OUTPUT REGULATION OF NONLINEAR NEUTRAL SYSTEMS

Emilia Fridman[1]

Department of Electrical Engineering, Tel-Aviv University
Ramat-Aviv, Tel-Aviv 69978, Israel
emilia@eng.tau.ac.il

Summary. Output regulation of neutral type nonlinear systems is considered. Regulator equations are derived, which generalize Francis-Byrnes-Isidori equations to the case of neutral systems. It is shown that, under standard assumptions, the regulator problem is solvable if and only if these equations are solvable. In the linear case, the solution of these equations is reduced to linear matrix equations.

Keywords: time-delay systems, nonlinear systems, output regulation, regulator equations, center manifold

1 Introduction

One of the most important problems in control theory is that of controlling the output of the system so as to achieve asymptotic tracking of prescribed trajectories. This problem of output regulation has been studied by many authors (see e.g. a survey paper by Byrnes and Isidori [2] and the references therein). In the linear case, Francis [4] showed that the solvability of a multivariable regulator problem corresponds to the solvability of a system of two linear matrix equations. In the nonlinear case, Isidori and Byrnes [11] proved that the solvability of the output regulation problem is equivalent to the solvability of a set of partial differential and algebraic equations. This set of partial differential and algebraic equations is now known as the *regulator equations* or *Francis-Isidori-Byrnes equations*.

For linear *infinite-dimensional* control systems a solution of the regulator problem was introduced by Schumacher [13] and Byrnes *et al.* [3], where a *Hilbert* space was used as a state space. The case of the bounded input and output operators was considered. In the case of systems with time-delay it means that there are *no discrete delays* in the *control input, controller output* and *measured output*. The solution was given in terms of the operator regulator equations.

The solution of the output regulation problem for retarded type systems was obtained recently in [6], where a *Banach* space was used as a state-space. In the present paper we generalize the results of [6] to the neutral type case. Our solution is based on the application of the center manifold theory. The existence, smoothness and the attractiveness of the center manifold for neutral type systems were proved by Hale [8] (see also [9], chapter 10.2). A partial differential equation for the function, determining the center manifold for such system was derived in [14], [5], [1]. In the present paper, we consider output regulation of *nonlinear* systems with *state, controller output* and *measured output delays*. As for the systems of retarded type [6], the problem is solvable iff certain regulator equations are solvable. These equations consist of partial differential equations for a center manifold of the closed-loop neutral system and of an algebraic equation. In the linear case the solution of these equations is reduced to linear matrix equations.

Notations. R^m is the Euclidean space with the norm $|\cdot|$ and $C^m[a,b]$ is the Banach space of continuous functions $\phi : [a,b] \to R^m$ with the supremum norm $||\cdot||$.

A function $f : X \to Y$, where X and Y are Banach spaces, is a C^k function if it has k continuous Frechet derivatives.

Denote by $x_t(\theta) = x(t+\theta)$ $(\theta \in [-h;0])$.

$L_2([-h,0],R^n)$ is the Hilbert space of square integrable R^n valued functions with the corresponding norm.

$W^{1,2}([-h,0],R^n)$ is the Sobolev space of absolutely continuous R^n valued functions on $[-h,0]$ with square integrable derivatives.

The transpose of a matrix M is written M'.

2 Problem Formulation

We consider a nonlinear system modelled by equations of the form

$$\frac{d}{dt}Dx_t = f(x_t, u(t), w(t)), \quad e(t) = g(x_t, w(t)) \tag{1a,b}$$

where $x(\theta) = \phi(\theta), \theta \in [-h,0]$, with state $x(t) \in R^n$, initial function $\phi \in C^n[-h,0]$, control input $u(t) \in R^m$, exogenous input $w(t) \in R^r$ and tracking error $e(t) \in R^p$. The linear bounded operator $D : C^n[-h,0] \to R^n$ is represented in the form of Stieltjes integral [9]:

$$D\phi = \phi(0) - \int_{-h}^{0} d[\xi(\theta)]\phi(\theta),$$

with $n \times n$-matrix function ξ of bounded variations.

We assume

H0: The following conditions hold:

(i) ξ is nonatomic at zero, i.e. $Var_{[-s,0]}\xi(\cdot) \to 0$ for $t \to 0$;

(ii) D is the stable operator, i.e. the equation $Dx_t = 0$ is asymptotically stable.

