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On robust stability of linear neutral systems with time-varying delays

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The application of the direct Lyapunov method to the stability analysis of neutral systems with time-varying delays usually encounters a restrictive assumption on the function in the right side of the differential equation. This function is supposed to satisfy the Lipschitz condition with respect to the delayed state derivative with a constant less than 1. In the present paper, we extend the input–output approach to consider the stability of neutral type systems with uncertain time-varying delays and norm-bounded uncertainties. The assumption on the Lipschitzian constant can then be avoided. Sufficient stability criteria are derived in the frequency domain and the time domain, where the descriptor discretized Lyapunov–Krasovskii functional is applied. As a by-product, new necessary conditions for neutral-delay-independent/retarded-delay-dependent stability criteria are obtained. The method can be easily extended to L_2-gain analysis and can be applied to design problems.

Keywords: neutral system; time-varying delay; input–output approach; Lyapunov–Krasovskii method.

1. Introduction

There are two main methods for the stability analysis of linear systems with delay: the direct Lyapunov method and the input–output approach, based on the small-gain theorem (see, e.g. Gu *et al.*, 2003). The latter approach has been applied to robust stability analysis of linear 'retarded-type' systems with norm-bounded uncertainties and uncertain time delays in Huang & Zhou (2000), Gu *et al.* (2003), Kao & Lincoln (2004) and Fridman & Shaked (2006).

The application of the direct Lyapunov method to neutral type systems with 'constant neutral delays' requires a well-known assumption on the stability of the difference equation (Hale & Lunel, 1993; Kolmanovskii & Myshkis, 1999; Niculescu, 2001). Necessary and sufficient conditions for the stability of this difference equation are given in Hale & Lunel (1993) in the frequency domain. It was shown in Fridman (2002) that the descriptor approach to neutral systems with constant delays (see, e.g. Fridman, 2001) implies the stability of the difference equation and thus avoids a verification of the above assumption. In the case of *time-varying neutral delays*, the situation becomes more difficult and the only known assumption for the application of direct Lyapunov method may be rather conservative: the sum of the norms of the matrices, which multiply the delayed state derivatives, should be less than 1 (El'sgol'ts & Norkin, 1973; Kolmanovskii & Myshkis, 1999). The latter assumption becomes especially restrictive in the case of an uncertain neutral part or in the case of multiple delays in the derivative of the state.

In the present paper, a new method is developed for the stability analysis of linear 'neutral systems with time-varying delays and norm-bounded uncertainties' in the neutral part. To the best of the author's knowledge, this is the first method that avoids the restrictive assumption mentioned above.

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The presented method extends the input–output approach to uncertain neutral systems with time-varying delays and it leads to sufficient frequency-domain stability conditions. Also some ‘necessary conditions’ for the feasibility of the resulting stability conditions are deduced. To illustrate the efficiency of the new method for neutral systems, the time-domain results are derived via the *descriptor discretized Lyapunov functional* method (Fridman, 2006). The latter method combines the discretized Lyapunov functional method of Gu (1997) with the descriptor model transformation (Fridman, 2001) and it efficiently solves design problems (for the first time for the discretized method).

The descriptor discretized Lyapunov functional method is chosen because of the power of the discretized method (which, differently to the simple Lyapunov functional methods, can analyse systems which are not stable without delays). We note that in Fridman (2006), the case of constant neutral delays and fast-varying state delays (without any constraints on the delay derivative) was studied. In the present paper, we extend the method of Fridman (2006) to the case of slowly varying delays in the state and in the state derivative (where the delay derivative is bounded from above by a constant, which is less than 1).

Notation: Throughout the paper, the superscript ‘$\top$’ stands for matrix transposition, $\mathcal{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$, means that P is symmetric and positive definite. We also denote $x_t(\theta) = x(t+\theta)(\theta \in [-h-\mu, 0])$. The symmetric elements of the symmetric matrix will be denoted by $*$. L_2 is the space of square-integrable functions v: $[0, \infty) \to C^n$ with the norm $\|v\|_{L_2} = \left[\int_0^\infty \|v(t)\|^2 \mathrm{d}t\right]^{1/2}$. $\|A\|$ denotes the Euclidean norm of an $n \times n$ (real or complex) matrix A, which is equal to the maximum singular value of A. For a transfer function matrix of a stable system $G(s)$, $s \in C$,

$$\|G\|_\infty = \sup_{-\infty<\omega<\infty} \|G(\mathrm{i}\omega)\|, \quad \mathrm{i} = \sqrt{-1}.$$

$\sigma(B)$ is the spectral radius of matrix B (i.e. the maximum absolute value of its eigenvalues).

2. Retarded-delay-dependent/neutral-delay-independent stability conditions

2.1 *Problem formulation*

Consider a linear system

$$\dot{x}(t) = (A_0 + H\Delta E_0)x(t) + (A_1 + H\Delta E_1)x(t-\tau(t)) + (F + H\Delta E_2)\dot{x}(t-g(t)), \tag{2.1}$$

where $x(t) \in \mathcal{R}^n$, A_0, A_1 , F and E_i, $i = 0, 1, 2$, are constant matrices. $\Delta(t)$ is a time-varying uncertain $n \times n$ matrix that satisfies

$$\Delta^\top(t)\Delta(t) \leqslant I_n. \tag{2.2}$$

The uncertain delays $\tau(t)$ and $g(t)$ are differentiable functions of the form

$$\tau(t) = h + \eta(t), \quad |\eta(t)| \leqslant \mu \leqslant h, \quad \dot{\tau}(t) \leqslant d < 1, \quad \dot{g}(t) \leqslant f < 1, \tag{2.3}$$

with the known bounds μ, d and f. The neutral delay $g(t)$ does not have bound other than the derivative bound.

If $\tau(t) = g(t)$, then the stability criterion may be applied with $f = d$. For simplicity, we consider a single delay $g(t)$. The presented results may be easily ‘generalized to the case of any finite number of delays $g(t)$’.

