

## RESEARCH ARTICLE

# Finite frequency $H_\infty$ control of singularly perturbed Euler-Lagrange systems: An artificial delay approach

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## Summary

In this paper, we show that small artificial delays in the feedback loops operating in different time scales may stabilize singularly perturbed systems (SPSs). An artificial delay approach is proposed for the robust stabilization and  $L_2$ -gain analysis of SPSs in the finite frequency domain. A two-time-scale delayed static output feedback controller is designed, in which the controller gains are formulated via a linear matrix inequality (LMI) algorithm. A distinctive feature of the proposed algorithm is setting controller parameters as free variables, which increases the degrees of freedom in controller design and leads to more flexibility in solving LMIs. Moreover, the proposed method is further extended to analyze the finite frequency system specifications of SPSs. The  $L_2$ -gain performance analysis is conducted for parameter-independent subsystems in their dominant frequency ranges, and the disturbance attenuation level of the original high-order system is then estimated. Finally, the efficiency of the proposed design method is verified in an active suspension system subject to multiple finite frequency disturbance.

## KEYWORDS

artificial time delays, disturbance attenuation, finite frequency, singular perturbations, static output feedback

## 1 | INTRODUCTION

Singular perturbation theory has been most suitable for modeling multiple-time-scale physical phenomena occurring at disparate time scales.<sup>1–5</sup> The multiscale modeling and Lagrangian-Euler methods have been successfully applied in flexible manipulators<sup>6</sup> and elastic joint robots.<sup>7</sup> A distinctive feature of singularly perturbed systems (SPSs) is the simultaneous presence of slow and fast transients in the dynamical response to system inputs. Some “parasitic” parameters, such as small mass and time constants, may increase the order of system models and easily result in the ill-conditioned numerical problems.<sup>3–5</sup> Based on the mathematical framework of singular perturbations, control engineers construct the reduced-order subsystems in different time scales for handling the slow and fast dynamics, respectively. These reduction methods have been already used in the design of efficient composite control algorithms for SPSs. Recently, the problem of robust control with finite frequency domain specifications for SPSs has been intensively investigated by many researchers.<sup>8–12</sup> The reasons are twofold: (i) many physical systems are sensitive to finite frequency disturbance that

generates vibrations at specific frequencies and (ii) slow and fast modes are sensitive to finite frequency external disturbance. Among them, the static output feedback (SOF) is of fundamental importance, because the physically available measurements usually comprise only part of the state variables.<sup>13,14</sup> However, some types of systems can be only stabilized via SOFs with delays.<sup>15,16</sup>

Some classes of systems, such as chains of integrators or inverted pendulums that cannot be stabilized by memoryless SOFs, can be stabilized by using SOFs with delays.<sup>17,18</sup> Recently, the design of delayed controllers has been intensively investigated, which takes advantage of the stabilizing effect of time delays to achieve the robust stability of practical systems.<sup>13,14,19–23</sup> Among them, a simple Lyapunov-based method was proposed in the work of Fridman and Shaikhet<sup>20</sup> for analyzing stability and system specifications. Based on the Taylor expansion with the integral remainder, a model transformation method was used for approximating the output derivatives in the works of Fridman et al.<sup>19,21,23</sup> In the work of Ramírez and Sipahi,<sup>24</sup> a derivative-free multiple-delay proportional-retarded protocol is designed for the fast consensus in a large-scale multiagent system. It is noted that, in the aforementioned works, the controller gains were set to be fixed or prescribed in order to deal with bilinear matrix inequalities (BMIs) via feasible linear matrix inequalities (LMIs). Thus, the conservativeness of the existing approaches comes from two aspects: the choice of controller parameters and the high dimensionality of LMIs. It is not easy for the pre-selection of controller parameters to guarantee that the closed-loop system matrix is Hurwitz for the high-order system with singular perturbation parameters.

In the literature, studies of the artificial time-delay approaches of SPSs is currently at an early stage. This paper suggests a design of a two-time-scale delayed SOF controller for the robust stabilization and performance analysis of SPSs by inserting multiple time-delays in the feedback loops running at different time scales. The key idea is to construct an artificial two-time-scale state feedback controller with full state information and then to formulate the approximations of immeasurable states or output derivatives. First, we present a singularly perturbed Euler-Lagrange formulation of physical systems. Then, a delayed SOF controller with two time scales is designed to directly deal with stability constraints and finite frequency specifications of SPSs without weighting functions or frequency gridding. Different from most existing papers, this work presents an LMI algorithm for designing the artificial time-delay controller for a two-time-scale system, which avoids the pre-formulation of controller parameters and provides a more user-friendly platform for control engineers. Moreover, the proposed method is further extended for analyzing the finite frequency system specifications of parameter-independent subsystems and estimating the disturbance attenuation level of the original high-order SPS. The example of an active suspension system is used to verify the effectiveness and merits of the proposed design method subject to finite frequency disturbance.

Throughout this paper,  $j$  stands for the imaginary unit  $\sqrt{-1}$ .  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are used to denote the  $n$ -dimensional Euclidean space and complex space, respectively.  $\mathbb{R}^{n \times m}$  denotes the set of  $n \times m$  real matrices.  $\mathbb{H}_n$  represents the set of  $n \times n$  Hermitian matrices. Given matrix  $X \in \mathbb{C}^{n \times n}$ , its singular value is defined as  $\sigma(X)$ .  $A \otimes B$  stands for the Kronecker product of matrices  $A$  and  $B$ . The superscripts  $T$  and  $*$  denote the matrix transpose and the complex transpose, respectively.  $\text{He}(M) = M + M^T$  denotes the Hermitian part of a square matrix  $M$ . The convex hull of points  $(A_1, A_2, \dots, A_N)$  is denoted as  $\text{col}\{A_1, A_2, \dots, A_N\}$ .  $\text{diag}\{M_1, M_2, \dots, M_L\}$  denotes the matrix with  $M_1, M_2, \dots, M_L$  as diagonal blocks. Function  $\text{int}(\cdot)$  is a integral function.  $L_2[0, \infty)$  is a space of square integrable Lebesgue functions  $f : [0, \infty) \rightarrow \mathbb{R}^n$  with the norm  $\|f\|_{L_2} = [\int_0^\infty \|f(t)\|^2 dt]^{1/2}$ . Unless explicitly stated, the matrices are assumed to have compatible dimensions for algebraic operation.

## 2 | PROBLEM FORMULATION AND PRELIMINARIES

A nominal singularly perturbed Euler-Lagrange system is considered as follows:

$$\begin{cases} \ddot{\zeta}(t) = A_{11}\zeta(t) + A_{12}\dot{\zeta}(t) + A_{13}\eta(t) + A_{14}\dot{\eta}(t) + B_{u1}u(t), \\ \epsilon^2\ddot{\eta}(t) = A_{21}\zeta(t) + A_{22}\dot{\zeta}(t) + A_{23}\eta(t) + A_{24}\dot{\eta}(t) + B_{u2}u(t), \end{cases} \quad (1)$$

where  $\zeta(t) \in \mathbb{R}^n$  and  $\eta(t) \in \mathbb{R}^m$  are the slow and fast states, respectively,  $u(t) \in \mathbb{R}^u$  is the control input, and  $\epsilon$  is the singular perturbation parameter that satisfies  $0 < \epsilon \ll 1$ .

Define new variables as

$$x(t) = \text{col}\{\zeta(t), \dot{\zeta}(t)\} \triangleq \text{col}\{x_1(t), x_2(t)\}, \quad z(t) = \text{col}\{\eta(t), \epsilon\dot{\eta}(t)\} \triangleq \text{col}\{z_1(t), z_2(t)\}.$$

Taking the external disturbance into consideration, system (1) can be represented in the standard singularly perturbed form

$$\begin{cases} \dot{x}(t) = \bar{A}_{11}x(t) + \bar{A}_{12}z(t) + \bar{B}_{u1}u(t) + \bar{B}_{w1}w(t), \\ \epsilon \dot{z}(t) = \bar{A}_{21}x(t) + \bar{A}_{22}z(t) + \bar{B}_{u2}u(t) + \bar{B}_{w2}w(t), \end{cases} \quad (2)$$

where  $w(t) \in \mathbb{R}^{n_w}$  is the external disturbance,  $E_\epsilon = \text{diag}\{I_{2n}, \epsilon I_{2m}\}$ , and

$$\begin{aligned} \bar{A}_{11} &= \begin{bmatrix} 0 & I_n \\ A_{11} & A_{12} \end{bmatrix}, & \bar{A}_{12} &= \begin{bmatrix} 0 & 0 \\ A_{13} & A_{14} \end{bmatrix}, & \bar{B}_{u1} &= \begin{bmatrix} 0 \\ B_{u1} \end{bmatrix}, \\ \bar{A}_{21} &= \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, & \bar{A}_{22} &= \begin{bmatrix} 0 & I_m \\ A_{23} & A_{24} \end{bmatrix}, & \bar{B}_{u2} &= \begin{bmatrix} 0 \\ B_{u2} \end{bmatrix}. \end{aligned} \quad (3)$$

Assuming that  $A_{23}$  is invertible, the inversion of  $\bar{A}_{22}$  can be obtained as

$$\bar{A}_{22}^{-1} = \begin{bmatrix} -A_{23}^{-1}A_{24} & A_{23}^{-1} \\ I_m & 0 \end{bmatrix}.$$

It is observed that the following state feedback controller with full state information can successfully stabilize an SPS in (2):

$$u(t) = \bar{K}_1x_1(t) + \bar{K}_2x_2(t) + K_1z_1(t) + K_2z_2(t), \quad (4)$$

where  $\bar{K}_1 \in \mathbb{R}^{n_u \times n}$ ,  $\bar{K}_2 \in \mathbb{R}^{n_u \times n}$ ,  $K_1 \in \mathbb{R}^{n_u \times m}$ , and  $K_2 \in \mathbb{R}^{n_u \times m}$  are controller gains. Note that controller (4) is inherently a PD controller.

In the setup under consideration, only  $x_1(t)$  and  $z_1(t)$  are measurable. In order to estimate the differentiation items  $x_2(t)$  and  $z_2(t)$ , the stabilizing delay items can be used by introducing the time lags  $h_1$  and  $\epsilon h_2$  in system states  $x_1(t)$  and  $z_1(t)$ , respectively. The following approximation based on the Taylor expansion with integral remainder is utilized:

$$x_2(t) = \frac{1}{h_1} (x_1(t) - x_1(t - h_1)) + \frac{1}{h_1} R_x(t|h_1) \quad (5)$$

$$z_2(t) = \frac{1}{\epsilon h_2} (z_1(t) - z_1(t - \epsilon h_2)) + \frac{1}{\epsilon h_2} R_z(\epsilon, t|h_2), \quad (6)$$

where the error items are written as

$$\begin{aligned} R_x(t|h_1) &= \int_{t-h_1}^t (s-t+h_1)x_3(s)ds, & R_z(\epsilon, t|h_2) &= \int_{t-\epsilon h_2}^t (s-t+\epsilon h_2)z_3(s)ds \\ x_3(t) &= \dot{x}_2(t), & z_3(t) &= \dot{z}_2(t). \end{aligned}$$

The following controller, which is referred to as a delayed SOF controller,<sup>21</sup> is formulated as

$$u(t, \epsilon) = \bar{K}_1x_1(t) + \frac{1}{h_1} \bar{K}_2 (x_1(t) - x_1(t - h_1)) + K_1z_1(t) + \frac{1}{\epsilon h_2} K_2 (z_1(t) - z_1(t - \epsilon h_2)). \quad (7)$$

*Remark 1.* The reason for the choice of the two-time-scale control structure in (7) is to achieve the simultaneous control of slow and fast states, and the controller gains can be obtained based on the design of reduced-order subsystems.

