Frechet derivatives exist. If there exist positive scalars $\chi_1 < \chi_2$, α, μ_1, μ_2 such that

$$\chi_1 \|\phi_t(0)\|^2 \le V(\phi_t, \dot\phi_t) \le \chi_2 \|\phi_t(\cdot)\|^2_{\mathcal{W}}, \tag{11a}$$

and

$$\dot V(\phi_t, \dot\phi_t) \le -2\alpha \left(V(\phi_t, \dot\phi_t) - \mu_1 \|w(t)\|^2 \right), \tag{11b}$$

$$\|G(t, \epsilon, w)\|^2 \le \mu_2 V(\phi_t, \dot\phi_t) \tag{11c}$$

for any $t \ge 0$, then the system (8a) with performance output (8b) is $\mathcal{L}_\infty$-stable with performance level $\gamma = \sqrt{\mu_1 \mu_2}$ for any delay $0 \le \tau(t) \le h$.

The following design theorem guides the construction of observers of the form (3) operating at specified performance levels.

Theorem 1. *Let the scalars h, η, α, ρ be fixed. If there exist matrices $P_0 = P_0^\top \succ 0 \in \mathbb{R}^{n_x \times n_x}$, $P_1 \in \mathbb{R}^{n_x \times n_x}$, $R = R^\top \succ 0 \in \mathbb{R}^{n_x \times n_x}$, $S = S^\top \succ 0 \in \mathbb{R}^{n_x \times n_x}$, $N \in \mathbb{R}^{n_x \times n_x}$, $Y_1 \in \mathbb{R}^{n_x \times n_y}$ an incremental multiplier matrix $M = M^\top \in \mathbb{R}^{(n_q+n_f)\times(n_q+n_f)}$ for the nonlinearity f, and a positive scalar $\mu > 0$ such that the following inequalities hold:*

$$\Xi + \Gamma^\top M \Gamma \preceq 0 \tag{12a}$$

$$\begin{bmatrix} R & N \\ \star & R \end{bmatrix} \succeq 0 \tag{12b}$$

$$\begin{bmatrix} P_0 & H^\top \\ \star & \mu I \end{bmatrix} \succeq 0 \tag{12c}$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & \star & \star & \star & \star & \star \\ \Xi_{21} & \Xi_{22} & \star & \star & \star & \star \\ \Xi_{31} & 0 & \Xi_{33} & \star & \star & \star \\ \Xi_{41} & \Xi_{42} & \Xi_{43} & \Xi_{44} & \star & \star \\ \Xi_{51} & \Xi_{52} & 0 & 0 & -2\alpha I & \star \\ \Xi_{61} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

with

$$\Xi_{11} = A^\top P_1 + P_1^\top A + 2\alpha P_0 + S - e^{-2\alpha h} R,$$
$$\Xi_{21} = P_0 - P_1 + \rho P_1^\top A, \quad \Xi_{22} = -\rho(P_1 + P_1^\top) + h^2 R,$$
$$\Xi_{31} = e^{-2\alpha h} N^\top, \quad \Xi_{33} = -e^{-2\alpha h}(S + R),$$
$$\Xi_{41} = C^\top Y_1^\top + e^{-2\alpha h}(R - N)^\top,$$
$$\Xi_{42} = \rho C^\top Y_1^\top, \quad \Xi_{43} = e^{-2\alpha h}(R^\top - N),$$
$$\Xi_{44} = e^{-2\alpha h}\left(N + N^\top - 2R\right), \quad \Xi_{51} = -(B^\top P_1 + D^\top Y_1^\top),$$
$$\Xi_{52} = -\rho(B^\top P_1 + D^\top Y_1^\top), \quad \Xi_{61} = B_f^\top P_1,$$

and

$$\Gamma = \begin{bmatrix} C_q & 0 & 0 & 0 & -D_q & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

then the observer (3) characterized by the gain $L_1 = P_1^{-\top} Y_1$ has $\mathcal{L}_\infty$-stable error dynamics with a performance output $z = H\epsilon$ operating at a performance level $\gamma = \sqrt{\mu}$ for any delay $0 \le \tau(t) \le h$.

Proof. As in Fridman (2014b, Proposition 1), let

$$V_1 = \|\epsilon(t)\|^2_{P_0}, \tag{14a}$$

$$V_2 = \int_{t-h}^{t} e^{-2\alpha(t-s)} \|\epsilon(s)\|^2_S \, ds, \tag{14b}$$

$$V_3 = h \int_{-h}^{0} \int_{t+r}^{t} e^{-2\alpha(t-s)} \|\dot\epsilon(s)\|^2_R \, ds\, dr. \tag{14c}$$

By interchanging the limits of integration, we get

$$V_3 = h \int_{t-h}^{t} (h + s - t) e^{-2\alpha(t-s)} \|\dot\epsilon(s)\|^2_R \, ds.$$