The exogenous input is generated by an autonomous dynamical system of the form

$$\dot{w}(t) = s(w(t)) \tag{2}$$

The functions $f : V \to R^n$, $s : W \to R^r$, $g : Y \to R^p$ are smooth (i.e. C^∞) mappings, where $V \subset C^n[-h, 0] \times R^m \times R^r$, $W \subset R^r$, $Y \subset C^n[-h, 0] \times R^r$ are some neighborhoods of the origin of the corresponding spaces. We assume that $f(0, 0, 0) = 0$, $s(0) = 0$, $g(0, 0) = 0$. Thus, for $u = 0$, the system (1a) has an equilibrium state $(x, w) = (0, 0)$ with zero error (1b).

A solution of (1) with initial value $x_0 \in C^n[-h, 0]$ is a continuous function taking $[-h, A), A > 0$ into R^n such that $D(x_t)$ is continuously differentiable and satisfies (1) for $t \in (0, A)$. Assumption H0 (i) guarantees the existence and the uniqueness of the solution to initial value problem for (1), where $u(t)$ and $w(t)$ are continuous functions [9]. Assumption H0 (ii) guarantees that the characteristic equation corresponding to the linear system

$$\frac{d}{dt} Dx_t = Lx_t,$$

where $L : C^n[-h, 0] \to R^n$ is a linear bounded operator, has a finite number of roots with nonnegative real part.

We consider both, a state-feedback and an error-feedback regulator problems.

Problem 1 (State-Feedback Regulator Problem): Find a state-feedback control law

$$u(t) = \alpha(x_t, w(t)), \tag{3}$$

where $\alpha : Y \to R^m$ is a $C^k (k \geq 2)$ function and $\alpha(0, 0) = 0$ such that :

1a) the equilibrium $x(t) \equiv 0$ of

$$\frac{d}{dt} Dx_t = f(x_t, \alpha(x_t, 0), 0),$$

is exponentially stable;

1b) there exists a neighborhood $Y \subset C^n[-h, 0] \times W$ of the origin such that, the solution of the closed-loop system

$$\frac{d}{dt} Dx_t = f(x_t, \alpha(x_t, w(t)), w(t)), \quad \dot{w}(t) = s(w(t)) \tag{4}$$

satisfies

$$\lim_{t \to \infty} g(x_t, w(t)) = 0. \tag{5}$$

Problem 2 (Error-Feedback Regulator Problem): Find an error-feedback controller

$$u = \Theta(z_t), \quad \frac{d}{dt}\bar{D}z_t = \eta(z_t, e(t)), \quad z(t) \in R^{\nu} \tag{6}$$

with C^k functions $\eta : Z_0 \to R^{\nu}$ and $\Theta : Z_1 \to R^m$, where $Z_0 \subset C^{\nu}[-h,0] \times R^p$, $Z_1 \subset C^{\nu}[-h,0]$ are some neighborhoods of the origin, such that:

2a) the equilibrium $(x(t), z(t)) \equiv 0$ of

$$\frac{d}{dt}Dx_t = f(x_t, \Theta(z_t), 0), \quad \frac{d}{dt}\bar{D}z_t = \eta(z_t, g(x_t, 0))$$

is exponentially stable;

2b) there exists a neighborhood $Z \subset C^n[-h,0] \times C^{\nu}[-h,0] \times W$ of the origin such that, the solution of the closed-loop system

$$\tfrac{d}{dt}Dx_t = f(x_t, \Theta(z_t), w(t)), \quad \tfrac{d}{dt}\bar{D}z_t = \eta(z_t, g(x_t, w(t))), \quad \dot{w}(t) = s(w(t)) \tag{7}$$

satisfies (5).

3 Linearized Problem and Assumptions

Using Taylor expansion in the neighborhood of the origin of the Banach space $C^n[-h,0] \times R^m \times R^r$, we obtain the following approximation of the smooth function f:

$$f(x_0, u, w) = \mathrm{A}x_0 + Bu + Pw + O(x_0, u, w)^2,$$

where the linear bounded operator $[\mathrm{A},\ B,\ P] : C^n[-h,0] \times R^m \times R^r \to R^n$ is a Frechet derivative of f at the origin. The function $O(\cdot)^2$ vanishes at the origin with its first-order Frechet derivative. Similarly, smooth functions g, α, Θ and η can be represented in the form

$$\begin{aligned} &g(x_0, w) = \mathrm{C}x_0 + Qw + O(x_0, w)^2, \\ &\alpha(x_0, w) = \mathrm{K}x_0 + Lw(t) + O(x_0, w)^2, \\ &\Theta(z_0) = \mathrm{H}z_0 + O(z_0)^2, \quad \eta(z_0, e) = \mathrm{F}z_0 + Ge + O(z_0, e)^2, \end{aligned}$$