Consider (5) and (6), Equation (8) is further represented as

$$u(t, \epsilon) = \bar{K}x(t) + Kz(t) - \frac{1}{h_1} \bar{K}_2 R_x(t|h_1) - \frac{1}{\epsilon h_2} K_2 R_z(\epsilon, t|h_2). \quad (8)$$

Due to the limitations of measurements, controller (4) cannot be directly implemented in system (2). To this end, we resort to designing a delayed SOF controller (8) to approximately function as a state feedback controller in order to stabilize system (2).

*Remark 2.* The proposed delayed controller (8) depends on both singular perturbation parameter  $\epsilon$  and small time delays  $h_1$  and  $h_2$ . The two-time-scale characteristic of (8) leads to two different types of Taylor's formulation: one is

$\epsilon$ -dependent in (6) and the other is  $\epsilon$ -independent in (5). In contrast to the work of Fridman and Shaikhet,<sup>21</sup> the Taylor expansion formulations in (5) and (6) enable the control engineers to use past measurements in different time-scales for the formulation of control laws.

Substituting (7) into system (2) yields

$$\begin{cases} \dot{x}(t) = \mathcal{A}_{11}x(t) + \mathcal{A}_{12}z(t) - \frac{1}{h_1}\mathcal{A}_{11}^d x_1(t-h_1) - \frac{1}{\epsilon h_1}\mathcal{A}_{12}^d z_1(t-\epsilon h_2) + \bar{B}_{w1}w(t), \\ \epsilon \dot{z}(t) = \mathcal{A}_{21}x(t) + \mathcal{A}_{22}z(t) - \frac{1}{h_1}\mathcal{A}_{21}^d x_1(t-h_1) - \frac{1}{\epsilon h_2}\mathcal{A}_{22}^d z_1(t-\epsilon h_2) + \bar{B}_{w2}w(t), \end{cases} \quad (9)$$

where

$$\begin{aligned} \mathcal{A}_{11} &= \bar{A}_{11} + \bar{B}_{u1} \left( \bar{K}_1 + \frac{1}{h_1} \bar{K}_2 \right) [I_n \quad 0], & \mathcal{A}_{12} &= \bar{A}_{12} + \bar{B}_{u1} \left( K_1 + \frac{1}{\epsilon h_2} K_2 \right) [I_m \quad 0], \\ \mathcal{A}_{21} &= \bar{A}_{21} + \bar{B}_{u2} \left( \bar{K}_1 + \frac{1}{h_1} \bar{K}_2 \right) [I_n \quad 0], & \mathcal{A}_{22} &= \bar{A}_{22} + \bar{B}_{u2} \left( K_1 + \frac{1}{\epsilon h_2} K_2 \right) [I_m \quad 0], \\ \mathcal{A}_{11}^d &= \bar{B}_{u1} \bar{K}_2, & \mathcal{A}_{12}^d &= \bar{B}_{u1} K_2, & \mathcal{A}_{21}^d &= \bar{B}_{u2} \bar{K}_2, & \mathcal{A}_{22}^d &= \bar{B}_{u2} K_2. \end{aligned}$$

Considering (5), (6), and (8), system (9) can be further represented as

$$\begin{cases} \dot{x}(t) = \bar{\mathcal{A}}_{11}x(t) + \bar{\mathcal{A}}_{12}z(t) - \frac{1}{h_1}\mathcal{A}_{11}^d R_x(t|h_1) - \frac{1}{\epsilon h_2}\mathcal{A}_{12}^d R_z(\epsilon, t|h_2) + \bar{B}_{w1}w(t), \\ \epsilon \dot{z}(t) = \bar{\mathcal{A}}_{21}x(t) + \bar{\mathcal{A}}_{22}z(t) - \frac{1}{h_1}\mathcal{A}_{21}^d R_x(t|h_1) - \frac{1}{\epsilon h_2}\mathcal{A}_{22}^d R_z(\epsilon, t|h_2) + \bar{B}_{w2}w(t), \end{cases} \quad (10)$$

where

$$\begin{aligned} \bar{\mathcal{A}}_{11} &= \bar{A}_{11} + \bar{B}_{u1} \bar{K}, & \bar{\mathcal{A}}_{21} &= \bar{A}_{21} + \bar{B}_{u2} \bar{K}, & \bar{K} &= [\bar{K}_1 \quad \bar{K}_2], \\ \bar{\mathcal{A}}_{12} &= \bar{A}_{12} + \bar{B}_{u1} K, & \bar{\mathcal{A}}_{22} &= \bar{A}_{22} + \bar{B}_{u2} K, & K &= [K_1 \quad K_2]. \end{aligned}$$

Next, the separation of time scales is conducted for the model simplification and finite frequency disturbance attenuation. By similar arguments to the work of Anastassiou and Dragomir,<sup>25</sup> the bounds of integral remainders are given as

$$|R_z(\epsilon, t|h_2)| \leq \frac{\epsilon^2 h_2^2}{2} \text{ess sup}_{t \in [t-\epsilon h_2, t]} |z_3(t)|.$$

Thus, it is easy to verify that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} R_z(\epsilon, t|h_1, h_2) = 0. \quad (11)$$

Consider (11), and the following quasi-steady state of the fast state is obtained by setting  $\epsilon = 0$  in (10):

$$z_s(t) = -\bar{\mathcal{A}}_{22}^{-1} \bar{\mathcal{A}}_{21} x(t) + \frac{1}{h_1} \bar{\mathcal{A}}_{22}^{-1} \mathcal{A}_{21}^d \tilde{R}_x(t|h_1, h_2) - \bar{\mathcal{A}}_{22}^{-1} \bar{B}_{w2} w_s(t), \quad (12)$$

where  $w_s(t)$  is the low frequency part of  $w(t)$ , and

$$\tilde{R}_x(t|h_1) = \int_{t-h_1}^t (s-t+h_1) x_{3s}(s) ds.$$

By replacing  $z(t)$  by  $z_s(t)$  in (10), the slow subsystem is obtained as

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) - \frac{1}{h_1} \mathcal{A}_{rs} \tilde{R}_x(t|h_1) + \mathcal{B}_{ws} w_s(t), \quad (13)$$

where  $x_s(t)$  is the slow component of  $x(t)$ , and

$$\mathcal{A}_s = \bar{\mathcal{A}}_{11} - \bar{\mathcal{A}}_{12} \bar{\mathcal{A}}_{22}^{-1} \bar{\mathcal{A}}_{21}, \quad \mathcal{A}_{rs} = \mathcal{A}_{11}^d - \bar{\mathcal{A}}_{12} \bar{\mathcal{A}}_{22}^{-1} \mathcal{A}_{21}^d, \quad \mathcal{B}_{ws} = \bar{B}_{w1} - \bar{\mathcal{A}}_{12} \bar{\mathcal{A}}_{22}^{-1} \bar{B}_{w2}.$$

Letting  $z_f(t) = z(t) - z_s(t)$ , the fast subsystem is represented as

$$\dot{z}_f(\tau) = \bar{A}_{22}z_f(\tau) - \frac{1}{h_2}\bar{A}_{22}^d\bar{R}_z(\tau|h_2) + \bar{B}_{w2}w_f(\tau), \quad (14)$$

where  $\tau$  is the fast time scale satisfying  $t = \epsilon\tau$ ,  $z_f(\tau)$  is the fast component of  $z(\tau)$ ,  $w_f(\tau)$  is the high frequency external disturbance, and

$$\bar{R}_z(\tau|h_2) = \int_{\tau-h_2}^{\tau} (s' - \tau + h_2)z_{3f}(s')ds'.$$

*Remark 3.* From the representations of subsystems (13) and (14), we can see that  $K_1$  and  $K_2$  are used to control fast states, while  $\bar{K}_1$ ,  $\bar{K}_2$ ,  $K_1$ , and  $K_2$  are related with system performance of slow subsystem (13).

Based on the slow-fast decomposition, the states and disturbance input of the resulting closed-loop system (10) are approximated as

$$x(t, \epsilon) = x_s(t) + O(\epsilon), \quad z(t, \epsilon) = z_s(t) + z_f(\epsilon, t) + O(\epsilon), \quad w(t, \epsilon) = w_s(t) + w_f(\epsilon, t) + O(\epsilon),$$

for all finite  $t \geq 0$  and all  $\epsilon \in (0, \epsilon^*]$ .

Now, we are ready to present the objective of this work: to choose appropriate controller gains  $K_1$ ,  $K_2$ ,  $\bar{K}_1$ , and  $\bar{K}_2$  to achieve the internal stability of the closed-loop system (9) with attenuating the influence of finite frequency external disturbance  $w(t)$  on the measurement output  $z(t)$ .

In frequency domain, the disturbance attenuation problem of system (9) can be decomposed into those of subsystems subject to finite frequency disturbance. Moreover, two target frequency regions are defined for representing the finite frequency specifications

$$\Lambda_l = \{\omega \mid \omega_1 < \omega < \omega_2\}, \quad \Lambda_h = \{\varpi \mid \varpi_1 < \varpi < \varpi_2\}, \quad (15)$$

where  $\omega$  and  $\varpi$  are frequency scales with the relationship of  $\varpi = \epsilon\omega$ ,  $\omega_1$ ,  $\omega_2$ ,  $\varpi_1$ , and  $\varpi_2$  are the cutoff frequencies operating in different frequency scales. In the following, we provide a time-domain characterization of the input-output stability of the decoupled subsystems subject to finite frequency disturbance.  $\Lambda_l$  and  $\Lambda_h$  are used to characterize the their dominant frequency ranges of subsystems.

**Definition 1** (See the work of Iwasaki and Hara<sup>26</sup>).

For the given disturbance attenuation indices  $\gamma_s$  and  $\gamma_f$ , finite frequency disturbance is said to be locally attenuated by  $\gamma$  if the slow subsystem (13) and the fast subsystem (14) are internally stable, and

$$\int_0^{\infty} x_{s1}^T(t)x_{s1}(t)dt \leq \gamma_s^2 \int_0^{\infty} w_s^T(t)w_s(t)dt, \quad (16)$$

$$\int_0^{\infty} z_{f1}^T(\tau)z_{f1}(\tau)d\tau \leq \gamma_f^2 \int_0^{\infty} w_f^T(\tau)w_f(\tau)d\tau, \quad (17)$$

for all solutions of (13) and (14) with  $w_s \in L_2[0, \infty)$  and  $w_f \in L_2[0, \infty)$  such that

$$\int_0^{\infty} (\omega_1 x_s(t) + j\dot{x}_s(t)) (\omega_2 x_s(t) + j\dot{x}_s(t))^T dt \leq 0, \quad (18)$$

$$\int_0^{\infty} (\varpi_1 z_f(\tau) + j\dot{z}_f(\tau)) (\varpi_2 z_f(\tau) + j\dot{z}_f(\tau))^T d\tau \leq 0. \quad (19)$$

The disturbance attenuation index  $\gamma$  of the whole system (10) will be estimated based on the system parameters and the values of  $\gamma_s$  and  $\gamma_f$ .

Note that the equivalence between the formulation of the frequency sets shown in (15) and inequalities (18) and (19) can be investigated by using Parseval's theorem in the work of Zhou and Doyle.<sup>27</sup>

Before ending this section, the following lemmas are used for developing our main results.

