Automatica 69 (2016) 201–209

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Decentralized networked control of systems with local networks: A time-delay approach[☆]

Dror Freirich, Emilia Fridman

School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel

ARTICLE INFO

Article history:
Received 12 May 2015
Received in revised form
25 January 2016
Accepted 15 February 2016

Keywords:
Networked control systems
Time-delay
Scheduling protocols
Decentralized control
Lyapunov–Krasovskii method

ABSTRACT

This paper develops the time-delay approach to large-scale networked control systems (NCSs) with multiple local communication networks connecting sensors, controllers and actuators. The local networks operate asynchronously and independently of each other in the presence of variable sampling intervals, transmission delays and scheduling protocols (from sensors to controllers). The communication delays are allowed to be greater than the sampling intervals. A novel Lyapunov–Krasovskii method is presented for the exponential stability analysis of the closed-loop large-scale system. In the case of networked control of a single plant our results lead to simplified conditions in terms of reduced-order linear matrix inequalities (LMIs) comparatively to the recent results in the framework of time-delay systems. Polytopic type uncertainties in the system model can be easily included in the analysis. Numerical examples from the literature illustrate the efficiency of the results.

© 2016 Elsevier Ltd. All rights reserved.

1. Introduction

Networked Control Systems are systems with spatially distributed sensors, actuators and controller nodes which exchange data over a communication data channel (Antsaklis & Baillieul, 2007). It is important to provide a stability and performance certificate that takes into account the network imperfections (such as variable sampling intervals, variable communication delays, scheduling protocols, etc.). The hybrid system approach has been applied to nonlinear NCSs under Try-Once-Discard (TOD) and Round-Robin (RR) scheduling protocols in Heemels, Teel, van de Wouw, and Nesic (2010), Nesic and Teel (2004), Walsh, Ye, and Bushnell (2002), where variable sampling intervals and *small communication delays* (that are smaller than the sampling intervals) have been considered. Recently the time-delay approach to NCSs (see e.g. Fridman, 2014; Fridman, Seuret, & Richard, 2004; Gao, Chen, & Lam, 2008) was extended to networked systems under TOD and RR protocols that allowed to treat *large communication delays* (Liu, Fridman, & Hetel, 2012, 2015).

It is common place in industry that the total plant to be controlled consists of a large number of interacting subsystems (Lunze, 1992). Usually the control of the plant is designed in a decentralized manner with local control stations allocated to individual subsystems. Most papers on NCSs assume that there is one controller and one global communication network. However, in the control of large-scale systems it is more efficient to use local controllers and local networks instead of the global ones. This leads to large-scale NCSs with independent and asynchronous local networks. Another application of NCSs with asynchronous local networks is platoons of vehicles that communicate wirelessly without timing coordination between members of the whole string (Heemels, Borgers, van de Wouw, Nesic, & Teel, 2013).

Decentralized networked control of large-scale interconnected systems with local independent networks was studied in the framework of hybrid systems (Borgers & Heemels, 2014; Heemels et al., 2013), where variable sampling or/and small communication delays were taken into account. Distributed estimation in the presence of *synchronous* sampling of local networks and RR protocol was recently analyzed in Ugrinovskii and Fridman (2014) in the framework of time-delay approach.

The goal of this paper is to extend the time-delay approach to decentralized NCS with multiple local communication networks connecting sensors, controllers and actuators. The local networks operate asynchronously and independently of each other in the presence of variable sampling intervals, transmission delays and scheduling protocols (from sensors to controllers). The

☆ This work was partially supported by Israel Science Foundation (Grant Nos. 754/10 and 1128/14) and by Kamea Fund of Israel. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Antonis Papachristodoulou under the direction of Editor Christos G. Cassandras.

E-mail addresses: drorfrc@gmail.com (D. Freirich), emilia@eng.tau.ac.il (E. Fridman).

http://dx.doi.org/10.1016/j.automatica.2016.02.037
0005-1098/© 2016 Elsevier Ltd. All rights reserved.

communication delays are allowed to be greater than the sampling intervals. Note that direct extension of the switched system modeling under RR protocol of Liu et al. (2012) to large-scale system would lead to numerous LMIs. The Lyapunov–Krasovskii method of Liu et al. (2015) developed for hybrid time-delay models of the closed-loop systems under TOD and RR protocols involves complicated conditions on the derivative and on the jumps of Lyapunov functionals that cannot be directly extended to large-scale systems.

In the present paper a novel Lyapunov–Krasovskii method is suggested for the exponential stability analysis of the closed-loop large-scale system. In the case of networked control of a single plant our results lead to simplified conditions in terms of reduced-order LMIs comparatively to the recent results (Liu et al., 2012, 2015). Numerical examples from the literature illustrate the efficiency of the results.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. The space of functions $\phi:[a,b]\to\mathbb{R}^n$, which are absolutely continuous on $[a,b]$ and have square integrable first-order derivatives, is denoted by $W[a,b]$ with the norm $\|\phi\|_W=\|\phi\|_{W[a,b]}=\max_{\theta\in[a,b]}|\phi(\theta)|+\left[\int_a^b|\dot\phi(s)|^2ds\right]^{\frac{1}{2}}$. $\mathbb{Z}_+$ and $\mathbb{N}$ denote the set of non-negative integers and positive integers, respectively. *MATI* and *MAD* denote maximum allowable transmission interval and maximum allowable delay, respectively. Denote by δ_{nm} the Kronecker delta meaning $\delta_{nm}=0,\ n\neq m$ and $\delta_{nn}=1\ (n,m\in\mathbb{N})$. Throughout the paper the subscript or superscript j stands for a subsystem index, while subscript i denotes the sensor index.

2. Problem formulation

Consider the system in Fig. 1, consisting of M physically coupled linear continuous-time plants Pj, controlled by M local controllers Cj ($j=1,\dots,M$). The dynamics of the plants Pj are given by subsystems:

$$\begin{aligned}&\dot x_j(t)=A_jx_j(t)+\sum_{l\neq j}F_{lj}x_l(t)+B_ju_j(t),\quad t\geq 0,\\&x_j(0)=x_{0j},\end{aligned}\tag{1}$$

where $j=1\dots M$ is the subsystem index, $x_j(t)\in\mathcal{R}^{n^j}$ is the state, $u_j(t)\in\mathcal{R}^{m^j}$ is the control input, A_j,B_j and F_{lj} are matrices of appropriate dimensions. Subsystem j has several nodes (N_j distributed sensors, a controller node and an actuator node) connected via a local communication network. The measurements are given by

$$y_{ij}(t)=C_{ij}x_j(t)\in\mathbb{R}^{n_i^j},\quad i=1,\dots,N_j,\qquad \sum_{i=1}^{N_j}n_i^j=n_y^j.$$

The jth subsystem is assumed to have an independent sequence of sampling instants

$$0=s_0^j<s_1^j<\dots<s_k^j<\dots,\qquad \lim_{k\to\infty}s_k^j=\infty$$

with bounded sampling intervals $s_{k+1}^j-s_k^j\leq MATI_j$. At each s_k^j, one of the outputs $y_{ij}(s_k^j)\in\mathbb{R}^{n_i^j}$ is transmitted via the sensor network to controller Cj.

Suppose that data loss is not possible and that the transmission of the information over the networks from sensors to actuators is subject to a variable roundtrip delay η_k^j. Then $t_k^j=s_k^j+\eta_k^j$ is the updating time instant of the Zero-Order Hold (ZOH). Communication delay is assumed to be bounded $\eta_k^j\in\left[\eta_m^j,\eta_M^j\right]$, where $\eta_M^j\triangleq MAD_j$. Differently from Borgers and Heemels (2014), we do not restrict the network delays to be small with $t_k^j=s_k^j+\eta_k^j<s_{k+1}^j$, i.e. $\eta_k^j<s_{k+1}^j-s_k^j$. As in Naghshtabrizi, Hespanha, and Teel (2010) we allow the delay to be non-small provided that the old sample cannot get to the same destination (same controller or same actuator) after the most recent one. We suppose that the controllers and the actuators are event-driven (in the sense that they update their outputs as soon as they receive a new sample).

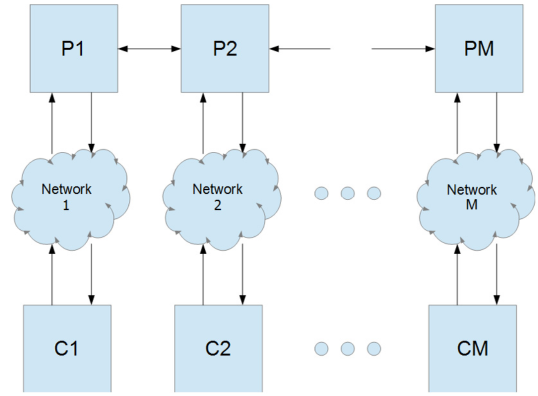

Fig. 1. Decentralized control of systems with local networks.