where the functions $O(\cdot)^2$ vanish at the origin with their first-order Frechet derivatives. The linear bounded operators $\mathrm{A} : C^n[-h,0] \to R^n$ and $\mathrm{C} : C^n[-h,0] \to R^p$ by Riesz theorem can be represented in the form of Stieltjes integrals [9]:

$$\mathrm{A}\phi = \int_{-h}^{0} d[\mu(\theta)]\phi(\theta), \quad \mathrm{C}\phi = \int_{-h}^{0} d[\zeta(\theta)]\phi(\theta), \tag{8}$$

with $n \times n$ and $p \times n$-matrix functions μ and ζ of bounded variations. A similar representation can be written for the linear bounded operators $\mathrm{K} : C^n[-h,0] \to R^m$, $\mathrm{H} : C^{\nu}[-h,0] \to R^m$ and $\mathrm{F} : C^{\nu}[-h,0] \to R^{\nu}$.

The linearized system is given by

$$\tfrac{d}{dt}Dx_t = \mathrm{A}x_t + Bu(t) + Pw(t), \quad \dot{w}(t) = Sw(t), \quad e(t) = \mathrm{C}x_t + Qw(t). \tag{9a-c}$$

The linearized state-feedback and error-feedback controllers have the form

$$u(t) = \mathrm{K}x_t + Lw(t) \tag{10}$$

and

$$u(t) = \mathrm{H}z_t, \quad \frac{d}{dt}\bar{D}z_t = \mathrm{F}z_t + Ge(t). \tag{11}$$

respectively.

Similarly to the case without delay [11] we assume the following:

H1. The exosystem (2) is neutrally stable (i.e. Lyapunov stable in forward and backward time, and thus S has all its eigenvalues on the imaginary axis).

H2. The triple $\{D, \mathrm{A}, B\}$ is stabilizable, i.e. there exists a linear bounded operator $\mathrm{K} : C^n[-h, 0] \to R^m$ such that the system

$$\frac{d}{dt}Dx_t = (\mathrm{A} + B\mathrm{K})x_t \tag{12}$$

is asymptotically stable.

H3. The pair

$$\begin{bmatrix} \mathrm{A} & P \\ 0 & S \end{bmatrix}, \qquad [\mathrm{C} \quad Q]$$

is detectable in the following sense: there exists a $(n + r) \times p$-matrix G such that the system

$$\frac{d}{dt}\begin{bmatrix} D\tilde{z}_{1t} \\ \tilde{z}_2(t) \end{bmatrix} = \left\{ \begin{bmatrix} \mathrm{A} & P \\ 0 & S \end{bmatrix} + G[\mathrm{C} \quad Q] \right\} \begin{bmatrix} \tilde{z}_{1t} \\ \tilde{z}_2(t) \end{bmatrix}, \tag{13}$$

where $\bar{z}_1(t) \in R^n$, $\bar{z}_2(t) \in R^r$, is asymptotically stable.

We note that H2 is equivalent to the following condition [10]:

H2'. $rank \left[\lambda[I - \int_{-h}^{0} d[\xi(\theta)]e^{\lambda\theta}] - \int_{-h}^{0} d[\mu(\theta)]e^{\lambda\theta}, \ B \right] = n$ *for all* $\lambda \in C$ *with* $Re\lambda \geq 0$.

Similar condition equivalent to H3 can be written for the case of $Cx_t = C_0x(t)$, where C_0 is a constant matrix. Some sufficient conditions for H2 and for finding a stabilizing controller $u(t) = K_0x(t)$ or $u(t) = K_1x(t - h)$ may be found e.g. in [7] (see also references therein) in terms of linear matrix inequalities. Similar sufficient conditions may be derived for H3.

4 Solution of the Regulator Problems

4.1 Center manifold of the closed-loop system.

The solution of the output regulation problem is based on the center manifold theory [8], [9].

Lemma 1. *Let H0 hold. Assume that all eigenvalues of S are on the imaginary axis and that for some $\alpha(x_t, w)$ condition 1a) holds. Then the closed-loop system (4) has a local center manifold $x_t(\theta) = \pi(w(t))(\theta),\ \theta \in [-h, 0]$, where $\pi : W_0 \to C^n[-h, 0]$ $(0 \in W_0 \subset W \subset R^r)$ is a C^k mapping with $\pi(0)(\theta) \equiv 0$. The center manifold is locally attractive, i.e. satisfies*

$$||x_t - \pi(w(t))|| \le Me^{-at}||x_0 - \pi(w(0))||, \quad M > 0,\ a > 0 \tag{14}$$

for all $x_0, w(0)$ sufficiently close to 0 *and all* $t \ge 0$.