Let $V \triangleq \sum_{i=1}^{3} V_i$ be a Lyapunov–Krasovskii functional candidate. Since P_0, R and S are positive definite, V is positive definite. Also, $\underline{\lambda}(P_0)\|\phi_t(0)\|^2 \le V$, and

$$V \le \left(\bar\lambda(P_0) + h\bar\lambda(S) + h^2 \bar\lambda(R)\right) \|\phi_t(\cdot)\|^2_{\mathcal{W}},$$

which implies that $\chi_1 = \underline{\lambda}(P_0)$, and $\chi_2 = \bar\lambda(P_0) + h\bar\lambda(S) + h^2\bar\lambda(R)$, in (11a). In order to evaluate the time-derivative of V on the trajectories of the observer error dynamics (4), we perform the following computations. Using the general Leibniz integral rule yields

$$\dot{V}_1 = 2\epsilon(t)^\top P_0 \dot{\epsilon}(t), \tag{15a}$$

$$\begin{aligned}\dot{V}_2 &= \|\epsilon(t)\|_S^2 - e^{-2\alpha h}\|\epsilon(t-h)\|_S^2 \\ &\quad - 2\alpha \int_{t-h}^{t} e^{-2\alpha(t-s)}\|\epsilon(s)\|_S^2\, ds,\end{aligned} \tag{15b}$$

$$\begin{aligned}\dot{V}_3 &= h^2\|\dot{\epsilon}(t)\|_R^2 - he^{-2\alpha h}\int_{t-h}^{t}\|\dot{\epsilon}(s)\|_R^2\, ds \\ &\quad - 2\alpha h \int_{-h}^{0}\int_{t+r}^{t} e^{-\alpha(t-s)}\|\dot{\epsilon}(s)\|_R^2\, ds\, dr.\end{aligned} \tag{15c}$$

Note that the error dynamics (4) can be re-written as $A\epsilon + L_1 C\epsilon_\tau + B_f\Delta f - (B + L_1 D)w - \dot{\epsilon} = 0$, where $\epsilon_\tau = \epsilon(t-\tau(t))$ and Δf is defined in (5). We use the descriptor method proposed in Fridman and Shaked (2002). This method involves treating $\dot{\epsilon}$ as an auxiliary state, not parameter instead of replacing the dynamics (4) directly into $\dot{V}_1$. That is,

$$\begin{aligned}\dot{V}_1 &= 2e(t)^\top P_0\dot{\epsilon}(t) + 2(\epsilon^\top(t)P_1^\top + \rho\dot{\epsilon}^\top(t)P_1^\top)(A\epsilon + L_1 C\epsilon_\tau \\ &\quad + B_f\Delta f - (B+L_1D)w - \dot{\epsilon}).\end{aligned}$$

Next, we use Jensen's inequality (see Fridman, 2014a, Lemma 3.4) and inequality (12b) to bound the second term of $\dot{V}_3$ as follows:

$$- he^{-2\alpha h}\int_{t-h}^{t}\|\dot{\epsilon}(s)\|_R^2\, ds \le -e^{-2\alpha h}\begin{bmatrix} r_1 \\ r_2\end{bmatrix}^\top \begin{bmatrix} R & N \\ \star & R\end{bmatrix}\begin{bmatrix} r_1 \\ r_2\end{bmatrix},$$

where $r_1 = \epsilon - \epsilon_\tau$ and $r_2 = \epsilon_\tau - \epsilon_h$, where $\epsilon_h = \epsilon(t-h)$. Let

$$\xi^\top = \begin{bmatrix}\epsilon^\top & \dot{\epsilon}^\top & \epsilon_h^\top & \epsilon_\tau^\top & w^\top & \Delta f^\top\end{bmatrix}.$$

Applying a congruence transform on the matrix inequality (12a) with ξ, we obtain

$$0 \ge \xi^\top\left(\Xi + \Gamma^\top M\Gamma\right)\xi = \xi^\top \Xi\xi + \xi^\top\Gamma^\top M\Gamma\xi.$$

Note that

$$\Gamma\xi = \begin{bmatrix} C_q\epsilon - D_q w \\ f(t,\hat{q}) - f(t,q)\end{bmatrix} = \begin{bmatrix}\Delta q \\ \Delta_f\end{bmatrix},$$

where Δq and Δf are defined in (6) and (5), respectively. Since M is an incremental multiplier matrix for f, we conclude that $\xi^\top\Gamma^\top M\Gamma\xi \ge 0$. Hence,

$$\begin{aligned}0 \ge \xi^\top\Xi\,\xi &= \sum_{i=1}^{3}\left(\dot{V}_i + 2\alpha V_i\right) - 2\alpha\|w\|^2 \\ &= \dot{V} + 2\alpha(V - \|w\|^2).\end{aligned}$$

Taking the Schur complement of (12c), we get $\mu\epsilon^\top P_0\epsilon \ge \epsilon^\top H^\top H\epsilon = \|z\|^2$. But $\mu > 0$ and $V \ge \epsilon^\top P_0\epsilon$, hence the conditions of Lemma 1 are satisfied with $\mu_1 = 1$ and $\mu_2 = \mu$, and that completes the proof. □

Remark 1. The inequalities (12) are linear matrix inequalities in M, N, P_0, P_1, R, S, Y and μ when the positive scalars α, ρ, η and h are fixed.