Assume the following assumption:

A1 There exist M gain matrices $K_j=\begin{bmatrix}K_{1j}&\cdots&K_{N_jj}\end{bmatrix}$, $K_{ij}\in\mathbb{R}^{m^j\times n_i^j}$ such that the matrices $A_j+B_jK_jC_j$ are Hurwitz, where $C_j=\begin{bmatrix}C_{1j}^T&\cdots&C_{N_jj}^T\end{bmatrix}^T$.

Remark 1. The assumption **A1** means that the "nominal system" $\dot x_j=A_j+B_ju_j$ is stabilizable by a static output-feedback $u_j=K_jC_jx_j$. Note that in the case of only one network (from sensors to controller) in each subsystem, the presented results can be easily adapted to decentralized observer-based control of large-scale systems as shown for the case of one plant in Liu et al. (2012, 2015).

We will consider TOD and RR protocols that orchestrate the sensor data transmission to the controller. Denote $J=\{1,\dots,M\}$, $J_{RR}=\{j\in J|\ j\text{th subsystem is under RR}\}$ and $J_{TOD}=\{j\in J|\ j\text{th subsystem is under TOD}\}$. Note that if for some j there is no scheduling from the sensors ($N_j=1$) to the controller we will refer to it as $j\in J_{RR}$, where $N_j=1$. Thus, $J=J_{RR}\bigcup J_{TOD}$. Denote by

$$\hat y_j(s_k^j)=\begin{bmatrix}\hat y_{1j}^T(s_k^j)&\cdots&\hat y_{N_jj}^T(s_k^j)\end{bmatrix}^T\in\mathbb{R}^{n_y^j}\tag{2}$$

the most recent output information submitted to the scheduling protocol of the jth subsystem (i.e. the most recent information at the jth controller side) at the sampling instant s_k^j. Then under **A1** the resulting static output-feedbacks are given by

$$\begin{aligned}&u_j(t)=\sum_{i=1}^{N_j}K_{ij}\hat y_{ij}(s_k^j),\quad t\in[t_k^j,t_{k+1}^j),\\&k\in\mathbb{Z}_+,\ j=1\dots M.\end{aligned}\tag{3}$$

Denote

$$\begin{aligned}&T\triangleq\max\{\{t_{N_j-1}^j\}_{|j\in J_{RR}},\{t_0^j\}_{|j\in J_{TOD}}\},\\&x(t)=col\{x_1(t),\dots,x_M(t)\}.\end{aligned}\tag{4}$$

We will define next a notion of solution to the closed-loop system (1), (3) and justify its existence. Monotonically increasing for each $j = 1, \ldots, M$ sequences of updating times $t_0^j < t_1^j < \cdots$ can be reordered in one monotonically increasing sequence $\tilde{t}_0 < \tilde{t}_1 < \cdots$, where $\tilde{t}_{k^*} = T$ for some $k^* \in \mathbb{Z}_+$. For any initial condition $x_T \triangleq x(T + \cdot) \in W[-T, 0]$, by applying the step method for $t \in [\tilde{t}_k, \tilde{t}_{k+1}]$ ($k \geq k^*$) one can show that there exists a unique absolutely continuous function $x : [T, \infty) \to \mathbb{R}^{\sum_{j=1}^M n_j}$ satisfying (1), (3) almost for all $t \geq T$. This function is called a solution of (1), (3) initialized by x_T.

Definition 1. The closed-loop large-scale system (1), (3) is called exponentially stable with a decay rate $\alpha_0 > 0$ if for any initial condition $x_T \in W[-T, 0]$ there exists $c > 0$ such that solutions of the system initiated by x_T satisfy the following inequality

$$|x(t)| \leq ce^{-\alpha_0(t-T)}\|x_T\|_W, \quad \forall t \geq T. \tag{5}$$

Our objective is to derive sufficient conditions for the exponential stability of the closed-loop system (1), (3).

3. NCSs under scheduling protocols

As mentioned in the previous section, at each sampling instant s_k^j, one of the system nodes $i \in \{1, \ldots, N_j\}$ is active, that is only one of $\hat{y}_{ij}(s_k^j)$ values is updated with the recent output $y_{ij}(s_k^j)$. Let $i_k^{*j} \in \{1, \ldots, N_j\}$ denote the active output node at the sampling instant s_k^j, which will be chosen due to RR or TOD scheduling protocols (to be defined hereafter). Then

$$\hat{y}_{ij}(s_k^j) = \begin{cases} y_{ij}(s_k^j), & i = i_k^{*j}, \\ \hat{y}_{ij}(s_{k-1}^j), & i \neq i_k^{*j}. \end{cases} \tag{6}$$

For simplicity we will omit j in i_k^{*j}.

3.1. RR protocol and the closed-loop model

The periodic choice of i_k^* corresponds to **RR protocol**. Under RR scheduling the measurements are sent in a periodic manner one after another. Then the components of the most recent output on the controller side $\hat{y}_j^T(s_k^j)$ given by (2) can be presented as

$$\hat{y}_{ij}(s_k^j) = y_{ij}(s_{k-\Delta_k^i}^j), \quad i = 1, \ldots, N_j$$

with some $\Delta_k^i \in \{0, \ldots, N_j - 1\}$. Following the time-delay approach to NCS denote

$$\tau_{ij}(t) = t - s_{k-\Delta_k^i}^j, \quad t \in [t_k^j, t_{k+1}^j).$$

We have

$$\begin{aligned} \eta_m^j &\leq \tau_{ij}(t) \leq t_{k+1}^j - s_{k-\Delta_k^i}^j = s_{k+1}^j - s_{k-\Delta_k^i}^j + \eta_{k+1}^j \\ &\leq (\Delta_k^i + 1) \cdot MATI_j + MAD_j \\ &\leq N_j \cdot MATI_j + MAD_j \triangleq \tau_M^j. \end{aligned}$$

Therefore, for $t \geq t_{N_j-1}^j$ (when all the measurements are transmitted at least once) the static output-feedback (3) under RR protocol can be presented as

$$u(t) = \sum_{i=1}^{N_j} K_{ij} y_{ij}(t - \tau_{ij}(t)), \quad t \geq t_{N_j-1}^j. \tag{7}$$

The resulting closed-loop model is a system with multiple delays

$$\begin{aligned} \dot{x}_j(t) &= A_j x_j(t) + \sum_{i=1}^{N_j} A_{ij} C_{ij} x_j(t - \tau_{ij}(t)) \\ &\quad + \sum_{l \neq j} F_{lj} x_l(t), \quad A_{ij} = B_j K_{ij}, \ t \geq t_{N_j-1}^j, \end{aligned} \tag{8}$$

where $\tau_{ij}(t) \in [\eta_m^j, \tau_M^j]$.

Note that under **A1** for $\tau_{ij} = 0$ and $F_{lj} = 0$, there exist K_{ij} such that (7) is exponentially stable. Then for small enough τ_{ij} the system (7) with the same K_{ij} is input-to-state stable (where $x_{l|l\neq j}$ are the inputs).

Remark 2. A more accurate model of the closed-loop system under RR protocol was presented in Liu et al. (2012) in the form of switched N_j subsystems with ordered multiple delays. Our simplified model (one system instead of N_j, but with independent delays from the maximum delay interval $[\eta_m^j, \tau_M^j]$) leads to reduced-order LMI conditions.

3.2. TOD protocol and the closed-loop model

In TOD protocol the choice of i_k^* at the sampling instant s_k^j depends on the transmission error

$$\mathcal{E}_{ij}(s_k^j) = \hat{y}_{ij}(s_{k-1}^j) - y_{ij}(s_k^j), \quad i \in \{1, \ldots, N_j\}.$$

The output node i with the greatest weighted error $\mathcal{E}_{ij}(s_k^j)$ will be granted the access to the network.