The stability analysis of systems with a neutral time-varying delay is a classical problem (El'sgol'ts & Norkin, 1973). In the robust stability context, where the parameter uncertainties and the variation of the time delay are taken into account, this problem becomes especially important. In the past, robust stability of (2.1) with time-varying delay was studied under the restrictive assumption (Lien, 2005)

$$\|F\| + \|H\|\|E_2\| < 1, \tag{2.4}$$

where $\|\cdot\|$ is any matrix norm. The latter assumption (which becomes especially restrictive in the case of multiple neutral delays) allows the direct Lyapunov method to be applied where it is assumed that the right side of (2.1) (denoted by $f(t, x_t, \dot{x}_t)$) satisfies the Lipschitz condition with respect to $\dot{x}_t$ with a constant less than 1 (see Kolmanovskii & Myshkis, 1999, p. 336). Inequality (2.4) guarantees the latter Lipschitz condition since

$$\begin{aligned}\|f(t, x_t, \dot{x}_t) - f(t, x_t, \dot{\bar{x}}_t)\| &= \|(F + H\Delta E_2)[\dot{x}(t - g(t)) - \dot{\bar{x}}(t - g(t))]\| \\ &\leqslant (\|F\| + \|H\|\|E_2\|)\|\dot{x}(t - g(t)) - \dot{\bar{x}}(t - g(t))\|.\end{aligned}$$

In some papers (see, e.g. Park, 2002) from the fact that $\dot{V} < 0$, where $V > 0$ is a Lyapunov functional, it is directly concluded that the neutral system is asymptotically stable (without any references to the corresponding Lyapunov theorems). Unfortunately, such a conclusion is not correct.

In the present paper, we develop the input–output approach to the stability analysis of uncertain neutral type systems with time-varying delays. We first derive frequency-domain stability conditions and then deduce their implications. Further, we find time-domain conditions by applying descriptor discretized Lyapunov functional method. Finally, we illustrate the efficiency of the proposed method by numerical examples. The new method gives tools for solution of different robust control problems for a wide class of neutral systems, where the condition (2.4) is removed.

2.2 *Frequency-domain stability conditions*

Representing

$$x(t - \tau(t)) = x(t - h) - \int_{t-h-\eta(t)}^{t-h} \dot{x}(s)\mathrm{d}s$$

and applying the input–output approach (see Gu *et al.*, 2003 and the references therein), we consider the following forward system:

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 x(t - h) + \mu A_1 v_1(t) + F v_2(t) + H v_4(t), \\ y_1(t) &= \dot{x}(t), \quad y_2(t) = \frac{1}{\sqrt{1-f}}\dot{x}(t), \quad y_3(t) = \frac{1}{\sqrt{1-d}}x(t), \\ y_4(t) &= E_0 x(t) + E_1 x(t - h) + \mu E_1 v_1(t) + E_2 v_2(t),\end{aligned} \tag{2.5a–d}$$

with the feedback

$$\begin{aligned}v_1(t) &= -\tfrac{1}{\mu}\int_{-h-\eta(t)}^{-h} y_1(t + s)\mathrm{d}s, \quad v_2(t) = \sqrt{1-f}\, y_2(t - g(t)), \\ v_3(t) &= \sqrt{1-d}\, y_3(t - \tau(t)), \quad v_4(t) = \Delta y_4(t).\end{aligned} \tag{2.6}$$

Similarly to He *et al.* (2004), we add to (2.5a) the left side of the equation

$$C[x(t-h)+\mu v_1(t)-v_3(t)] = C\left[x(t-h) - \int_{-h-\eta(t)}^{-h} \dot{x}(t+s)\mathrm{d}s - x(t-\tau(t))\right] = 0,$$

where C is an arbitrary $n \times n$ matrix (which is equivalent to the parameterized model transformation Niculescu, 2001). We represent the forward system in the following parameterized form:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + (A_1+C)x(t-h) + \mu(A_1+C)v_1(t) + Fv_2(t) \\ &\quad - Cv_3(t) + Hv_4(t), \\ y_1(t) &= \dot{x}(t), \quad y_2(t) = \frac{1}{\sqrt{1-f}}\dot{x}(t), \quad y_3(t) = \frac{1}{\sqrt{1-d}}x(t), \\ y_4(t) &= E_0 x(t) + E_1 x(t-h) + \mu E_1 v_1(t) + E_2 v_2(t). \end{aligned} \tag{2.7a–d}$$

Note that $C=0$ corresponds to the moderately varying delay $\tau(t)$ with $\dot{\tau} \leqslant 1$ (Fridman & Shaked, 2006), while $C = -A_1$ corresponds to τ-independent/$\dot{\tau}$-dependent result. The input–output model (2.7a–d), (2.6) and the results of the present paper are appropriate also to the system (2.1) with delay $h \geqslant 0$ and non-negative $\eta \in [0, \mu]$. In the latter case, $C=0$ corresponds to systems with fast-varying delay τ (i.e. without any constraints on $\dot{\tau}$).

In the case of a retarded system with $F=0$ and with $\dot{\tau} \leqslant 1$, the input–output model (2.7a–d), (2.6) has been introduced in Fridman & Shaked (2006), where C, v_i *and* $y_i, i = 2, 3$, were taken to be zero. Moreover, y_1 and y_2 correspond to the descriptor method (Fridman, 2001), where $\dot{x}(t)$ appears in the derivative of the Lyapunov functional.

We assume the following.

(**A1**) The (parameterized) nominal system

$$\dot{x}(t) = A_0 x(t) + (A_1+C)x(t-h) \tag{2.8}$$

is asymptotically stable.

