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Robust H_∞ Control of Distributed Delay Systems With Application to Combustion Control

L. Xie, E. Fridman, and U. Shaked

***Abstract*—This note is concerned with the H_∞ analysis and synthesis of linear distributed delay systems. An efficient stability and $\mathcal{L}_2$-gain criterion is established. It is based on a recent approach to the analysis and design of linear time delay systems which represents the system in an equivalent descriptor form. The obtained criterion is used to derive an efficient state-feedback control design which stabilizes the distributed delay system and achieves a guaranteed disturbance attenuation level in spite of a polytopic uncertainty in the system parameters. The new method is applied to the robust stabilization and control of combustion in rocket motor chambers.**

***Index Terms*—Combustion control, distributed delay, H_∞-control, linear matrix inequality (LMI), time delay systems.**

I. Introduction

It is well-known (see, e.g., [1]–[3]) that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving good stability criteria for delay systems. The same is true concerning for bounded real criteria. The general form of the Lyapunov–Krasovskii functional leads to a complicated system of Riccati type partial differential equations [4], [5] or inequalities [6]. Special forms of Lyapunov–Krasovskii functionals lead to simpler (but more conservative) delay-independent [7]–[10] and delay-dependent [8]–[11] sufficient conditions. Recently, a delay-dependent bounded real lemma (BRL) has been derived which considerably reduces the conservatism entailed in previous results. A new type of Lyapunov–Krasovskii functional is introduced in [12] which is based on an equivalent "descriptor form" representation of the system. Developing the BRL using the latter functional, a significant reduction in the overdesign entailed in all existing methods is achieved, mainly due to the fewer bounds needed to derive the lemma.

The models used in [12] include multiple point time delays but exclude distributed delays. Our interest in the latter stems from the works of [15] and [16] that derived a linearized model with distributed delay for the feeding system and the combustion chamber in a liquid monopropellant rocket motor with pressure feeding. In this model a nonsteady flow was assumed and nonuniform lags were taken into account.

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In the present note we formulate two types of linear systems with distributed time delay and for each model we develop a BRL based on the new approach of [12]. Since the latter requires the least overbounding, the results obtained will be the least conservative. Further, a state-feedback H_∞ control of linear systems with distributed delays and polytopic uncertainty in system parameters is given. We also apply our theory to the problem of stabilizing combustion in rocket motor chambers. We obtain proportional and PI state-feedback controls that stabilize the system and ensure a guaranteed bound for its disturbance attenuation level, in spite of uncertainty in the process parameters.

Notation: Throughout this note the superscript "T" stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The space of functions in $\mathcal{R}^q$ that are square integrable over $[0, \infty)$ is denoted by $\mathcal{L}_2^q[0, \infty)$.

II. $\mathcal{L}_2$-Gain Analysis of Linear Distributed Time Delay Systems

We consider the following two models of linear systems with distributed time delay

1) *The "convolution-type" model:* The system is described by:

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t-h) + \int_{-d}^{0} A_d(s)x(t+s)\,ds\\ &\quad + B_1w(t),\\ x(t) &= 0 \quad \forall t \in [-\max\{h, d\}, 0],\\ z(t) &= col\{C_0x(t), C_1x(t-h)\}\end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $w(t) \in \mathcal{L}_2^q[0, \infty)$ is the exogenous disturbance signal and $z(t) \in \mathcal{R}^p$ is the state combination (objective function signal) to be attenuated. The time delays h and d are assumed to be known. The matrices A_i, C_i, $i = 0, 1$ and B_1 are constant matrices of appropriate dimensions and $A_d(t)$ is a continuous matrix on $[-d, 0)$.

2) *The "summation-type" model:* This model is described by

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t-h) + \int_{t-d}^{t} A_d(s)x(s)\,ds + B_1w(t),\\ x(t) &= 0 \quad \forall t \in [-\max\{h, d\}, 0],\\ z(t) &= col\{C_0x(t), C_1x(t-h)\}\end{aligned} \tag{2}$$

where the vectors x, w, z and the matrices A_i, C_i, $i = 0, 1$ and B_1 are the same as in the "convolution-type" model and A_d is a continuous matrix on $[-d, \infty)$.

