# An Improved Delay-Dependent $H_\infty$ Filtering of Linear Neutral Systems

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***Abstract*—An improved delay-dependent $H_\infty$ filtering design is proposed for linear, continuous, time-invariant systems with time delay. The resulting filter is of the Luenberger observer type, and it guarantees that the $H_\infty$-norm of the system, relating the exogenous signals to the estimation error, is less than a prescribed level. The filter is based on the application of the descriptor model transformation and Park's inequality for the bounding of cross terms. The advantage of the new filtering scheme is clearly demonstrated via simple examples.**

## I. Introduction

The $H_\infty$ filtering problem for linear systems with delay-dependent [1]–[3] and (more conservative) delay-independent [4], [5] designs have received a lot of attention recently. The prevailing methods are based on bounded real lemmas (BRLs) in terms of Riccati algebraic equations or linear matrix inequalities (LMIs), which guarantee a prescribed attenuation level. Recently, a new approach to $H_\infty$ filtering has been introduced [6]. This approach is based on representing the system by a descriptor type model [7] and on deriving a BRL for the corresponding adjoint system. The new BRL was found to be very efficient, and it considerably reduced the achievable attenuation level as compared with other results reported in the literature. By assuming a Leunberger-type estimator [8], the new BRL was applied to the resulting estimation error system. In spite of the advantage of the new filter design, it still entails a significant amount of conservatism stemming from the overbounding of mixed terms in the proof of the BRL in [6].

A new overbounding technique has recently been proposed that produces tighter bounds [9]. In the present paper, this technique is applied to reduce the overdesign entailed in the approach of [6]. The treatment is also extended to the more general class of neutral-type systems with multiple delays. It is shown, via simple examples, that the resulting schemes significantly improve the estimation results.

**Notation**: Throughout the paper, the superscript "$T$" stands for matrix transposition, $\mathcal{R}^n$ denotes the $n$-dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$, means that $P$ is symmetric and positive definite. The space of functions in $\mathcal{R}^q$ that are square integrable over $[0\ \infty)$ is denoted by $\mathcal{L}_2^q[0,\ \infty)$, and $col\{a,\ b\}$ denotes $\begin{bmatrix} a \\ b \end{bmatrix}$.

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## II. Problem Formulation

Consider the following system:

$$\dot{x}(t) - F\dot{x}(t-g) = \sum_{i=0}^{2} A_i x(t-h_i) + Bw(t), \quad x=0,\ \forall t\le 0 \tag{1a,b}$$

where $x(t)\in\mathcal{R}^n$ is the system state vector, $w(t)\in\mathcal{L}_2^q[0,\ \infty)$ is the exogenous disturbance signal, and $h_0=0$, $h_i>0$, $i=1$, 2, and $g>0$ are constant time delays. The matrices $A_0$, $A_1$, $A_2$, $F$, and $B$ are constant matrices of appropriate dimensions. For simplicity only, two delays $h_1$ and $h_2$ and one $g$ are considered; however, the results can easily be generalized to any finite number of delays: $h_1,\ldots,h_k$ and $g_1,\ldots,g_m$.

Equation (1) describes a system of neutral type since it contains derivatives in delayed states. In the case of $F=0$, (1) is a retarded-type system (see, e.g., [10]). Neutral systems are encountered in the modeling of lossless transmission lines, or in dynamical processes, including steam and water pipes (see, e.g., [10] and the references therein). Unlike retarded systems, linear neutral systems may be destabilized by small changes of the delay [10].

To guarantee robustness of the results with respect to small changes of delay, we assume that the difference equation $\mathcal{D}x_t = x(t)-Fx(t-g)=0$ is asymptotically stable for all values of $g$ or, equivalently, the following.

