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Brief paper

Predictor-based networked control under uncertain transmission delays ☆

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ABSTRACT

We consider state-feedback predictor-based control of networked control systems with large time-varying communication delays. We show that even a small controller-to-actuators delay uncertainty may lead to a non-small residual error in a networked control system and reveal how to analyze such systems. Then we design an event-triggered predictor-based controller with sampled measurements and demonstrate that, depending on the delay uncertainty, one should choose various predictor models to reduce the error due to triggering. For the systems with a network only from a controller to actuators, we take advantage of the continuously available measurements by using a continuous-time predictor and employing a recently proposed switching approach to event-triggered control. By an example of an inverted pendulum on a cart we demonstrate that the proposed approach is extremely efficient when the uncertain time-varying network-induced delays are too large for the system to be stabilizable without a predictor.

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1. Introduction

In networked control systems (NCSs), which are comprised of sensors, controllers, and actuators connected through a communication medium, transmitted signals are sampled in time and are subject to time-delays. Most existing papers on NCSs study robust stability with respect to small communication delays (see, e.g., Antsaklis & Baillieul, 2004; Fridman, Seuret, & Richard, 2004; Gao, Chen, & Lam, 2008; Liu & Fridman, 2012). To compensate large transport delays, predictor-based approach can be employed. So far this was done only for sampled-data control with *known constant delays* (Karafyllis & Krstic, 2012; Mazenc & Normand-Cyrot, 2013). In this paper we develop predictor-based sampled-data control for *unknown time-varying delays*.

There are several works that study robustness (w.r.t. delay uncertainty) of a predictor-based *continuous-time* controller (Bekiaris-Liberis & Krstic, 2013; Karafyllis & Krstic, 2013; Li, Zhou, & Lin, 2014; Yue & Han, 2005). In these works the residual error that appears due to delay uncertainty can be made arbitrary small by reducing the upper bound of the uncertainty. However, this is not true for *sampled-data systems*, where an arbitrary small delay uncertainty may lead to a non-vanishing error (because the terms that appear in the residual error may belong to different sampling intervals).

In this work we study an NCS with two networks: from sensors to a controller and from the controller to actuators. Both networks introduce large time-varying delays. We assume that the messages sent from the sensors are time stamped (Zhang, Branicky, & Phillips, 2001). This allows to calculate the sensors-to-controller delay. The controller-to-actuators delay is assumed to be unknown but belongs to a known delay interval. We use a state-feedback predictor, which is calculated on the controller side, to partially compensate both delays. By extending the time-delay modeling of NCSs (Fridman, 2014; Fridman et al., 2004; Gao et al., 2008), we present the system in a form suitable for analysis. Using a proper Lyapunov–Krasovskii functional, we derive LMI-based conditions for the stability analysis and design that guarantee the desired decay rate of convergence.

As the next step we introduce an event-triggering mechanism (Heemels, Johansson, & Tabuada, 2012; Tabuada, 2007) into predictor-based networked control. The event-triggering condition is checked on a controller side and allows to reduce the amount of control signals sent through a controller-to-actuators network. We demonstrate that it is reasonable to choose different predictor models for a zero and non-zero controller-to-actuators delay uncertainty. Finally, we consider predictor-based event-triggered control with continuous-time measurements and

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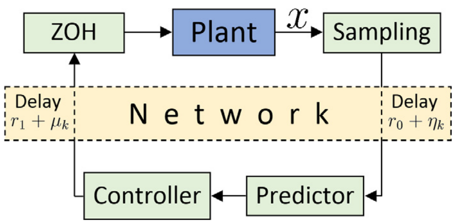

Fig. 1. NCS with a predictor.