The two models are equivalent in the case where the matrix A_d is independent of time. We first derive the stability and the BRL criteria for the "convolution-type" model.

A. Delay-Dependent BRL for the "Convolution-Type" Model

Consider the system (1). For a prescribed scalar $\gamma > 0$, we define the performance index

$$\mathcal{J}(w) = \int_0^\infty (z^Tz - \gamma^2w^Tw)\,d\tau. \tag{3}$$

We are looking for a BRL that depends on delays h and d. Following [12], we represent (1) in the equivalent descriptor form

$$\begin{aligned}\dot{x}(t) &= y(t)\\ y(t) &= A_0x(t) + A_1x(t-h) + \int_{-d}^{0} A_d(s)x(t+s)\,ds\\ &\quad + B_1w(t)\end{aligned} \tag{4}$$

which is further equivalent to the following descriptor system:

$$\begin{aligned}\dot{x}(t) &= y(t)\\ y(t) &= (A_0+A_1)x(t) - A_1\int_{t-h}^{t} y(\tau)\,d\tau\\ &\quad + \int_{-d}^{0} A_d(s)x(t+s)\,ds + B_1w(t).\end{aligned} \tag{5}$$

We adopt the following Lyapunov-Krasovskii functional for the system (5):

$$\begin{aligned}V(t) &= [x^T(t) \quad y^T(t)]EP\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}\\ &\quad + \int_{-h}^{0}\int_{t+\theta}^{t} y^T(\tau)Ry(\tau)\,d\tau\,d\theta + \int_{-d}^{0}\int_{t}^{t-\theta}\\ &\quad \cdot x^T(\tau+\theta)A_d^T(\theta)R_dA_d(\theta)x(\tau+\theta)\,d\tau\,d\theta\end{aligned} \tag{6}$$

where

$$E = \begin{bmatrix}I_n & 0\\ 0 & 0\end{bmatrix} \quad P = \begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix}, \quad P_1 > 0, \quad R,\, R_d > 0. \tag{7a, b}$$

The following result is then obtained.

Theorem 2.1: Consider the system of (1). The system is asymptotically stable and for a prescribed $\gamma > 0$, the cost function (3) satisfies $\mathcal{J}(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0,\,\infty)$, if there exist $n \times n$-matrices $0 < P_1,\, P_2,\, P_3,\, R$ and R_d that satisfy the linear matrix inequality (LMI) shown in (8), shown at the bottom of the page, where

$$\begin{aligned}\Phi \triangleq (A_0+A_1)^TP_2 + P_2^T(A_0+A_1)\\ + \int_{-d}^{0} A_d^T(\theta)R_dA_d(\theta)\,d\theta.\end{aligned} \tag{9}$$

Proof: We note that

$$[x^T \quad y^T]EP\begin{bmatrix}x\\ y\end{bmatrix} = x^TP_1x$$

and, hence, differentiating V_1, the first term of (6) with respect to t gives us

$$\begin{aligned}\frac{dV(t)}{dt} &= \frac{d}{dt}\left\{[x^T(t) \quad y^T(t)]EP\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}\right\}\\ &= 2x^T(t)P_1\dot{x}(t)\\ &= 2[x^T(t) \quad y^T(t)]P^T\begin{bmatrix}y(t)\\ -y(t)+\dot{x}(t)\end{bmatrix}.\end{aligned} \tag{10}$$

Substituting (5) into (10) and denoting $\xi \triangleq col\{x(t),\, y(t),\, w(t)\}$ we obtain (11), as shown at the bottom of the page, where

$$\begin{aligned}\eta_h(t) &\triangleq -2\int_{t-h}^{t}[x^T(t) \quad y^T(t)]P^T\begin{bmatrix}0\\ A_1\end{bmatrix}y(s)\,ds\\ &\le h[x^T(t) \quad y^T(t)]P^T\begin{bmatrix}0\\ A_1\end{bmatrix}R^{-1}[0 \quad A_1^T]\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}\\ &\quad + \int_{t-h}^{t} y^T(s)Ry(s)\,ds\end{aligned} \tag{12}$$