**A1)** $F$ is a Schur–Cohn stable matrix, i.e., all the eigenvalues of $F$ are inside the unit circle.

Given the measurement equation

$$y(t) = Cx(t) + D_{21}w(t) \tag{2}$$

where $y(t)\in\mathcal{R}^r$ is the measurement vector and the matrices $C$ and $D_{21}$ are constant matrices of appropriate dimension, a filter of the following Luenberger observer form is sought:

$$\dot{\hat{x}}(t) - F\dot{\hat{x}}(t-g) = \sum_{i=0}^{2} A_i\hat{x}(t-h_i) + K\left(y(t)-C\hat{x}(t)\right). \tag{3}$$

This form of the observer is known to produce an estimation error that is independent of the system trajectory, and it only depends on the initial condition of the system state [8]. It is therefore widely used in many practical estimation applications (e.g Kalman filtering).

The filter of (3) must ensure that the performance index

$$J(w) = \int_0^\infty (z^T z - \gamma^2 w^T w) d\tau \quad (4)$$

is negative $\forall w(t) \in \mathcal{L}_2^q[0, \infty]$ for a prescribed value of $\gamma$. The signal $z(t) \in \mathcal{R}^p$ is the state combination to be estimated and is given by

$$z(t) \triangleq L(x(t) - \hat{x}(t)) \quad (5)$$

where $L$ is a constant matrix.

## III. Delay-Dependent $H_\infty$ Filtering

From (1)–(3), it follows that the estimation error

$$e(t) = x(t) - \hat{x}(t) \quad (6)$$

is described by the following model:

$$\begin{aligned} \dot{e}(t) - F\dot{e}(t-g) &= (A_0 - KC)e(t) + \sum_{i=1}^{2} A_i e(t-h_i) \\ &\quad + (B - KD_{21})w(t) \\ z(t) &= Le(t). \end{aligned} \quad (7a,b)$$

The problem then becomes one of finding the filter gain $K$ such that the $H_\infty$-norm of the system of (7) will be less than a prescribed value of $\gamma$.

### A. $H_\infty$-Norm of the "Adjoint" System

Using the arguments of [6], it can be shown that the $H_\infty$-norms of the system described by (7) and the following system are equal:

$$\begin{aligned} \dot{\xi}(t) - F^T\dot{\xi}(t-g) &= \left(A_0^T - C^T K^T\right)\xi(t) + \sum_{i=1}^{2} A_i^T \xi(t-h_i) \\ &\quad + L^T \tilde{z}(t), \quad \xi(t) = 0, \quad \forall t \le 0 \\ \tilde{w}(t) &= \left(B^T - D_{21}^T K^T\right)\xi(t) \end{aligned} \quad (8a,b)$$

where $\xi(t) \in \mathcal{R}^n$, $\tilde{z}(t) \in \mathcal{R}^p$, and $\tilde{w}(t) \in \mathcal{R}^q$. Note that the latter system represents the forward adjoint of (7) (as defined in [13, vol. 1]).

Following [7], we represent (8a) in the form of the equivalent descriptor system

$$\begin{aligned} \dot{\xi}(t) &= \zeta(t) \\ 0 &= -\zeta(t) + F^T\zeta(t-g) + (A_0^T - C^T K^T)\xi(t) \\ &\quad + \sum_{i=1}^{2} A_i^T \xi(t-h_i) + L^T\tilde{z}(t), \quad \xi(t) = 0, \quad \forall t \le 0. \end{aligned}$$

Since $\xi(t-h_i) = \xi(t) - \int_{t-h_i}^t \dot{\xi}(s)ds$, the latter system is equivalent to the following one:

$$\begin{aligned} \dot{\xi}(t) &= \zeta(t) \\ 0 &= -\zeta(t) + F^T\zeta(t-g) + \left(\sum_{i=0}^{2} A_i^T - C^T K^T\right)\xi(t) \\ &\quad - \sum_{i=1}^{2} A_i^T \int_{t-h_i}^{t} \zeta(s)ds + L^T\tilde{z}(t) \\ \xi(t) &= 0, \quad t \le 0. \end{aligned} \quad (9)$$