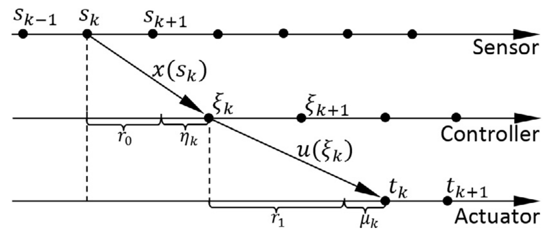

Fig. 2. Time-delays and updating times.

sampled control signals sent through a controller-to-actuators network. Such systems naturally appear when a visually observed vehicle is controlled through a wireless network. To take advantage of the continuously available measurements, we use a continuous-time predictor (Artstein, 1982; Kwon & Pearson, 1980; Mazenc & Normand-Cyrot, 2013) and a recently proposed switching approach to event-triggered control (Selivanov & Fridman, in press).

By an example of an inverted pendulum on a cart we demonstrate that the proposed approach is extremely efficient when the uncertain time-varying network-induced delays are too large for the system to be stabilizable without a predictor. Moreover, the considered event-triggering mechanism allows to significantly reduce the network workload.

2. Networked control employing predictor

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0 \tag{1}$$

with the state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, and constant matrices A, B of appropriate dimensions for which there exists $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is a Hurwitz matrix. Let $\{s_k\}$ be sampling instants such that

$$0 = s_0 < s_1 < s_2 < \cdots, \qquad \lim_{k\to\infty} s_k = \infty, \qquad s_{k+1} - s_k \leq h.$$

At each sampling time s_k the state $x(s_k)$ is transmitted to a controller, where a control signal is calculated and transmitted to actuators (see Fig. 1). We assume that the controller and the actuators are event-driven (update their outputs as soon as they receive new data). Both state and control signals are subject to network-induced delays $r_0 + \eta_k$ and $r_1 + \mu_k$, respectively. Thus, the controller updating times are $\xi_k = s_k + r_0 + \eta_k$ and the actuators updating times are $t_k = \xi_k + r_1 + \mu_k$, where $k \in \mathbb{Z}_+$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ (see Fig. 2). Here r_0 and r_1 are known constant transport delays, η_k and μ_k are time-varying delays such that

$$0 \leq \eta_k \leq \eta_M, \quad 0 \leq \mu_k \leq \mu_M, \quad \xi_k \leq \xi_{k+1}, \quad t_k \leq t_{k+1}. \tag{2}$$

We assume that the sensors and controller clocks are synchronized and together with $x(s_k)$ the time stamp s_k is transmitted so that the value of $\eta_k = \xi_k - s_k - r_0$ can be calculated on the controller side at time ξ_k. Delay uncertainty μ_k is assumed to be unknown. Note that we do not require $\eta_k + \mu_k$ to be less than the sampling interval but the sequences $\{\xi_k\}$ and $\{t_k\}$ of updating times should be increasing.

Define $u(\xi) = 0$ for $\xi < \xi_0$. Then (1) transforms to

$$\begin{aligned} \dot{x}(t) &= Ax(t), & t &\in [0, t_0), \\ \dot{x}(t) &= Ax(t) + Bu(\xi_k), & t &\in [t_k, t_{k+1}), \ k \in \mathbb{Z}_+. \end{aligned} \tag{3}$$

To construct a predictor-based controller for (3), define

$$v(\xi) \triangleq \begin{cases} 0, & \xi < \xi_0, \\ u(\xi_k), & \xi \in [\xi_k, \xi_{k+1}), \ k \in \mathbb{Z}_+ \end{cases} \tag{4}$$

and consider the change of variable (Artstein, 1982; Kwon & Pearson, 1980)

$$z(t) \triangleq e^{A(r_0+r_1)}x(t) + \int_{t-r_1}^{t+r_0} e^{A(t+r_0-\theta)}Bv(\theta)\,d\theta, \tag{5}$$