Proof: The closed-loop system (4) has the form

$$\begin{aligned} &\dot{w}(t) = Sw(t) + O(w(t))^2, \\ &\tfrac{d}{dt}Dx_t = (\mathrm{A} + B\mathrm{K})x_t + (P + BL)w(t) + O(x_t, w(t))^2. \end{aligned} \tag{15a,b}$$

By assumption, the zeros of the characteristic equation corresponding to (12) are in C^-, and the eigenvalues of the matrix S are on the imaginary axis.

It is well-known (see e.g.[8]) that according to this dichotomy, the space $R^r \times C^n[-h, 0]$ of the initial values of the linear system

$$\dot{w}(t) = Sw(t), \quad \frac{d}{dt}Dx_t = (\mathrm{A} + B\mathrm{K})x_t + (P + BL)w(t), \tag{16}$$

can be decomposed as a direct sum $R^r \times C^n[-h, 0] = \mathcal{P} \oplus \mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ are invariant sub-spaces of the solutions of (16), in the sense that for all initial conditions from $\mathcal{P}$ ($\mathcal{Q}$), solutions of (16) satisfy $\{w(t), x_t\} \in \mathcal{P}$ $(\{w(t), x_t\} \in \mathcal{Q})$ for all $t \ge 0$. Moreover, $\mathcal{P}$ is an r-dimensional and corresponds to solutions of (16) of the form $p(t)e^{\lambda t}$, where $p(t)$ is a polynomial in t and λ is an eigenvalue of S. The space $\mathcal{Q}$ corresponds to exponentially decaying solutions of (16). By Theorem 2.1 of [9] (p. 314) the system (15) has a local smooth center manifold $x_0 = \pi(w)$. The flow on this manifold is governed by (15a). By Theorem 2.2 of [9] (p.216) this manifold is locally attractive. □

The function π which determines a center manifold of (4) can be considered as a function of one variable $\pi : W_0 \to C^n[-h, 0]$ in the Banach space or a function of two variables $\pi : W_0 \times [-h, 0] \to R^n$ in the Euclidean space. Further we find relation between the smoothness properties in both considerations by introducing two classes of functions:

Class $\mathcal{M}_1$ of C^1 functions $\pi : W_0 \to C^n[-h, 0] (W_0 \subset R^r)$, satisfying the following conditions:

(i) For each $w \in W_0$ there exists a continuous in $\theta \in [-h, 0]$ partial derivative $\frac{\partial \pi(w)(\theta)}{\partial \theta} \stackrel{\Delta}{=} \gamma(w)(\theta)$;

(ii) The function $\gamma : W_0 \to C^n[-h, 0]$ is continuous.

Class $\mathcal{M}_2$ of functions $\psi : W_0 \to C^n[-h, 0]$ such that the functions $\bar{\psi}(w, \theta) \stackrel{\Delta}{=} \psi(w)(\theta),\ \bar{\psi} : W_0 \times [-h, 0] \to R^n$ are continuously differentiable.

Proposition 1. *[6]* $\mathcal{M}_1 = \mathcal{M}_2$.

Lemma 2. *Assume H0. A* C^1 *mapping* $\pi : W_0 \to C^n[-h, 0],\ \pi(0) = 0$ *defines a center manifold* $x_t(\theta) = \pi(w(t))(\theta),\ \theta \in [-h, 0]$ *of (4) if and only if* $\pi \in \mathcal{M}_1$ *and* $\forall w \in W_0,\ \forall \theta \in [-h, 0]$ *it satisfies the following system of partial differential equations*

$$\begin{aligned} &\tfrac{\partial \pi(w)(\theta)}{\partial w} s(w) = \tfrac{\partial \pi(w)(\theta)}{\partial \theta} \\ &\tfrac{\partial [D\pi(w)]}{\partial w} s(w) = f(\pi(w), \alpha(\pi(w), w), w). \end{aligned} \tag{17a,b}$$

Proof. Note that for a C^1 mapping $\pi : W_0 \to C^n[-h, 0]$ and for $w(t)$, satisfying (2), we find that for each $\theta \in [-h, 0]$

$$\frac{d}{dt}[\pi(w(t))(\theta)] = \frac{\partial \pi(w(t))(\theta)}{\partial w} s(w(t)). \tag{18}$$