Definition 2 (*TOD Protocol*)**.** Let $Q_{i,j} > 0$ ($i = 1, \ldots, N_j$) be some weighting matrices. At the sampling instant s_k^j, the *weighted TOD protocol* is a protocol for which the active output node with the index i_k^* ($k \in \mathbb{Z}_+$) is defined as any index that satisfies

$$|\sqrt{Q_{i_k^*,j}}\mathcal{E}_{i_k^*}(s_k^j)|^2 \geq |\sqrt{Q_{i,j}}\mathcal{E}_{ij}(s_k^j)|^2, \quad i = 1, \ldots, N_j. \tag{9}$$

Here the weighting matrices $Q_{1,j}, \ldots, Q_{N_j,j}$ are variables to be designed. Then the feedback can be presented as

$$\begin{aligned} u_j(t) &= K_{i_k^*} y_{i_k^* j}(s_k^j) + \sum_{i=1, i \neq i_k^*}^{N_j} K_{ij} \hat{y}_{ij}(s_{k-1}^j), \\ &\quad t \in [t_k^j, t_{k+1}^j), \ k \in \mathbb{Z}_+ \end{aligned} \tag{10}$$

with $u_j(t) = 0, \ 0 \leq t < t_0^j$.

Note that for K_{ij} from **A1** and small enough $MATI_j$ and MAD_j, the closed-loop system (1), (10) is input-to-state stable, where $x_{l|l\neq j}$ are the inputs (cf. Remark 6). Denote

$$\tau_j(t) = t - s_k^j, \quad t \in [t_k^j, t_{k+1}^j), \ k \in \mathbb{Z}_+.$$

Then

$$\eta_m^j \leq \tau_j(t) \leq MATI_j + MAD_j \triangleq \tau_M^j.$$

In order to obtain the impulsive closed-loop model we define as in Liu et al. (2015) the piecewise-continuous error

$$e_{ij}(t) = \mathcal{E}_{ij}(s_k^j), \quad t \in [t_k^j, t_{k+1}^j), \ i = 1 \ldots N_j,$$

where we assume $\hat{y}_{ij}(s_{-1}^j) = 0$, implying $e_{ij}(t_0^j) = -y_{ij}(s_0^j)$. Then the closed-loop model has the following continuous dynamics:

$$\begin{aligned} \dot{x}_j(t) &= A_j x(t) + A_{1j} C_j x_j(t - \tau_j(t)) \\ &\quad + \sum_{i=1, i \neq i_k^*}^{N_j} B_{ij} e_{ij}(t) + \sum_{l \neq j} F_{lj} x_l(t), \\ &\quad \dot{e}_{ij}(t) = 0, \quad i = 1 \ldots N_j, \ t \geq t_0^j, \\ &\quad A_{1j} = B_j K_j, \ B_{ij} = B_j K_{ij}. \end{aligned} \tag{11}$$

Similar to Liu et al. (2015) we obtain for $i = i_k^*$

$$e_{ij}(t_{k+1}^j) = \hat{y}_{ij}(s_k^j) - y_{ij}(s_{k+1}^j) = C_{ij}[x_j(s_k^j) - x_j(s_{k+1}^j)],$$

and for $i \neq i_k^*$

$$e_{ij}(t_{k+1}^j) = \hat{y}_{ij}(s_{k-1}^j) - y_{ij}(s_{k+1}^j) = e_{ij}(t_k^j) + C_{ij}[x_j(s_k^j) - x_j(s_{k+1}^j)].$$

Thus, the delayed reset system is given by

$$\begin{aligned} x_j(t_{k+1}^j) &= x_j(t_{k+1}^{j-}), \\ e_{ij}(t_{k+1}^j) &= [1-\delta(i,i_k^*)]e_{ij}(t_k^j) \\ &\quad + C_{ij}[x_j(t_k^j - \eta_k^j) - x_j(t_{k+1}^j - \eta_{k+1}^j)], \\ &\quad i = 1,\ldots,N_j,\ k \in \mathbb{Z}_+, \end{aligned} \tag{12}$$

where δ is Kronecker delta. Summarizing, (11)–(12) is the hybrid model of the NCS.

Note that in our model the first updating time t_0^j corresponds to the time instant when the first data is received by the actuator. We define $x_j(t) = x_j(0)$ for $t < 0$. Thus the initial conditions for (11)–(12) are given by

$$x_j(t_0^j + \cdot) \in W[-\tau_M^j, 0], \qquad e_j(t_0^j) = -C_j x_j(t_0^j - \eta_0^j). \tag{13}$$

3.3. Lyapunov-based analysis under RR protocol

Assume that the jth subsystem (1) is under RR protocol, i.e. $j \in J_{RR}$. Consider the closed-loop model (8) and the following Lyapunov functional:

$$\begin{aligned} V_j(t) &= x_j^T(t)P_j x_j(t) + V_{0j}(t) + V_{1j}(t), \\ V_{0j}(t) &= \sum_{i=1}^{N_j}\Big[\int_{t-\eta_m^j}^{t} e^{2\alpha(s-t)} x_j^T(s) C_{ij}^T S_{0i,j} C_{ij} x_j(s) ds \\ &\quad + \eta_m^j \int_{-\eta_m^j}^{0}\int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}_j^T(s) C_{ij}^T R_{0i,j} C_{ij}\dot{x}_j(s) ds d\theta\Big], \\ V_{1j}(t) &= \sum_{i=1}^{N_j}\Big[\int_{t-\tau_M^j}^{t-\eta_m^j} e^{2\alpha(s-t)} x_j^T(s) C_{ij}^T S_{1i,j} C_{ij} x_j(s) ds \\ &\quad + h_j \int_{-\tau_M^j}^{-\eta_m^j}\int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}_j^T(s) C_{ij}^T R_{1i,j} C_{ij}\dot{x}_j(s) ds d\theta\Big], \\ h_j &\triangleq (\tau_M^j - \eta_m^j), \quad \alpha > 0,\ m = 0, 1, \\ P_j &> 0, \quad S_{mi,j} > 0, \quad R_{mi,j} > 0, \end{aligned} \tag{14}$$

where we define (for simplicity) $x_j(t) = x_{0j}$, for $t < 0$. Note that differently from conventional Lyapunov functionals for the stability of systems with interval delays (see e.g. Liu et al., 2012, Park, Ko, & Jeong, 2011), the one given by (14) contains C_{ij} in integral terms with the reduced-order matrices $S_{mi,j}$ and $R_{mi,j}$. The latter matrices will be decision variables of the resulting LMIs.

Proposition 1. *Consider the jth subsystem given by (8). Given tuning parameters $\alpha > \varepsilon > 0$ and $(M-1)$ $n_l \times n_l$ $(l \neq j)$ matrices $P_l > 0$, let there exist a $n_j \times n_j$ matrix $P_j > 0$, $n_i^j \times n_i^j$ matrices $R_{0i,j} > 0, R_{1i,j} > 0, S_{0i,j} > 0, S_{1i,j} > 0$ and $W_{i,j}$ $(i = 1 \ldots N_j)$ that satisfy*

$$\Gamma_{i,j} = \begin{bmatrix} R_{1i,j} & W_{i,j} \\ * & R_{1i,j} \end{bmatrix} \geq 0, \quad i = 1 \ldots N_j, \tag{15}$$

and

$$\hat{\Sigma}_j = \begin{bmatrix} \Sigma_j & \Xi_j \\ * & \Pi_j \end{bmatrix} < 0, \tag{16}$$

where $\Sigma_j = \begin{bmatrix} \Phi & D_2^T C_j^T H_j \\ * & -H_j \end{bmatrix}$ *and*