**Lemma 1** (Jensen's inequality<sup>16,21</sup>).

Denote  $G = \int_a^b f(s)x(s)ds$ , where  $a \leq b$ ,  $f : [a, b] \rightarrow [0, \infty)$ ,  $x(s) \in \mathbb{R}^n$  and the integration concerned is well defined. Then, for any matrix  $R \in \mathbb{C}^{n \times n}$  satisfying  $R > 0$ , the following inequality holds:

$$G^T R G \leq \int_a^b f(\theta)d\theta \int_a^b f(s)x^T(s)R x(s)ds.$$

**Lemma 2** (S-procedure<sup>26,28</sup>).

Let  $\Theta \in \mathbb{C}^{n \times n}$  and  $M \in \mathbb{C}^{n \times n}$  be Hermitian matrices, and  $\zeta \in \mathbb{C}^n$  be a vector. The following conditions are equivalent:

- (1)  $\exists \tau \in \mathbb{R}$  such that  $\tau \geq 0$ ,  $\Theta + \tau M \leq 0$ ; and
- (2)  $\zeta^T \Theta \zeta < 0$  for all  $\zeta \neq 0$  such that  $\zeta^T M \zeta > 0$ .

### 3 | FINITE FREQUENCY APPROACH FOR THE DESIGN OF AN ARTIFICIAL DELAY CONTROLLER

#### 3.1 | Internal stability

We present the internal stability analysis of the closed-loop system (10) for  $w(t) = 0$ . The internal stability problem of an SPS can be decomposed into two separate stability problems: one for the slow subsystem (13) and the other for the fast subsystem (14). Condition  $w(t) = 0$  implies that  $w_s(t) = 0$ ,  $w_f(\tau) = 0$ .

Then, the multiplier method will be used to represent the derived delay-dependent stability criteria in the form of LMIs, which leads to a significant reduction of conservatism with respect to the existing approaches. The following theorem investigates the internal stability of subsystem (14) with no high frequency disturbance input, ie,  $w_f(\tau) = 0$ .

**Theorem 1.** For given small positive scalars  $\delta_1$  and  $\tilde{\delta}_1$  and a small constant delay  $h_2$ , a delayed SOF controller (8) can stabilize subsystem (14), if there exist symmetric matrices  $P_{f1} \in \mathbb{R}^{m \times m}$ ,  $P_{f3} \in \mathbb{R}^{m \times m}$ ,  $L_f \in \mathbb{R}^{n_u \times n_u}$ ,  $R_f \in \mathbb{R}^{n_u \times n_u}$ , and matrices  $P_{f2} \in \mathbb{R}^{m \times m}$ ,  $\mathcal{K}_{f1} \in \mathbb{R}^{n_u \times m}$ , and  $\mathcal{K}_{f2} \in \mathbb{R}^{n_u \times m}$ , such that the following LMIs are satisfied:

$$\begin{bmatrix} \text{He}(P_f \bar{A}_{22} + \chi_2) & -\frac{1}{h_2} P_f \bar{B}_{u2} & \bar{A}_{22}^T D_f^T P_{f3} + \chi_1 & 0 \\ \star & -4R_f & -\frac{1}{\phi} B_{u2}^T P_{f3} & 0 \\ \star & \star & -\text{He}(P_{f3}) & \mathcal{K}_{f2}^T B_{u2}^T \\ \star & \star & \star & -\text{He}(B_{u2}^T P_{f3}) + h_2^4 R_f \end{bmatrix} < 0, \quad (20)$$

$$\begin{bmatrix} -\delta_1 I_m & \star \\ P_{f2} B_{u2} & -I_m \end{bmatrix} < 0, \quad \begin{bmatrix} -\tilde{\delta}_1 I_m & \star \\ P_{f3} B_{u2} - B_{u2} L_f & -I_m \end{bmatrix} < 0, \quad (21)$$

$$\begin{bmatrix} P_{f1} & P_{f2} \\ \star & P_{f3} \end{bmatrix} > 0, \quad (22)$$

where  $D_f = [0 \ I_m]$ , and

$$P_f = \begin{bmatrix} P_{f1} & P_{f2} \\ \star & P_{f3} \end{bmatrix}, \quad \chi_1 = \begin{bmatrix} \mathcal{K}_{f1}^T B_{u2}^T \\ \mathcal{K}_{f2}^T B_{u2}^T \end{bmatrix}, \quad \chi_2 = \begin{bmatrix} 0 & 0 \\ B_{u2} \mathcal{K}_{f1} & B_{u2} \mathcal{K}_{f2} \end{bmatrix}. \quad (23)$$

*Proof.* We investigate the asymptotical stability of subsystem (14). Inspired from the work of Fridman and Shaikhet,<sup>21</sup> the following Lyapunov-Krasovskii functional is used:

$$V_f(\tau) = V_{f1}(\tau) + V_{f2}(\tau), \quad (24)$$

wherein we choose  $V_{f1}(\tau) = z_f^T(\tau) P_f z_f(\tau)$ ,  $P_f = P_f^T > 0$ . Differentiating  $V_{f1}(\tau)$  with respect to the trajectories of system (14) is written as

$$\begin{aligned} \dot{V}_{f1}(\tau) &= 2z_f^T(\tau) P_f \dot{z}_f(\tau), \\ &= z_f^T(\tau) \text{He}(P_f \bar{A}_{22}) z_f(\tau) - \frac{2}{h_2} z_f^T(\tau) P_f \mathcal{A}_{22}^d \bar{R}_z(\tau | h_2), \end{aligned}$$

where

$$\tilde{R}_z(\tau|h_2) = \int_{\tau-h_2}^{\tau} (s' - \tau + h_2)z_{3f}(s')ds'.$$

To compensate the items  $\tilde{R}_z(\tau|h_2)$  in (25), the formulation of  $V_{f2}(\tau)$  is given as

$$V_{f2}(\tau) = h_2^2 \int_{\tau-h_2}^{\tau} (s' - \tau + h_2)^2 z_{3f}^T(s')K_2^T R_f K_2 z_{3f}(s')ds',$$

with  $R_f > 0$ . Moreover, the derivative of  $V_{f2}(\tau)$  with respect to time scale  $\tau$  is given as

$$\dot{V}_{f2}(\tau) = h_2^4 z_{3f}^T(\tau)K_2^T R_f K_2 z_{3f}(\tau) - 2h_2^2 \int_{\tau-h_2}^{\tau} (s' - \tau + h_2)z_{3f}^T(s')K_2^T R_f K_2 z_{3f}(s')ds'. \quad (25)$$

Based on the Jensen's inequality in Lemma 1, the following relaxed technique is used:

$$-2h_2^2 \int_{\tau-h_2}^{\tau} (s' - \tau + h_2)z_{3f}^T(s')K_2^T R_f K_2 z_{3f}(s')ds' \leq -4\tilde{R}_z^T(\tau|h_2)K_2^T R_f K_2 \tilde{R}_z(\tau|h_2). \quad (26)$$

Define  $D_f = [0 \ I_m]$ . From (14), it is easy to obtain that, when  $w_f(\tau) = 0$ ,

$$\begin{aligned} z_{3f}(\tau) &= D_f \dot{z}_f(\tau), \\ &= \left[ D_f \bar{A}_{22} \quad -\frac{1}{h_2} D_f \bar{B}_{u2} \right] \phi(\tau), \end{aligned} \quad (27)$$

where  $\phi(\tau) = \text{col}\{z_f(\tau), K_2 \tilde{R}_{z2}(\tau|h_2)\}$ .

By using (26) and (27) in (25), we arrive at

$$\begin{aligned} \dot{V}_f(\tau) &= \dot{V}_{f1}(\tau) + \dot{V}_{f2}(\tau) \leq \phi^T(\tau) \begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_{u2} K) & -\frac{1}{h_2} P_f \bar{B}_{u2} \\ \star & -4R_f \end{bmatrix} \phi(\tau) \\ &\quad - \phi^T(\tau) \begin{bmatrix} \bar{A}_{22}^T D_f^T + K^T \bar{B}_{u2}^T D_f^T \\ -\frac{1}{h_2} \bar{B}_{u2}^T D_f^T \end{bmatrix} \left( -\hat{R}_f^{-1} \right)^{-1} \begin{bmatrix} \bar{A}_{22}^T D_f^T + K^T \bar{B}_{u2}^T D_f^T \\ -\frac{1}{h_2} \bar{B}_{u2}^T D_f^T \end{bmatrix}^T \phi(\tau), \end{aligned}$$

where  $\hat{R}_f = K_2^T (h_1^4 R_f) K_2$ .

By virtue of the Schur complement formula, condition  $\dot{V}_f(\tau) < 0$  is equivalent to

$$\begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_{u2} K) & -\frac{1}{h_2} P_f \bar{B}_{u2} & \bar{A}_{22}^T D_f^T + K^T \bar{B}_{u2}^T D_f^T \\ \star & -4R_{f2} & -\frac{1}{h_2} \bar{B}_{u2}^T D_f^T \\ \star & \star & -\hat{R}_f^{-1} \end{bmatrix} < 0. \quad (28)$$

In order to obtain an LMI, matrix  $P_f$  is partitioned as

$$P_f = \begin{bmatrix} P_{f1} & P_{f2} \\ \star & P_{f3} \end{bmatrix}.$$

Define a transformation matrix as  $\mathcal{T}_{f1} = \text{diag}\{I_{2n}, I_{2n}, I_{2n}, P_{f3}\}$ . Performing a congruent transformation  $\mathcal{T}_{f1}$  on (28) yields

$$\begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_{u2} K) & -\frac{1}{h_2} P_f \bar{B}_{u2} & \bar{A}_{22}^T D_f^T P_{f2} + K^T \bar{B}_{u2}^T D_f^T P_{f3} \\ \star & -4R_{f2} & -\frac{1}{h_2} \bar{B}_{u2}^T D_f^T P_{f3} \\ \star & \star & -P_{f3} \hat{R}_f^{-1} P_{f3} \end{bmatrix} < 0. \quad (29)$$

The following inequality is derived based on the definition of positive definite matrix:

$$(M - N^{-1})^T N (M - N^{-1}) \geq 0,$$

where  $N$  is a positive definite matrix, which is equivalent to

$$M^T N^{-1} M - \text{He}(M) + N \geq 0. \quad (30)$$

Let  $M = P_{f3}$  and  $N = \hat{R}_f$  in (30), we construct

$$-P_{f3} \hat{R}_f^{-1} P_{f3} \leq -\text{He}(P_{f3}) + \hat{R}_f. \quad (31)$$

Consider (31) together with  $D_f \bar{B}_{u2} = B_{u2}$ , BMI (29) becomes

$$\begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_{u2} K) & -\frac{1}{h_2} P_f \bar{B}_{u2} & \bar{A}_{22}^T D_f^T P_{f3} + K^T \bar{B}_{u2}^T D_f^T P_{f3} \\ \star & -4R_f & -\frac{1}{h_2} B_{u2}^T P_{f3} \\ \star & \star & -\text{He}(P_{f3}) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ K_2^T \end{bmatrix} \left( -\bar{R}_f^{-1} \right)^{-1} \begin{bmatrix} 0 & 0 & K_2 \end{bmatrix} < 0,$$

with  $\bar{R}_f = h_2^4 R_f$ , which is equivalent to

$$\begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_{u2} K) & -\frac{1}{h_2} P_f \bar{B}_{u2} & \bar{A}_{22}^T D_f^T P_{f3} + K^T \bar{B}_{u2}^T D_f^T P_{f3} & 0 \\ \star & -4R_{f2} & -\frac{1}{h_2} B_{u2}^T P_{f3} & 0 \\ \star & \star & -\text{He}(P_{f3}) & K_2^T \\ \star & \star & \star & -\bar{R}_f \end{bmatrix} < 0, \quad (32)$$

based on the Schur complement lemma. Note that (32) is a BMI on  $K_1$ ,  $K_2$ ,  $P_f$ ,  $R_{f1}$ , and  $R_{f2}$ , which is not a convex optimal problem. Thus, a BMI may not be efficiently solved by MATLAB.