Let $v^\top = [v_1^\top \quad \cdots \quad v_4^\top]$ and $y^\top = [y_1^\top \quad \cdots \quad y_4^\top]$. Assume that $y_i(t) = 0, \forall t \leqslant 0, i = 1, \ldots, 4$. The following holds:

$$\|v_i\|_{L_2} \leqslant \|y_i\|_{L_2}, \quad i = 1, \ldots, 4. \tag{2.9}$$

The forward system (2.7a–d) can be written as $y = Gv$ with transfer matrix

$$\begin{aligned} G(s) &= \begin{bmatrix} sI_n & \frac{1}{\sqrt{1-f}}sI_n & \frac{1}{\sqrt{1-d}}I_n & E_0^\top + E_1^\top \mathrm{e}^{-hs} \end{bmatrix}^\top \\ &\quad \times (sI - A_0 - (A_1+C)\mathrm{e}^{-hs})^{-1}[\mu(A_1+C) \quad F \quad -C \quad H] \\ &\quad + \begin{bmatrix} 0_{3n\times n} & 0_{3n\times n} & 0_{3n\times 2n} \\ \mu E_1 & E_2 & 0_{n\times 2n} \end{bmatrix}. \end{aligned} \tag{2.10}$$

By the small-gain theorem (see, e.g. Gu *et al.*, 2003), the system (2.1) is input–output stable (and thus asymptotically stable, since the nominal system is time invariant) if $\|G\|_\infty < 1$. A stronger result may be obtained by scaling G.

THEOREM 1 Consider (2.1) with delays given by (2.3), where $\eta(t)$ and $g(t)$ are differentiable functions. If there exists an $n \times n$ matrix C such that A1 holds and there exist non-singular $n \times n$ matrices X_i, $i = 1, 2, 3$, and a scalar $r \neq 0$ such that

$$\|G_X\|_\infty < 1, \quad G_X(s) = \mathrm{diag}\{X_1, X_2, X_3, rI_n\} G(s)\, \mathrm{diag}\{X_1^{-1}, X_2^{-1}, X_3^{-1}, r^{-1}I_n\}, \tag{2.11}$$

then (2.1) is input–output stable.

REMARK 1 In Section 2.4 below sufficient conditions for the feasibility of A1 and (2.11) will be derived in terms of linear matrix inequalities (LMIs). The free matrices C, X_i and the scalar r of (2.11) will be related to the decision variables of the LMIs (see Remark 3 below).

2.3 *Implications of the frequency-domain stability conditions*

Now, we are in a position to formulate some necessary conditions for the feasibility of (2.11).

PROPOSITION 1 If (2.11) holds, then

(i) the eigenvalues of $\mu(A_1 + C)$ are inside the unit circle ($\mu\sigma(A_1 + C) < 1$), i.e. the difference equation

$$x(t) - \mu(A_1 + C)x(t - g_0) = 0, \tag{2.12}$$

with constant delay g_0 is stable;

(ii) the eigenvalues of $\frac{1}{\sqrt{1-f}}F$ are inside the unit circle, i.e. the difference equation

$$x(t) - \frac{1}{\sqrt{1-f}} F x(t - g) = 0, \tag{2.13}$$

with constant delay g is stable;

(iii) $\sigma_0 < 1$, where

$$\sigma_0 \triangleq \sup\left\{\sigma\left(\mu(A_1 + C)\mathrm{e}^{\mathrm{i}\theta_0} + \frac{1}{\sqrt{1-f}} F\, \mathrm{e}^{\mathrm{i}\theta}\right) : \theta_0, \theta \in [0, 2\pi]\right\}, \tag{2.14}$$

which is equivalent to the delay-independent stability of the following difference equations with constant delays g_0 and g (Hale & Lunel, 2003, Theorem 6.1, p. 286):

$$x(t) - \mu(A_1 + C)x(t - g_0) - \frac{1}{\sqrt{1-f}} F x(t - g) = 0. \tag{2.15}$$

Proof. We will prove (iii) only. The proof of (i) and (ii) is similar.

Assume that (2.11) holds. Then,

$$\begin{aligned}&\left\|\begin{bmatrix} X_1 \\ \frac{X_2}{\sqrt{1-f}}\end{bmatrix}[\mu(A_1+C)X_1^{-1} \quad FX_2^{-1}]\right\| \\ &\leqslant \sup_{-\infty<\omega<\infty}\left\| \mathrm{i}\omega\begin{bmatrix} X_1 \\ \frac{X_2}{\sqrt{1-f}}\end{bmatrix}(\mathrm{i}\omega I_n - A_0 - (A_1+C)\mathrm{e}^{-h\mathrm{i}\omega})^{-1}\right. \\ &\left.\times[\mu(A_1+C)X_1^{-1} \quad FX_2^{-1}]\right\| \leqslant \|G_X\|_\infty < 1. \end{aligned} \tag{2.16}$$

Let λ and a be an eigenvalue and an eigenvector of $\left(\mu(A_1+C)\mathrm{e}^{\mathrm{i}\theta_0}+\frac{F}{\sqrt{1-f}}\mathrm{e}^{\mathrm{i}\theta}\right)$, i.e.

$$\left(\mu(A_1+C)\mathrm{e}^{\mathrm{i}\theta_0}+\frac{F}{\sqrt{1-f}}\mathrm{e}^{\mathrm{i}\theta}\right)a=\lambda a.$$