$$\begin{aligned} \Phi &= D_1^T(2\alpha P_j + C_j^T \hat{S}_{0j} C_j) D_1 + \left(D_2^T P_j D_1 + D_1^T P_j D_2\right) \\ &\quad - \rho_m D_3^T(\hat{S}_{0j} - \hat{S}_{1j}) D_3 - \rho_m D_4^T \hat{R}_{0j} D_4 - \rho D_5^T \hat{S}_{1j} D_5 \\ &\quad - \rho D_6^T \hat{\Gamma}_j D_6, \qquad H_{i,j} = \eta_m^2 R_{0i,j} + (\tau_M - \eta_m)^2 R_{1i,j}, \\ \rho_m &= e^{-2\alpha\eta_m^j}, \quad \rho = e^{-2\alpha\tau_M^j}, \qquad H_j = diag\{H_{1,j},\ldots,H_{N_j,j}\}, \\ \hat{S}_{pj} &= diag\{S_{p1,j},\ldots,S_{pN_j,j}\}, \quad p = 0, 1, \\ \hat{R}_{0j} &= diag\{R_{01,j},\ldots,R_{0N_j,j}\}, \\ \hat{\Gamma}_j &= diag\{\Gamma_{1,j},\ldots,\Gamma_{N_j,j}\}, \qquad D_1 = [I_{n^j}\ 0_{n^j \times 3n_y^j}], \\ D_2 &= \left[A_j, [0\ 1\ 0] \otimes A_{1j}, \ldots, [0\ 1\ 0] \otimes A_{N_j j}\right], \\ D_3 &= \left[0_{n_y^j \times n^j} \quad diag\left\{[1\ 0\ 0] \otimes I_{n_1^j}, \ldots, [1\ 0\ 0] \otimes I_{n_{N_j}^j}\right\}\right], \\ D_4 &= \left[C_j \quad diag\left\{[-1\ 0\ 0] \otimes I_{n_1^j}, \ldots, [-1\ 0\ 0] \otimes I_{n_{N_j}^j}\right\}\right], \\ D_5 &= \left[0_{n_y^j \times n^j} \quad diag\left\{[0\ 0\ 1] \otimes I_{n_1^j}, \ldots, [0\ 0\ 1] \otimes I_{n_{N_j}^j}\right\}\right], \\ D_6 &= \left[0_{2n_y^j \times n^j} \quad diag\left\{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \otimes I_{n_1^j}, \ldots, \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \otimes I_{n_{N_j}^j}\right\}\right], \\ \Xi_j^T &= \mathcal{F}_j^T \left[P_j D_1 \quad C_j^T H_j\right], \\ \mathcal{F}_j &= row_{\,l=1,\ldots,M}\{F_{lj},\ l \neq j\} \\ \Pi_j &= -\frac{2\varepsilon}{M-1} diag_{\,l=1,\ldots,M}\{P_l,\ l \neq j\}. \end{aligned}$$

Then the Lyapunov functional $V_j(t)$ given by (14) satisfies the following inequality along the solutions of (8):

$$\dot{V}_j(t) + 2\alpha V_j(t) \leq \frac{2\varepsilon}{M-1}\sum_{l \neq j} x_l^T(t) P_l x_l(t), \quad t \geq t_{N_j-1}^j. \tag{17}$$

Moreover, in the case where the jth subsystem (8) is independent of other subsystems (i.e. $F_{lj|l\neq j} = 0$, $l = 1,\ldots,M$), if $\Sigma_j < 0$, then (8) is exponentially stable with a decay rate α.

Proof. We follow the standard arguments for the exponential stability analysis via Krasovskii method (see e.g. Fridman, 2014, Park et al., 2011). Differentiating $\dot{V}_j(t)$ we have

$$\begin{aligned} \dot{V}_j(t) + 2\alpha V_j(t) &\leq 2\alpha x_j^T(t) P_j x_j(t) + 2\dot{x}_j^T(t) P_j x_j(t) \\ &\quad + \sum_{i=1}^{N_j} |\sqrt{S_{0i,j}} C_{ij} x_j(t)|^2 - |\sqrt{S_{0i,j}} e^{-\alpha\eta_m^j} C_{ij} x_j(t-\eta_m^j)|^2 \\ &\quad + \rho_m |\sqrt{S_{1i,j}} C_{ij} x_j(t-\eta_m^j)|^2 - \rho |\sqrt{S_{1i,j}} C_{ij} x_j(t-\tau_M^j)|^2 \\ &\quad - \eta_m^j \int_{t-\eta_m^j}^{t} |\sqrt{R_{0i,j}} e^{\alpha(s-t)} C_{ij}\dot{x}_j(s)|^2 ds \\ &\quad - h_j \int_{t-\tau_M^j}^{t-\eta_m^j} |\sqrt{R_{1i,j}} e^{\alpha(s-t)} C_{ij}\dot{x}_j(s)|^2 ds + |\sqrt{H_{i,j}} C_{ij}\dot{x}_j(t)|^2. \end{aligned}$$

By Jensen's inequality

$$\eta_m^j \int_{t-\eta_m^j}^{t} |\sqrt{R_{0i,j}} e^{\alpha(s-t)} C_{ij}\dot{x}_j(s)|^2 ds \geq \rho_m |\sqrt{R_{0i,j}} C_{ij}(x_j(t) - x_j(t-\eta_m^j))|^2,$$

Table 1
Example 1: max. value of MATI for a given $\eta_M \triangleq MAD$.

MATI	η_m		
	0	0.01	0.02
Liu et al. (2012)	$0.023-\eta_M$	$0.029-\eta_M$	$0.035-\eta_M$
Liu et al. (2015)	$0.025-\eta_M$	$0.031-\eta_M$	$0.036-\eta_M$
Proposition 1	$0.022-\frac{\eta_M}{2}$	$0.0221-\frac{\eta_M}{2}$	$0.0223-\frac{\eta_M}{2}$

Table 2
Example 1: max. value of MATI for $MAD = 0.024$.

MATI	η_m			Dec. vars	LMI rows
	0	0.01	0.02		
Liu et al. (2012)	–	0.005	0.011	146	236
Liu et al. (2015)	0.001	0.007	0.012	72	208
Proposition 1	0.01	0.0101	0.0103	42	108

whereas under (15) by arguments of Park et al. (2011)

$$e^{2\alpha\tau_M^j} h_j \int_{t-\tau_M^j}^{t-\eta_m^j} e^{-2\alpha\tau_M^j} |\sqrt{R_{1i,j}} e^{\alpha(s-t)} C_{ij}\dot{x}_j(s)|^2 ds \geq \begin{bmatrix} C_{ij}(x_j(t-\eta_m^j) - x_j(t-\tau_{ij}(t))) \\ C_{ij}(x_j(t-\tau_{ij}(t)) - x_j(t-\tau_M^j)) \end{bmatrix}^T \times \Gamma_{i,j} \begin{bmatrix} C_{ij}(x_j(t-\eta_m^j) - x_j(t-\tau_{ij}(t))) \\ C_{ij}(x_j(t-\tau_{ij}(t)) - x_j(t-\tau_M^j)) \end{bmatrix}.$$

Denote

$$\zeta_j(t) = \left[x_j^T(t), \left[x_j(t-\eta_m^j)\, x_j(t-\tau_{1j}(t))\, x_j(t-\tau_M^j)\right]^T C_{1j}^T, \ldots, \left[x_j(t-\eta_m^j)\, x_j(t-\tau_{N_jj}(t))\, x_j(t-\tau_M^j)\right]^T C_{N_jj}^T\right]^T,$$
$$X_j(t) = col_{\,l=1,\ldots,M}\{x_l(t), l \neq j\}.$$

Substituting $\dot{x}_j(t) = D_2\zeta_j(t) + \mathcal{F}_j X_j(t)$ we arrive at

$$\dot{V}_j(t) + 2\alpha V_j(t) - \frac{2\varepsilon}{M-1}\sum_{l\neq j} x_l^T(t) P_l x_l(t) \leq \begin{bmatrix} \zeta_j^T(t) & X_j^T(t) \end{bmatrix} \begin{bmatrix} \Phi & D_1^T P_j \mathcal{F}_j \\ * & \Pi_j \end{bmatrix} \begin{bmatrix} \zeta_j(t) \\ X_j(t) \end{bmatrix} + [D_2\zeta_j(t) + \mathcal{F}_j X_j(t)]^T C_j^T H_j C_j [D_2\zeta_j(t) + \mathcal{F}_j X_j(t)].$$

Then, by Schur's complement, (16) implies (17).

For the case of the single jth subsystem, $\Sigma_j < 0$ implies $\dot{V}_j(t) + 2\alpha V_j(t) \leq 0$, i.e. by comparison principle

$$x_j^T(t) P_j x_j(t) \leq V_j(t) \leq e^{-2\alpha(t-t_{N_j-1}^j)} V_j(t_{N_j-1}^j).$$

The latter guarantees the exponential stability since $V_j(t_{N_j-1}^j) \leq \gamma_j \|x_{t_{N_j-1}^j}\|_W$ for some $\gamma_j > 0$. □

Remark 3. Under **A1** there exists $P_j > 0$ such that

$$P_j\left(A_j + \sum_{i=1}^{N_j} A_{ij}C_{ij}\right) + \left(A_j + \sum_{i=1}^{N_j} A_{ij}C_{ij}\right)^T P_j < 0.$$

Then, by standard arguments for delay-dependent conditions (Fridman, 2014), for small enough τ_{ij} and $\alpha > 0$ there exist $R_{0i,j} > 0, R_{1i,j} > 0, S_{0i,j} > 0, S_{1i,j} > 0$ and $W_{i,j}$ $(i = 1 \ldots N_j)$ that satisfy (15) and $\Xi_j < 0$ with the same P_j. Therefore, by Schur complements, (16) is feasible for given $\varepsilon > 0$ and small enough F_{lj}.