where $t \geq 0$. We set $z(t) = 0$ for $t < 0$. If $\mu_M = 0$, i.e. controller-to-actuators delay is constant, (4), (5) is the state prediction, namely, $z(t) = x(t + r_0 + r_1)$. If $\mu_k \not\equiv 0$ to obtain the precise state prediction one needs to integrate (3), where $t_k = \xi_k + r_1 + \mu_k$ depends on μ_k. Since μ_k is unknown, we use the prediction (4), (5) that is imprecise for $\mu_k \not\equiv 0$. By substituting (3) for $\dot{x}(t)$ we obtain

$$\begin{aligned} &\dot{z}(t) = Az(t) + Bv(t + r_0) - e^{A(r_0+r_1)}Bv(t - r_1), \quad t \in [0, t_0), \\ &\dot{z}(t) = Az(t) + Bv(t + r_0) + e^{A(r_0+r_1)}B\left[u(\xi_k) - v(t - r_1)\right], \\ &\quad t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}_+. \end{aligned} \tag{6}$$

Consider the following control law

$$u(\xi_k) \triangleq Kz(s_k) = K\left[e^{A(r_0+r_1)}x(s_k) + \int_{\xi_k-\eta_k-r_0-r_1}^{\xi_k-\eta_k} e^{A(\xi_k-\eta_k-\theta)}Bv(\theta)\,d\theta\right], \quad k \in \mathbb{Z}_+. \tag{7}$$

Since η_k is available to the controller at time ξ_k, the control signal (7) can be calculated. Moreover, no numerical difficulties arise while calculating the integral term in (7) with a piecewise constant $v(\theta)$ given by (4).

We analyze (4)–(7) using the time-delay approach to NCSs (Fridman, 2014; Fridman et al., 2004; Gao et al., 2008). According to (4), (7), $v(t + r_0) = Kz(s_k)$ whenever $t + r_0 \in [\xi_k, \xi_{k+1})$, that is, when $t \in [\xi_k - r_0, \xi_{k+1} - r_0)$. If $t < \xi_0 - r_0$ then $v(t + r_0) = 0 = Kz(t - \eta_0)$. Therefore,

$$v(t + r_0) = Kz(t - \tau(t)), \quad t \in \mathbb{R}, \tag{8}$$

where

$$\tau(t) = \begin{cases} \eta_0, & t < \xi_0 - r_0, \\ t - s_k, & t \in [\xi_k - r_0, \xi_{k+1} - r_0), \ k \in \mathbb{Z}_+. \end{cases}$$

Note that for $t \geq \xi_0 - r_0$

$$0 \leq \tau(t) \leq \max_k\{(s_{k+1} + r_0 + \eta_{k+1}) - r_0 - s_k\} \leq h + \eta_M.$$

By similar reasoning we obtain

$$\begin{aligned} \dot{z}(t) &= Az(t) + BKz(t - \tau(t)) \\ &\quad + e^{A(r_0+r_1)}BK[z(t - \tau_2(t)) - z(t - \tau_1(t))], \quad t \geq 0, \end{aligned} \tag{9}$$

with

$$z(0) = e^{A(r_0+r_1)}x(0), \qquad z(t) = 0 \quad \text{for } t < 0, \tag{10}$$

$$\begin{aligned} &\tau(t) \triangleq \begin{cases} \eta_0, & t < \xi_0 - r_0, \\ t - s_k, & t \in [\xi_k - r_0, \xi_{k+1} - r_0), k \in \mathbb{Z}_+, \end{cases} \\ &\tau_1(t) \triangleq \begin{cases} r_1 + r_0 + \eta_0, & t \in [0, t_0 - \mu_0), \\ t - s_k, & t \in [t_k - \mu_k, t_{k+1} - \mu_{k+1}), k \in \mathbb{Z}_+, \end{cases} \\ &\tau_2(t) \triangleq \begin{cases} r_0 + r_1 + \eta_0 + \mu_0, & t \in [0, t_0), \\ t - s_k, & t \in [t_k, t_{k+1}), k \in \mathbb{Z}_+, \end{cases} \\ &0 \leq \tau(t) \leq \bar{\tau} \triangleq h + \eta_M, \\ &r_0 + r_1 \leq \tau_1(t) \leq \tau_2(t) \leq \tau_M \triangleq r_0 + r_1 + h + \eta_M + \mu_M. \end{aligned} \tag{11}$$