5. Unknown input estimation

In this section, we present sufficient conditions for reconstructing components of the unknown input signal

$$v = \mathcal{H}w, \tag{16}$$

where $\mathcal{H} \in \mathbb{R}^{n_v\times n_w}$ is known. From $\mathcal{H}$, we can compute $\mathcal{G}$ from

$$\mathcal{G}\begin{bmatrix} B \\ D\end{bmatrix} = \mathcal{H},$$

which will be used to guarantee the unknown input estimation quality.

We make the following assumption on the disturbance input w and state x.

Assumption 1. The exogenous input $w \in \mathcal{L}_\infty$ is absolutely continuous and its time-derivative $\dot{w} \in \mathcal{L}_\infty$.

Assumption 2. The derivative of the state of the plant (1) is bounded, that is, $\dot{x} \in \mathcal{L}_\infty$.

The class of nonlinearities considered in this section have q as their sole argument, and conform to the following condition.

Assumption 3. The function $f \triangleq f(q)$ is differentiable and there is a scalar κ_1 such that $\|\mathcal{D}f(q)\| \le \kappa_1$ for any $q \in \mathbb{R}^{n_q}$.

To proceed, the class of time-varying delays is restricted.

Assumption 4. The time delay τ is absolutely continuous, and there exists a known scalar $\mathcal{K} > 0$ such that $|\dot{\tau}(t)| \le \mathcal{K}$ for any $t \in \mathbb{R}$.

5.1. Main unknown input estimation result

Suppose an estimate of the unknown input v be given by

$$\hat{v} \triangleq \mathcal{G}\begin{bmatrix} L_1(\hat{y} - y) \\ -(\hat{y} - y)\end{bmatrix}. \tag{17}$$

Herein, we demonstrate that if certain linear matrix inequalities are satisfied, then we can guarantee an ultimate bound on the unknown input estimation error norm $\|\hat{v} - v\|$ by leveraging the notion of $\mathcal{L}_\infty$-stability with specified performance for time delay systems.

Theorem 2. *Suppose Assumptions 1–4 hold. Let scalars h, α, $\mathcal{K}$, ρ be fixed. Suppose there exist matrices $P_0 = P_0^\top \succ 0$, P_1, Y_1, $Q = Q^\top \succ 0$, $R = R^\top \succ 0$, $S = S^\top \succ 0$, N, an incremental multiplier matrix M for the nonlinearity f, and scalars $\nu > 0$, $\mu > 0$ such that the following conditions, along with (12b) and (12c) with $H = I$, hold:*

$$\left[\begin{array}{cc|cc} \Xi & \star & \star & \star \\ \Phi_0 & -2\alpha I & \star & \star \\ \hline \Phi_1 & 0 & -\nu I & 0 \\ \Phi_2 & 0 & 0 & -\nu I\end{array}\right] + \left[\begin{array}{c|c}\hat{\Gamma}^\top M\hat{\Gamma} & 0 \\ \hline 0 & 0\end{array}\right] \preceq 0 \tag{18}$$

where the matrix Ξ has been defined in (13),

$$\hat{\Gamma} = \begin{bmatrix} C_q & 0 & 0 & 0 & -D_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0\end{bmatrix}, \tag{19}$$

and

$$\left[\begin{array}{c}\Phi_0 \\ \hline \Phi_1 \\ \hline \Phi_2\end{array}\right] = \left[\begin{array}{cccc} B_f^\top P_1 & \rho B_f^\top P_1 & 0 & 0 \\ \hline -\nu\mathcal{K}I & 0 & 0 & 0 \\ \hline 0 & C^\top Y_1^\top & \rho C^\top Y_1^\top & 0\end{array}\right]. \tag{20}$$

Then the observer (3) with gain $L_1 = P_1^{-\top}Y_1$ generates an estimate $\hat{v}$ defined in (17) that satisfies

$$\limsup_{t\to\infty}\|\hat{v} - v\| \le \gamma_1\|w(\cdot)\|_\infty + \gamma_2\|\dot{w}(\cdot)\|_\infty + \gamma_3\|\dot{x}(\cdot)\|_\infty, \tag{21}$$

for any delay $0 \le \tau(t) \le h$, where

$$\begin{aligned}\gamma_1 &= \|\mathcal{G}\|\left(\kappa_1\|B_f\|\|D_q\|\right. \\ &\quad \left. + \sqrt{\mu}\left(\|A\| + \|C\| + \kappa_1\|B_f\|\|C_q\|\right)\right),\end{aligned} \tag{22a}$$

$$\gamma_2 = \sqrt{\mu}\|\mathcal{G}\|(1 + \kappa_2\|D_q\|), \tag{22b}$$

$$\gamma_3 = \sqrt{\mu}\,\kappa_2\|\mathcal{G}\|\|C_q\|. \tag{22c}$$

Proof. Let $\Delta f = f(\hat{q}) - f(q)$, and recall that

$$\hat{y} - y = C\epsilon_\tau - Dw. \tag{23}$$

We begin by rewriting the error dynamics (4) and output error equation (23) as

$$\begin{bmatrix} L_1(\hat{y}-y) - Bw \\ -(\hat{y}-y) - Dw \end{bmatrix} = \begin{bmatrix} \dot{\epsilon} - A\epsilon - B_f \Delta f \\ -C\epsilon_\tau \end{bmatrix}.$$