Remark 4. The LMIs of Proposition 1 and of Theorem 1 (see Section 4) are affine in the system matrices. Therefore, in the case of system matrices from an uncertain time-varying polytope one have to solve these LMIs simultaneously for all the vertices of the polytope applying the same decision matrices.

Example 1 (*Geromel, Korogui, & Bernussou, 2007*, *Liu et al., 2015*)**.** Consider an inverted pendulum mounted on a small cart. The linearized model can be written as (1) with one subsystem ($F_{lj} = 0, M = 1$), where $A_1 = E^{-1}A_f$, $B_1 = E^{-1}B_0$ and where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3/2 & -1/4 \\ 0 & 0 & -1/4 & 1/6 \end{bmatrix},$$
$$A_f = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(f_c+f_b) & f_b/2 \\ 0 & 5/2 & f_b/2 & -f_b/3 \end{bmatrix},$$
$$B_0^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

Here $f_c(t) \in [0.15, 0.25]$ and $f_b(t) \in [0.15, 0.25]$ are uncertain parameters. Thus A_1 belongs to uncertain polytope, defined by four vertices A_1^p $(p = 1, \ldots, 4)$ corresponding to $f_c/f_b = 0.15$ and $f_c/f_b = 0.25$. The pendulum can be stabilized by a state feedback $u_1(t) = K_1x_1(t)$ with

$$K_1 = [11.2062 \ -128.8597 \ 10.7823 \ -22.2629].$$

Suppose that the state variables are not accessible simultaneously. Consider the case of $N_1 = 2$ measurements, where $C_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $C_{21} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

For the values of η_m given in Table 1, we apply Remark 4, where LMI (15) and 4 LMIs (16) corresponding to 4 vertices A_1^p (with A_1 substituted by A_1^p) are solved with the same decision variables and with $\alpha = 0.015$. Table 2 presents the MATI for a given $MAD = 0.024$ (the case of large communication delay). It is observed that under RR protocol the LMI conditions of Proposition 1 possess essentially less decision variables and are given in terms of smaller LMIs than Liu et al. (2012, 2015) though in some cases guarantee the exponential stability for larger MATI.

3.4. Lyapunov-based analysis under TOD protocol

In this section we assume that the jth subsystem (1) is under TOD scheduling protocol, i.e. $j \in J_{TOD}$. Consider the closed-loop model (11)–(12) and the following Lyapunov functional:

$$V_j^e(t) = V_j(t) + \sum_{i=1}^{N_j} e_{ij}^T(t) Q_{i,j} e_{ij}(t) + \mathcal{W}_j^e(t), \tag{18}$$

where

$$\mathcal{W}_j^e(t) = 2\alpha(t_k^j - t) e_{i_k^*j}^T(t) Q_{i_k^*,j} e_{i_k^*j}(t) + \sum_{i=1, i\neq i_k^*}^{N_j} \frac{t_k^j - t}{t_{k+1}^j - t_k^j} e_{ij}^T(t) U_{i,j} e_{ij}(t),$$
$$V_j(t) = \tilde{V}_j(t) + V_j^G(t),$$
$$V_j^G = \sum_{i=1}^{N_j} h_j \int_{s_k^j}^t e^{2\alpha(s-t)} |\sqrt{G_{i,j}} C_{ij}\dot{x}_j(s)|^2 ds,$$

$$\begin{aligned}\tilde{V}_j(t) &= x_j^T(t)P_jx_j(t) \\ &+ \int_{t-\eta_m^j}^{t} e^{2\alpha(s-t)}x_j^T(s)C_j^TS_{0j}C_jx_j(s)ds \\ &+ \int_{t-\tau_M^j}^{t-\eta_m^j} e^{2\alpha(s-t)}x_j^T(s)C_j^TS_{1j}C_jx_j(s)ds \\ &+ \eta_m^j\int_{-\eta_m^j}^{0}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{x}_j^T(s)C_j^TR_{0j}C_j\dot{x}_j(s)dsd\theta \\ &+ h_j\int_{-\tau_M^j}^{-\eta_m^j}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{x}_j^T(s)C_j^TR_{1j}C_j\dot{x}_j(s)dsd\theta,\end{aligned}$$

$P_j > 0,\ S_{0j} > 0,\ S_{1j} > 0,\ R_{0j} > 0,\ R_{1j} > 0,$
$G_{i,j} > 0,\ Q_{i,j} > 0,\ U_{i,j} > 0,\ \alpha > 0,$
$h_j = \tau_M^j - \eta_m^j,\quad i = 1\dots N_j,\ t \in [t_k^j, t_{k+1}^j),\ k \in \mathbb{Z}_+.$

Remark 5. Differently from Liu et al. (2015), the Lyapunov functional (18) contains novel negative terms $\mathcal{W}_j^e(t)$ that essentially simplifies the exponential stability analysis of the hybrid system. Indeed, denoting $\hat{V}_j^e(t) = V_j^e(t)_{|\mathcal{W}_j^e=0}$ we have

$$\begin{aligned}&\dot{V}_j^e(t) + 2\alpha V_j^e(t) \\ &\le \dot{\hat{V}}_j^e(t) + 2\alpha\hat{V}_j^e(t) - \frac{1}{\tau_M^j - \eta_m^j}\sum_{i=1,i\neq i_k^*}^{N_j}|\sqrt{U_{i,j}}e_{ij}(t)|^2 \\ &\quad - 2\alpha|\sqrt{Q_{i_k^*}}e_{i_k^*}(t)|^2,\quad t\in[t_k^j, t_{k+1}^j),\end{aligned} \tag{19}$$

$$\begin{aligned}&V_j^e(t_{k+1}^j) - V_j^e(t_{k+1}^{j\,-}) \le \hat{V}_j^e(t_{k+1}^j) - \hat{V}_j^e(t_{k+1}^{j\,-}) \\ &\quad + \sum_{i=1,i\neq i_k^*}^{N}|\sqrt{U_{i,j}}e_{ij}(t_k^j)|^2 + 2\alpha h_j|\sqrt{Q_{i_k^*}}e_{i_k^*}(t_k^j)|^2.\end{aligned}$$

In Liu et al. (2015) the stability of (11)–(12) with $F_{lj} = 0$ is guaranteed if the right-hand sides of (19) are non-positive along the system for some $\alpha > 0$. By using the novel functional (18), under the same LMIs as in Liu et al. (2015) up to the order reduction due to C_j in $\tilde{V}_j(t)$ (see LMIs of Proposition 2) we will guarantee that V_e^j is positive, does not grow at t_k^j and satisfies

$$\dot{V}_j^e(t) + 2\alpha V_j^e(t) \le 0,\quad t\in[t_k^j, t_{k+1}^j) \tag{20}$$

along the jth hybrid system with $F_{lj} = 0$. The inequality (20) immediately implies the exponential stability of the jth hybrid subsystem, that essentially simplifies the proof of the stability (which is crucial for the extension of the results to large-scale hybrid systems).

The terms

$$e_{ij}^T(t)Q_{i,j}e_{ij}(t) \equiv e_{ij}^T(t_k^j)Q_{i,j}e_{ij}(t_k^j),\quad t\in[t_k^j, t_{k+1}^j)$$

are piecewise-constant, $\tilde{V}(t)$ presents a Lyapunov functional (with reduced-order decision matrices) for systems with interval delays $\tau_j(t) \in [\eta_m^j, \tau_M^j]$. The piecewise-continuous in time term V_j^G has been introduced in Liu et al. (2015) to cope with the delays in the reset conditions:

$$\begin{aligned}&V_j^G(t_{k+1}^j) - V_j^G(t_{k+1}^{j-}) \\ &= \sum_{i=1}^{N_j}h_j\int_{s_{k+1}^j}^{t_{k+1}^j} e^{2\alpha(s-t_{k+1}^j)}|\sqrt{G_{i,j}}C_{ij}\dot{x}_j(s)|^2ds \\ &\quad - \sum_{i=1}^{N_j}h_j\int_{s_k^j}^{t_{k+1}^{j-}} e^{2\alpha(s-t_{k+1}^j)}|\sqrt{G_{ij}}C_{ij}\dot{x}_j(s)|^2ds \\ &\le -\sum_{i=1}^{N_j}h_je^{-2\alpha\tau_M^j}\int_{s_k^j}^{s_{k+1}^j}|\sqrt{G_{i,j}}C_{ij}\dot{x}_j(s)|^2ds \\ &\le -\sum_{i=1}^{N_j}e^{-2\alpha\tau_M^j}|\sqrt{G_{i,j}}C_{ij}[x(s_k^j) - x(s_{k+1}^j)]|^2\end{aligned} \tag{21}$$

where we applied Jensen's inequality (see e.g. Gu, Kharitonov, & Chen, 2003). The function $V_j^e(t)$ is thus continuous and differentiable over $[t_k^j, t_{k+1}^j)$. The following lemma gives sufficient conditions for the positivity of $V_j^e(t)$ and for the fact that it does not grow in the jumps t_k^j:

Lemma 1. *Given a tuning parameter $\alpha > 0$, let there exist matrices $0 < Q_{i,j} \in \mathbb{R}^{n_i^j\times n_i^j}$, $0 < U_{i,j} \in \mathbb{R}^{n_i^j\times n_i^j}$ and $0 < G_{i,j} \in \mathbb{R}^{n_i^j\times n_i^j}$, $i = 1\dots N_j$ that satisfy the LMIs*

$$\Omega_{ij} \triangleq \begin{bmatrix} -\dfrac{1-2\alpha(\tau_M^j - \eta_m^j)}{N_j - 1}Q_{i,j} + U_{i,j} & Q_{i,j} \\ * & Q_{i,j} - G_{i,j}e^{-2\alpha\tau_M^j}\end{bmatrix} < 0,\quad i = 1\dots N_j. \tag{22}$$

Then $V_j^e(t)$ of (18) is positive in the sense that

$$V_j^e(t) \ge \beta\left(|x_j(t)|^2 + |e_j(t)|^2\right),\quad t\ge t_0^j,\quad e_j(t) \triangleq col\{e_{1,j}(t),\dots,e_{N_jj}(t)\} \tag{23}$$

for some $\beta > 0$. Moreover, $V_j^e(t)$ does not grow in the jumps along (11)–(12):

$$\Theta \triangleq V_j^e(t_{k+1}^j) - V_j^e(t_{k+1}^{j-}) \le 0. \tag{24}$$

Proof. It can be seen that (22) implies

$$\alpha(\tau_M^j - \eta_m^j) < 0.5 \quad\text{and}\quad U_{i,j} < Q_{i,j}$$

yielding the positivity of $V_j^e(t)$.

We show next that $V_j^e(t)$ does not grow in the jumps. Since $\tilde{V}_j(t_{k+1}^j) = \tilde{V}_j(t_{k+1}^{j-})$ and $e_j(t_{k+1}^{j-}) = e_j(t_k^j)$, we obtain by taking into account (21)

$$\begin{aligned}\Theta &= \sum_{i=1}^{N_j}\left[|\sqrt{Q_{i,j}}e_{ij}(t_{k+1}^j)|^2 - |\sqrt{Q_{i,j}}e_{ij}(t_k^j)|^2\right] \\ &\quad + 2\alpha(t_{k+1}^j - t_k^j)|\sqrt{Q_{i_k^*,j}}e_{i_k^*j}(t_k^j)|^2 \\ &\quad + \sum_{i\neq i_k^*}|\sqrt{U_{i,j}}e_{ij}(t_k^j)|^2 + V_j^G(t_{k+1}^j) - V_j^G(t_{k+1}^{j-}) \\ &\le |\sqrt{Q_{i_k^*,j}}e_{i_k^*j}(t_{k+1}^j)|^2 + \sum_{i=1,i\neq i_k^*}^{N_j}\Big[|\sqrt{Q_{i,j}}e_{ij}(t_{k+1}^j)|^2 \\ &\quad - |\sqrt{Q_{i,j} - U_{i,j}}e_{ij}(t_k^j)|^2\Big] - [1 - 2\alpha h_j]|\sqrt{Q_{i_k^*,j}}e_{i_k^*j}(t_k^j)|^2 \\ &\quad - \sum_{i=1}^{N_j}e^{-2\alpha\tau_M^j}|\sqrt{G_{i,j}}C_{ij}[x(s_k^j) - x(s_{k+1}^j)]|^2.\end{aligned}$$

Under TOD

$$-|\sqrt{Q_{i_k^*,j}}e_{i_k^*j}(t_k^j)|^2 \le -\frac{1}{N_j - 1}\sum_{i\neq i_k^*}|\sqrt{Q_{i,j}}e_{ij}(t_k^j)|^2.$$

Denote $\zeta_i = col\{e_{ij}(t_k^j), C_{ij}[x_j(s_k^j) - x_j(s_{k+1}^j)]\}$. Then, employing (12) we arrive at

$$\begin{aligned}\Theta \le & -\left|\sqrt{G_{i_k^*,j}e^{-2\alpha\tau_M^j} - Q_{i_k^*,j}}C_{i_k^*j}[x_j(s_k^j) - x_j(s_{k+1}^j)]\right|^2 \\ & + \sum_{i\neq i_k^*}\Big[\Big|\sqrt{Q_{i,j}}\left[C_{ij}[x_j(s_k^j) - x_j(s_{k+1}^j)] + e_{ij}(t_k^j)\right]\Big|^2 \\ & - \left|\sqrt{Q_{i,j} + \frac{1-2\alpha h_j}{N_j - 1}Q_{i,j} - U_{i,j}}e_{ij}(t_k^j)\right|^2 \\ & - e^{-2\alpha\tau_M^j}|\sqrt{G_{i,j}}C_{ij}[x(s_k^j) - x(s_{k+1}^j)]|^2\Big] \\ = & -\left|\sqrt{G_{i_k^*,j}e^{-2\alpha\tau_M^j} - Q_{i_k^*,j}}C_{i_k^*j}[x_j(s_k^j) - x_j(s_{k+1}^j)]\right|^2 \\ & + \sum_{i\neq i_k^*}\zeta_i^T\Omega_{ij}\zeta_i.\end{aligned}$$

Therefore, under (22) $\Theta \le 0$. □

By applying Lemma 1 and by modifying derivations of Liu et al. (2015) for $F_{lj} \neq 0$ we arrive at.

Proposition 2. *Consider the jth hybrid subsystem (11)–(12). Given tuning parameters $\alpha > \varepsilon > 0$ and $(M-1)$ $n_l \times n_l$ $(l \neq j)$ matrices $P_l > 0$, let there exist a $n_j \times n_j$ matrix $P_j > 0$, $n_y^j \times n_y^j$ matrices $R_{0j} > 0, R_{1j} > 0, S_{0j} > 0, S_{1j} > 0, W_j$, and $n_i^j \times n_i^j$ matrices $Q_{i,j} > 0, U_{i,j} > 0, G_{i,j} > 0$ $(i = 1\dots N_j)$ that satisfy the LMIs (22) and*

$$\Gamma_j = \begin{bmatrix} R_{1j} & W_j \\ W_j^T & R_{1j}\end{bmatrix} \ge 0, \quad \Sigma_j^i < 0, \ i = 1\dots N_j, \tag{25}$$

where

$$\Sigma_j^i = \begin{bmatrix}\phi_{ij} & D_2^TC_j^TH_j \\ * & -H_j\end{bmatrix},$$

$$\begin{aligned}\phi_{ij} = & D_1^T(2\alpha P_j + C_j^TS_{0j}C_j)D_1 + \left(D_2^TP_jD_1 + D_1^TP_jD_2\right) \\ & - \rho_m D_3^T(S_{0j} - S_{1j})D_3 - \rho_m D_4^TR_{0j}D_4 - \rho D_5^TS_{1j}D_5 \\ & - \rho D_6^T\Gamma_jD_6 + D_7^T\Psi_i^jD_7 + D_8^T\Pi_jD_8,\end{aligned}$$

$$H_j = \eta_m^2R_{0,j} + (\tau_M - \eta_m)^2R_{1,j} + h_j\cdot diag\{G_{1,j},\dots,G_{N_j,j}\},$$

$$\Psi_j^i = diag\{\psi_1^j,\dots,\psi_{r\neq i}^j,\dots\}, \qquad \psi_r^j \triangleq 2\alpha Q_{r,j} - \frac{1}{h_j}U_{r,j},$$

$$\Pi_j = \frac{-2\varepsilon}{M-1}diag\{P_l, l\neq j\}, \qquad \rho_m = e^{-2\alpha\eta_m^j}, \rho = e^{-2\alpha\tau_M^j},$$

$$D_1 = [I_{n^j}\ 0_{n^j\times(4n_y^j - n_i^j + n - n^j)}], \qquad D_2 = [A_j\ [0\ 1\ 0]\otimes A_{1j}\ B_jK_j^i\ \mathcal{F}_j],$$