and

$$\begin{aligned}\eta_d(t) &\triangleq -2\int_{-d}^{0}[x^T(t) \quad y^T(t)]P^T\begin{bmatrix}0\\ A_d(s)\end{bmatrix}x(t+s)\,ds\\ &\le d[x^T(t) \quad y^T(t)]P^T\begin{bmatrix}0\\ I\end{bmatrix}R_d^{-1}[0 \quad I]\,P\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}\\ &\quad + \int_{-d}^{0} x^T(t+s)A_d^T(s)R_dA_d(s)x(t+s)\,ds.\end{aligned} \tag{13}$$

$$\begin{bmatrix}\Phi & P_1 - P_2^T + (A_0+A_1)^TP_3 & P_2^TB_1 & hP_2^TA_1 & dP_2^T & C_0^T & C_1^T\\ P_1 - P_2 + P_3^T(A_0+A_1) & -P_3 - P_3^T + hR & P_3^TB_1 & hP_3^TA_1 & dP_3^T & 0 & 0\\ B_1^TP_2 & B_1^TP_3 & -\gamma^2I & 0 & 0 & 0 & 0\\ hA_1^TP_2 & hA_1^TP_3 & 0 & -hR & 0 & 0 & 0\\ dP_2 & dP_3 & 0 & 0 & -dR_d & 0 & 0\\ C_0 & 0 & 0 & 0 & 0 & -I & 0\\ C_1 & 0 & 0 & 0 & 0 & 0 & -I\end{bmatrix} < 0 \tag{8}$$

$$\begin{aligned}\frac{dV(t)}{dt} + z^T(t)z(t) - \gamma^2w^T(t)w(t) &= \xi^T\begin{bmatrix}P^T\begin{bmatrix}0 & I\\ (A_0+A_1) & -I\end{bmatrix} + \begin{bmatrix}0 & (A_0^T+A_1^T)\\ I & -I\end{bmatrix}P & P^T\begin{bmatrix}0\\ B_1\end{bmatrix}\\ [0 \quad B_1^T]\,P & -\gamma^2I_q\end{bmatrix}\xi\\ &\quad + x^T(t)\int_{-d}^{0} A_d^T(\theta)^TR_dA_d(\theta)\,d\theta x(t) + hy^T(t)Ry(t) - \int_{t-h}^{t} y^T(\tau)Ry(\tau)\,d\tau\\ &\quad - \int_{-d}^{0} x^T(t+\theta)A_d^T(\theta)R_dA_d(\theta)x(t+\theta)d\theta + \eta_h(t) + \eta_d(t) + z^T(t)z(t)\end{aligned} \tag{11}$$

The stability of the system readily follows from the fact that by (11)–(13), taking $z \equiv 0$, $B_1 = 0$ and $w \equiv 0$, the following holds:

$$\frac{dV(t)}{dt} \le [x^T(t) \quad y^T(t)] \left[\mathrm{diag}\{\Phi,\, hR\} + P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \cdot R^{-1} [0 \quad A_1^T] P + P^T \begin{bmatrix} 0 \\ I \end{bmatrix} R_d^{-1} [0 \quad I] P \right] \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The derivative of V with respect to t will, therefore, be negative and the system will thus be stable, if the matrix block that is built of the first, second, fourth, and fifth row and column blocks in (8) is negative–definite.

From (4), and due to the asymptotic stability of $x(t)$, it follows that $x(t)$ is square integrable on $[0, \infty)$. By noting that

$$\begin{aligned}\int_0^\infty z^T z\, dt &= \int_0^\infty x^T(\tau) C_0^T C_0 x(\tau)\, d\tau + \int_0^\infty x^T(\tau - h) C_1^T C_1 x(\tau - h)\, d\tau \\ &= \sum_{i=0}^{1} \int_0^\infty x^T(\tau) C_i^T C_i x(\tau)\, d\tau\end{aligned}$$

and substituting (12) and (13) into (11) we obtain (by Schur complements) that $\mathcal{J} < 0$ if the following LMI holds:

$$\Gamma \triangleq \begin{bmatrix} \Psi & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & hP^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & dP^T \begin{bmatrix} 0 \\ I \end{bmatrix} \\ [0 \quad B_1^T] P & -\gamma^2 I_q & 0 & 0 \\ h[0 \quad A_1^T] P & 0 & -hR & 0 \\ d[0 \quad I] P & 0 & 0 & -dR_d \end{bmatrix} < 0 \quad (14)$$

where

$$\Psi \triangleq P^T \begin{bmatrix} 0 & I \\ A_0 + A_1 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^T + A_1^T \\ I & -I \end{bmatrix} P + \begin{bmatrix} C_0^T C_0 + C_1^T C_1 + \int_{-d}^0 A_d^T(\theta) R_d A_d(\theta)\, d\theta & 0 \\ 0 & hR \end{bmatrix}.$$