Necessity: Let a C^1 mapping $\pi : W_0 \to C^n[-h, 0]$ determine a center manifold of (15). Then there exists $\delta > 0$ such that $x_t(\theta) = \pi(w(t))(\theta)$ satisfies (4) for $t \in [-\delta, \delta]$ and, hence

$$\begin{aligned} &\tfrac{\partial x_t(\theta)}{\partial t} = \tfrac{\partial x_t(\theta)}{\partial \theta}, \quad x_0 = \phi, \quad \theta \in [-h, 0], \quad t \in [-\delta, \delta], \\ &\tfrac{\partial D x_t}{\partial t} = f(x_t, \alpha(x_t, w(t)), w(t)), \quad \dot{w}(t) = s(w(t)). \end{aligned} \tag{19}$$

Substituting $x_t = \pi(w(t)),\ w(0) = w,\ t \in [-\delta, \delta]$ into (19) and setting further $t = 0$, we obtain that for all $w \in W_0$, $\pi(w)(\theta)$ is differentiable in $\theta \in [-h, 0]$ and π satisfies (17). The function $\frac{\partial \pi}{\partial \theta} : W_0 \to C^n[-h, 0]$ is continuous since the left hand side of (17a) has the same property.

Sufficiency: let a C^1 mapping $\pi : W_0 \to C^n[-h, 0]$ satisfy (17). Substitute $w = w(t)$ into (17), where $w(t)$ is a solution of (2), then $x_t = \pi(w(t))$ satisfies (19) (and thus (4)) and therefore π determines the invariant manifold of (4). □

Remark 1. Approximate solution to (17) can be found in a form of series expansions in the powers of w (similarly to [8], [14], [1]).

4.2 State-Feedback Regulator Problem

Applying Lemmas 1 and 2, we obtain regulator equations by using arguments of [11].

Lemma 3. *Under H0 and H1 assume that for some* $\alpha(x_t, w)$ *condition 1a) holds. Then, condition 1b) is also fulfilled iff there exists a* $C^k (k \geq 2)$ *mapping* $\pi : W_0 \to C^n[-h, 0],\ \pi(0) = 0$ *satisfying (17) and the algebraic equation*

$$g(\pi(w), w) = 0. \tag{20}$$

Proof is similar to [6].

Theorem 1. *Under H0, H1 and H2, the state-feedback regulator problem is solvable if and only if there exist* $C^k (k \geq 2)$ *mappings* $x_0(\theta) = \pi(w)(\theta)$, *with* $\pi \in \mathcal{M}_1,\ \pi(0)(\theta) = 0$, *and* $u = c(w)$, *with* $c(0) = 0$, *both defined in a neighborhood* $W \subset R^r$ *of the origin, satisfying the conditions* $\forall w \in W_0,\ \forall\theta \in [-h, 0]$

$$\begin{array}{l} \frac{\partial\pi(w)(\theta)}{\partial w}s(w) = \frac{\partial\pi(w)(\theta)}{\partial\theta}, \\ \frac{\partial[D\pi(w)]}{\partial w}s(w) = f(\pi(w), c(w), w), \\ g(\pi(w), w) = 0. \end{array} \tag{21a-c}$$

Suppose that π *and* c *satisfy (21), then the state-feedback*

$$u = \alpha(x_t, w(t)) = c(w(t)) + K[x_t - \pi(w(t))], \tag{22}$$

where K *is a stabilizing gain which is defined in H2, solves the state-feedback regulator problem.*

Proof. The necessity follows immediately from Lemma 3. For the sufficiency consider the state-feedback (22). This choice satisfies 1a), since

$$f(x_t, \alpha(x_t, 0), 0) = (\mathrm{A} + B\mathrm{K})x_t + O(x_t)^2.$$

Moreover, by construction

$$\alpha(\pi(w), w) = c(w)$$

and therefore, (21a), (21b) reduce to (17). From (21c) by Lemma 2 it follows that condition 1b) is also fulfilled. □

4.3 Error-Feedback Regulator Problem

Applying Lemmas 1 and 2 to the system (7), we obtain the following:

Lemma 4. *Let H0 hold. Assume that all eigenvalues of* S *are on the imaginary axis and that for some* $\theta(z_t)$ *and* $\eta(z_t, e)$ *condition 2a) holds. Then*
(i) *the closed-loop system (7) has a local center manifold* $x_t(\theta) = \pi(w(t))(\theta)$, $z_t(\theta) = \sigma(w(t))(\theta)$, *where* $\pi : W_0 \to C^n[-h, 0],\ \sigma : W_0 \to C^\nu[-h, 0]$ $(0 \in W_0 \subset W \subset R^r)$ *are* C^k *mappings with* $\pi(0)(\theta) \equiv 0,\ \sigma(0)(\theta) \equiv 0$*;*
(ii) *the center manifold is locally attractive, i.e. satisfies*