Pre-multiplying throughout by $\mathcal{G}$ and recalling the definition of $\hat{v}$ in (17), we obtain

$$\hat{v} - v = \mathcal{G} \begin{bmatrix} \dot{\epsilon} - A\epsilon - B_f \Delta f \\ -C\epsilon_\tau \end{bmatrix}.$$

Hence,

$$\begin{aligned} \limsup_{t\to\infty} \|\hat{v}(t) - v(t)\| \le \|\mathcal{G}\| \Big(&\|A\| \limsup_{t\to\infty} \|\epsilon(t)\| \\ &+ \|B_f\| \limsup_{t\to\infty} \|\Delta f\| \\ &+ \limsup_{t\to\infty} \|\dot{\epsilon}(t)\| \\ &+ \|C\| \limsup_{t\to\infty} \|\epsilon_\tau(t)\| \Big). \end{aligned} \tag{24}$$

The rest of the proof will leverage the notion of $\mathcal{L}_\infty$-stability with specified performance to bound each limit superior on the right-hand side of (24).

We note that a feasible solution to the matrix inequalities (18), (12b), and (12c) imply that there exists a feasible solution to the matrix inequalities (12) with $H = I$. Applying Theorem 1, we conclude that there exists an observer of the form (3) with gain $L_1 = P_1^{-\top} Y_1$ that satisfies

$$\limsup_{t\to\infty} \|\epsilon(t)\| \le \sqrt{\mu} \|w(\cdot)\|_\infty \tag{25}$$

for any delay $0 \le \tau(t) \le h$. Since this is true for arbitrarily large t and τ is bounded, we also have

$$\limsup_{t\to\infty} \|\epsilon_\tau(t)\| \le \sqrt{\mu} \|w(\cdot)\|_\infty. \tag{26}$$

We now show that $\|\Delta f\|$ is ultimately bounded. It follows from Assumption 3, and the definition of Δq in (6) that

$$\|\Delta f\| \le \kappa_1 \|\Delta q\| \le \kappa_1 \|C_q \epsilon - D_q w\|.$$

Using (25), it is clear that

$$\limsup_{t\to\infty} \|\Delta f\| \le \kappa_1 \left(\sqrt{\mu} \|C_q\| + \|D_q\| \right) \|w(\cdot)\|_\infty. \tag{27}$$

It remains to show that $\dot{\epsilon}$ is ultimately bounded. We demonstrate this by taking the time-derivative of the error system and noting that this 'error-derivative system' is an uncertain system, with $\dot{\tau}$ behaving like a norm-bounded uncertainty. With this insight, we can verify that a feasible solution to (18) results in a finite ultimate bound on $\dot{\epsilon}$ for the uncertain error-derivative system, since w, $\dot{w}$ and $\dot{x}$ are bounded by assumption.

Let

$$\eta \triangleq \dot{\tau}/\mathcal{K}. \tag{28}$$

By Assumption 4,

$$|\eta(t)| \le 1 \tag{29}$$

for any $t \in \mathbb{R}$. Next, we take the time-derivative of the error system (4), which yields the error-derivative system

$$\ddot{\epsilon} = A\dot{\epsilon} + (L_1 + \Delta L) C \dot{\epsilon}_\tau + B_f \frac{d\Delta f}{dt} - (B + L_1 D)\dot{w}, \tag{30}$$

where

$$\Delta L \triangleq \Delta L(t) = -\dot{\tau} L_1 = -\mathcal{K}\eta(t) L_1.$$

Since the quantity $\dot{\tau}$ (equivalently η) is not readily available and $|\eta| \le 1$, we treat the quantity ΔL as a norm-bounded uncertainty in the sense of Petersen (1987). We also note that

$$\begin{aligned} \frac{d\Delta f}{dt} &= \frac{d}{dt}(f(\hat{q}) - f(q)) = \mathcal{D}f(\hat{q})\dot{\hat{q}} - \mathcal{D}f(q)\dot{q} \\ &= \mathcal{D}f(\hat{q})(\dot{\hat{q}} - \dot{q}) + \left(\mathcal{D}f(\hat{q}) - \mathcal{D}f(q)\right)\dot{q}. \end{aligned}$$