Finally, the LMI (8) results from the latter LMI by expansion of the block matrices. □

Remark 1: The affinity of (8) in d and h implies that once Theorem 2.1 holds for d_1 and h_1 it also holds for all $d \in [0, d_1]$ and $h \in [0, h_1]$.

Remark 2: When $A_d(s)$ is a constant matrix, i.e., $A_d(s) \equiv A_d$, $\forall s \in [-d, 0]$, Φ in (9) can be rewritten as

$$\Phi = (A_0 + A_1)^T P_2 + P_2(A_0 + A_1) + dA_d^T R_d A_d.$$

In this situation, the LMI (8) is readily implementable. On the other hand, if one chooses $R_d = \epsilon I$, where ϵ is a positive scaling parameter, we have

$$\Phi = (A_0 + A_1)^T P_2 + P_2(A_0 + A_1) + \epsilon \int_{-d}^0 A_d^T(\theta) A_d(\theta)\, d\theta$$

and the LMI is implementable for any time-varying $A_d(s)$, $s \in [-d, 0]$. In general, one may choose to discretize the integral and implement the LMI as follows:

$$\Phi = (A_0 + A_1)^T P_2 + P_2(A_0 + A_1) + \Delta \sum_{i=0}^{N-1} A_d^T(-d + i\Delta) R_d A_d(-d + i\Delta)$$

where $\Delta = d/N$ and N is a positive integer.

Remark 3: Theorem 2.1 presents an efficient LMI algorithm for checking the stability and $\mathcal{L}_2$-gain of distributed delay systems. The result is derived using minimum number of boundings and is expected to be the least conservative.

B. Delay-Dependent BRL for the "Summation-Type" Model

We consider the system (2). Also here, for a prescribed scalar $\gamma > 0$, we seek a condition that will verify the stability of the system and will guarantee that $\mathcal{J}(w)$ of (3) will be negative for all nonzero $w(t) \in \mathcal{L}_2^q[0, \infty)$.

We apply the following Lyapunov–Krasovskii functional

$$\begin{aligned}V(t) = [x^T(t) \quad y^T(t)] EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &+ \int_{-h}^0 \int_{t+\theta}^t y^T(\tau) R y(\tau)\, d\tau\, d\theta \\ &+ \int_{-d}^0 \int_{t+\theta}^t x^T(\tau) A_d^T(\tau) R_d A_d(\tau) x(\tau)\, d\tau\, d\theta \end{aligned} \quad (15)$$

where E, the structure of P and the positive definite matrices of R and R_d are defined in (7). Following the same lines of the proof of Theorem 2.1, we readily obtain the following BRL.

Theorem 2.2: Consider the system of (2). The system is asymptotically stable and for a prescribed $\gamma > 0$, the cost function (3) achieves $\mathcal{J}(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$, if there exist $n \times n$-matrices $0 < P_1$, P_2, P_3, R and R_d that satisfy the LMI of (8) $\forall t \in [0, \infty)$, with

$$\Phi \triangleq (A_0 + A_1)^T P_2 + P_2^T (A_0 + A_1) + dA_d^T(t) R_d A_d(t). \quad (16)$$

Remark 4: In the case where A_d depends on time, the validity of the latter LMI should be guaranteed all over $[0, \infty)$. If $A_d(t)$ is a smooth function of time, an appropriate grid of time instants may be chosen for which the LMI is to be solved.

C. The Case of Constant A_d

If A_d is a constant matrix the two models are equivalent and indeed the two expressions for Φ in (9) and (16) are then identical.

In view of Remark 2, and because the combustion control problem to be solved below has a constant A_d, we shall assume in the sequel, for simplicity, that $A_d(s) \equiv A_d$, $\forall s \in [-d, 0]$.