A slack variable  $L_f \in \mathbb{R}^{n_u \times n_u}$  is introduced that satisfies  $P_f \bar{B}_{u2} = \bar{B}_{u2} L_f$ , and then, we can obtain

$$P_{f2} B_{u2} = 0, \quad P_{f3} B_{u2} = B_{u2} L_f,$$

such that

$$\begin{aligned} P_f \bar{B}_{u2} K &= \bar{B}_{u2} L_f K = \begin{bmatrix} 0 & 0 \\ B_{u2} L_f K_1 & B_{u2} L_f K_2 \end{bmatrix}, \\ K^T \bar{B}_{u2}^T D_f^T P_{f3} &= \begin{bmatrix} K_1^T B_{u2}^T P_{f3} \\ K_2^T B_{u2}^T P_{f3} \end{bmatrix} = \begin{bmatrix} K_1^T L_f^T B_{u2}^T \\ K_2^T L_f^T B_{u2}^T \end{bmatrix}. \end{aligned} \quad (33)$$

Define a transformation matrix as

$$\mathcal{T}_{f2} = \text{diag}\{I_{2n}, I_{2n}, I_{2n}, I_{2n}, P_{f3} B_{u2}\}.$$

Premultiply and postmultiply (32) with  $\mathcal{T}_{f2}$  and  $\mathcal{T}_{f2}^T$ , respectively, yield

$$\begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_{u2} K) & -\frac{1}{h_2} P_f \bar{B}_{u2} & \bar{A}_{22}^T D_f^T P_{f3} + \chi_1 & 0 \\ \star & -4R_f & -\frac{1}{h_2} B_{u2}^T P_{f3} & 0 \\ \star & \star & -\text{He}(P_{f3}) & K_2^T B_{u2}^T P_{f3} \\ \star & \star & \star & -P_{f3} B_{u2} \bar{R}_f^{-1} B_{u2}^T P_{f3} \end{bmatrix} < 0, \quad (34)$$

where  $\chi_1 = K^T \bar{B}_{u2}^T D_f^T P_{f3}$ .



Setting  $M = B_{u2}^T P_{f3}$  and  $N = \bar{R}_f$  in (35), we have

$$-P_{f3} B_{u2} \bar{R}_f^{-1} B_{u2}^T P_{f3} \leq -\text{He}(B_{u2}^T P_{f3}) + \bar{R}_f. \quad (35)$$

Consider (35), the sufficient condition for the feasibility of (34) is written as

$$\begin{bmatrix} \text{He}(P_f \bar{A}_{22} + \chi_2) & -\frac{1}{h_2} P_f \bar{B}_{u2} & \bar{A}_{22}^T D_f^T P_{f3} + \chi_1 & 0 \\ \star & -4R_f & -\frac{1}{h_2} B_{u2}^T P_{f3} & 0 \\ \star & \star & -\text{He}(P_{f3}) & K_2^T B_{u2}^T P_{f3} \\ \star & \star & \star & -\text{He}(B_{u2}^T P_{f3}) + \bar{R}_f \end{bmatrix} < 0, \quad (36)$$

where

$$\chi_2 = \begin{bmatrix} 0 & 0 \\ B_{u2} L_f K_1 & B_{u2} L_f K_2 \end{bmatrix}, \quad \chi_1 = \begin{bmatrix} K_1^T L_f^T B_{u2}^T \\ K_2^T L_f^T B_{u2}^T \end{bmatrix}.$$

The following multipliers are defined to convert (36) into an LMI:

$$\mathcal{K}_{f1} = L_f K_1, \quad \mathcal{K}_{f2} = L_f K_2.$$

such that

$$\chi_1 = \begin{bmatrix} \mathcal{K}_{f1}^T B_{u2}^T \\ \mathcal{K}_{f2}^T B_{u2}^T \end{bmatrix}, \quad \chi_2 = \begin{bmatrix} 0 & 0 \\ B_{u2} \mathcal{K}_{f1} & B_{u2} \mathcal{K}_{f2} \end{bmatrix},$$

which can be further represented as (20). Moreover, the following constraints are introduced:

$$\begin{aligned} (P_{f2} B_{u2})^T (P_{f2} B_{u2}) &< \delta_1 I_m, \\ (P_{f3} B_{u2} - B_{u2} L_f)^T (P_{f3} B_{u2} - B_{u2} L_f) &< \tilde{\delta}_1 I_m, \end{aligned}$$

where  $\delta_1$  and  $\tilde{\delta}_1$  are sufficiently small positive scalars, which are equivalent to (21) based on the Schur complement lemma.  $\square$

After solving Theorem 1, the fast controller gain  $K$  can be calculated. Thus,  $\bar{A}_{12}$  and  $\bar{A}_{22}$  that depend on  $K$  are known matrices. Letting  $C_m = \bar{A}_{12} \bar{A}_{22}^{-1}$ , we have

$$\mathcal{A}_s = \bar{A}_{11} - C_m \bar{A}_{21} = \bar{A}_s + \bar{B}_{us} \bar{K}, \quad \mathcal{A}_{rs} = \bar{B}_{us} \bar{K}_2, \quad \mathcal{B}_{ws} = \bar{B}_{ws},$$

where

$$\bar{A}_s = \bar{A}_{11} - C_m \bar{A}_{21}, \quad \bar{B}_{us} = \bar{B}_{u1} - C_m \bar{B}_{u2} = \text{col}\{0, B_{us}\}, \quad \bar{B}_{ws} = \bar{B}_{w1} - C_m \bar{B}_{w2}.$$

Thus, with fast controller, the slow subsystem (13) is of the same structure as the fast subsystem (14).

We can directly extend the result in Theorem 1 to the slow subsystem (13). Then, the following theorem presents the sufficient conditions for the internal stability of the slow subsystem (13) with  $w_s(t) = 0$ .

**Theorem 2.** For given small positive scalars  $\delta_2$  and  $\tilde{\delta}_2$  and a small constant delay  $h_1$ , a delayed SOF controller (8) can stabilize subsystem (13), if there exist symmetric matrices  $L_s \in \mathbb{R}^{n_u \times n_u}$ ,  $R_s \in \mathbb{R}^{n_u \times n_u}$ ,  $P_{s1} \in \mathbb{R}^{n \times n}$ , and  $P_{s3} \in \mathbb{R}^{n \times n}$ , and matrices  $P_{s2} \in \mathbb{R}^{n \times n}$ ,  $\mathcal{K}_{s1} \in \mathbb{R}^{n_u \times n}$ , and  $\mathcal{K}_{s2} \in \mathbb{R}^{n_u \times n}$ , such that the following LMIs hold:

$$\begin{bmatrix} \text{He}(P_s \bar{A}_s + \chi_4) & -\frac{1}{h_1} \bar{B}_{us} & \bar{A}_s^T D_s^T P_{s3} + \chi_3 & 0 \\ \star & -4R_s & -\frac{1}{h_1} B_{us}^T P_{s3} & 0 \\ \star & \star & -\text{He}(P_{s3}) & \mathcal{K}_{s2}^T B_{us}^T \\ \star & \star & \star & -\text{He}(B_{us}^T P_{s3}) + \bar{R}_s \end{bmatrix} < 0, \quad (37)$$

$$\begin{bmatrix} -\delta_2 I_n & \star \\ P_{s2} B_{us} & -I_n \end{bmatrix} < 0, \quad \begin{bmatrix} -\tilde{\delta}_2 I_n & \star \\ P_{s3} B_{us} - B_{us} L_s & -I_n \end{bmatrix} < 0, \quad (38)$$

$$\begin{bmatrix} P_{s1} & P_{s2} \\ \star & P_{s3} \end{bmatrix} > 0, \quad (39)$$

where  $\phi = h_2 - h_1$ ,  $\bar{R}_s = h_1^4 R_{s1} + h_2^4 R_{s2}$ ,  $D_s = [0 \ I_m]$ , and

$$P_s = \begin{bmatrix} P_{s1} & P_{s2} \\ \star & P_{s3} \end{bmatrix}, \quad \chi_3 = \begin{bmatrix} \mathcal{K}_{s1}^T B_{us}^T \\ \mathcal{K}_{s2}^T B_{us}^T \end{bmatrix}, \quad \chi_4 = \begin{bmatrix} 0 & 0 \\ B_{us} \mathcal{K}_{s1} & B_{us} \mathcal{K}_{s2} \end{bmatrix}.$$

The proof of the internal stability of the slow subsystem (13) is similar to that of Theorem 1, which is omitted here.

*Remark 4.* The design methods in the work of Fridman and Shaikhet<sup>21</sup> depend on the pre-selection of  $K_i$ , which results from full state feedback that guarantees the stabilization and the values of delay that guarantees the LMIs feasibility. Moreover, it is shown that the resulting LMIs are feasible for such choice of gains and small enough delays. In this paper, we provide LMIs for the direct design of the delayed controller gains. The example in Section 6 illustrates the efficiency of the presented design method.

*Remark 5.* Note that BMI (34) is in the standard form of the work of He and Wang.<sup>29</sup> Compared with the iterative LMI method in the aforementioned work,<sup>29</sup> our method is derived based on the relaxed technique, which avoids formulating the initial values, investigating the stopping criterion, and proving the uniform convergence of iterations.

#### 4 | $L_2$ -GAIN PERFORMANCE ANALYSIS IN FINITE FREQUENCY DOMAIN

It has been mentioned in the work of Kokotovic et al<sup>1</sup> that SPSs are sensitive to low frequency and high frequency external disturbance. The closed-loop SPS with the proposed delayed controller is designed to be robust to two different types of disturbance  $w_s(t)$  and  $w_f(\tau)$ . The frequency bands of  $w_s(t)$  and  $w_f(\tau)$  are characterized by using the cutoff frequencies  $\omega_1, \omega_2, \varpi_1$ , and  $\varpi_2$ . Thus, the controller gains  $\bar{K}_1, \bar{K}_2$  and  $K_1, K_2$  are separately designed based on the slow and fast performance specifications, which will finally arrive at a physically realizable composite controller (8).

First, we present the sufficient conditions for the high frequency disturbance attenuation of system (14).