The following Lyapunov–Krasovskii functional has been suggested in [7] and [11]:

$$\begin{aligned} V(t) &= \left[\xi^T(t)\ \zeta^T(t)\right] EP \begin{bmatrix} \xi(t) \\ \zeta(t) \end{bmatrix} \\ &\quad + \sum_{i=1}^{2} \int_{t-h_i}^{t} \xi^T(\tau) S_i \xi(\tau) d\tau + \int_{t-g}^{t} \zeta^T(\tau) U \zeta(\tau) d\tau \\ &\quad + \sum_{i=1}^{2} \int_{-h_i}^{0} \int_{t+\theta}^{t} \zeta^T(s)[0\ A_i] R_i \begin{bmatrix} 0 \\ A_i^T \end{bmatrix} \zeta(s) d\tau d\theta \end{aligned} \quad (10)$$

where

$$E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$$
$$P_1 > 0, \quad U > 0, \quad S_i > 0, \quad R_i > 0. \quad (11a,b)$$

The first term of (10) corresponds to the descriptor system (see, e.g., [15] and [16]), the third corresponds to the delay-independent conditions with respect to the discrete delays of $\zeta$, whereas the second and the fourth terms correspond to the delay-dependent conditions with respect to the distributed delays (with respect to $h_1$ and $h_2$).

Based on a similar functional, a BRL was derived in [6] that provided an LMI sufficiency condition for the $H_\infty$-norm of (8) to be less than $\gamma$. This condition, though still efficient compared with other methods in the literature, is still conservative, due to the bounding of a mixed term in the proof of the BRL in [6]. Recently, an improved BRL was proposed by [11], which considerably reduces the overdesign entailed in the over bounding of the above mixed term. It is based on the fact that for any $2n \times 2n$-matrices $R_i > 0$ and $M_i$, the following inequality holds (see [9]):

$$\begin{aligned} -2\int_{t-h_i}^{t} b^T(s)a(s)ds &\le \int_{t-h_i}^{t} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \\ &\quad \times \begin{bmatrix} R_i & R_i M_i \\ M_i^T R_i & \Upsilon_i \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \end{aligned} \quad (12)$$

for $a(s) \in \mathcal{R}^{2n}$, and $b(s) \in \mathcal{R}^{2n}$ defined for $s \in [t-h_i, t]$. Here, $\Upsilon_i \triangleq (M_i^T R_i + I)R_i^{-1}(R_i M_i + I)$.

In the proof of the BRL in [6], $M_i = 0$ was chosen. Taking $M_i \neq 0$, the following result is obtained (see [11]).

*Lemma 1:* Consider the system of (7). Given $\gamma > 0$ and $K \in \mathcal{R}^{n \times r}$, the cost function (4) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all positive delay $g$, if there exist $n \times n$-matrices $0 < P_1, P_2, P_3, S_i = S_i^T, U = U^T$, and $2n \times 2n$-matrices $W_i, R_i = R_i^T$, and $i = 1, 2$ that satisfy the LMI in (13), shown at the bottom of the page, where $P$ is given by (11), and for $i = 1, 2$

$$\begin{aligned}\Phi_i &= W_i + P\\ \Psi &\triangleq P^T \begin{bmatrix} 0 & I \\ \sum_{i=0}^2 A_i^T - C^T K^T & -I \end{bmatrix}\\ &\quad + \begin{bmatrix} 0 & \sum_{i=0}^2 A_i - KC \\ I & -I \end{bmatrix} P\\ &\quad + \begin{bmatrix} \sum_{i=1}^2 S_i & 0 \\ 0 & U + \sum_{i=1}^2 h_i [0\ A_i] R_i \begin{bmatrix} 0 \\ A_i^T \end{bmatrix} \end{bmatrix}\\ &\quad + \sum_{i=1}^2 W_i^T \begin{bmatrix} 0 & 0 \\ A_i^T & 0 \end{bmatrix} + \sum_{i=1}^2 \begin{bmatrix} 0 & A_i \\ 0 & 0 \end{bmatrix} W_i.\end{aligned}$$