It follows from (8) that if there exists a solution to the LMI then the resulting P_3 cannot possibly be singular since $P_3 + P_3^T$ must be negative definite. It thus follows that if there exists a solution to the inequality (14) the resulting P will be nonsingular. Denoting, therefore

$$P^{-1} \triangleq Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}$$

an alternative representation for the BRL of Theorem 2.1 can be derived. Multiplying Γ in (14) from the left and the right by $\mathrm{diag}\{Q^T, I\}$ and $\mathrm{diag}\{Q, I\}$, respectively, we obtain the following.

Corollary 2.3: Consider the system of (1) with $A_d(s) \equiv A_d$, $\forall s \in [-d, 0]$. The system is asymptotically stable and for a prescribed $\gamma > 0$, the cost function (3) achieves $\mathcal{J}(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$, if there exist $n \times n$-matrices $0 < Q_1$, Q_2, Q_3, $\overline{R}$ and $\overline{R}_d$ that satisfy the following LMI as shown in (17) at the bottom of the next page, where "*" denotes entry which can be deduced from the symmetry of the matrix.

The BRL of Corollary 2.3 was derived for the system (1) where the system matrices A_0, A_1, A_d, C_0, C_1, and B_1 are all known. However, since the LMI of (17) is affine in the system matrices, the corollary can be used to derive a criterion that will guarantee the required attenuation level in the case where the system matrices are not exactly known and they reside within a given polytope.

Denoting

$$\Omega(t) = \begin{bmatrix} A_0 & A_1 & A_d \\ B_1 & C_0 & C_1 \end{bmatrix}$$

we assume that for all $t \in [0, \infty)$ $\Omega(t) \in \mathcal{C}o\{\Omega_i,\ i = 1, \ldots, N\}$, namely

$$\Omega(t) = \sum_{i=1}^{N} f_i(t)\Omega_i \text{ for some } 0 \le f_i(t) \le 1,\ \sum_{i=1}^{N} f_i(t) = 1 \qquad (18)$$

where the N vertices of the polytope are described by

$$\Omega_i = \begin{bmatrix} A_0^{(i)} & A_1^{(i)} & A_d^{(i)} \\ B_1^{(i)} & C_0^{(i)} & C_1^{(i)} \end{bmatrix}.$$

We readily obtain the following.

Corollary 2.4: Consider the system of (1) that satisfies (18). The system is asymptotically stable and for a prescribed $\gamma > 0$, the cost function (3) achieves $\mathcal{J}(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all Ω within the polytope of (18) if there exist $n \times n$-matrices $0 < Q_1$, $Q_2^{(i)}$, $Q_3^{(i)}$, $\overline{R}_d^{(i)}$, $i = 1, \ldots, N$ and $\overline{R} > 0$ that satisfy the following set of LMIs for $i = 1, \ldots, N$; see (19), as shown at the bottom of the page.

Remark 5: It should be remarked that by explicitly exploiting the fact that A_d is constant, we may obtain another BRL based on existing result of [12] for point time delay systems. In fact, by defining $y = \int_0^t x(\tau)\, d\tau$, (2) can be rewritten as a point time-delay system

$$\begin{aligned} \dot{\xi}(t) &= \tilde{A}_0\xi(t) + \tilde{A}_1\xi(t-h) + \tilde{A}_d\xi(t-d) + \tilde{B}_1 w(t) \\ z(t) &= col\{\tilde{C}_0 x(t),\ \tilde{C}_1 x(t-h)\} \end{aligned}$$

where $\xi(t) = col\{x(t),\ y(t)\}$ and

$$\tilde{A}_0 = \begin{bmatrix} A_0 & A_d \\ I_n & 0 \end{bmatrix} \quad \tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_d = \begin{bmatrix} 0 & -A_d \\ 0 & 0 \end{bmatrix}$$
$$\tilde{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad \tilde{C}_0 = [\,C_0 \quad 0\,] \quad \tilde{C}_1 = [\,C_1 \quad 0\,].$$