From (6), we get $\dot{\hat{q}} - \dot{q} = C_q\dot{\epsilon} - D_q\dot{w}$ and $\dot{q} = C_q\dot{x} + D_q\dot{w}$. Hence,

$$\ddot{\epsilon} = A\dot{\epsilon} + (L_1 + \Delta L_1(t))C\dot{\epsilon}_\tau + B_f\tilde{f}(t,\tilde{q}) + B_{\tilde{w}}\tilde{w}, \tag{31a}$$

$$\tilde{f}(t,\tilde{q}) = \mathcal{D}f(\hat{q}(t))\tilde{q}, \tag{31b}$$

$$\tilde{q} = C_q\dot{\epsilon} - D_q\dot{w}, \tag{31c}$$

where $\tilde{w} = \begin{bmatrix} \dot{w} & \tilde{w}_2 \end{bmatrix}$ with

$$\tilde{w}_2 = \left(\mathcal{D}f(\hat{q}) - \mathcal{D}f(q)\right)\dot{q}, \tag{32}$$

and $B_{\tilde{w}} = \begin{bmatrix} -B - L_1 D & B_f \end{bmatrix}$. Since M is an incremental multiplier for f, we know from D'Alto and Corless (2013, Lemma 4.4) that

$$\begin{bmatrix} \tilde{q} \\ \mathcal{D}f(\hat{q})\tilde{q} \end{bmatrix}^\top M \begin{bmatrix} \tilde{q} \\ \mathcal{D}f(\hat{q})\tilde{q} \end{bmatrix} \ge 0$$

for all $\hat{q}, \tilde{q} \in \mathbb{R}^{n_q}$. Hence, M is also an incremental multiplier matrix for $\tilde{f}$ defined in (31b). Henceforth, we consider the error-derivative system (31) as an uncertain nonlinear system with state $\dot{\epsilon}$, nonlinearity $\tilde{f}$, and exogenous input $\tilde{w}$. Since, by Assumptions 1 and 2, $\dot{w}$ and $\dot{x}$ are bounded, we have,

$$\|\dot{q}\| \le \|C_q\|\|\dot{x}\| + \|D_q\|\|\dot{w}\|, \tag{33}$$

that is, $\dot{q}$ is bounded. Also, from Assumption 3, we know that $\|\mathcal{D}f(\hat{q}) - \mathcal{D}f(q)\| \le \kappa_2$, which implies that $\|\tilde{w}_2\| \le \kappa_2\left(\|C_q\|\|\dot{x}\| + \|D_q\|\|\dot{w}\|\right)$. Consequently,

$$\|\tilde{w}\| \le \kappa_2\|C_q\|\|\dot{x}\| + \left(1 + \kappa_2\|D_q\|\right)\|\dot{w}\|. \tag{34}$$

Next, we demonstrate that solving (18) implies that the uncertain nonlinear system (31) with norm-bounded uncertainty ΔL is $\mathcal{L}_\infty$-stable with a specified performance level for any delay $0 \le \tau(t) \le h$. To this end, we take Schur complements of (18). Since $\nu > 0$, the inequality (18) is equivalent to

$$\begin{bmatrix} \Xi & \star \\ \Phi_0 & -2\alpha I \end{bmatrix} + \hat{\Gamma}^\top M \hat{\Gamma} + \frac{1}{\nu}\Phi_1^\top\Phi_1 + \frac{1}{\nu}\Phi_2^\top\Phi_2 \preceq 0, \tag{35}$$

where Φ_0, Φ_1 and Φ_2 are defined in (20).

Let $\tilde{\Phi}_1 = \begin{bmatrix} -\mathcal{K}I & 0 & 0 & 0 \end{bmatrix}$. Then, we get $(1/\nu)\Phi_1^\top\Phi_1 + (1/\nu)\Phi_2^\top\Phi_2 = \nu\tilde{\Phi}_1^\top\tilde{\Phi}_1 + (1/\nu)\Phi_2^\top\Phi_2$. Using Young's inequality, we know that for any $\nu > 0$ and any $|\eta| \le 1$,

$$\nu\tilde{\Phi}_1^\top\tilde{\Phi}_1 + \frac{1}{\nu}\Phi_2^\top\Phi_2 \succeq \eta\left(\tilde{\Phi}_1^\top\Phi_2 + \Phi_2^\top\tilde{\Phi}_1\right). \tag{36}$$

Substituting (36) into (35), we obtain

$$\begin{aligned} 0 &\succeq \begin{bmatrix} \Xi & \star \\ \Phi_0 & -2\alpha I \end{bmatrix} + \hat{\Gamma}^\top M \hat{\Gamma} + \eta\left(\tilde{\Phi}_1^\top\Phi_2 + \Phi_2^\top\tilde{\Phi}_1\right) \\ &= \hat{\Xi} + \hat{\Gamma}^\top M \hat{\Gamma}, \end{aligned} \tag{37}$$

where

$$\hat{\Xi} = \begin{bmatrix} \Xi_{11} & \star & \star & \star & \star & \star & \star \\ \Xi_{21} & \Xi_{22} & \star & \star & \star & \star & \star \\ \Xi_{31} & 0 & \Xi_{33} & \star & \star & \star & \star \\ \tilde{\Xi}_{41} & \tilde{\Xi}_{42} & \Xi_{43} & \Xi_{44} & \star & \star & \star \\ \Xi_{51} & \Xi_{52} & 0 & 0 & -2\alpha I & \star & \star \\ \Xi_{61} & 0 & 0 & 0 & 0 & 0 & \star \\ \Xi_{71} & \Xi_{72} & 0 & 0 & 0 & 0 & -2\alpha I \end{bmatrix}$$