*Remark 1:* For

$$W_i = -P, \quad R_i = \frac{\varepsilon I_{2n}}{h_i}, \quad i = 1, 2 \tag{14}$$

the LMI of (13) produces, for $\varepsilon \to 0^+$, the following BRL condition that is delay-independent:

$$\begin{bmatrix} \hat{\Psi} & * & * & * & * & * \\ [0\ L]P & -\gamma^2 I & * & * & * & * \\ [0\ A_1]P & 0 & -S_1 & * & * & * \\ [0\ A_2]P & 0 & 0 & -S_2 & * & * \\ [0\ F]P & 0 & 0 & 0 & -U & * \\ [(B - KD_{21})^T\ 0] & 0 & 0 & 0 & 0 & -I_p \end{bmatrix} < 0 \tag{15}$$

where

$$\hat{\Psi} = P^T \begin{bmatrix} 0 & I \\ A_0^T - C^T K^T & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0 - KC \\ I & -I \end{bmatrix} P + \begin{bmatrix} \sum_{i=1}^2 S_i & 0 \\ 0 & U \end{bmatrix}.$$

### B. Case of Instantaneous Measurements

Restricting the discussion to the case of $W_i = \varepsilon_i P$, $i = 1, 2$, where $\varepsilon_i \in \mathcal{R}$ is a scalar parameter, enables a LMI formulation. Note that for $\varepsilon_i = 0$, (13) implies the delay-dependent conditions of [7] and [17], whereas for $\varepsilon_i = -1$, (13) yields the delay-independent condition of Remark 1. It is obvious from the requirement of $0 < P_1$, and the fact that the (2,2) block in $\Psi$ is negative definite, that $-(P_3 + P_3^T)$ must be negative definite, and thus, $P$ is nonsingular. Defining

$$P^{-1} \triangleq Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad \Delta \triangleq \mathrm{diag}\{Q,\ I_{q+p+7n}\}. \tag{16a,b}$$

Equation (13) is multiplied by $\Delta^T$ and $\Delta$ on the left and on the right, respectively. Applying Schur's formula to the quadratic term in $Q$, the following inequality results:

$$G_1 [I_{2n}\ 0] + \begin{bmatrix} I_{2n} \\ 0 \end{bmatrix} G_1^T - G_2 < 0 \tag{17}$$

where

$$\begin{aligned} G_1 = col\,\Big\{&\Xi, [0\ L], h_1(\varepsilon_1 + 1)I, h_2(\varepsilon_2 + 1)I\\ &\varepsilon_1 [0\ A_1], \varepsilon_2 [0\ A_2], [0\ F]\\ &\left[(B - KD_{12})^T\ 0\right] Q, [I_n\ 0]Q\\ &[I_n\ 0]Q, h_1 \begin{bmatrix} 0 & 0 \\ 0 & A_1^T \end{bmatrix} Q, h_2 \begin{bmatrix} 0 & 0 \\ 0 & A_2^T \end{bmatrix} Q \Big\}\\ G_2 = \mathrm{diag}\,\Big\{&0, \gamma^2 I_p, h_1 R_1, h_2 R_2, S_1, S_2, U, I_q\\ &S_1^{-1}, S_2^{-1}, U_1^{-1}, h_1 R_1^{-1}, -h_2 R_2^{-1}\Big\}\end{aligned}$$

and

$$\Xi = \begin{bmatrix} 0 & I_n \\ \sum_{i=0}^2 A_i^T + \sum_{i=1}^2 \varepsilon_i A_i^T - C^T K^T & -I_n \end{bmatrix} Q.$$

Denoting $Q_1 K$ by $Y$, we obtain the following.

*Theorem 1:* Consider the system of (1) and the cost function of (4). For a prescribed $0 < \gamma$, $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ if for some prescribed scalars $\varepsilon_1, \varepsilon_2 \in R$, there exist $Q_1 > 0, \bar{S}_1, \bar{S}_2, \bar{U}, Q_2, Q_3, \in \mathcal{R}^{n \times n}$, $\bar{R}_1, \bar{R}_2 \in \mathcal{R}^{2n \times 2n}$, and $Y \in \mathcal{R}^{n \times r}$ that satisfy the following LMI:

$$G_3 [I_{2n}\ 0] + \begin{bmatrix} I_{2n} \\ 0 \end{bmatrix} G_3^T - G_4 < 0 \tag{18a}$$