III. State-Feedback Control

In this section, we shall apply the results of the previous section to the problem of robust state-feedback control of systems with distributed delay. Given the system $\mathcal{S}(\overline{A}_0,\ \overline{A}_1,\ A_d,\ B_1,\ B_2,\ \overline{C}_1,\ D_{12})$

$$\begin{aligned} \dot{x}(t) &= \overline{A}_0 x(t) + \overline{A}_1 x(t-h) + \int_{-d}^{0} A_d x(t+s)\, ds \\ &\quad + B_1 w(t) + B_2 u(t), \\ z &= col\{\overline{C}_1 x,\ D_{12} u\}, \\ x(t) &= 0 \quad \forall t \in [-\max\{h,\ d\},\ 0] \end{aligned} \qquad (20)$$

where x and w are defined in Section II, $u \in \mathcal{R}^{\ell}$ is the control input, $\overline{A}_0$, $\overline{A}_1$, A_d, B_1, B_2 are constant matrices of appropriate dimension, z is the objective vector, $\overline{C}_1 \in \mathcal{R}^{p\times n}$ and $D_{12} \in \mathcal{R}^{r\times \ell}$. As noted in Remark 2, the case when A_d is time varying over $[-d, 0]$ can be handled similarly.

Denoting

$$\Omega_s(t) = \begin{bmatrix} \overline{A}_0 & \overline{A}_1 & A_d \\ B_1 & B_2 & \overline{C}_1\, D_{12} \end{bmatrix}$$

we assume that for all $t \in [0, \infty)$ $\Omega_s(t) \in \mathcal{C}o\{\Omega_i,\ i = 1, \ldots, N\}$, namely

$$\Omega_s(t) = \sum_{i=1}^{N} f_i(t)\Omega_{s,i} \text{ for some } 0 \le f_i(t) \le 1,\ \sum_{i=1}^{N} f_i(t) = 1 \qquad (21)$$

where the N vertices of the polytope are described by

$$\Omega_{s,i} = \begin{bmatrix} \overline{A}_0^{(i)} & \overline{A}_1^{(i)} & A_d^{(i)} \\ B_1^{(i)} & B_2^{(i)} & \overline{C}_1^{(i)}\, D_{12}^{(i)} \end{bmatrix}.$$

For a prescribed scalar $\gamma > 0$, we consider the performance index of (3) and seek a state-feedback gain matrix K which, via the control law $u(t) = Kx(t)$, achieves, within the polytope of (21), $\mathcal{J}(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$.

$$\begin{bmatrix} Q_2 + Q_2^T & Q_3 - Q_2^T + Q_1(A_0^T + A_1^T) & 0 & 0 & 0 & Q_1 C_0^T & Q_1 C_1^T & dQ_1 A_d^T & hQ_2^T \\ * & -Q_3 - Q_3^T & B_1 & hA_1\overline{R} & d\overline{R}_d & 0 & 0 & 0 & hQ_3^T \\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h\overline{R} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -d\overline{R}_d & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -d\overline{R}_d & 0 \\ * & * & * & * & * & * & * & * & -h\overline{R} \end{bmatrix} < 0 \qquad (17)$$

$$\begin{bmatrix} Q_2^{(i)} + Q_2^{(i)T} & Q_3^{(i)} - Q_2^{(i)T} + Q_1(A_0^{(i)} + A_1^{(i)})^T & 0 & 0 & 0 & Q_1 C_0^{(i)T} & Q_1 C_1^{(i)T} & dQ_1 A_d^{(i)T} & hQ_2^{(i)T} \\ * & -Q_3^{(i)} - Q_3^{(i)T} & B_1^{(i)} & hA_1^{(i)}\overline{R} & d\overline{R}_d^{(i)} & 0 & 0 & 0 & hQ_3^{(i)T} \\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h\overline{R} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -d\overline{R}_d^{(i)} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -d\overline{R}_d^{(i)} & 0 \\ * & * & * & * & * & * & * & * & -h\overline{R} \end{bmatrix} < 0. \qquad (19)$$

Substituting $u = Kx$ into (20), we obtain the structure of (1) with

$$\begin{aligned} A_0 &= \overline{A}_0 + B_2 K, \quad A_1 = \overline{A}_1 \\ C_0^T C_0 &= \overline{C}_1^T \overline{C}_1 + K^T D_{12}^T D_{12} K. \end{aligned} \tag{22}$$

Denoting $Y = KQ_1$, we apply Corollary 2.3 on the latter system and obtain the following.