$$||x_t - \pi(w(t))|| + ||z_t - \sigma(w(t))|| \leq Me^{-at}(||x_0 - \pi(w(0))|| + ||z_0 - \sigma(w(0))||), \quad M > 0,\ a > 0 \tag{23}$$

for all $x_0, z_0, w(0)$ *sufficiently close to* 0 *and all* $t \geq 0$.
(iii) C^1 *mappings* $\pi : W_0 \to C^n[-h, 0],\ \pi(0)(\theta) = 0,\ \sigma : W_0 \to C^\nu[-h, 0]$, $\sigma(0)(\theta) = 0$ *define a center manifold* $x_t(\theta) = \pi(w(t))(\theta),\ z_t(\theta) = \sigma(w(t))(\theta),\ \theta \in [-h, 0]$ *of (7) if and only if* $\pi : W_0 \times [-h, 0] \to R^n,\ \sigma : W_0 \times [-h, 0] \to R^\nu$ *are continuously differentiable functions and* $\forall w \in W_0,\ \forall\theta \in [-h, 0]$ *they satisfy the following system of partial differential equations*

$$\begin{array}{ll}\frac{\partial\pi(w)(\theta)}{\partial w}s(w)=\frac{\partial\pi(w)(\theta)}{\partial\theta}, & \frac{\partial\sigma(w)(\theta)}{\partial w}s(w)=\frac{\partial\sigma(w)(\theta)}{\partial\theta},\\ \frac{\partial[D\pi(w)]}{\partial w}s(w)=f(\pi(w),\theta(\sigma(w)),w), & \frac{\partial[\bar{D}\sigma(w)]}{\partial w}s(w)=\eta(\sigma(w),0).\end{array} \tag{24a-d}$$

Remark 2. In the case when $z(t)=col\{z_1(t),z_2(t)\}$, where z_2 appears in (7) without delay and thus $col\{z_{1t}(\theta),z_2(t)\}=col\{\sigma_1(w(t))(\theta),\ \sigma_2(w(t))\}$, (24b) holds only for $\sigma=\sigma_1$.

Similarly to Lemma 3, the following lemma can be proved

Lemma 5. *Under H0 and H1, assume that for some $\Theta(z_t)$ and $\eta(z_t,e)$ condition 2a) holds. Then, condition 2b) is also fulfilled iff there exist $C^k(k\geq 2)$ mappings $\pi:W_0\to C^n[-h,0]$, $\pi(0)=0$, $\sigma:W_0\to C^\nu[-h,0]$, $\sigma(0)=0$ satisfying (24) and the algebraic equation (20).*

From the latter lemmas we deduce a necessary and sufficient condition for the solvability of the error-feedback regulator problem

Theorem 2. *Under H0-H3, the error-feedback regulator problem is solvable if and only if there exist $C^k(k\geq 2)$ mappings $x_0(\theta)=\pi(w)(\theta)$, with $\pi\in\mathcal{M}_1$, $\pi(0)(\theta)=0$, and $u=c(w)$, with $c(0)=0$, both defined in a neighborhood $W\subset R^r$ of the origin, satisfying the conditions (21) $\forall w\in W,\ \forall\theta\in[-h,0]$.*

Suppose that π and c satisfy (21), and that a linear bounded operator H $:C^n[-h,0]\to R^m$ is such that the system

$$\frac{d}{dt}Dx_t=(A+BH)x_t \tag{25}$$

is asymptotically stable. Then the error-feedback (6), where

$$\begin{array}{l}z(t)=col\{z_1(t),z_2(t)\},\ \eta=col\{\eta_1,\eta_2\},\ \bar{D}=diag\{D,I\},\\ u=\Theta(z_t)=c(z_2(t))+H[z_{1t}-\pi(z_2(t))],\\ \eta_1(z_{1t},z_2(t),e(t))=f(z_{1t},\Theta(z_t),z_2(t))-G_1(h(z_{1t},z_2(t))-e(t)),\\ \eta_2(z_{1t},z_2(t),e(t))=s(z_2(t))-G_2(h(z_{1t},z_2(t))-e(t)),\end{array} \tag{26}$$

and where $G=col\{G_1,G_2\}$ is defined in H3, solves the regulator problem.