$$\begin{bmatrix} \Psi & * & * & * & * & * & * & * \\ [0\ L]P & -\gamma^2 I & * & * & * & * & * & * \\ h_1 \Phi_1 & 0 & -h_1 R_1 & * & * & * & * & * \\ h_2 \Phi_2 & 0 & 0 & -h_2 R_2 & * & * & * & * \\ [0\ A_1] W_1 & 0 & 0 & 0 & -S_1 & * & * & * \\ [0\ A_2] W_2 & 0 & 0 & 0 & 0 & -S_2 & * & * \\ [0\ F]P & 0 & 0 & 0 & 0 & 0 & -U & * \\ [(B - KD_{21})^T\ 0] & 0 & 0 & 0 & 0 & 0 & 0 & -I_p \end{bmatrix} < 0 \tag{13}$$

where

$$G_3 = col\Big\{\hat{\Xi}, [0\ L], h_1(\varepsilon_1+1)\bar{R}_1, h_2(\varepsilon_2+1)\bar{R}_2$$
$$\varepsilon_1\bar{S}_1[0\ A_1], \varepsilon_2\bar{S}_2[0\ A_2], [Q_1\ 0], [Q_1\ 0]$$
$$\left[(Q_1B - YD_{12})^T\ 0\right], \bar{U}[0\ F], [Q_2\ Q_3]$$
$$h_1\begin{bmatrix}0\\ A_1^T\end{bmatrix}[Q_2\ Q_3], h_2\begin{bmatrix}0\\ A_2^T\end{bmatrix}[Q_2\ Q_3]\Big\}$$
$$G_4 = \mathrm{diag}\left\{0, \gamma^2 I_p, h_1\bar{R}_1, h_2\bar{R}_2, \bar{S}_1, \bar{S}_2\right.$$
$$\left.\bar{S}_1, \bar{S}_2, I, U, U, h_1\bar{R}_1, h_2\bar{R}_2\right\} \qquad \text{(18b,c)}$$

and where

$$\hat{\Xi} = \begin{bmatrix} Q_2 & Q_3 - Q_2^T + Q_1\left(\sum_{i=0}^2 A_i + \sum_{i=1}^2 \varepsilon_i A_i\right) - YC \\ 0 & -Q_3 \end{bmatrix}.$$

The filter gain is then given by

$$K = Q_1^{-1}Y. \qquad (19)$$

Note that in the latter LMI $\bar{S}_1$, $\bar{S}_2$, $\bar{U}$, $\bar{R}_1$, and $\bar{R}_2$ are the inverses of $S_1, S_2, U, R_1$, and $R_2$ of (17), respectively. If this LMI possesses a solution for $h_1 > 0$ and $h_2 > 0$, then because of the special dependence of its matrix entries on the delay length, it will also posses a solution for all $0 < \bar{h}_i < h_i$, $i = 1, 2$.

*Remark 2:* The result of Theorem 1 applies the tuning parameters $\varepsilon_1$ and $\varepsilon_2$. The question arises about how to find the optimal combination of these parameters. One way to address the tuning issue is to choose for a cost function the parameter $t_{min}$ that is obtained while solving the feasibility problem using Matlab's LMI toolbox. This scalar parameter is positive in cases where the combination of the tuning parameters is one that does not allow a feasible solution to the set of LMIs considered. Applying a numerical optimization algorithm, such as the program **fminsearch** in the optimization toolbox of Matlab [18] to the above cost function, a locally convergent solution to the problem is obtained. If the resulting minimum value of the cost function is negative, the tuning parameters that solve the problem are found. In the examples we solved, the single tuning parameter $\varepsilon_1 = \varepsilon_2 = \varepsilon$ achieved results that are quite close to those obtained by the fminsearch program.

The result of Theorem 1 is applied to the following example.

*Example 1:* Consider the same system as found in [12] (for $F = 0$) to which a state-feedback has been applied. Assuming that the measurement equation is the same as in (2), an observer that achieves a minimum estimation level is sought. The matrices corresponding to (1), (2), and (5) are as follows:

$$F = \begin{bmatrix} f & 0 \\ 0 & f\end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}$$
$$A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9\end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0\end{bmatrix}$$
$$L = [1\ 0], \quad C = [0\ 1]$$
$$D_{21} = [0\ 0.01], \quad h = 0.999 \text{ s}.$$

Consider first $f = 0$. Note that the system is unstable. Using the method of [6], a minimum value of $\gamma = 22.8784$ was obtained with a filter gain matrix of $K = [4790\ 18\,139]^T$.