with $\tilde{\Xi}_{41} = C^\top(L + \Delta L(t))^\top P_1$, $\tilde{\Xi}_{42} = C^\top(L + \Delta L(t))^\top \rho P_1$, $\Xi_{71} = B_f^\top P_1$, $\Xi_{72} = \rho B_f^\top P_1$, and $Y = P_1^\top L$. The other sub-matrices of Ξ are identical to those in (13) for the system (31). The terms $\tilde{\Xi}_{41}$ and $\tilde{\Xi}_{42}$ reflect the change in Ξ due to the presence of the norm-bounded uncertainty term $(L + \Delta L(t))C\dot{\epsilon}_\tau$. Since the inequality (18) is equivalent to (37) for the uncertain nonlinear system (31), applying Theorem 1 implies that a feasible solution to (18) guarantees that

$$\limsup_{t\to\infty} \|\dot{\epsilon}(t)\| \le \sqrt{\mu}\|\tilde{w}(\cdot)\|_\infty \tag{38}$$

for any delay $0 \le \tau(t) \le h$.

Replacing the right hand side of (24) with the corresponding bounds in (25), (26), (27), and (38), we obtain the desired performance bounds (21) for any delay $0 \le \tau(t) \le h$. This completes the proof. □

Remark 2. One can pose the unknown input estimation problem as a generalized eigenvalue problem to yield good performance. For example, if solving

$$\mu^\star := \arg\min \mu \text{ subject to the constraints (18)}$$

yields a small value of $\mu^\star$, then the ultimate unknown input estimation error is expected to be small.

This work extends our results on unknown input estimation for the delay-free case in Chakrabarty, Corless et al. (2017) in the following two ways. The additional challenge of non-constant time delays in the measurement equation (1b) leads to novel analysis and design via existing Lyapunov–Krasovskii techniques. We also present a novel estimation method for unknown input estimation based on the performance analysis of the error derivative system (30). Specifically for time-delay case an additional term $-\dot{\tau}L_1C\dot{e}_\tau$ has to be taken into account. This challenge is overcome by treating this as a norm-bounded uncertainty and leveraging this insight to construct new conditions detailed in Theorem 2.

5.2. Some special cases

Based on the main unknown input estimation result of this paper (Theorem 2), we can formulate simpler LMIs when additional conditions are satisfied. In particular, we consider the cases when (i) the time delay $\tau(t)$ is constant, and, (ii) the error dynamics are linear.

5.2.1. Case I: Constant time delay

We now consider the special case when the time delay τ is constant, that is, $\dot{\tau} \equiv 0$. The time derivative system of the error dynamics of the observer (3) becomes

$$\ddot{\epsilon} = A\dot{\epsilon} + L_1C\dot{\epsilon}_\tau + B_f\frac{d\Delta f}{dt} - (B + L_1D)\dot{w}. \tag{39}$$

Note that, unlike (30), this derivative system is not uncertain, which yields the following result.

Corollary 1. *Suppose Assumptions 1–4 hold. Let the scalars h, α, $\mathcal{K}$ and ρ be fixed. Suppose there exist matrices $P_0 = P_0^\top \succ 0$, $P_1 \succ 0$, Y_1, $Q = Q^\top \succ 0$, $S = S^\top \succ 0$, an incremental multiplier matrix M for the nonlinearity f, and a scalar $\mu > 0$ such that the following conditions, along with (12c) with $H = I$ hold:*

$$\begin{bmatrix} \Xi & \star \\ \Phi_0 & -2\alpha I \end{bmatrix} + \hat{\Gamma}^\top M \hat{\Gamma} \preceq 0 \tag{40}$$

where the matrix Ξ is defined in (13), $\hat{\Gamma}$ is defined in (19), and Φ_0 is defined in (20). Then the observer (3) with gain $L_1 = P_1^{-\top}Y_1$ generates an estimate $\hat{v}$ defined in (17), which satisfies (21) for any constant delay $0 \le \tau \le h$, where γ_1, γ_2 and γ_3 are described in (22).

Proof. Taking Schur complements of (40) yields (37) with $\Delta L = 0$. The inequality (12b) is not required for the case of constant delays (Fridman, 2014a). The rest of the proof follows arguments used in the proof of Theorem 2. □

5.2.2. Case II: Linear error dynamics

For $f = 0$, we have a system of the form

$$\dot{x} = Ax + Bw + g(t, u, y) \tag{41a}$$

$$y = Cx_\tau + Dw, \tag{41b}$$

with the symbols bearing the same meaning as in (1). The corresponding unknown input observer has the structure

$$\dot{\hat{x}} = A\hat{x} + L(\hat{y} - y) + g(t, u, y) \tag{42a}$$

$$\hat{y} = C\hat{x}_\tau \tag{42b}$$

that yields the error dynamics

$$\dot{\epsilon} = A\epsilon + LC\epsilon_\tau - (B + LD)w. \tag{43}$$

Taking the derivative of (43) with respect to t and recalling Assumption 4 yields the uncertain, linear time-delay system

$$\ddot{\epsilon} = A\dot{\epsilon} + (L + \Delta L(t))C\dot{\epsilon}_\tau - (B + LD)\dot{w}, \tag{44}$$

with $\Delta L(t) = -\dot{\tau}L = -\mathcal{K}\eta(t)L$ by (28). Note that η satisfies (29). The following corollary provides LMIs to reconstruct the unknown input w using the observer (42).