$$K_j^i = row_{\,r=1,\dots,N_j}\{K_{rj},\ r\neq i\}, \qquad \mathcal{F}_j = row_{\,l=1,\dots,M}\{F_{1j}, l\neq j\},$$

$$D_3 = \left[0_{n_y^j\times n^j} \quad [1\ 0\ 0]\otimes I_{n_y^j} \quad Z_{ij}\right],$$

$$D_4 = \left[C_j \quad [-1\ 0\ 0]\otimes I_{n_y^j} \quad Z_{ij}\right],$$

$$D_5 = \left[0_{n_y^j\times n^j} \quad [0\ 0\ 1]\otimes I_{n_y^j} \quad Z_{ij}\right], \qquad D_7 = \left[0\ 0\ 0\ 0\ I_{n_y^j - n_i^j}\ 0\right],$$

$$D_6 = \left[0_{2n_y^j\times n^j} \quad \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}\otimes I_{n_y^j} \quad \begin{bmatrix}1\\1\end{bmatrix}\otimes Z_{ij}\right],$$

$$D_8 = \left[0\ I_{n-n^j}\right],$$

$$Z_{ij} \triangleq 0_{n_y^j\times(n_y^j - n_i^j + n - n^j)}.$$

Then the Lyapunov functional $V_j^e(t)$ given by (18) is positive (i.e. (23)) holds), it does not grow in the jumps (i.e. (24) holds) and satisfies the following inequality along (11):

$$\dot V_j^e(t) + 2\alpha V_j^e(t) \le \frac{2\varepsilon}{M-1}\sum_{l\neq j}x_l^T(t)P_lx_l(t)$$
$$t\in[t_k^j, t_{k+1}^j),\ k\in\mathbb{Z}_+. \tag{26}$$

Proof. Denote

$$\zeta_j(t) = \left[x_j^T(t), \left[x_j(t-\eta_m^j) \quad x_j(t-\tau_{1j}(t)) \quad x_j(t-\tau_M^j)\right]^T C_j^T\right]^T,$$
$$\xi_{ij}(t) = col_{\,i=1,\dots,N_j}\{e_{rj}(t), r\neq i\},$$
$$X_j(t) = col_{\,l=1,\dots,M}\{x_l(t), l\neq j\}.$$

By using arguments of Proposition 1, where

$$\dot x_j^T(t) = \left[\zeta_j^T(t) \quad \xi_{i_k^*j}^T(t) \quad X_j^T(t)\right]D_2^T,$$

we arrive at

$$\dot V_j^e(t) + 2\alpha V_j^e(t) - \frac{2\varepsilon}{M-1}\sum_{l\neq j}x_l^T(t)P_lx_l(t)$$
$$\le \left[\zeta_j^T(t) \quad \xi_{i_k^*j}^T(t) \quad X_j^T(t)\right]\Phi_{i_k^*j}\begin{bmatrix}\zeta_j(t)\\ \xi_{i_k^*j}(t)\\ X_j(t)\end{bmatrix} + \dot x_j^T(t)C_j^TH_jC_j\dot x_j(t),$$
$$t\in[t_k^j, t_{k+1}^j),\ k\in\mathbb{Z}_+.$$

Then, by Schur's complement, (25) implies (26). □

Remark 6. Under **A1**, for $F_{lj} = 0$ the LMIs (22) and (25) are feasible (see Remark 3.3 of Liu et al., 2015). Then for given $\varepsilon > 0$ and small enough F_{lj} the above LMIs are feasible.

4. Decentralized networked control

Consider the decentralized NCS given by (1) where every plant Pj is controlled over a communication network and is either under RR or under TOD scheduling protocols. The controllers u_j are given by (7) and (10) respectively. We are in a position to formulate the main result:

Theorem 1. *Given tuning parameters $\alpha > \varepsilon > 0$, let there exist $n_j \times n_j$ matrices $P_j > 0$ $(j\in J)$, $n_i^j \times n_i^j$ matrices $R_{0i,j} > 0, R_{1i,j} > 0, S_{0i,j} > 0, S_{1i,j} > 0$ and $W_{i,j}$ $(i = 1\dots N_j, j \in J_{RR})$ that satisfy the LMIs (15) and (16) for all $j\in J_{RR}$, and $n_y^j \times n_y^j$ matrices $R_{0j} > 0, R_{1j} > 0, S_{0j} > 0, S_{1j} > 0, W_j$, and $n_i^j \times n_i^j$ matrices $Q_{i,j} > 0, U_{i,j} > 0, G_{i,j} > 0$ $(i = 1\dots N_j, j\in J_{TOD})$ that satisfy the LMIs (22) and (25) for all $j \in J_{TOD}$. Then the closed-loop large-scale system (1), (3) is exponentially stable with a decay rate $\alpha_0 = \alpha - \varepsilon$.*

Proof. Let the LMIs of the theorem be feasible. We choose the following Lyapunov functional for the large-scale system (1), (3):

$$V(t) = \sum_{j\in J_{RR}}V_j(t) + \sum_{j\in J_{TOD}}V_j^e(t), \quad t\ge 0,$$

where $\{V_j(t)\}_{j\in J_{RR}}$ is given by (14) and $\{V_j^e(t)\}_{j\in J_{TOD}}$ is given by (18). Define $x(t) = x(0)$ for $t < 0$ and denote

$$\Delta_{TOD} = \{t\ge 0|\ t = t_k^j,\ j\in J_{TOD}, k\in\mathbb{Z}_+\}.$$

Let T be given by (4).

We apply further Propositions 1 and 2. Then for some constants $0 < \beta_m < \beta_M$ V satisfies the following bounds:

$$\beta_m\Big[|x(t)|^2 + \sum_{j\in J_{TOD}}|e_j(t)|^2\Big]$$
$$\le V(t) \le \beta_M\Big[\|x_t\|^2_{W[-\tau_M,0]} + \sum_{j\in J_{TOD}}|e_j(t)|^2\Big], \tag{27}$$

where $\tau_M = \max_{j\in J} \tau_M^j$. Moreover, by summing in $j = 1, \dots, M$ the inequalities (17) and (26) we obtain that

$$\dot V(t) + 2\alpha V(t) \le 2\varepsilon \sum_{l=1}^{M} x_l^T(t) P_l x_l(t)$$

for all $t \ge T$ and $t \notin \Delta_{TOD}$, implying

$$\dot V(t) + 2(\alpha - \varepsilon)V(t) \le 0, \quad \forall t \ge T,\ t \notin \Delta_{TOD}. \tag{28}$$

Additionally we have

$$V(t) - V(t^-) \le 0, \quad \forall t \ge T, \quad t \in \Delta_{TOD} \tag{29}$$

along (1), (3). The inequalities (28) and (29) yield

$$V(t) \le e^{-2\alpha_0(t-T)} \cdot V(T), \quad t \ge T. \tag{30}$$

Then from (27), (30) for some $\gamma > 0$ we have

$$|x(t)|^2 + \sum_{j\in J_{TOD}} |e_j(t)|^2 \le \gamma e^{-2\alpha_0(t-T)} \Big[\|x_T\|^2_{W[-T,0]} + \sum_{j\in J_{TOD}} |e_j(T)|^2 \Big], \quad t \ge T. \tag{31}$$

We will show next that for some $\gamma_0 > 0$

$$\sum_{j\in J_{TOD}} |e_j(T)|^2 \le \gamma_0 \|x_T\|^2_{W[-T,0]}. \tag{32}$$

Indeed, from (24) and (26) we obtain that for some $\gamma_1 > 0$ the following holds for $t \in [t_k^j, t_{k+1}^j)$:

$$\begin{aligned} V_e^j(t) &\le e^{-2\alpha(t-t_k^j)} V_e^j(t_k^j) + \gamma_1 \int_{t_k^j}^{t} e^{-2\alpha(t-s)} |x(s)|^2 ds \\ &\le \cdots \le e^{-2\alpha(t-t_0^j)} V_e^j(t_0^j) + \gamma_1 \int_{t_0^j}^{t} e^{-2\alpha(t-s)} |x(s)|^2 ds. \end{aligned} \tag{33}$$

Taking into account the initial conditions (13) and $x(t) = x(0)$ for $t < 0$ we arrive at $V_e^j(t_0^j) \le \beta_{jM} \|x(t_0^j + \cdot)\|_{W[-t_0^j,0]}$ with some $\beta_{jM} > 0$. Moreover, $\beta_{jm}|e^j(t)|^2 \le V_e^j(t)$ for some $\beta_{jm} > 0$ that together with (33) yield (32). The inequalities (31) and (32) imply (5) with $c = \sqrt{\gamma + \gamma_0}$. □

Remark 7. The inequalities (31), (32) imply the exponentially converging bound on the errors $e_j(t), j \in J_{TOD}$ meaning the exponential stability of the large-scale hybrid system given by (11)–(12) for $j \in J_{TOD}$ and by (8) for $j \in J_{RR}$.