**Theorem 3.** *Given the cutoff frequencies  $\varpi_1$  and  $\varpi_2$ , and a positive scalar  $\gamma_f$ , subsystem (14) with  $K_1$  and  $K_2$  can achieve the finite frequency specification (17) with respect to (19), if there exist symmetric matrices  $P_{f1} \in \mathbb{R}^{m \times m}$ ,  $P_{f3} \in \mathbb{R}^{m \times m}$ ,  $L_f \in \mathbb{R}^{n_u \times n_u}$ ,  $P_f \in \mathbb{R}^{2m \times 2m}$ ,  $Q_f \in \mathbb{R}^{2m \times 2m}$ ,  $R_f \in \mathbb{R}^{n_u \times n_u}$ , and matrices  $P_{f2} \in \mathbb{R}^{m \times m}$ ,  $\mathcal{K}_{f1} \in \mathbb{R}^{n_u \times m}$ , and  $\mathcal{K}_{f2} \in \mathbb{R}^{n_u \times m}$ , such that LMIs (21)-(22) hold, and*

$$\begin{bmatrix} \bar{\Theta}_{f1} & \bar{\Theta}_{f2} & P_f \bar{B}_{w2} & P_f + j\varpi_c Q_f & \bar{A}_{22}^T D_f^T P_{f3} + \chi_1 & 0 \\ \star & \bar{\Theta}_{f3} & 0 & 0 & -\frac{1}{h_2} B_{u2}^T P_{f3} & 0 \\ \star & \star & -\gamma_f^2 I_m & 0 & \bar{B}_{w2}^T D_f^T P_{f3} & 0 \\ \star & \star & \star & -\varpi_1 \varpi_2 Q_f & 0 & 0 \\ \star & \star & \star & \star & -\text{He}(P_{f3}) & \mathcal{K}_{f2}^T B_{u2}^T \\ \star & \star & \star & \star & \star & -\text{He}(B_{u2}^T P_{f3}) + 4R_f \end{bmatrix} < 0, \quad (40)$$

where  $\varpi_c = 0.5(\varpi_1 + \varpi_2)$ , matrices  $P_f, R_{f1}, R_{f2}, \bar{R}_f, \chi_1$ , and  $\chi_2$  have been given in Theorem 1, and

$$\bar{\Theta}_{f1} = \text{He}(P_f \bar{A}_{22} + \chi_2) + E_f^T E_f - Q_f, \quad \bar{\Theta}_{f2} = -\frac{1}{h_2} P_f \bar{B}_{u2}, \quad \bar{\Theta}_{f3} = -4R_f, \quad E_f = [I_m \ 0]^T. \quad (41)$$

*Proof.* We shall investigate the  $L_2$ -gain performance of the closed-loop subsystem (14) under zero initial conditions. The performance index can be denoted as

$$J_f = \|z_{f1}(\tau)\|_2^2 - \gamma_f^2 \|w_f(\tau)\|_2^2.$$

The attenuation of high frequency disturbance  $w_f(\tau)$  is considered, which aims to realize the system specification (17) subject to (19).

Taking the Lyapunov functional  $V_f(\tau)$  in (24) into account, we can obtain that

$$J_f \leq \int_0^\infty z_{f1}^T(\tau)z_{f1}(\tau) - \gamma_f^2 w_f^T(\tau)w_f(\tau) + \dot{V}_f(\tau) d\tau. \quad (42)$$

When  $w_f(\tau) \neq 0$ , the derivative of  $V_f(\tau)$  is obtained as

$$\begin{aligned} \dot{V}_f(\tau) \leq & \xi^T(\tau) \begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_{u2} K) & -\frac{1}{h_2} P_f \bar{B}_{u2} & P_f \bar{B}_{w2} & 0 \\ \star & -4R_f & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \xi(\tau) \\ & - \xi^T(\tau) \begin{bmatrix} \bar{A}_{22}^T D_f + K^T \bar{B}_{u2}^T D_f \\ -\frac{1}{h_2} \bar{B}_{u2}^T D_f^T \\ \bar{B}_{w2}^T D_f^T \\ 0 \end{bmatrix} \left( -\hat{R}_f^{-1} \right) \begin{bmatrix} \bar{A}_{22}^T D_f^T + K^T \bar{B}_{u2}^T D_f^T \\ -\frac{1}{h_2} \bar{B}_{u2}^T D_f^T \\ \bar{B}_{w2}^T D_f^T \\ 0 \end{bmatrix}^T \xi(\tau), \end{aligned} \quad (43)$$

where  $\xi(\tau) = \text{col}\{\varphi(\tau), w_f(\tau), z_f(\tau)\}$ . Combining (43) with (42), we have

$$J_f \leq \int_0^\infty \xi^T(\tau) \Theta_f \xi(\tau) d\tau,$$

where

$$\begin{aligned} \Theta_f = & \begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_{u2} K) + E_f^T E_f & -\frac{1}{h_2} P_f \bar{B}_{u2} & P_f \bar{B}_{w2} & 0 \\ \star & -4R_f & 0 & 0 \\ \star & \star & -\gamma_f^2 I_{n_w} & 0 \\ \star & \star & \star & 0 \end{bmatrix} \\ & + \begin{bmatrix} \bar{A}_{22}^T D_f + K^T \bar{B}_{u2}^T D_f \\ -\frac{1}{h_2} \bar{B}_{u2}^T D_f^T \\ \bar{B}_{w2}^T D_f^T \\ 0 \end{bmatrix} \left( -\hat{R}_f^{-1} \right) \begin{bmatrix} \bar{A}_{22}^T D_f^T + K^T \bar{B}_{u2}^T D_f^T \\ -\frac{1}{h_2} \bar{B}_{u2}^T D_f^T \\ \bar{B}_{w2}^T D_f^T \\ 0 \end{bmatrix}^T. \end{aligned}$$

Denote  $\bar{J}_f = \int_0^\infty \xi^T(\tau) \Theta_f \xi(\tau) d\tau$ . Following the similar argument in the work of Sun et al,<sup>9</sup> it is readily verified that

$$\bar{J}_f = \frac{1}{2\pi} \int_{-\infty}^\infty \xi^T(\omega) \Theta_f \xi(\omega) d\omega,$$

based on the Parseval's theorem. It follows from the S-procedure in Lemma 2 that the following condition:

$$\xi^T(\omega) (\Theta_f + M_f) \xi(\omega) < 0, \quad U_2 \xi(\omega) = 0, \quad (44)$$

can guarantee that

$$\xi^T(\omega) \Theta_f \xi(\omega) < 0, \quad \text{and} \quad \xi^T(\omega) M_f \xi(\omega) \geq 0,$$

where  $M \in \mathbf{M}_f$ , and

$$\mathbf{M}_f = \{F^T(\Phi \otimes P_f + \Psi \otimes Q_f)F : P_f, Q_f \in \mathbb{H}_{2m}\}, \quad (45)$$

with

$$F = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}. \quad (46)$$

Then, a frequency set is defined as  $\mathbf{W}_f = \{\varepsilon \in \mathbb{C} : \varepsilon \neq 0, \varepsilon^T M_f \varepsilon \geq 0, \exists M_f \in \mathbf{M}_f\}$ , which can be used to express the prescribed finite frequency region.<sup>9,26</sup>

In the work of Iwasaki and Hara,<sup>26</sup> the frequency set  $\Lambda_h$  can be specified by setting

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} -1 & j\varpi_c \\ -j\varpi_c & -\varpi_1 \varpi_2 \end{bmatrix}.$$

From (45) and (46), we can reformulate (44) as

$$\Theta_f + M_f \leq \begin{bmatrix} \text{He}(P_f \bar{A}_{22} + P_f \bar{B}_u K) + E_f^T E_f & -\frac{1}{h_2} P_f \bar{B}_u & P_f \bar{B}_{w2} & 0 \\ \star & -4R_f & 0 & 0 \\ \star & \star & -\gamma_f^2 I_m & 0 \\ \star & \star & \star & 0 \end{bmatrix} + \begin{bmatrix} \bar{A}_{22}^T D_f + K^T \bar{B}_u^T D_f \\ -\frac{1}{h_2} B_u^T \\ \bar{B}_{w2}^T D_f \\ 0 \end{bmatrix} \left( -\hat{R}_f^{-1} \right) \begin{bmatrix} \bar{A}_{22}^T D_f + K^T \bar{B}_u^T D_f \\ -\frac{1}{h_2} B_u^T \\ \bar{B}_{w2}^T D_f \\ 0 \end{bmatrix}^T + \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}^T \begin{bmatrix} -Q_f & P_f + j\omega_c Q_f \\ \star & -\varpi_1 \varpi_2 Q_f \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}. \quad (47)$$

Condition (47) is equivalent to

$$\begin{bmatrix} \bar{\Theta}_{f1} & \bar{\Theta}_{f2} & P_f \bar{B}_{w2} & P_f + j\omega_c Q_f & \bar{A}_{22}^T D_f + K^T \bar{B}_u^T D_f \\ \star & \bar{\Theta}_{f3} & 0 & 0 & -\frac{1}{h_2} B_u^T P_f \\ \star & \star & -\gamma_f^2 I_m & 0 & \bar{B}_{w2}^T D_f \\ \star & \star & \star & -\varpi_1 \varpi_2 Q_f & 0 \\ \star & \star & \star & \star & -\hat{R}_f^{-1} \end{bmatrix} < 0, \quad (48)$$

where  $\bar{\Theta}_{fi}$  is shown in (41). Following the same line with Theorem 1, the sufficient condition for the feasibility of (48) is given as LMI (40). This completes the proof.  $\square$

Now, the following theorem presents the sufficient conditions for the attenuation of low frequency external disturbance.

**Theorem 4.** Given the cutoff frequencies  $\omega_1$  and  $\omega_2$  and a positive scalar  $\gamma_s$ , subsystem (13) with  $\bar{K}_1$  and  $\bar{K}_2$  can achieve the finite frequency specification (16) with respect to (18), if there exist symmetric matrices  $P_{s1} \in \mathbb{R}^{n \times n}$ ,  $P_{s3} \in \mathbb{R}^{n \times n}$ ,  $L_f \in \mathbb{R}^{n_u \times n_u}$ ,  $P_s \in \mathbb{R}^{2n \times 2n}$ ,  $Q_s \in \mathbb{R}^{2m \times 2m}$ ,  $R_s \in \mathbb{R}^{n_u \times n_u}$  and matrices  $P_{s2} \in \mathbb{R}^{n \times n}$ ,  $\mathcal{K}_{s1} \in \mathbb{R}^{n_u \times n}$ , and  $\mathcal{K}_{s2} \in \mathbb{R}^{n_u \times n}$ , such that LMIs (38)-(39) hold, and

$$\begin{bmatrix} \bar{\Theta}_{s1} & \bar{\Theta}_{s2} & P_s \bar{B}_{ws} & P_s + j\omega_c Q_s & \bar{A}_s^T D_s^T P_{s3} + \chi_3 & 0 \\ \star & \bar{\Theta}_{s3} & 0 & 0 & -\frac{1}{h_1} B_{us}^T P_{s3} & 0 \\ \star & \star & -\gamma_s^2 I_n & 0 & \bar{B}_{ws}^T D_s^T P_{s3} & 0 \\ \star & \star & \star & -\omega_1 \omega_2 Q_s & 0 & 0 \\ \star & \star & \star & \star & -\text{He}(P_{s3}) & \mathcal{K}_{s2}^T B_{us}^T \\ \star & \star & \star & \star & \star & -\text{He}(B_{us}^T P_{s3}) + \bar{R}_s \end{bmatrix} < 0, \quad (49)$$

where  $\omega_c = 0.5(\omega_1 + \omega_2)$ , matrices  $P_s$ ,  $R_{s1}$ ,  $R_{s2}$ ,  $\bar{R}_s$ ,  $\chi_3$ , and  $\chi_4$  have been given in Theorem 2, and

$$\bar{\Theta}_{s1} = \text{He}(P_s \bar{A}_s + \chi_4) + E_s^T E_s - Q_s, \quad \bar{\Theta}_{s2} = -\frac{1}{h_1} P_s \bar{B}_{us}, \quad \bar{\Theta}_{s3} = -4R_{s2}, \quad E_s = [I_n \quad 0]^T. \quad (50)$$

*Proof.* For the frequency set  $\Lambda_l$ , matrices  $\Phi$  and  $\Psi$  can be specified as

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1 \omega_2 \end{bmatrix}.$$

The following proof is similar to that of Theorem 3, which is omitted here.  $\square$

## 5 | ESTIMATION FOR THE DISTURBANCE ATTENUATION LEVEL AND THE UPPER BOUND OF SINGULAR PERTURBATION PARAMETER

In previous sections, controller gains are formulated based on the reduced-order models of plant dynamics

$$K_1 = L_f^{-1} \mathcal{K}_{f1}, \quad K_2 = L_f^{-1} \mathcal{K}_{f2}, \quad \bar{K}_1 = L_s^{-1} \mathcal{K}_{s1}, \quad \bar{K}_2 = L_s^{-1} \mathcal{K}_{s2}.$$

We will investigate whether the obtained controller gains  $\bar{K}_1$ ,  $\bar{K}_2$ ,  $K_1$ , and  $K_2$  are effective for the original two-time-scale system (10). The accurate knowledge of the stability bound of singular perturbed parameters and disturbance attenuation index is very important for control engineers, which indicate the application range of the proposed method. The disturbance attenuation level and the upper bound of singular perturbation parameter are estimated based on frequency-domain transfer functions.