Theorem 3.1: Consider the system of (20) that satisfies (21). The system is stabilizable and for a prescribed $\gamma > 0$, the cost function (3) achieves $\mathcal{J}(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all Ω within the polytope of (21) if there exist $n \times n$-matrices $0 < Q_1, Q_2^{(i)}, Q_3^{(i)}, \overline{R}_d^{(i)}, i = 1, \ldots, N, 0 < \overline{R}$ and $Y \in \mathcal{R}^{\ell \times n}$ that satisfy the following set of LMIs, for $i = 1, \ldots, N$, see (23) shown at the bottom of the page. The state-feedback gain is then given by $K = YQ_1^{-1}$.

The latter result has been obtained for the proportional feedback rule. If an integral action is also allowed, the model of (5) can be used. Applying the state-feedback result of [12] to the latter system we obtain the following.

Theorem 3.2: Consider the system of (20) and (21). For a prescribed $\gamma > 0$, there exists a PI controller $u(t) = K_1 x(t) + K_2 \int_0^t x(\tau)\, d\tau$ that stabilizes the system and achieves $\mathcal{J}(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all Ω within the polytope of (21) if there exist $2n \times 2n$-matrices $0 < Q_1, Q_2^{(i)}, Q_3^{(i)}, i = 1, \ldots, N, 0 < \overline{R}_d, 0 < \overline{R}$ and $Y \in \mathcal{R}^{\ell \times 2n}$ that satisfy the following set of LMIs for $i = 1, \ldots, N$; see (24), as shown at the bottom of the page, where the augmented matrices are defined in Remar 5 and $\tilde{B}_2^{(i)T} = [B_2^{(i)T} \quad 0]$.

The feedback gains are then given by $[K_1 \quad K_2] = YQ_1^{-1}$.

IV. An Example of Combustion Control

In this section, we shall demonstrate the application of the results in Theorems 3.1 and 3.2 to robust stabilization and control of combustion in rocket motor chambers.

We consider a liquid monopropellant rocket motor with a pressure feeding system. Assuming nonsteady flow and taking nonuniform lag into account, a linearized model of the feeding system and the combustion chamber equations has been obtained by [15], [16], and [13]. Their model is of the form (20) with

$$x(t) = col\{x_1(t), x_2(t), x_3(t), x_4(t)\}$$

where $x_1(t), x_2(t)$ and $x_3(t)$ are, respectively, the relative deviations of the instantaneous combustion chamber pressure, the instantaneous mass flow upstream of the capacitance and the instantaneous mass rate of the injected propellant from their steady values, and $x_4(t)$ is the ratio between the deviation of the instantaneous pressure in a special place in the feeding line from its value in steady operation and twice the injector pressure drop in steady operation.

The model matrices are

$$\overline{A}_0 = \begin{bmatrix} \rho - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\zeta J} \\ -0.5\dfrac{p}{(1-\zeta)J} & 0 & -\dfrac{1}{(1-\zeta)J} & \dfrac{1}{(1-\zeta)J} \\ 0 & \dfrac{1}{E_e} & -\dfrac{1}{E_e} & 0 \end{bmatrix}$$

$$\overline{A}_1 = 0 \quad A_d = \begin{bmatrix} -\dfrac{\rho}{d} & 0 & \dfrac{1}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ \dfrac{1}{\zeta J} \\ 0 \\ 0 \end{bmatrix} \quad \overline{C}_1 = [1 \quad 0 \quad 0 \quad 0] \quad D_{12} = 1$$

where ζ is the fractional length for pressure supply, J is the line inertia, E_e is the line elasticity parameter, p is the ratio of steady-state pressure

$$\begin{bmatrix} Q_2^{(i)} + Q_2^{(i)T} & Q_3^{(i)} - Q_2^{(i)T} + Q_1(\overline{A}_0^{(i)T} + \overline{A}_1^{(i)T}) + Y^T B_2^{(i)T} & 0 & & 0 & Q_1 \overline{C}_1^{(i)T} & Y^T D_{12}^{(i)T} & dQ_1 A_d^{(i)T} & hQ_2^{(i)T} \\ * & -Q_3^{(i)} - Q_3^{(i)T} & B_1^{(i)} & h\overline{A}_1^{(i)}\overline{R} & d\overline{R}_d^{(i)} & 0 & 0 & 0 & hQ_3^{(i)T} \\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h\overline{R} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -d\overline{R}_d^{(i)} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -d\overline{R}_d^{(i)} & 0 \\ * & * & * & * & * & * & * & * & -h\overline{R} \end{bmatrix} < 0. \tag{23}$$