From (2.16), it follows that

$$\begin{aligned}|\lambda|\left\|\begin{bmatrix} X_1 a \\ \frac{X_2 a}{\sqrt{1-f}}\end{bmatrix}\right\| &= \left\|\begin{bmatrix} X_1 \\ \frac{X_2}{\sqrt{1-f}}\end{bmatrix}\lambda a\right\| \\ &= \left\|\begin{bmatrix} X_1 \\ \frac{X_2}{\sqrt{1-f}}\end{bmatrix}[\mu(A_1+C)X_1^{-1} \quad FX_2^{-1}]\begin{bmatrix} X_1\,\mathrm{e}^{\mathrm{i}\theta_0}a \\ \frac{X_2}{\sqrt{1-f}}\mathrm{e}^{\mathrm{i}\theta}a\end{bmatrix}\right\| \\ &< \left\|\begin{bmatrix} X_1\,\mathrm{e}^{\mathrm{i}\theta_0}a \\ \frac{X_2}{\sqrt{1-f}}\mathrm{e}^{\mathrm{i}\theta}a\end{bmatrix}\right\| = \left\|\begin{bmatrix} X_1 a \\ \frac{X_2}{\sqrt{1-f}}a\end{bmatrix}\right\|. \end{aligned} \tag{2.17}$$

Hence, $|\lambda| < 1$. This completes the proof. □

REMARK 2 We note that the existing LMI stability criteria, derived via different direct Lyapunov methods, may usually be recovered via an input–output approach (similar to Zhang *et al.*, 2001; Gu *et al.*, 2003). In this case, the LMI conditions (which give sufficient conditions for the frequency-domain condition (2.11) and for A1) are feasible if the difference equation (2.15) is stable, where C is related to the decision variables of these LMIs (as, e.g. in (2.34) below).

Necessary conditions for delay-dependent stability via different model transformations of linear retarded type systems with 'small' delays (where $h = 0$ and $\eta \in [0, \mu]$) were found in Gu & Niculescu (2001) and Kharitonov & Melchor-Aguilar (2002), where additional dynamics of the transformed systems were analysed. A simple condition of Gu & Niculescu (2001) (for the constant delay case) $\mu\sigma(A_1) < 1$ coincides with (i) for the case of fast-varying delay. However, for $h = \mu$ and $\dot{\tau} \leqslant 1$ (i) guarantees the stability on the double interval $\tau(t) \in [0, 2\mu]$. A simple condition of Kharitonov & Melchor-Aguilar (2002) $\sum_{k=1}^{m}\mu_k\|A_k\| < 1$ for the system

$$\dot{x}(t)=\sum_{k=1}^{m}A_k x(t-\tau_k(t)), \quad \tau_k(t)\in[0,\mu_k], \quad \dot{\tau}_k < 1,$$

is more restrictive than (i) in the case of $\dot{\tau}_k \leqslant 1$, where the latter has the form

$$\sup\left\{\sigma\left(\sum_{k=1}^{m}\mu_k A_k\,\mathrm{e}^{\mathrm{i}\theta_k}\right): \theta_k \in [0, 2\pi]\right\} < 1$$

and guarantees the stability on the double intervals $\tau_k \in [0, 2\mu_k]$ for $h_k = \mu_k$.

2.4 *Time-domain criterion for robust stability*

We will derive sufficient conditions for (2.11) by using the complete Lyapunov–Krasovskii functional (LKF):

$$\begin{aligned} V(x_t) &= x^{\top}(t)P_1x(t) + 2x^{\top}(t)\int_{-h}^{0} Q(\xi)x(t+\xi)\mathrm{d}\xi + \int_{-h}^{0}\int_{-h}^{0} x^{\top}(t+s)R(s,\xi)\mathrm{d}s\,x(t+\xi)\mathrm{d}\xi \\ &\quad + \int_{-h}^{0} x^{\top}(t+\xi)S(\xi)x(t+\xi)\mathrm{d}\xi, \quad P_1 > 0, \end{aligned} \tag{2.18}$$

which for $S \equiv 0$ corresponds to necessary and sufficient conditions for stability of the nominal system (2.8). The time-domain results will be derived via the discretized Lyapunov functional method (Gu, 2003), since this powerful method (differently from simple Lyapunov functionals) can be applied to the systems, which are not stable without delays. The new descriptor discretized Lyapunov method of Fridman (2006) will be used, which allows to solve the design problems.

Since

$$\int_{0}^{\infty} v^{\top}(t)\bar{X}v(t)\mathrm{d}t \leqslant \int_{0}^{\infty} y^{\top}(t)\bar{X}y(t)\mathrm{d}t,$$
$$\bar{X} = \mathrm{diag}\{\mu R_a, (1-f)U, (1-d)S_a, \rho I_n\},$$

the following condition along with (2.7a–d)

$$\begin{aligned} \mathcal{W} \triangleq\; & \frac{\mathrm{d}}{\mathrm{d}t}V(x_t) + (1-f)y_2^{\top}(t)Uy_2(t) + \mu y_1^{\top}(t)R_a y_1(t) \\ & +(1-d)y_3^{\top}(t)S_a y_3(t) + \rho y_4^{\top}(t)y_4(t) - (1-f)v_2^{\top}(t)Uv_2(t) \\ & -\mu v_1^{\top}(t)R_a v_1(t) - (1-d)v_3^{\top}(t)S_a v_3(t) - \rho v_4^{\top}(t)v_4(t) \\ & < -\varepsilon(\|x(t)\|^2 + \|\dot{x}(t)\|^2 + \|v(t)\|^2), \quad \varepsilon > 0, \end{aligned} \tag{2.19}$$

for some $n \times n$ matrices $R_a > 0, U > 0, S_a$ and a scalar $\rho > 0$ guarantees the asymptotic stability of (2.1) (Gu *et al.*, 2003). Note that (2.19) guarantees A1. Therefore, in the time domain we do not assume the asymptotic stability of the nominal system.