Denote a new vector as  $y(t) = \text{col}\{x_1(t), z_1(t)\}$ . Under zero initial conditions, (9) is written as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^1 \tilde{\mathcal{A}}_{1i} x(t - h_i) + \sum_{i=0}^1 \tilde{\mathcal{A}}_{2i} z(t - \epsilon \tilde{h}_i), \\ \epsilon \dot{z}(t) &= \sum_{i=0}^1 \tilde{\mathcal{A}}_{3i} x(t - h_i) + \sum_{i=0}^1 \tilde{\mathcal{A}}_{4i} z(t - \epsilon \tilde{h}_i), \end{aligned} \quad (51)$$

where  $h_0 = 0$ ,  $\tilde{h}_0 = 0$ ,  $\tilde{h}_1 = h_2$ , and

$$\begin{aligned} \tilde{\mathcal{A}}_{10} &= \mathcal{A}_{11}, \quad \tilde{\mathcal{A}}_{11} = -\frac{1}{h_1} \mathcal{A}_{11}^d E_s^T, \quad \tilde{\mathcal{A}}_{20} = \mathcal{A}_{12}, \quad \tilde{\mathcal{A}}_{21} = -\frac{1}{\epsilon h_2} \mathcal{A}_{12}^d E_f^T, \\ \tilde{\mathcal{A}}_{30} &= \mathcal{A}_{21}, \quad \tilde{\mathcal{A}}_{31} = -\frac{1}{h_1} \mathcal{A}_{21}^d E_s^T, \quad \tilde{\mathcal{A}}_{40} = \mathcal{A}_{22}, \quad \tilde{\mathcal{A}}_{41} = -\frac{1}{\epsilon h_2} \mathcal{A}_{22}^d E_f^T. \end{aligned}$$

The following theorem estimates the stability bound and disturbance attention index of SPS (9).

**Theorem 5.** *If subsystems (13) and (14) are internally stable and satisfy control system specifications (16) and (17) subject to constraints (18) and (19), then there exists a positive scalar  $\epsilon^*$  such that the original system (9) is robustly and asymptotically stable with the disturbance attenuation level  $\gamma$ , ie,*

$$\|T(s, \epsilon)\|_\infty < \gamma, \quad (52)$$

and  $\epsilon^*$  can be estimated following the method below.

(1) Search for  $\epsilon_1^*$  that satisfies

$$\left\| \left( \epsilon s I_n - \tilde{\mathcal{A}}_{40} \right)^{-1} \left( \sum_{i=1}^2 \tilde{\mathcal{A}}_{4i} \exp(-\epsilon \tilde{h}_i s) \right) \right\|_\infty < 1, \quad \forall \epsilon \in (0, \epsilon_1^*].$$

(2) Search for  $\epsilon_2^*$  that satisfies

$$\|H_s(s, \epsilon)\|_\infty < 1, \quad \forall \epsilon \in (0, \epsilon_2^*],$$

where

$$\begin{aligned} H_s(s, \epsilon) &= \epsilon s M_s(s) \left( \sum_{i=0}^n \tilde{\mathcal{A}}_{2i} \exp(-\epsilon \tilde{h}_i s) \right) \left( \sum_{i=0}^n \tilde{\mathcal{A}}_{4i} \exp(-\epsilon \tilde{h}_i s) \right)^{-1} \\ &\quad \left( \epsilon s I_m - \sum_{i=0}^n \tilde{\mathcal{A}}_{4i} \exp(-h_i s) \right)^{-1} \left( \sum_{i=0}^n \tilde{\mathcal{A}}_{3i} \exp(-\epsilon \tilde{h}_i s) \right), \\ M_s(s) &= \left\{ s I_n - \left[ \sum_{i=0}^2 \tilde{\mathcal{A}}_{1i} \exp(-h_i s) - \sum_{i=0}^2 \tilde{\mathcal{A}}_{2i} \exp(-\epsilon \tilde{h}_i s) \left( \sum_{i=0}^2 \tilde{\mathcal{A}}_{4i} \exp(-\epsilon \tilde{h}_i s) \right)^{-1} \sum_{i=0}^2 \tilde{\mathcal{A}}_{3i} \exp(-h_i s) \right] \right\}^{-1}, \end{aligned}$$

and  $\epsilon^* = \min(\epsilon_1^*, \epsilon_2^*)$ . Then, the disturbance attenuation index of the whole system (51) over the target frequency range  $\Lambda_t = \Lambda_l \cup \Lambda_h$  can be given by

$$\gamma = \max \left\{ \gamma_s + \sigma_{\max} \left( E_f^T \left( \sum_{i=0}^2 \tilde{\mathcal{A}}_{4i} \right)^{-1} \bar{B}_{w2} \right), \gamma_f \right\},$$

where  $\sigma_{\max}(\cdot)$  denotes the maximum singular value of a matrix.

*Proof.* From the work of Pan et al.,<sup>30</sup> the stability of the slow and fast subsystems (13) and (14) can guarantee the asymptotical stability of the original system (9) for  $\forall \epsilon \in (0, \epsilon^*]$ .

First, we investigate the stability bound of system (51), which is characterized by the range  $(0, \epsilon^*]$ . Then, the problem of the formulation of the stability bound is converted into the estimation of the upper bound of singular perturbation parameters. By setting  $\epsilon = 0$  in (9), we have

$$\begin{cases} sX_s(s) - x_s(0) = \sum_{i=0}^1 \tilde{\mathcal{A}}_{1i} \exp(-h_i s) X_s(s) + \sum_{i=0}^1 \tilde{\mathcal{A}}_{2i} Z_s(s), \\ 0 = \sum_{i=0}^1 \tilde{\mathcal{A}}_{3i} \exp(-h_i s) X_s(s) + \sum_{i=0}^1 \tilde{\mathcal{A}}_{4i} Z_s(s). \end{cases} \quad (53)$$

Then, it can be derived that

$$X_s(s) = M_s(s)x_s(0),$$

where

$$M_s(s) = \left\{ sI_n - \left[ \sum_{i=0}^2 \tilde{\mathcal{A}}_{1i} \exp(-h_i s) - \sum_{i=0}^2 \tilde{\mathcal{A}}_{2i} \left( \sum_{i=0}^2 \tilde{\mathcal{A}}_{4i} \right)^{-1} \sum_{i=0}^2 \tilde{\mathcal{A}}_{3i} \exp(-h_i s) \right] \right\}^{-1}.$$

Based on the time-scale techniques, we can obtain the counterpart of fast subsystem

$$Z_f(s) = M_f(s, \epsilon)z_f(0),$$

where

$$M_f(s, \epsilon) = \left\{ \epsilon sI_m - \sum_{i=0}^2 \tilde{\mathcal{A}}_{4i} \exp(-\epsilon \tilde{h}_i s) \right\}^{-1}.$$

Then, the upper bound of system (51) can be obtained by using the result in the work of Pan et al.<sup>30</sup>

Next, we investigate the  $H_\infty$  disturbance attention capability of the original system (9). Due to the presence of the singular perturbation parameter  $\epsilon$ , it can be seen that  $T(s, \epsilon)$  is a two-frequency-scale transfer function matrix, which can be decomposed into two parts. The establishment of (16) subject to (18) implies

$$\begin{aligned} \|T_s(j\omega)\|_\infty &< \gamma_s, \quad \omega \in \Lambda_l, \\ \|T_f(j\varpi)\|_\infty &< \gamma_f, \quad \varpi \in \Lambda_h. \end{aligned}$$

Under zero initial conditions, the transfer function matrix of fast subsystem is obtained by using Laplace transformations

$$T_f(p) = E_f^T \left( pI_m - \sum_{i=0}^2 \tilde{\mathcal{A}}_{4i} \exp(-h_i p) \right)^{-1} \bar{B}_{w2},$$

where  $p = \epsilon s$ . Similarly, we have

$$T_s(s) = E_s^T M_s(s) \bar{B}_{w1}.$$

The singular perturbation analysis of  $T(s, \epsilon)$  from the input  $w(t)$  to the output  $y(t)$  yields

$$T(s, p) = T_s(s) + T_f(p),$$

such that

$$\|T(s, p)\|_\infty \leq \|T_s(s)\|_\infty + \|T_f(p)\|_\infty.$$

In the frequency set  $\Lambda_l$ , the frequency  $s$  is significant, that is,  $s = O(1)$  and  $p = O(\epsilon)$ , and

$$\begin{aligned} \|T(s, p)\|_\infty &\leq \|T_s(s)\|_\infty + \|T_f(0)\|_\infty, \\ &\leq \gamma_s + \sigma_{\max} \left( E_f^T \left( \sum_{i=0}^2 \tilde{\mathcal{A}}_{4i} \right)^{-1} \bar{B}_{w2} \right), \quad s = j\omega, \quad \omega \in \Lambda_l. \end{aligned} \quad (54)$$

Similarly, we can obtain that  $s \rightarrow \infty, p = O(1)$ , in the frequency set  $\Lambda_h$ , such that

$$\lim_{s \rightarrow \infty} M(s) = 0, \quad T(s, p) = T_f(p).$$

Thus, we have

$$\|T(s, p)\|_\infty = \|T_f(p)\|_\infty \leq \gamma_f, \quad p = j\varpi, \quad \varpi \in \Lambda_h. \quad (55)$$

By integrating (54) with (55), it can be seen that conditions (16) and (17) subject to (18) and (19) can guarantee the establishment of (52). Then, the disturbance attenuation index  $\gamma$  is given as

$$\gamma = \max \left\{ \gamma_s + \sigma_{\max} \left( E_f^T \left( \sum_{i=0}^2 \tilde{\mathcal{A}}_{4i} \right)^{-1} \bar{B}_{w2} \right), \gamma_f \right\}.$$

This completes the proof.  $\square$

*Remark 6.* There are two types of methods to compute the stability bound of SPSs: one is based on transfer function matrices such as those in other works<sup>3-5,30-32</sup> and the other is the use of  $\epsilon$ -dependent Lyapunov functions and the resulting LMIs such as those in the works of Fridman<sup>15</sup> and Yang and Zhang.<sup>33</sup> In this paper, we could also use Lyapunov functionals depending the singular perturbation parameter  $\epsilon$ , but this might result in the higher-order LMIs. Here, we prefer to use the simplest reduced-order LMIs. The controller design of this work is based on the slow-fast decomposition method. The upper bound of singular perturbation parameters is estimated based on the transfer function matrix of system (51). The proposed design method consists of two sequent steps: (1) the formulation of controller gains based on subsystems and (2) the estimation of  $\epsilon^*$  and  $\gamma$  based on the closed-loop SPS (51).