$$\begin{bmatrix} Q_2^{(i)} + Q_2^{(i)T} & Q_3^{(i)} - Q_2^{(i)T} + Q_1(\tilde{A}_0^{(i)} + \tilde{A}_1^{(i)} + \tilde{A}_d^{(i)})^T + Y^T \tilde{B}_2^{(i)T} & 0 & 0 & 0 & Q_1 \tilde{C}_1^{(i)T} & Y^T D_{12}^{(i)T} & dQ_2^{(i)T} & hQ_2^{(i)T} \\ * & -Q_3 - Q_3^T & \tilde{B}_1^{(i)} & h\tilde{A}_1^{(i)}\overline{R} & d\tilde{A}_d^{(i)}\overline{R}_d & 0 & 0 & dQ_3^{(i)T} & hQ_3^{(i)T} \\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h\overline{R} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -d\overline{R}_d & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -d\overline{R}_d & 0 \\ * & * & * & * & * & * & * & * & -h\overline{R} \end{bmatrix} < 0 \tag{24}$$

and steady-state injector pressure drop and ρ is the pressure exponent of the combustion process.

The nominal value of the pressure exponent ρ is 1. We consider the following uncertainty in ρ: $\rho \in [1 - \delta\rho_{\max}, 1 + \delta\rho_{\max}]$. Let $\zeta = 0.1$, $p = 1$, $J = 2$ and $E_e = 1$. Then, the system can be described by a polytope of two vertices with

$$\overline{A}_0^{(1)} = \begin{bmatrix} -\delta\rho_{\max} & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ -0.5556 & 0 & -0.5556 & 0.5556 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\overline{A}_0^{(2)} = \begin{bmatrix} \delta\rho_{\max} & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ -0.5556 & 0 & -0.5556 & 0.5556 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\overline{A}_d^{(1)} = \begin{bmatrix} -\frac{1-\delta\rho_{max}}{d} & 0 & \frac{1}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{A}_d^{(2)} = \begin{bmatrix} -\frac{1+\delta\rho_{max}}{d} & 0 & \frac{1}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\delta\rho_{\max} = 0.15$ and $d = 1$ sec. By applying Theorem 3.1 the minimum value of γ which admits a solution to the LMIs is $\gamma_{\min} = 14$. The solution for this value of γ provides the robust state-feedback control law of $u(t) = Kx(t)$ with $K = [68.8632 \quad -2.7737 \quad -48.7091 \quad -11.7800]$. This state-feedback controller is applied to the system and the frequency response of the resulting transference between w and z is plotted. Since the inequalities in Theorem 3.1 guarantee the stability of the closed-loop, the peak value of the latter plot represents the actual bound that is achieved for the disturbance attenuation in the system. In our case the height of the peak was 4.75.

Applying, on the other hand, the result of Theorem 3.2 a minimum value of $\gamma_{\min} = 22$ was obtained with $K_1 = [61.095 \quad -2.7302 \quad -40.251 \quad -10.298]$ and $K_2 = [-7.0411 \times 10^{-12} \quad -1.4372 \times 10^{-11} \quad -1.1798 \times 10^{-11} \quad -1.6519 \times 10^{-11}]$. The latter result implies that the integral part of the feedback is not used and that, in fact, the resulting controller is a standard proportional feedback. The frequency plots for this controller showed that an attenuation bound of 4.6 is actually achieved.

V. CONCLUSION

A comprehensive theory for dealing with linear systems with both point and distributed time delays is introduced. Two models of distributed delays have been considered. Based on the recent approach of [12], efficient BRLs have been obtained for both models. The criteria obtained provide sufficient conditions for stability and for achieving a prescribed attenuation level. The conservatism of the results stems from the bounding of the terms in (12) and (13). The overdesign that is entailed in all the recent methods for controlling distributed delay systems is proportional to the number of bounding involved. The descriptor approach we used to derive our results applies the minimum number of bounding. The conservatism of these results is therefore the minimum. We have also applied the result to the combustion control design in rocket motor chambers.

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