On the other hand, applying Theorem 1 for $h = 0.999$ s, a minimum value of $\gamma = 0.0823$ was achieved by using $\varepsilon = -0.28$. The resulting filter gain was $K = 10^4[6.158\ 6.1594]^T$. Furthermore, although it was impossible to obtain a solution for $h \geq 1$, using the method of [6], it was found that, by applying the LMI of Theorem 1, a solution for all $h \leq 1.295$ was available. For, say, $h = 1.25$ s and $\varepsilon = -0.33$, a minimum value of $\gamma = 0.61$, with $K = 10^5[2.2354\ 2.2358]^T$ was obtained.

For $f = 0.2$, the maximum value of $h$ for which there exists a solution to (18) remains 1.295 s. The minimum achievable $\gamma$ for $h = 1.25$ s also remains $\gamma = 0.61$. The only thing that changed, in comparison with the solution found for $f = 0$, is the resulting filter gain. For $f = 0.2$, we obtain, for $\varepsilon = -0.33$, $K = 10^5[3.0709\ 3.0715]^T$

### C. Case of Delayed Measurements

The above results were obtained for the case where no delay is encountered in the measurement. In case the measurement includes delayed state information of the form

$$y(t) = col\left\{C_0x(t), C_1x(t-h_1)\right\} + D_{21}w(t) \qquad (20)$$

where $C_0 \in \mathcal{R}^{r_1\times n}$ and $C_1 \in \mathcal{R}^{r_2\times n}$ are constant matrices and $r_1 + r_2 = r$, an additional component is placed in series with the delayed component of $y$. The state space model of this component is given by

$$\dot{\eta}(t) = -\rho I_{r_2}\eta(t) + [0\ \rho I_{r_2}]y(t) \qquad (21)$$

for $1 \ll \rho$. Denoting the augmented state vector by $\xi(t) = col\{x(t), \eta(t)\}$, the augmented system is then described by

$$\dot{\xi}(t) - \tilde{F}\dot{\xi}(t-g) = \sum_{i=0}^{2}\tilde{A}_i\xi(t-h_i) + \tilde{B}w \qquad (22)$$

where

$$\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & -\rho I_{r_2}\end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ \rho C_1 & 0\end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} A_2 & 0 \\ 0 & 0\end{bmatrix}$$
$$\tilde{F} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad \text{and} \quad \tilde{B} = \begin{bmatrix} B \\ \rho[0\ I_{r_2}]D_{21}\end{bmatrix}.$$

The following augmented filter should be considered:

$$\dot{\hat{\xi}}(t) - \tilde{F}\dot{\hat{\xi}}(t-g) = \sum_{i=0}^{2}\tilde{A}_i\hat{\xi}(t-h_i)$$
$$+\tilde{K}\left(col\left\{[I_{r_1}\ 0]\,y(t), \eta(t)\right\} - \tilde{C}\hat{\xi}(t)\right) \qquad (23)$$

where

$$\tilde{C} = \begin{bmatrix} C_0 & 0 \\ 0 & I_{r_2}\end{bmatrix}.$$

The resulting estimation error vector is denoted by $\tilde{e}(t) = \xi(t) - \hat{\xi}(t)$ with the following state space representation:

$$\dot{\tilde{e}}(t) - \tilde{F}\dot{\tilde{e}}(t-g) = (\tilde{A}_0 - \tilde{K}\tilde{C})\tilde{e}(t) + \sum_{i=1}^{2}\tilde{A}_i\tilde{e}(t-h_i)$$
$$+\tilde{B}w(t) - \tilde{K}\begin{bmatrix} I_{r_1} & 0 \\ 0 & 0\end{bmatrix}D_{21}w(t). \qquad (24)$$