Differentiating $V(x_t)$ along the trajectories of (2.7a–d), we obtain that $\dot V$ is given by

$$\dot V(x_t) = 2\dot x^\top(t)\left[P_1 x(t) + \int_{-h}^{0} Q(\xi)x(t+\xi)\mathrm{d}\xi\right] + 2x^\top(t)\int_{-h}^{0} Q(\xi)\dot x(t+\xi)\mathrm{d}\xi$$
$$+2\int_{-h}^{0}\int_{-h}^{0} \dot x^\top(t+s)R(s,\xi)\mathrm{d}s\, x(t+\xi)\mathrm{d}\xi + 2\int_{-h}^{0}\dot x^\top(t+\xi)S(\xi)x(t+\xi)\mathrm{d}\xi. \quad (2.20)$$

Adding to $\dot V(x_t)$ the right side of the expression

$$0 = 2[x^\top(t)P_2^\top \quad \dot x^\top(t)P_3^\top] \times \begin{bmatrix} A_0x(t) - \dot x(t) + (A_1+\bar C)x(t-h) + \mu(A_1+\bar C)v_1(t) + Fv_2(t) - \bar C v_3(t) + Hv_4(t) \\ A_0x(t) - \dot x(t) + (A_1+C)x(t-h) + \mu(A_1+C)v_1(t) + Fv_2(t) - C v_3(t) + Hv_4(t)\end{bmatrix}, \quad (2.21)$$

where P_2 and P_3 are $n\times n$ matrices, which is equivalent to descriptor model transformation of Fridman (2001) and integrating by parts in (2.20), we find

$$\dot V(x_t) = \zeta^\top \Xi\zeta + 2\dot x^\top(t)\int_{-h}^{0} Q(\xi)x(t+\xi)\mathrm{d}\xi$$
$$-\int_{-h}^{0}\int_{-h}^{0} x^\top(t+\xi)\left(\frac{\partial}{\partial\xi}R(\xi,\theta) + \frac{\partial}{\partial\theta}R(\xi,\theta)\right)x(t+\theta)\mathrm{d}\theta\,\mathrm{d}\xi$$
$$+2x^\top(t)\int_{-h}^{0}[-\dot Q(\xi) + R(0,\xi)]x(t+\xi)\mathrm{d}\xi$$
$$-2x^\top(t-h)\int_{-h}^{0} R(-h,\theta)x(t+\theta)\mathrm{d}\theta - \int_{-h}^{0} x^\top(t+\xi)\dot S(\xi)x(t+\xi)\mathrm{d}\xi$$
$$+2[x^\top(t)P_2^\top \quad \dot x^\top(t)P_3^\top]\begin{bmatrix}\mu(A_1+\bar C)v_1(t) + Fv_2(t) - \bar C v_3(t) + Hv_4(t) \\ \mu(A_1+C)v_1(t) + Fv_2(t) - Cv_3(t) + Hv_4(t)\end{bmatrix}, \quad (2.22)$$

where

$$\zeta = \begin{bmatrix} x(t) \\ \dot x(t) \\ x(t-h)\end{bmatrix}, \quad \Xi = \begin{bmatrix} \Psi & \begin{bmatrix} P_2^\top(A_1+\bar C) \\ P_3^\top(A_1+C)\end{bmatrix} - \begin{bmatrix} Q(-h) \\ 0\end{bmatrix} \\ * & -S(-h)\end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3\end{bmatrix},$$

$$\Psi = P^\top\begin{bmatrix} 0 & 0 \\ A_0 & -I\end{bmatrix} + \begin{bmatrix} 0 & A_0^\top \\ 0 & -I\end{bmatrix}P + \begin{bmatrix} Q(0)+Q^\top(0)+S(0) & 0 \\ 0 & 0\end{bmatrix}. \quad (2.23a–c)$$

We apply next the discretization of Gu (1997). Divide the delay interval $[-h,0]$ into N segments $[\theta_p,\theta_{p-1}]$, $p=1,\ldots,N$, of equal length $\bar h = h/N$, where $\theta_p = -p\bar h$. This divides the square $[-h,0]\times[-h,0]$ into $N\times N$ small squares $[\theta_p,\theta_{p-1}]\times[\theta_q,\theta_{q-1}]$. Each small square is further divided into two triangles.

The continuous matrix functions $Q(\xi)$ and $S(\xi)$ are chosen to be linear within each segment and the continuous matrix function $R(\xi,\theta)$ is chosen to be linear within each triangular:

$$Q(\theta_p+\alpha\bar{h}) = (1-\alpha)Q_p + \alpha Q_{p-1},$$
$$S(\theta_p+\alpha\bar{h}) = (1-\alpha)S_p + \alpha S_{p-1}, \quad \alpha\in[0,1],$$
$$R(\theta_p+\alpha\bar{h},\theta_q+\beta\bar{h}) = \begin{cases}(1-\alpha)R_{pq}+\beta R_{p-1,q-1}+(\alpha-\beta)R_{p-1,q}, & \alpha\geqslant\beta,\\ (1-\beta)R_{pq}+\alpha R_{p-1,q-1}+(\beta-\alpha)R_{p,q-1}, & \alpha<\beta.\end{cases} \tag{2.24}$$

Thus, the LKF is completely determined by P_1, Q_p, S_p, R_{pq}, $p,q=0,1,\ldots,N$.

The LKF condition $V(x_t)\geqslant\varepsilon\|x(t)\|^2$, $\varepsilon>0$, is satisfied (Gu *et al.*, 2003, p. 185) if $S_p>0$, $p=0,1,\ldots,N$, and

$$\begin{bmatrix} P_1 & \tilde{Q}\\ * & \tilde{R}+\tilde{S}\end{bmatrix} > 0, \tag{2.25}$$

where

$$\tilde{Q} = [Q_0 \quad Q_1 \quad \cdots \quad Q_N], \quad \tilde{S} = \operatorname{diag}\left\{\frac{1}{\bar{h}S_0},\frac{1}{\bar{h}S_1},\ldots,\frac{1}{\bar{h}S_N}\right\},$$
$$\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0N}\\ R_{10} & R_{11} & \cdots & R_{1N}\\ \vdots & \vdots & \ddots & \vdots\\ R_{N0} & R_{N1} & \cdots & R_{NN}\end{bmatrix}. \tag{2.26}$$