Proof. The necessity follows immediately from Lemma 5. For the sufficiency we note, that there exist a linear bounded operator H $:C^\nu[-h,0]\to R^m$ and a matrix $G=col\{G_1,G_2\}$ such that (25) and (13) are asymptotically stable. A standard calculation shows that for any $m\times r$-matrix K, the characteristic quasipolynomial that corresponds to the system

$$\begin{bmatrix}\frac{d}{dt}Dx_t\\ \frac{d}{dt}\bar{D}z_{1t}\\ \dot{z}_2(t)\end{bmatrix}=\begin{bmatrix}\mathrm{A} & B\mathrm{H} & BK\\ G_1\mathrm{C} & \mathrm{A}+B\mathrm{H}-G_1\mathrm{C} & P+BK-G_1Q\\ G_2\mathrm{C} & -G_2\mathrm{C} & S-G_2Q\end{bmatrix}\begin{bmatrix}x_t\\ z_{1t}\\ z_2(t)\end{bmatrix} \tag{27}$$

is equal to the product of the characteristic quasipolynomials that correspond to (25) and (13) respectively. Therefore, (27) is asymptotically stable.

Consider the error-feedback controller of (6), (26). The linearized system corresponding to the closed-loop system (7) has exactly the form of (27), where

$$K = \left[\frac{\partial c}{\partial w}\right]_{w=0} - \mathrm{H}\left[\frac{\partial \pi}{\partial w}\right]_{w=0}.$$

Thus requirement 2a) is satisfied. By construction $z_2(t)$ appears in (7) without delay and thus (21a)-(21b) imply (24) with $\sigma(w) = col\{\sigma_1(w), \sigma_2(w)\} = col\{\pi(w), w\}$, where in (24b) $\sigma = \sigma_1$. Thus requirement 2b) follows from Lemma 5. □

5 Linear Case.

5.1 Linear Regulator equations.

Consider the linear regulator problem (9). In the linear case the center manifold has a form $x_t = \Pi(\theta)w(t)$, where Π is an $n \times r$ matrix function continuously differentiable in $\theta \in [-h, 0]$. From Theorems 1 and 2 it follows, that the linear problem (9) is solvable iff there exists Π and an $m \times r$-matrix Γ that satisfy the following system

$$\begin{aligned} &\dot{\Pi}(\theta) = \Pi(\theta)S, \quad \theta \in [-h, 0], \\ &(D\Pi)S = \textstyle\int_{-h}^{0} d[\mu(\theta)]\Pi(\theta) + B\Gamma + P, \\ &\textstyle\int_{-h}^{0} d[\zeta(\theta)]\Pi(\theta) + Q = 0. \end{aligned} \tag{28a-c}$$

Eq. (28a) yields $\Pi(\theta) = \Pi(0)\exp S\theta$. Substituting the latter into (28b) and (28c), we obtain the following linear algebraic system for initial value $\Pi(0)$:

$$\begin{aligned} &[\Pi(0) - \textstyle\int_{-h}^{0} d[\xi(\theta)]\Pi(0)e^{S\theta}]S = \int_{-h}^{0} d[\mu(\theta)]\Pi(0)e^{S\theta} + B\Gamma + P, \\ &\textstyle\int_{-h}^{0} d[\zeta(\theta)]\Pi(0)e^{S\theta} + Q = 0. \end{aligned} \tag{29}$$

The latter system is a generalization of Francis equations [4] to the case of neutral systems.

We consider now a particular, but important in applications case of (9) with

$$\begin{aligned} Dx_t &= x(t) - \textstyle\sum_{i=1}^{k} D_i x(t-h_i) - \int_{-h}^{0} D_d(\theta)x(t+\theta)d\theta, \\ \mathrm{A}x_t &= \textstyle\sum_{i=0}^{k} A_i x(t-h_i) + \int_{-h}^{0} A_d(\theta)x(t+\theta)d\theta, \\ \mathrm{C}x_t &= \textstyle\sum_{i=0}^{k} C_i x(t-h_i) + \int_{-h}^{0} C_d(\theta)x(t+\theta)d\theta, \end{aligned} \tag{30}$$

where $0 = h_0 < h_1 < \ldots < h_k \le h$, D_d, A_d and C_d are piecewise continuous matrix functions and where D_i, A_i and C_i are constant matrices of the appropriate dimensions. In this case (29) has the form:

$$\begin{array}{l}[\Pi(0) - \sum_{i=1}^{k} D_i\Pi(0)e^{-Sh_i} - \int_{-h}^{0} D_d(\theta)\Pi(0)e^{S\theta}d\theta]S = \sum_{i=0}^{k} A_i\Pi(0)e^{-Sh_i} \\ + \int_{-h}^{0} A_d(\theta)\Pi(0)e^{S\theta}d\theta + B\Gamma + P, \\ \sum_{i=0}^{k} C_i\Pi(0)e^{-Sh_i} + \int_{-h}^{0} C_d(\theta)\Pi(0)e^{S\theta}d\theta + Q = 0.\end{array} \tag{31}$$

Theorem 3. *Under H0-H2, the linear state-feedback regulator problem (9) ((9) and (30)) is solvable if and only if there exist $n \times r$ and $m \times r$-matrices $\Pi(0)$ and Γ which solve the linear matrix equations (29) ((31)).*

In the case of error-feedback regulator problem, the similar result holds under H0-H3.