Corollary 2. *Suppose Assumptions 1 and 4 hold. Let the scalars h, α, $\mathcal{K}$ and ρ be fixed. Suppose there exist matrices $P_0 = P_0^\top \succ 0$, $P_1 \succ 0$, Y, $Q = Q^\top \succ 0$, $R = R^\top \succ 0$, $S = S^\top \succ 0$, N, and scalars $\nu > 0$, $\mu > 0$ such that the following inequalities, along with (12b) and (12c) with $H = I$, hold:*

$$\begin{bmatrix} \Xi & \star & \star \\ \Phi_1 & -\nu I & 0 \\ \Phi_2 & 0 & -\nu I \end{bmatrix} \preceq 0 \tag{45}$$

where the matrix Ξ has been defined in (13) and Φ_1, Φ_2 are described in (20). Then the observer (3) with gain $L = P_1^{-\top}Y$ generates an estimate $\hat{v}$ defined in (17), which satisfies

$$\limsup_{t\to\infty} \|\hat{v}(t) - v(t)\| \le \sqrt{\mu}\|\mathcal{G}\|(\|A\| + \|C\|)\|w(\cdot)\|_\infty + \sqrt{\mu}\|\mathcal{G}\|\|\dot{w}(\cdot)\|_\infty,$$

for any delay $0 \le \tau(t) \le h$.

Proof. We prove this by removing the matrices associated with the nonlinearity, namely, M, C_q, D_q, and B_f from the conditions (18) and Eq. (22). □

Next, we demonstrate that if, for the system (41), the pair (A, C) is detectable, then the performance level γ is finite for any performance output $z = H\epsilon$.

Theorem 3. *Consider the system (41). Suppose (A, C) is detectable. Then, for any performance output $z = H\epsilon$ and sufficiently small h, there exists an observer (42) with gain L such that the observer error dynamics (43) is $\mathcal{L}_\infty$-stable with performance level $\gamma < \infty$, for any $0 < \tau(t) \le h$.*

Proof. Since the pair (A, C) is detectable, this implies that there exists a matrix L such that $A + LC$ is Hurwitz. Consider the error dynamics (43) with $\tau \equiv 0$, namely $\dot{\epsilon} = (A + LC)\epsilon - (B + LD)w$ Since $A + LC$ is Hurwitz, from Krichman, Sontag, and Wang (2001, Proposition 2.6), we know that this implies the existence of two positive scalars $\delta_1 \in (0, \infty)$ and $\delta_2 \in (0, \|A + LC\|)$ such that $\|\epsilon(t)\| \le \delta_1 e^{-\delta_2 t}\|\epsilon(0)\| + \frac{\delta_1}{\delta_2}\|B + LD\|\|w(\cdot)\|_\infty$. Therefore, $\limsup_{t\to\infty}\|z(t)\| \le \frac{\delta_1}{\delta_2}\|H\|\|B + LD\|\|w(\cdot)\|_\infty$, which implies $\gamma = \|H\|\|B + LD\|\delta_1/\delta_2 < \infty$. As we have LMI-based sufficient conditions that are feasible for non-delay case, then by standard arguments for delay-dependent conditions via simple Lyapunov–Krasovskii functionals (Fridman, 2014a), the time-delay system will preserve a finite gain for small enough delays. □

6. Simulation results on enzyme kinetics

To demonstrate the performance of our proposed observer, we use a modification of an enzyme kinetic model for studying oscillations at the cellular level, fitted to experimental data in Goodwin (1965). A challenge in estimating the states of such cellular systems *in vitro* or *in vivo* is that measurements are generally delayed due to inherent time lags in the measurement apparatus/method. The delay is also non-constant, typically exhibiting low variance.

We incorporate these difficulties into the nonlinear dynamical system of enzyme kinetics as follows:

$$\dot{x}_1(t) = -x_1(t) + \frac{360}{43 + x_3^r} + w(t) \tag{46a}$$

$$\dot{x}_2(t) = x_1 - 0.6x_2 \tag{46b}$$

$$\dot{x}_3(t) = x_2 - 0.8x_3 \tag{46c}$$

$$y(t) = x_1(t - \tau(t)) + w(t). \tag{46d}$$

Here, x_1 represents the concentration of an enzyme whose synthesis rate is regulated by a metabolite x_3, and is regulated in turn by the metabolite x_2. The nonlinear term is akin to a Hill function, and r is chosen to be 10, as in Chis, Banga, and Balsa-Canto (2011) to induce oscillatory behavior in the cells. We made the following modifications to illustrate the performance of our method: (a) the unknown input $w(t)$ representing measurement noise and intracellular crosstalk, is added to the state x_1 and the output y; this w is generated from a uniform distribution with smoothing (to ensure differentiability of w) and lies in the range $[-1, 1]$; and, (b) the time delay signal $\tau(t) = 0.5 + 0.2\sin(0.2t)$ is introduced to reflect the realistic challenge of delayed measurements: τ is a positively biased sinusoidal signal bounded by $h = 0.7$ that satisfies $|\dot{\tau}| \le 0.04$.