To derive the LKF derivative condition, we note that

$$\dot{S}(\xi) = \frac{1}{\bar{h}(S_{p-1}-S_p)}, \quad \dot{Q}(\xi) = \frac{1}{\bar{h}(Q_{p-1}-Q_p)},$$
$$\frac{\partial}{\partial\xi}R(\xi,\theta)+\frac{\partial}{\partial\theta}R(\xi,\theta) = \frac{1}{\bar{h}(R_{p-1,q-1}-R_{pq})}. \tag{2.27}$$

We have

$$\mathcal{W} = \zeta_\upsilon^\top \Xi_\upsilon\zeta_\upsilon - \int_0^1 \phi^\top(\alpha)S_d\phi(\alpha)\mathrm{d}\alpha - \int_0^1\left[\int_0^1 \phi^\top(\alpha)R_d\phi(\beta)\mathrm{d}\alpha\right]\mathrm{d}\beta$$
$$+2\zeta^\top\int_0^1[D^s+(1-2\alpha)D^a]\phi(\alpha)\bar{h}\,\mathrm{d}\alpha, \tag{2.28}$$

where

$$\zeta_\upsilon^\top = [x^\top(t) \quad \dot{x}^\top(t) \quad x^\top(t-h) \quad \upsilon^\top(t)],$$

$$\Xi_v = \begin{bmatrix} \Xi & \mu \begin{bmatrix} P_2^\top A_1 \\ P_3^\top A_1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} Y_a^\top \\ 0 \end{bmatrix} & \begin{bmatrix} P_2^\top F \\ P_3^\top F \\ 0 \end{bmatrix} & \begin{bmatrix} -Y_a^\top \\ 0 \end{bmatrix} & \begin{bmatrix} P_2^\top H \\ P_3^\top H \\ 0 \end{bmatrix} \\ * & -\mu R_a & 0 & 0 & 0 \\ * & * & -(1-f)U & 0 & 0 \\ * & * & * & -(1-d)S_a & 0 \\ * & * & * & * & -\rho I_n \end{bmatrix}$$

$$+ \rho \begin{bmatrix} E_0^\top \\ 0_{n\times n} \\ E_1^\top \\ \mu E_1^\top \\ E_2^\top \\ 0 \end{bmatrix} \begin{bmatrix} E_0^\top \\ 0_{n\times n} \\ E_1^\top \\ \mu E_1^\top \\ E_2^\top \\ 0 \end{bmatrix}^\top, \tag{2.29}$$

$$\Xi = \begin{bmatrix} \Psi_d & \begin{bmatrix} P_2^\top A_1 \\ P_3^\top A_1 \end{bmatrix} - \begin{bmatrix} Q_N \\ 0 \end{bmatrix} + Y_a^\top \\ * & -S_N \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad Y_a^\top = \begin{bmatrix} P_2^\top \bar{C} \\ P_3^\top C \end{bmatrix},$$

$$\Psi_d = P^\top \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^\top \\ 0 & -I \end{bmatrix} P + \begin{bmatrix} Q_0 + Q_0^\top + S_0 + S_a & 0 \\ 0 & U + \mu R_a \end{bmatrix}, \tag{2.30}$$

$$\phi^\top(\alpha) = [x^\top(t - \bar{h} + \alpha\bar{h}) \quad x^\top(t - 2\bar{h} + \alpha\bar{h}) \quad \cdots \quad x^\top(t - N\bar{h} + \alpha\bar{h})],$$

$$S_d = \mathrm{diag}\{S_0 - S_1, S_1 - S_2, \ldots, S_{N-1} - S_N\},$$

$$R_d = \begin{bmatrix} R_{d11} & R_{d12} & \cdots & R_{d1N} \\ R_{d21} & R_{d22} & \cdots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \cdots & R_{dNN} \end{bmatrix}, \quad R_{dpq} = \bar{h}(R_{p-1,q-1} - R_{pq}),$$

$$D^s = [D_1^s \quad D_2^s \quad \cdots \quad D_N^s], \quad D^a = [D_1^a \quad D_2^a \quad \cdots \quad D_N^a],$$

$$D_p^s = \begin{bmatrix} \bar{h}/2(R_{0,p-1} + R_{0p}) - (Q_{p-1} - Q_p) \\ \bar{h}/2(Q_{p-1} + Q_p) \\ -\bar{h}/2(R_{N,p-1} + R_{Np}) \end{bmatrix} \quad \text{and}$$

$$D_p^a = \begin{bmatrix} -\bar{h}/2(R_{0,p-1} - R_{0p}) \\ -\bar{h}/2(Q_{p-1} - Q_p) \\ \bar{h}/2(R_{N,p-1} - R_{Np}) \end{bmatrix}. \tag{2.31a–i}$$

Applying Proposition 5.21 of Gu *et al.* (2003) to (2.28) and Schur complements to the last term of Ξ_v, we conclude that $\mathcal{W} < 0$ if the following LMI holds:

$$\begin{bmatrix} \Xi & \psi & \begin{bmatrix} P_2^\top F \\ P_3^\top F \\ 0 \end{bmatrix} & \begin{bmatrix} -Y_a^\top \\ 0 \end{bmatrix} & \begin{bmatrix} P_2^\top H \\ P_3^\top H \\ 0 \end{bmatrix} & \rho \begin{bmatrix} E_0^\top \\ 0 \\ E_1^\top \end{bmatrix} & D^s & D^a \\ * & -\mu R_a & 0 & 0 & 0 & \rho\mu E_1^\top & 0 & 0 \\ * & * & -(1-f)U & 0 & 0 & \rho E_2^\top & 0 & 0 \\ * & * & * & -(1-d)S_a & 0 & 0 & 0 & 0 \\ * & * & * & * & -\rho I_n & 0 & 0 & 0 \\ * & * & * & * & * & -\rho I_n & 0 & 0 \\ * & * & * & * & * & * & -R_d - S_d & 0 \\ * & * & * & * & * & * & * & -3S_d \end{bmatrix} < 0, \tag{2.32}$$

where

$$\psi = \mu \begin{bmatrix} P_2^\top A_1 \\ P_3^\top A_1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} Y_a^\top \\ 0 \end{bmatrix}. \tag{2.33}$$

We thus obtain the following.