Consider the case of (30) with the general controller output. We assume that the regulator problem for (9) without delay, i.e. for

$$\begin{array}{l}(I - \sum_{i=1}^{k} D_i)\dot{x}(t) = (\sum_{i=0}^{k} A_i)x(t) + Bu(t) + Pw(t), \\ \dot{w}(t) = Sw(t), \\ e(t) = (\sum_{i=0}^{k} C_i)x(t) + Qw(t)\end{array}$$

is solvable for all P and Q. This is equivalent (see e.g. [4]) to the following assumption

A1. $det\mathcal{G}_0(\lambda) \neq 0$ for all eigenvalues λ of S, where

$$\mathcal{G}_0(\lambda) = (\sum_{i=0}^{k} C_i)[\lambda(I - \sum_{i=1}^{k} D_i) - \sum_{i=0}^{k} A_i]^{-1}B.$$

Under A1 the linear regulator equations

$$(I - \sum_{i=1}^{k} D_i)\Pi_0 S = (\sum_{i=0}^{k} A_i)\Pi_0 + B\Gamma + P, \quad (\sum_{i=0}^{k} C_i)\Pi_0 + Q = 0,$$

where Π_0 and Γ are constant matrices, are solvable for all P and Q. Then, by the implicit function theorem for all small enough $h > 0$ (31) is solvable. We have:

Proposition 2. *Under H0-H2 and A1, the output regulation of (9) with (30) via state-feedback of (10) is achievable and the regulator equations (31) are solvable for all small enough h.*

6 Conclusions

The geometric theory of output regulation is generalized to nonlinear neutral type systems. It is shown that the state-feedback and the error-feedback regulator problems are solvable, under the standard assumptions on stabilizability and detectability of the linearized system, if and only if a set of regulator equations is solvable. This set consists of partial differential and algebraic equations. In the linear case these equations are reduced to the linear matrix equations.

The solvability of the nonlinear regulator equations and the approximate solutions to these equations are issues for the future study.

References

1. M. Ait Babram, O. Arino and M. Hbid, Computational scheme of a center manifold for neutral functional differential equations, J. Math. Anal. Appl. 258 (2001) 396-414.
2. C.I.Byrnes and A. Isidori, Output regulation for nonlinear systems: an overview, Int. J. of Rob. and Nonlin. Cont. 10 (2000) 323-337.
3. C.I.Byrnes, I. G. Lauko, D. S. Gilliam and V. I. Shubov, Output Regulation for Linear Distributed parameter systems, IEEE Trans. On Aut. Cont., 45 (2000) 2236-2252.
4. B.A. Francis, The Linear Multivariable Regulator Problem, SIAM J. on Cont. and Optim., 15 (1977) 486-505.
5. E. Fridman, Asymptotics of integral manifolds and decomposition of singularly perturbed systems of neutral type, Differential equations 26 (1990) 457-467.
6. E. Fridman, Output regulation of nonlinear systems with delay, Systems & Control Letters 50 (2003) 81-93.
7. E. Fridman and U. Shaked, A descriptor system approach to H_∞ control of time-delay systems, IEEE Trans. on Automat. Contr., 47 (2002) 253-270.
8. J. Hale, Critical cases for neutral functional differential equations, *J. Differential Eqns.* 10 (1971) 59-82.
9. J. Hale and S. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993 .
10. J. Hale and S. Verduyn Lunel, Strong stabilization of neutral functional differential equations, *IMA Journal of Mathematical Control and Information*, 19 (2002) 5-23.
11. A. Isidori and C.I. Byrnes, Output regulation of nonlinear systems, IEEE Trans. On Autom. Control 35 (1990) 131-140.
12. K. Murakami, Bifurcated periodic solutions for delayed Van der pol equation, Neural, Parallel & Scientific Computations 7 (1999) 1-16.
13. J. Schumacher, Finite-dimensional regulators for a class of infinite-dimensional systems, Systems & Control Letters, 3 (1983) 7-12.
14. V. Strygin and E. Fridman, Asymptotics of integral manifolds of singularly perturbed differential equations with retarded argument, Math. Nachr., 117 (1984) 83-109.