## 6 | PRACTICAL EXAMPLE: VEHICLE ACTIVE SUSPENSION SYSTEM

In this section, the proposed design method is applied in a vehicle active suspension system to improve the ride comfort and road handling performance. The following state-space model is considered:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{c_s}{m_s} & 0 & \frac{c_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{c_s}{m_u} & -\frac{k_u}{m_u} & -\frac{c_s+c_t}{m_u} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_s} \\ 0 \\ \frac{1}{m_u} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ -1 \\ \frac{c_t}{m_u} \end{bmatrix} w(t), \quad (56)$$

where  $m_s$  and  $m_u$  are the sprung and unsprung masses,  $c_s$  and  $k_s$  represent damping and stiffness of the active suspension system,  $k_t$  and  $c_t$  are compressibility and damping of the pneumatic tire,  $u(t)$  stands for the control force from the hydraulic actuator, and  $w(t)$  is the road disturbance.<sup>34,35</sup> The state variables are defined as

$$x_1(t) = z_s(t) - z_u(t), \quad x_2(t) = \dot{z}_s(t), \quad x_3(t) = z_u(t) - z_r(t), \quad x_4(t) = \dot{z}_u(t),$$

where  $x_1(t)$  denotes the suspension travel,  $x_2(t)$  is the car body velocity,  $x_3(t)$  represents the tire deflection, and  $x_4(t)$  is the wheel velocity.

Define a new variable as  $\tilde{x}_2(t) = x_2(t) - x_4(t)$ . The ratio between sprung mass and unsprung mass is sufficiently small to play the role as a singular perturbation parameter, ie,  $\epsilon = \frac{m_u}{m_s}$ . We transform model (56) into model (2):

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{\tilde{x}}_2(t) \\ \epsilon \dot{x}_3(t) \\ \epsilon \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \tilde{x}_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_{u1} \\ 0 \\ b_{u2} \end{bmatrix} u(t) + \begin{bmatrix} B_{w1} \\ B_{w2} \end{bmatrix} w(t), \quad (57)$$

where

$$\begin{aligned} a_{11} &= -\frac{k_s}{m_s} - \frac{k_u}{m_u}, & a_{12} &= -\frac{c_s}{m_s} - \frac{c_u}{m_u}, & a_{13} &= \frac{k_u}{m_u}, & a_{14} &= \frac{c_s + c_t}{m_u} + \frac{c_s}{m_s}, & b_{u1} &= \frac{1}{m_s} - \frac{1}{m_u}, \\ a_{21} &= \frac{k_s}{m_s}, & a_{22} &= \frac{c_s}{m_s}, & a_{23} &= -\frac{k_u}{m_s}, & a_{24} &= \frac{-c_s - c_t}{m_s}, & b_{u2} &= \frac{1}{m_s}, & B_{w1} &= \begin{bmatrix} 0 \\ -\frac{c_t}{m_u} \end{bmatrix}, & B_{w2} &= \begin{bmatrix} -1 \\ \frac{c_t}{m_s} \end{bmatrix}. \end{aligned}$$

In the experiment setup, the parameters of model (57) are given as

$$m_s = 1000 \text{ kg}, \quad m_u = 110 \text{ kg}, \quad k_s = 42720 \text{ kN/m}, \quad k_u = 10115 \text{ kN/m}, \quad c_s = 1095 \text{ Ns/m}, \quad c_t = 14.6 \text{ Ns/m},$$

which have been listed in the work of Li et al.<sup>36</sup> The singular perturbation parameter is obtained as  $\epsilon = 0.11$ . In the active suspension system, the ride quality and the road-holding property are the key factors to be considered. For the comfort of passengers in the vehicle, the frequency should be situated between 0.5 and 1.5 Hz.<sup>34,35</sup> Thus, the cutoff frequencies are formulated as

$$\omega_1 = 2\pi \times 0.5 \text{ rad/s}, \quad \omega_2 = 2\pi \times 1.5 \text{ rad/s}, \quad \varpi_1 = \epsilon\omega_1, \quad \varpi_2 = \epsilon\omega_2,$$

which covers the target frequency range 0.5-1.5 Hz.

(1) Controller design based on the slow and fast subsystems.

Choose  $\delta_1 = 0.01$ ,  $\bar{\delta}_1 = 0.01$ ,  $h_1 = 0.1$ ,  $h_2 = 0.3$ ,  $\gamma_f = 0.1$ . By solving LMIs (20)-(22) and LMI (40), the fast controller gains are obtained as

$$K_1 = 1.0098e + 5, \quad K_2 = 0.8795. \quad (58)$$

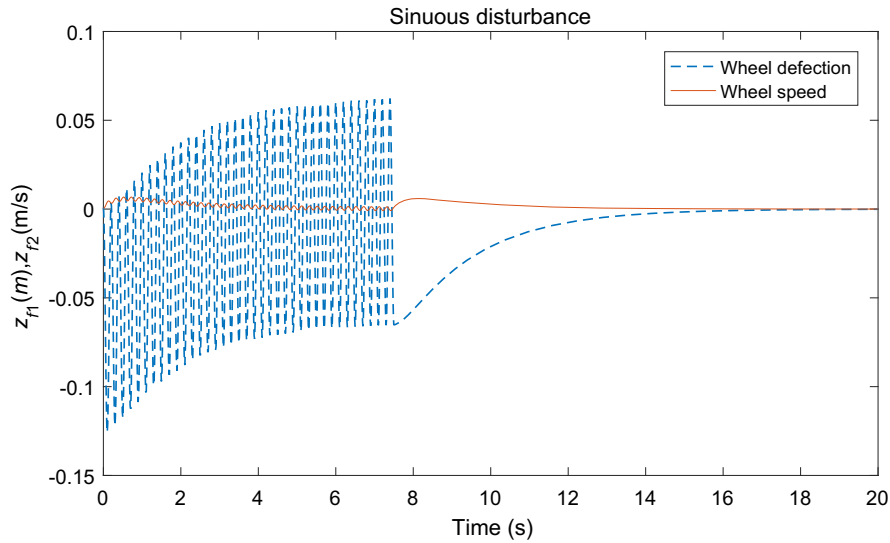
To evaluate the disturbance attenuation capability of the closed-loop fast subsystem (14), three types of disturbance are considered. First, the following sinuous disturbance signal is used to verify the effectiveness of our finite frequency delayed output feedback controller:

$$w(t) = \begin{cases} 0.5 \sin(2\pi \times 1.25t), & t \leq 30 \text{ s}, \\ 0, & t > 30 \text{ s}. \end{cases} \quad (59)$$

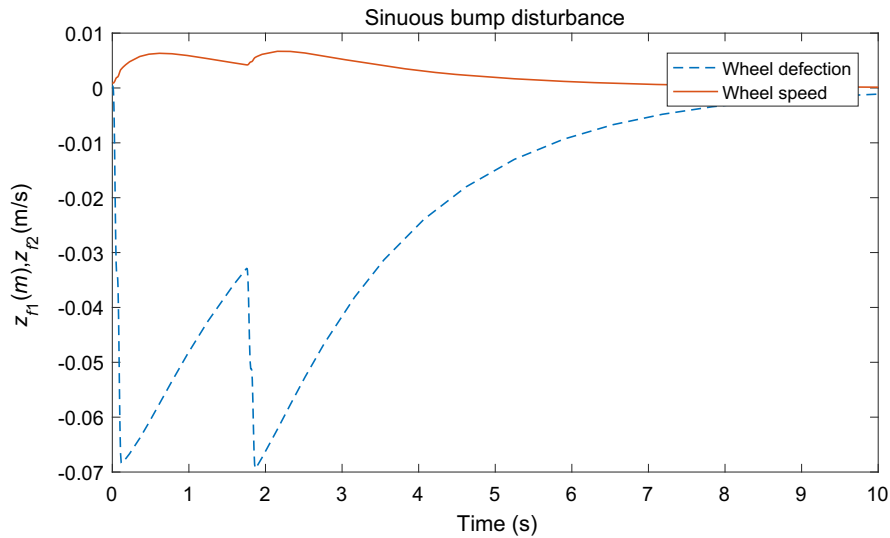
The time domain response of the closed-loop fast subsystem (14) subject to disturbance  $w_f(\tau)$  is shown in Figure 1. It can be seen that system states can reach their stable states in the presence of sinusoidal disturbance, and the effect of disturbance input has been suppressed to a low level. Then, the case of isolated bumps in a smooth road surface is taken into account, with the disturbance inputs given as

$$w(t) = \begin{cases} 0.9 \sin(2\pi t), & t \leq 0.5 \text{ s}, \\ 0.5 \sin(2\pi t), & 2 \leq t < 2.5 \text{ s}, \\ 0, & t \geq 2.5 \text{ s}. \end{cases} \quad (60)$$





**FIGURE 1** Time-domain response of fast subsystem subject to sinuous disturbance [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]



**FIGURE 2** Time-domain response of fast subsystem subject to sinuous bump disturbance [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

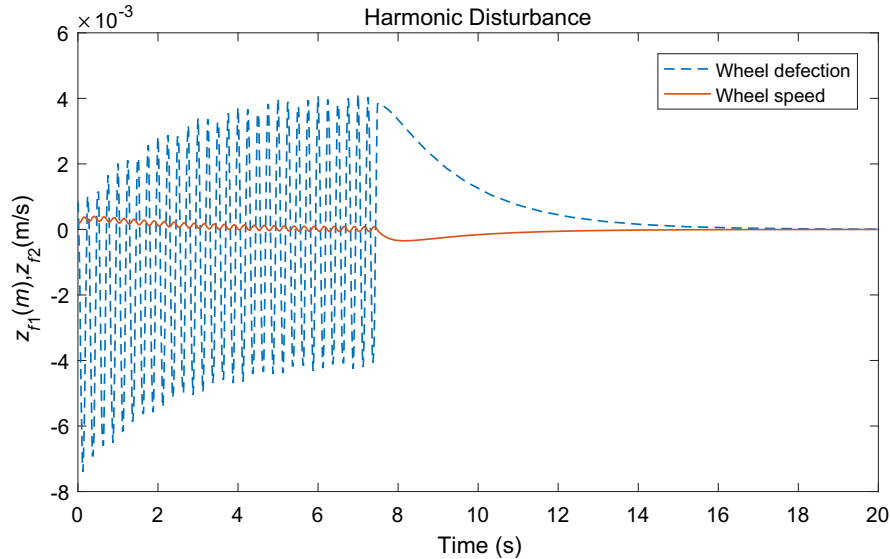
To further evaluate the effectiveness of fast controller (58), the case of harmonic disturbance is considered, with disturbance input being in the form of

$$w(t) = \begin{cases} 0.0254 \sin(2\pi t) - 0.005 \sin(10.5\pi t) + 0.001 \sin(21.5\pi t), & t \leq 7.8 \text{ s}, \\ 0, & t \geq 7.8 \text{ s}. \end{cases} \quad (61)$$

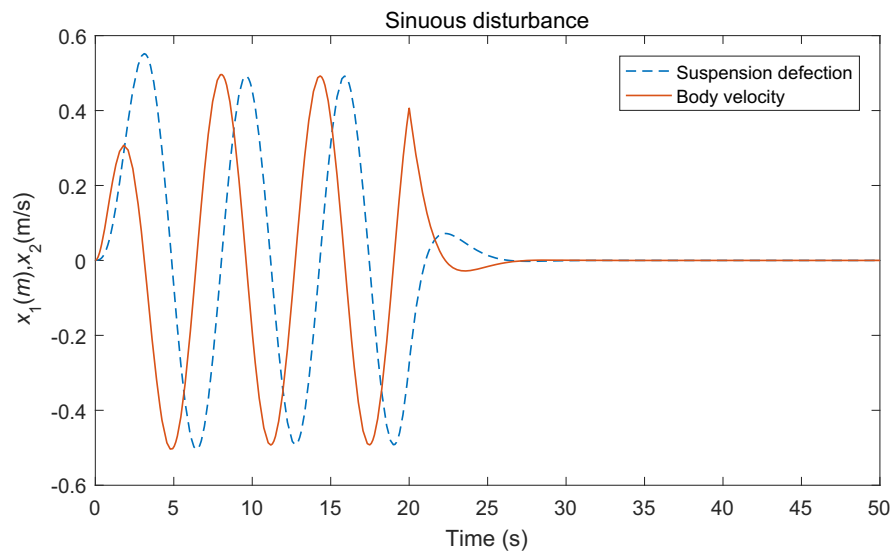
Figures 1 to 3 show the time-domain response of the closed-loop fast subsystem (14) with respect to sinuous, sinuous bump, and harmonic disturbance inputs, respectively. In all Figures, three types of disturbances can be attenuated to an acceptable level.