Example 2 (*Borgers & Heemels, 2014*). Consider two coupled inverted pendulums under the scenario of decentralized networked control, where $M = 2$, $N_j = 2$ or $N_j = 4$ $(j = 1, 2)$. The system matrices are given by

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2.9156 & 0 & -0.0005 & 0 \\ 0 & 0 & 0 & 1 \\ -1.6663 & 0 & 0.0002 & 0 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ -0.0042 \\ 0 \\ 0.0167 \end{bmatrix},$$

$$F_{12} = F_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0011 & 0 & 0.0005 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0003 & 0 & -0.0002 & 0 \end{bmatrix}$$

Table 3
Example 2: max. τ_M^j for $\eta_m^j = 0$ ($\tau_M^j(RR) = N_j \cdot MATI_j + MAD_j$, $\tau_M^j(TOD) = MATI_j + MAD_j$).

$\mathbb{N}$	2		4	
	τ_M^1	τ_M^2	τ_M^1	τ_M^2
Theorem 1 (RR)	0.0209	0.0074	0.0202	0.0073
Theorem 1 (TOD)	0.01	0.0039	0.0029	0.001

Table 4
Example 2: max. $MATI_j$ for $MAD_j = \eta_m^j = 0$ $(j = 1, 2)$.

$\mathbb{N}$	2		4	
	$MATI_1$	$MATI_2$	$MATI_1$	$MATI_2$
Borgers and Heemels (2014) TOD	both $< 2 \cdot 10^{-6}$		both $< 1 \cdot 10^{-6}$	
Theorem 1 (RR)	0.0104	0.0037	0.005	0.0018
Theorem 1 (TOD)	0.01	0.0039	0.0029	0.001

Table 5
Example 2: max. $MATI_j$ for $MAD_j = 0.005$, $\eta_m^j = 0$.

$\mathbb{N}$	2		4	
	$MATI_1$	$MATI_2$	$MATI_1$	$MATI_2$
Theorem 1 (RR)	0.0079	0.0012	0.0038	0.00057
Theorem 1 (TOD)	0.095	–	–	–

$$K_1 = [k_{11}\ k_{21}\ k_{31}\ k_{41}] = [11\,396\ 7196.2\ 573.96\ 1199.0],$$
$$K_2 = [k_{12}\ k_{22}\ k_{32}\ k_{42}] = [29\,241\ 18\,135\ 2875.3\ 3693.9].$$

In the case of $N_j = 2$ we consider

$$C_{1j} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad C_{2j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$K_{1j} = [k_{1j}\ k_{2j}], \qquad K_{2j} = [k_{3j}\ k_{4j}], \quad j = 1, 2.$$

In the case of $N_j = 4$, $C_{1j}, \dots, C_{4j}$ are the rows of I_4 and $K_{1j}, \dots, K_{4j}$ are the entries of K_j.

We analyze the exponential stability for $\eta_m^j = 0$ by applying LMI conditions of Theorem 1 with $\alpha = 0.015$ and $\varepsilon = 0.002$ for the case where both pendulums are either under RR or under TOD protocols (the resulting decay rate α_0 is 0.013). Maximum values of τ_M^j that preserve the stability are given in Table 3. Then for $MAD_j = 0$ and $MAD_j = 0.005$ ($MAD_j = 0.005$ is larger than max $MATI_j$ achieved in Borgers and Heemels (2014)) the resulting maximum $MATI_j$ that preserve the stability are given in Tables 4 and 5 respectively.

It is seen that the presented method leads to essentially larger values of maximum $MATI_j$ comparatively to Borgers and Heemels (2014) and allows large values of MAD_j. Moreover, our method is applicable in this example with a much stronger coupling. Thus, for $F_{12} = F_{21} = 40 \cdot A_{12}$ by Theorem 1 the stability is preserved e.g. for $MATI_j = MAD_j = 0.001 (j = 1, 2)$ (either under RR or under TOD protocols).

5. Conclusions

In this paper, a time-delay approach has been developed for the decentralized exponential stabilization of large-scale NCSs with local networks, where asynchronous variable sampling intervals, large bounded variable communication delays and RR/TOD scheduling protocols are taken into account. The presented novel Lyapunov–Krasovskii method leads to LMI conditions for the exponential stability of the closed-loop large-scale system. Being applied to the example of two coupled pendulums with local networks, our results are favorably compared to the existing ones. The presented new technique may be useful for decentralized control of microgrids with islanded generators. Future work may involve consideration of stochastic communication delays and observer-based networked control. The presented approach may be useful for asynchronous decentralized control in microgrids (Vasquez et al., 2010).

References

Antsaklis, P., & Baillieul, J. (2007). Special issue on technology of networked control systems. *Proceedings of the IEEE*, *95*(1), 5–8.

Borgers, D., & Heemels, W. (2014). Stability analysis of large-scale networked control systems with local networks: A hybrid small-gain approach. In *CST report, 2014.025*.

Fridman, E. (2014). *Systems and control: Foundations and applications*, *Introduction to time-delay systems: Analysis and control*. Birkhauser.

Fridman, E., Seuret, A., & Richard, J. P. (2004). Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, *40*(8), 1441–1446.

Gao, H., Chen, T., & Lam, J. (2008). A new delay system approach to network-based control. *Automatica*, *44*(1), 39–52.

Geromel, J., Korogui, R., & Bernussou, J. (2007). H_2 and H_∞ robust output feedback control for continuous time polytopic systems. *IET Control Theory & Applications*, *1*(5), 1541–1549.

Gu, K., Kharitonov, V., & Chen, J. (2003). *Stability of time-delay systems*. Boston: Birkhauser.

Heemels, W., Borgers, D., van de Wouw, N., Nesic, D., & Teel, A. 2013. Stability analysis of nonlinear networked control systems with asynchronous communication: a small-gain approach. In *Proceedings of the 52th IEEE conference on decision and control*.

Heemels, W., Teel, A., van de Wouw, N., & Nesic, D. (2010). Networked control systems with communication constraints: tradeoffs between transmission intervals, delays and performance. *IEEE Transactions on Automatic Control*, *55*(8), 1781–1796.

Liu, K., Fridman, E., & Hetel, L. (2012). Stability and L_2-gain analysis of networked control systems under Round–Robin scheduling: a time-delay approach. *Systems & Control Letters*, *61*(5), 666–675.

Liu, K., Fridman, E., & Hetel, L. (2015). Networked control systems in the presence of scheduling protocols and communication delays. *SIAM Journal on Control and Optimization*, *53*(4), 1768–1788.

Lunze, J. (1992). *Feedback control of large scale systems*. Prentice Hall PTR.

Naghshtabrizi, P., Hespanha, J. P., & Teel, A. R. (2010). Stability of delay impulsive systems with application to networked control systems. *Transactions of the Institute of Measurement and Control*, *32*(5), 511–528.

Nesic, D., & Teel, A. R. (2004). Input–output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, *49*(10), 1650–1667.

Park, P. G., Ko, J., & Jeong, C. (2011). Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, *47*, 235–238.

Ugrinovskii, V., & Fridman, E. (2014). A Round–Robin type protocol for distributed estimation with H_∞ consensus. *Systems & Control Letters*, *69*, 103–110.

Vasquez, J. C., Guerrero, J. M., Miret, J., Castilla, M., Vicuna, D., & García, L. (2010). Hierarchical control of intelligent microgrids. *IEEE Industrial Electronics Magazine*, *4*(4), 23–29.

Walsh, G. C., Ye, H., & Bushnell, L. G. (2002). Stability analysis of networked control systems. *IEEE Transactions on Control Systems Technology*, *10*(3), 438–446.

Dror Freirich received his B.Sc. in Computers Engineering from the Technion (IIT). He is finishing his studies towards an M.Sc. degree in Electrical Engineering at Tel Aviv University, Israel.

His research interests include stability and control of time-delay and networked control systems, as well as machine-learning applications.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics.

From 1986 to 1992 she was an assistant and associate professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), and KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control.

She has published more than 100 articles in international scientific journals.

She is the author of the monograph "Introduction to Time-Delay Systems: Analysis and Control" (Birkhauser, 2014).

In 2014 she was nominated as a Highly Cited Researcher by Thomson ISI. Currently she serves as an associate editor in Automatica and SIAM Journal on Control and Optimization.