Letting $\tilde{z} = \tilde{L}\tilde{e}$, $\tilde{L} = [L\ 0]$, and considering

$$J_1 = \int_0^\infty (\tilde{z}^T\tilde{z} - \gamma^2 w^T w)d\tau \qquad (25)$$

one could apply Lemma 1 to obtain $\tilde{K}$ via an LMI that corresponds to the one in Theorem 1. The problem is, however, that due to the $O(\rho)$ entries in $\tilde{A}$ and $\tilde{B}$, the restriction of $W_i = \varepsilon_i P$, $\varepsilon_i \in \mathcal{R}$, and $i = 1, 2$, which was made in order to obtain the LMI of Theorem 1, forces $\varepsilon_i$ to be $O(\rho^{-1})$, and thus, the solution that will be achieved for these scalars $\varepsilon_i$ will tend to the one obtained in [6].

In order to utilize the extra freedom provided by Park's overbounding method, diagonal matrices $\bar{\varepsilon}_i$ are sought that satisfy $W_i = \bar{\varepsilon}_i P$, $i = 1, 2$. Similarly to Theorem 1, the choice of $\bar{\varepsilon}_i = -I_{2n}$ leads to delay-independent conditions. Denoting

$$\bar{\varepsilon}_i = \mathrm{diag}\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i2}\},\ \bar{\varepsilon}_{i1},\ \bar{\varepsilon}_{i2} \in \mathcal{R}^{\bar{n}\times\bar{n}},\ i=1,2,\ \ \bar{n}=n+r$$
$$\tilde{D}_{12} = \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix} D_{21}$$

and applying the method of Section III-B, the following theorem is obtained.

*Theorem 2:* Consider the system of (20), (22), and (23) and the cost function (25). For a prescribed $0 < \gamma$ and for $\rho \gg 1$, $J_1 < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ if for some prescribed diagonal matrices $\bar{\varepsilon}_1, \bar{\varepsilon}_2 \in \mathcal{R}^{2\bar{n}\times 2\bar{n}}$, there exist $Q_1 > 0$, $\bar{S}_1, \bar{S}_2$, $\bar{U}$, $Q_2$, $Q_3 \in \mathcal{R}^{\bar{n}\times\bar{n}}$, $\bar{R}_1, \bar{R}_2 \in \mathcal{R}^{2\bar{n}\times 2\bar{n}}$, and $Y \in \mathcal{R}^{(\bar{n}+r)\times r}$ that satisfy

$$G_5[I_{2n}\ 0] + \begin{bmatrix} I_{2n} \\ 0 \end{bmatrix} G_5^T - G_4 < 0 \tag{26}$$

where

$$\begin{aligned} G_5 = col\Big\{&\tilde{\Xi}, [0\ L], h_1(\bar{\varepsilon}_1 + I)\bar{R}_1, h_2(\bar{\varepsilon}_2 + I)\bar{R}_2 \\ &\bar{S}_1[0\ \tilde{A}_1]\bar{\varepsilon}_1, \bar{S}_2[0\ \tilde{A}_2]\bar{\varepsilon}_2, [Q_1\ 0], [Q_1\ 0] \\ &\left[(Q_1\tilde{B} - Y\tilde{D}_{12})^T\ 0\right], \bar{U}[0\ F], [Q_2\ Q_3] \\ &h_1\begin{bmatrix} 0 \\ \tilde{A}_1^T \end{bmatrix}[Q_2\ Q_3], h_2\begin{bmatrix} 0 \\ \tilde{A}_2^T \end{bmatrix}[Q_2\ Q_3]\Big\} \end{aligned}$$

where $G_4$ is defined in (18c), and where

$$\tilde{\Xi} = \begin{bmatrix} Q_2 & Q_3 - Q_2^T + Q_1\left(\sum_{i=0}^{2} \tilde{A}_i + \sum_{i=1}^{2} \bar{\varepsilon}_i \tilde{A}_i\right) - Y\tilde{C} \\ 0 & -Q_3 \end{bmatrix}.$$

The filter gain is then given by

$$\tilde{K} = Q_1^{-1} Y. \tag{27}$$

*Remark 3:* The problem of choosing $\bar{\varepsilon}_{ij}$, $i, j = 1, 2$ is now more involved. One way to reduce the complexity is to choose zeros for those diagonal elements of $\bar{\varepsilon}_{i2}$ that correspond to the $O(\rho)$ elements in $\tilde{A}_i$ and the same scalar for all the other diagonal elements in $\bar{\varepsilon}_{ij}$, $i, j = 1, 2$.