Similarly, slow controller gains  $\bar{K}_1$  and  $\bar{K}_2$  will be designed to achieve the control performance (16) with respect to constraint (18). The corresponding design parameters in Theorems 2 and 4 are chosen as  $\delta_2 = 0.001$ ,  $\tilde{\delta}_2 = 0.001$ ,  $h_1 = 0.1$ ,  $h_2 = 0.3$ , and  $\gamma_s = 0.5$ . A feasible solution by solving LMIs (37)-(39) and LMI (49) is given as

$$\bar{K}_1 = 2.2555e + 04, \quad \bar{K}_2 = -6.1686. \quad (62)$$



**FIGURE 3** Time-domain response of fast subsystem subject to harmonic disturbance [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

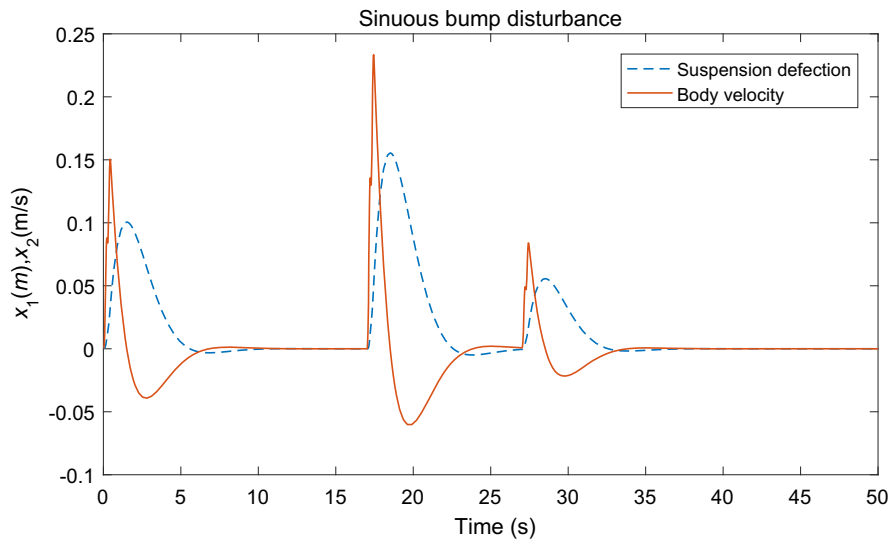


**FIGURE 4** Time-domain response of slow subsystem subject to sinuous disturbance [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

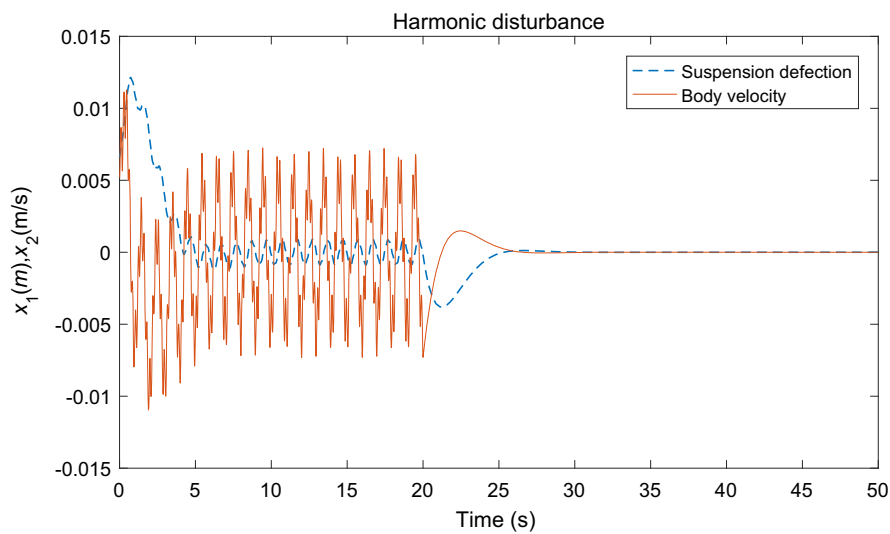
In Figures 4 to 6, it can be seen that the state trajectories for system (13) move toward the origin of the state space, even in the presence of sinuous, sinuous bump, and harmonic disturbance.

(2) Performance evaluation of the delayed output feedback controller in the SPS (1).

In order to evaluate the disturbance attenuation capability of finite frequency disturbance, sinuous bump oscillations are simultaneously imposed on the whole system (10) to validate the robustness of the designed controller (8). In Figure 7, it is observed that the bounded-input–bounded-output property (BIBO) is achieved, and the effects of disturbance inputs are attenuated to a satisfactory level. Figure 8 reveals the bode diagram of system (57) from road disturbance to the measured outputs. Compared with the passive case and full frequency case, the active system with finite frequency controller improves the disturbance attenuation capability near 1 Hz, which indicates the improvement of ride comfort of passengers. The first peak of the magnitude–frequency characteristic has been reduced to the lowest level with the aid of finite frequency delayed controller. Moreover, controller (8) cannot function as well as that of the finite frequency state



**FIGURE 5** Time-domain response of slow subsystem subject to sinuous bump disturbance [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]



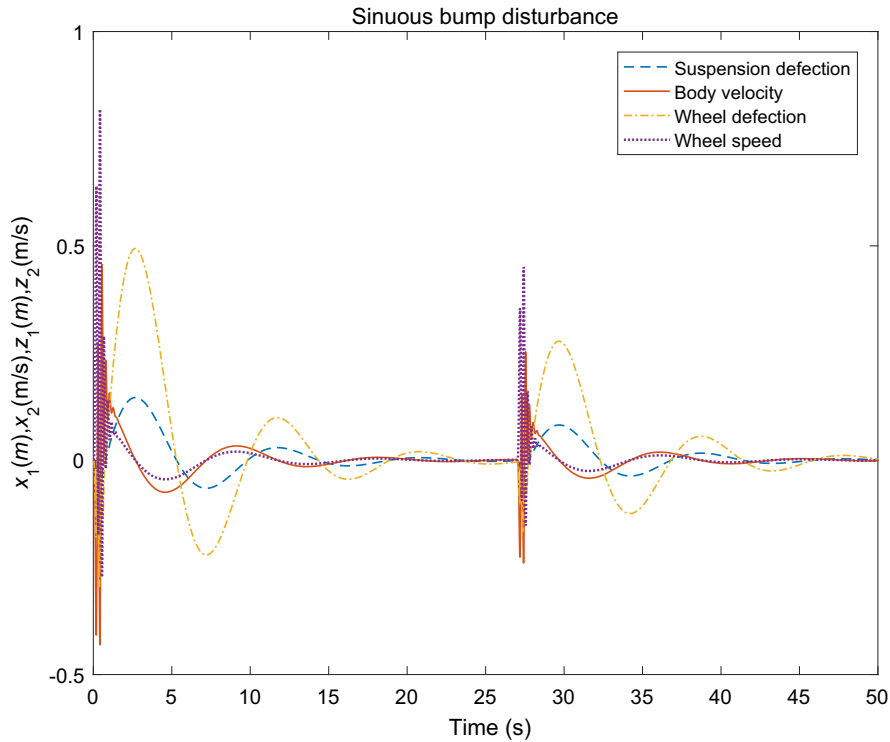
**FIGURE 6** Time-domain response of slow subsystem subject to harmonic disturbance [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

feedback controller in the work of Sun et al.<sup>9</sup> The reason behind this is the lack of measurements of  $x_2(t)$  and  $z_2(t)$ , which sacrifice part control quality for the approximation of unmeasurable system states.

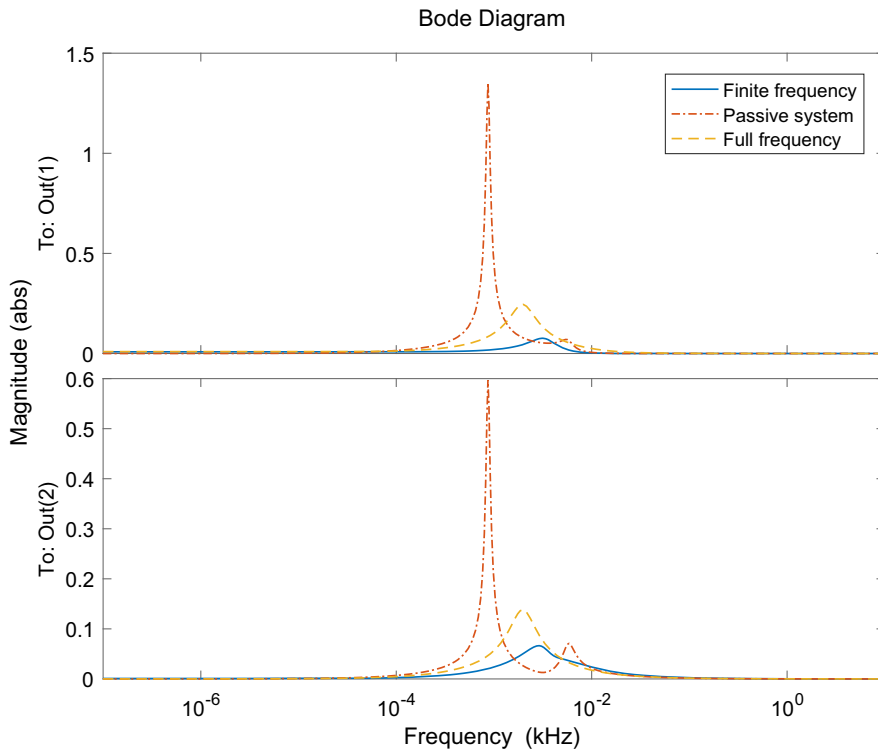
To further evaluate the performance of the delayed SOF controller, the value of disturbance attenuation index  $\gamma$  is estimated. This problem can be effectively tackled by using the command `mincx` in MATLAB (LMI toolbox). We have

$$\min \gamma_s = 0.22, \quad \min \gamma_f = 0.25.$$

Based on Theorem 5, the disturbance attenuation index is estimated as  $\gamma = 0.28$ . Moreover, the upper bound of the singular perturbation parameter  $\epsilon^*$  is given as  $\epsilon^* = 0.96$ , which reveals the effective application of our design method in SPSs. Otherwise, the stability or BIBO property of system (1) may be destroyed, or even results in the instability. Because  $\epsilon^* > \epsilon = 0.11$  in (57), the designed delayed SOF controller can be used in system (57).



**FIGURE 7** Time-domain response of the whole system subject to sinuous bump disturbance [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]



**FIGURE 8** The Bode plot of the whole system [Colour figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

## 7 | CONCLUSION

In this work, the problem of the delayed SOF stabilization of SPSs has been investigated by inserting artificial time delays in the feedback loops. Based on the singular perturbation analysis of a time-delay system, a two-stage design

procedure is put forward for the LMI formulation of controller gains, which avoids the pre-selection of controller parameters and the technique of bounding cross terms. First, a time-delay SPS is decomposed into slow and fast subsystems operating in different time scales. Then, the stability and performance analysis of an SPS is achieved by investigating those of parameter-independent subsystems, respectively. The disturbance attenuation index and the upper bound of singular perturbed parameter are estimated. The proposed design method is verified in an active suspension system to show its merits and effectiveness.

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