THEOREM 2 System (2.1) is asymptotically stable for all delays satisfying (2.3), if there exist $n \times n$ matrices $0 < P_1, P_2, P_3, R_a, S_a, Y_{1a}, Y_{2a}, US_p = S_p^\top, Q_p, R_{pq} = R_{qp}^\top, p = 0, 1, \ldots, N, q = 0, 1, \ldots, N$, and a scalar $\rho > 0$ such that LMIs (2.25) and (2.32) are satisfied with $Y_a = [Y_{1a} \quad Y_{2a}]$ and with the notation defined in (2.30), (2.26), (2.31b–i) and (2.33).

REMARK 3 Similar to Fridman & Shaked (2006), it can be shown that the time-domain conditions (2.32) are sufficient for the frequency-domain conditions (2.11), where $R_a = X_1^\top X_1, U = X_2^\top X_2$, $S_a = X_3^\top X_3$ and $\rho = r^2$. Since Proposition 1 gives necessary conditions for feasibility of (2.11), while (2.11) is necessary for feasibility of (2.32), the conditions of Proposition 1 are necessary for feasibility of (2.32). We note that the conditions of Proposition 1 follow immediately from the feasibility of (2.32). If LMI (2.32) is feasible, then the following LMI

$$\begin{bmatrix} -P_3 - P_3^\top + U + \mu R_a & P_3^\top F & \mu P_3^\top (A_1 + C) \\ * & -(1-f)U & 0 \\ * & * & -\mu R_a \end{bmatrix} < 0, \tag{2.34}$$
$$C = P_3^{-\top} Y_{2a}^\top$$

holds. LMI (2.32) guarantees the stability of the difference equations (2.13) and (2.15) with constant delays (Fridman, 2002).

REMARK 4 Since the LMIs (2.25) and (2.32) are affine in the system matrices, the results of Theorem 2 can be applied to systems with polytopic-type uncertainties by solving LMIs in the polytope vertices (Boyd *et al.*, 1994).

REMARK 5 Stability conditions in the case of $\dot{\tau}(t) \leqslant 1$ follow from (2.25) and (2.32) by choosing $Y_a = 0$ and $S_a \to 0$. In the case of fast-varying delays $\tau(t)$ (without any constraints on the derivative of $\tau(t)$), the stability conditions have the form of (2.25) and (2.32), where the term μR_a in Ξ should be multiplied by 2 and where $Y_a = 0$ and $S_a \to 0$.

REMARK 6 Following De Oliveira & Skelton (2001), He *et al.* (2004) and Suplin *et al.* (2004), we can introduce additional degrees of freedom by changing $[x^\top(t)P_2^\top \quad \dot{x}^\top(t)P_3^\top]$ in the right side of (2.21) by the full-order vector $\zeta_v^\top$, multiplied by the corresponding weighting matrices. This means that the right side of the following equation can be added to $\dot{V}(x_t)$ (additionally to (2.21)):

$$0 = 2[x^\top(t-h)P_4^\top \quad v_1^\top(t)P_5^\top \quad v_2^\top(t)P_6^\top \quad v_3^\top(t)P_7^\top \quad v_4^\top(t)P_8^\top]$$
$$\times \begin{bmatrix} A_0x(t) - \dot{x}(t) + (A_1 + C_4)x(t-h) + \mu(A_1 + C_4)v_1(t) + Fv_2(t) - C_4v_3(t) + Hv_4(t) \\ A_0x(t) - \dot{x}(t) + (A_1 + C_5)x(t-h) + \mu(A_1 + C_5)v_1(t) + Fv_2(t) - C_5v_3(t) + Hv_4(t) \\ A_0x(t) - \dot{x}(t) + (A_1 + C_6)x(t-h) + \mu(A_1 + C_6)v_1(t) + Fv_2(t) - C_6v_3(t) + Hv_4(t) \\ A_0x(t) - \dot{x}(t) + (A_1 + C_7)x(t-h) + \mu(A_1 + C_7)v_1(t) + Fv_2(t) - C_7v_3(t) + Hv_4(t) \\ A_0x(t) - \dot{x}(t) + (A_1 + C_8)x(t-h) + \mu(A_1 + C_8)v_1(t) + Fv_2(t) - C_8v_3(t) + Hv_4(t) \end{bmatrix}. \tag{2.35}$$

The LMI (2.32) in this case will take the form

$$\Phi_0 + \Phi_1 + \Phi_1^\top < 0, \tag{2.36}$$

where $S_N > 0, U > 0, R_a > 0, S_a > 0$ and Φ_0 is the matrix in the left side of (2.32), while

$$\Phi_1 = \begin{bmatrix} 0_{2n\times 2n} & 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} & 0 \\ P_4^\top[A_0 - I] & P_4^\top A_1 + Y_4^\top & \mu P_4^\top A_1 + \mu Y_4^\top & P_4^\top F & -Y_4^\top & P_4^\top H & 0 \\ P_5^\top[A_0 - I] & P_5^\top A_1 + Y_5^\top & \mu P_5^\top A_1 + \mu Y_5^\top & P_5^\top F & -Y_5^\top & P_5^\top H & 0 \\ P_6^\top[A_0 - I] & P_6^\top A_1 + Y_6^\top & \mu P_6^\top A_1 + \mu Y_6^\top & P_6^\top F & -Y_6^\top & P_6^\top H & 0 \\ P_7^\top[A_0 - I] & P_7^\top A_1 + Y_7^\top & \mu P_7^\top A_1 + \mu Y_7^\top & P_7^\top F & -Y_7^\top & P_7^\top H & 0 \\ P_8^\top[A_0 - I] & P_8^\top A_1 + Y_8^\top & \mu P_8^\top A_1 + \mu Y_8^\top & P_8^\top F & -Y_8^\top & P_8^\top H & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_i^\top = P_i^\top C_i, \quad i = 4, \ldots, 8.$$

The above weighting matrices may lead to some improvement.