The existence of a solution to the LMI of Theorem 2 guarantees that the filter built from the series connection of (21) and (23) will achieve the required performance as long as $0 < \rho$. Considering, however, $1 \ll \rho$ and denoting

$$\tilde{K} = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}$$

it follows from Theorem 2 that if the LMI is feasible, then the estimate of $x(t)$ is given by

$$\begin{aligned} \dot{\hat{x}}(t) - F\dot{\hat{x}}(t-g) = &\sum_{i=0}^{2} A_i \hat{x}(t-h_i) \\ &+ K_{00}\left([I_{r_1}\ 0]\, y(t) - C_0\hat{x}\right) + K_{01}(\eta - \hat{\eta}). \end{aligned}$$

When $1 \ll \rho$ and $r_1 = 0$ (namely, when all of the measurements are delayed), the latter equation, together with the one obtained from (23) for $\hat{\eta}$, lead to the following filter:

$$\begin{aligned} \dot{\hat{x}}(t) - F\dot{\hat{x}}(t-g) = &\sum_{i=0}^{2} A_i \hat{x}(t-h_i) \\ &+ \bar{K}\left(y(t-h_1) - C_1\hat{x}(t-h_1)\right) + \phi \end{aligned}$$

where

$$\bar{K} = (\rho I_r + K_{11})^{-1}\rho K_{01} + O(\rho^{-1}) \tag{28a,b}$$

and where $\phi = O(\rho^{-1}, s)$. The latter filter, with $\phi = 0$, will achieve the required estimation accuracy if $\rho$ is chosen to be large enough.

The use of the results of Theorem 2 are demonstrated by the following example.

*Example 2:* Consider the system of Example 1 with $f = 0$ and a delay of $h = 0.9$ *sec.*. The measurement equation is as in (20) with

$$C_0 = 0, \quad C_1 = [0\ 1], \quad r_2 = r = 1, \quad \text{and} \quad h_1 = h.$$

This example was solved in [6], where, for $\rho = 10^{10}$, a minimum value of $\gamma = 128.406$ was obtained for the gain matrix $\bar{K} = [-0.8450\ 0.2045]^T$. Applying the result of Theorem 2, for $\bar{\varepsilon}_{11} = -0.22 I_3$ and $\bar{\varepsilon}_{12} = \mathrm{diag}\{-.28,\ 0,\ 0\}$, a minimum value of $\gamma = 51.67$ is achieved for $\bar{K} = [-0.9054\ 0.2089]^T$. Taking $\bar{\varepsilon}_{12} = \mathrm{diag}\{-0.22,\ 0,\ 0\}$, as was suggested in Remark 2, a slightly higher minimal value of $\gamma = 54.45$ is obtained with $\bar{K} = [-0.9166\ 0.2094]^T$.

## IV. Conclusions

A solution to the problem of $H_\infty$ filtering for linear, continuous, time-invariant neutral systems with time delay has been presented. The solution procedure is based on applying an observer type filter, and it provides a sufficient condition for achieving a prescribed estimation accuracy. Since the results are only sufficient, the question arises as to how large an overdesign is entailed in this method and whether or not it is smaller than the one encountered in other designs appearing in the literature. To answer this question, one has to bear in mind that the filter designs are based, one way or another, on a related BRL that provides the sufficient condition for a system with delay to possess an $H_\infty$-norm that is less than a prescribed value. The overdesign of the corresponding filter design approach will therefore strongly depend on the conservatism of the BRL used. In this paper, the BRL utilized is less conservative than other finite-dimensional BRLs appearing in the literature, and it therefore provides a less conservative filtering solution.

The solution method in this paper is based on a neutral-type Luenberger estimator. In spite of the fact that the LMI of Theorem 1 is affine in the system matrices, it cannot be used to treat the case of polytopic uncertain system parameters.

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