We begin by rewriting the system (46) in the form (1). This yields the matrices

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -0.6 & 0 \\ 0 & 1 & -0.8 \end{bmatrix}, \; B_f = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; D = 1, \; C_q = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \; D_q = 0.$$

Note that since there is no control input or other redundant information, we have $g = 0$. With $q = x_3$, the nonlinearity can be written as $f(q) = 360/(43 + q^{10})$, and $g = 0$. Numerically, we obtain that the derivative of f is bounded by $\sigma_f = 14.514$, implying from D'Alto and Corless (2013, Section 6.1) that an incremental multiplier matrix for f is given by $M = \operatorname{diag}\left(\begin{bmatrix} k\sigma_f^2 & -k \end{bmatrix}\right)$, where $k > 0$ is a decision variable for the LMIs to be solved. We fix $\alpha = 0.05$, $\rho = 2$, and choose the known time-varying measurement delay illustrated in the top right plot of Fig. 1. Note that $\|w(\cdot)\|_\infty \le 1$ as shown in the bottom right plot of Fig. 1. Since we wish to estimate $v = w$, we deduce that $\mathcal{H} = 1$, and therefore, $\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$. We solve the LMIs in Theorem 2 using CVX (Grant & Boyd, 2008) to obtain $k = 372.2974$, $\nu = 1.83 \times 10^5$, $L_1 = \begin{bmatrix} -0.9971 & -0.0005 & 0.0000 \end{bmatrix}^\top$, and $\gamma = 0.0016$.

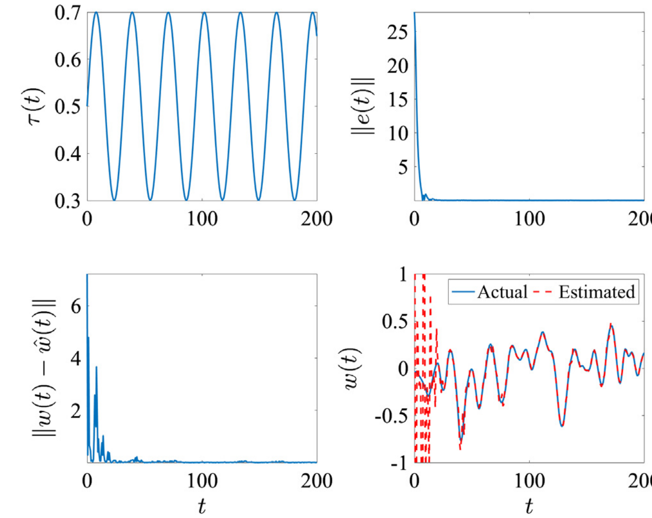

Fig. 1. (*Top right*) Decay of the state estimation error norm $\|e(t)\|$. (*Bottom left*) Decay of the unknown input estimation error norm $\|w(t) - \hat{w}(t)\|$. (*Top left*) Time-varying delay signal $\tau(t)$. (*Bottom right*) Unknown input $w(t)$ and its estimate.

The cellular oscillator and proposed observer is simulated in MATLAB. The result of our *in-silico* evaluation is shown in Fig. 1. As expected, the state estimation and unknown input estimation errors decay to small levels in spite of sharp changes in τ and w, and the unknown input estimation (bottom-right plot) is of satisfactory accuracy.

7. Conclusions

Many engineering applications involving cyber–physical systems or sensor signals arising from the internet-of-things produce measurable outputs that are delayed due to sensor lag or communication protocols such as handshaking. In this paper, we develop a systematic framework for developing observers that are capable of leveraging delayed measurements to provide state estimates in the presence of exogenous disturbances in the state and measurement vector fields, in addition to nonlinearities in the system structure. We express a wide range of nonlinearities via incremental quadratic constraints and generate computationally tractable conditions that are used to derive the observer gains. Furthermore, we provide sufficient conditions that enable the estimation of the unknown inputs acting on the system, despite measurement delays; this problem is very challenging and has remained relatively unexplored in the literature.

The suggested method has some limitations that pose open challenges. For example, there is no automatic method for deriving incremental multiplier matrices for a given nonlinearity, so some domain-specific knowledge is required to fix the structure of M. Another challenge is to relax the assumption of boundedness on $\dot{x}$, although this is not a practical concern. Advantages of our observer design strategy include simplicity and tractability of design, inherent graceful degradation that enables weakening of the derogatory effects of w in cases where w cannot be asymptotically estimated with arbitrary precision, and integrability into a wide range of systems arising in distributed autonomy, such as connected vehicles and medical internet-of-things.

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