REMARK 7 During the last few years, new techniques for adding free weighting matrices have been developed (see, e.g. He *et al.*, 2005; Parlakci, 2006 and the references therein). All these work consider simple Lyapunov functionals. The extension of these methods to the discretized Lyapunov functional method may be the topic of future research.

The LMI conditions via discretized Lyapunov functional method are known to be numerically complex, depending on the large number of decision variables (Gu, 1997; Gu *et al.*, 1997; Han, 2005). The descriptor discretized Lyapunov functional method (Fridman, 2006) adds to the existing method two more $n \times n$ decision variables P_2 and P_3, that allow, however, to solve the design problems. Simplification of the discretized Lyapunov functional method is another direction for future research.

EXAMPLE 1 Consider (2.1) with

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.1 \end{bmatrix},$$
$$H = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_0 = E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{2.37}$$

This system with $F = \mathrm{diag}\{0.1, 0.1\}$ and $E_2 = 0$ was analysed in Han (2005) by the discretized Lyapunov functional method, where the following upper bound on the constant delay was found for $N = 3$: $0 \leqslant g = \tau \leqslant 2.12$. Theorem 2 with $N = 3$ leads to a less restrictive bound for constant delays: $\tau \leqslant 2.4$ and all g.

Consider now the case of H and E_2 given by (2.37) and time-varying $g(t)$ and $\tau(t)$. Applying the Euclidean matrix norm, we find that $\|F\| + \|H\|\|E_2\| = 1.1 > 1$ and thus the existing (direct) Lyapunov methods cannot be applied. By Theorem 2 for $N = 3$, we find that for all $g(t)$ the system is asymptotically stable for $\tau(t)$ from the following intervals:

$$\begin{array}{lll} f = d = 0, & \mu = 0,\ h \leqslant 0.48, & 0 \leqslant \tau \leqslant 0.48, \\ f = d = 0.1, & h = \mu = 0.20, & 0 \leqslant \tau(t) \leqslant 0.40, \\ f = d = 0.5, & h = \mu = 0.08, & 0 \leqslant \tau(t) \leqslant 0.16. \end{array}$$

Note that in this example, the free weighting matrices $P_4, \ldots, P_8, Y_4, \ldots, Y_8$ of (2.36) do not improve the results.

EXAMPLE 2 (Michiels & Vyhlidal, 2005) Consider

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)) + F_1\dot{x}(t - g_1(t)) + F_2\dot{x}(t - g_2(t)), \tag{2.38}$$

where

$$F_1 = \begin{bmatrix} 0 & 0.2 & -0.4 \\ -0.5 & 0.3 & 0 \\ 0.2 & 0.7 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -0.3 & -0.1 & 0 \\ 0 & 0.2 & 0 \\ 0.1 & 0 & 0.4 \end{bmatrix},$$
$$A = \begin{bmatrix} -4.8 & 4.7 & 3 \\ 0.1 & 1.4 & -0.4 \\ 0.7 & 3.1 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 \\ 0.7 \\ 0.1 \end{bmatrix}.$$

It was shown in Michiels & Vyhlidal (2005) that there exists K that stabilizes (2.38) with constant delays $\tau \equiv 0.5$, $g_1 \equiv 0.7$ and $g_2 \equiv 1.7$. Note that in this example $\|F_1\| + \|F_2\| = 1.22 > 1$ and thus the existing (direct) Lyapunov methods cannot be applied to the case of 'time-varying' g_1 or g_2.

Consider now constant g_2 and time-varying $g_1(t)$ and $\tau(t)$ with $\dot{g}_1 \leqslant f, \dot{\tau} \leqslant d$. We choose $K = [-0.3626 \quad -6.7792 \quad 1.3247]$ (this gain was found by using the stabilization via descriptor discretized Lyapunov functional, Fridman, 2006) and we analyse the asymptotic stability of the resulting closed-loop system. By applying the extension of Theorem 2 and (2.36) with $N = 3$ to multiple delays g_1 and g_2, we obtain for all $g_1(t)$ and g_2 the following stability intervals for $\tau(t)$:

$$\begin{array}{llll} f = d = 0, & h \leqslant 0.12, & \mu = 0, & 0 \leqslant \tau \leqslant 0.12, \\ f = d = 0.1, & h = 0.07, & \mu = 0.03, & 0.04 \leqslant \tau(t) \leqslant 0.1. \end{array}$$

In this example, the weighting matrices $P_4, \ldots, P_8, Y_4, \ldots, Y_8$ improve the results; however, verification of the latter condition takes more computation time. Thus, without these matrices for $f = d = 0$ we find $0 \leqslant \tau \leqslant 0.04$.

3. Conclusions

The input–output approach is extended to the stability analysis of linear neutral type systems with uncertain time-varying delays and either norm-bounded or polytopic-type uncertainties. This allows a restrictive assumption to be avoided on the sum of the norms of the matrices in the neutral part to be less than 1. New sufficient and necessary stability criteria are derived in the frequency and in the time domains. These conditions are retarded-delay-dependent/neutral-delay-independent. The time-domain criterion is based on the descriptor discretized Lyapunov functionals, which is known to be efficient for the solution of design problems. The method can be extended to L_2-gain analysis.

The approach presented allows robust control theory to be developed to a wide class of neutral uncertain systems with multiple time-varying delays. It gives insight to further development of the direct Lyapunov method for neutral and more general descriptor systems with time-varying delays.

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