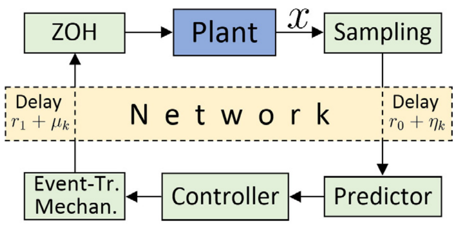

Fig. 3. NCS with a predictor and event-triggering mechanism.

Remark 1. If $\xi_k = \xi_{k+1}$ then $\tau(t) = t - s_{k-1}$ for $t \in [\xi_{k-1} - r_0, \xi_{k+1} - r_0)$ and it may seem that the bound $\tau(t) \le h + \eta_M$ can be violated. This is not the case, since $\xi_k = \xi_{k+1}$ implies $s_k + r_0 + \eta_k = s_{k+1} + r_0 + \eta_{k+1}$, that is, $\eta_{k+1} \le \eta_k - h$. Therefore, for $t \in [\xi_{k-1} - r_0, \xi_{k+1} - r_0)$

$$\begin{aligned}\tau(t) &\le \xi_{k+1} - r_0 - s_{k-1} = s_{k+1} + r_0 + \eta_{k+1} - r_0 - s_{k-1}\\ &\le (s_{k+1} - s_{k-1}) + (\eta_k - h) \le 2h + \eta_k - h = \eta_k + h.\end{aligned}$$

Similar explanation is valid for $\xi_k = \xi_{k+1} = \cdots = \xi_{k+d}$ and $t_k = t_{k+1} = \cdots = t_{k+d}$.

Remark 2. If $\mu_k \equiv 0$ then $\tau_1(t) = \tau_2(t)$ and (9) simplifies to

$$\dot z(t) = Az(t) + BKz(t - \tau(t)), \quad t \ge 0. \tag{12}$$

The system (12) is independent of r_0 and r_1. Therefore, the stability conditions for (12) are independent of r_0 and r_1: these delays are compensated by the predictor (4), (5). For $\mu_k \not\equiv 0$ the system (9) contains the residual error that appears due to impreciseness of the predictor (4), (5).

Remark 3. While studying robustness of a predictor for the time-delay $r + \mu(t)$ with the uncertainty $\mu(t) \le \mu_M$, usually the residual $e^{Ar}BK[z(t - r - \mu(t)) - z(t - r)]$ appears in the closed-loop system (Fridman, 2014; Karafyllis & Krstic, 2013). Since $\dot z$ is generally proved to be bounded, even for unstable A and large r by reducing μ_M one can retain this error small enough to preserve the stability. In a word, r can be made arbitrary large by decreasing μ_M. This does not hold for sampled-data systems: for arbitrary small $\mu_k > 0$ when $t \in [t_k - \mu_k, t_k)$ the arguments of $z(t - \tau_1(t))$ and $z(t - \tau_2(t))$ belong to different sampling intervals, namely, $(t - \tau_1(t)) - (t - \tau_2(t)) = s_k - s_{k-1}$ (if $t_k - \mu_k > t_{k-1}, k \ge 1$). Therefore, smallness of the residual in (9) for large $r = r_0 + r_1$ can be guaranteed only by reducing μ_M together with the maximum sampling interval h.

Stability conditions for the systems (9) and (12) follow from Theorem 1 and Proposition 1 of the next section.

3. Event-triggering with sampled measurements

To reduce the workload of a controller-to-actuators network, we incorporate an event-triggering mechanism (see Tabuada, 2007). The idea is to send only those control signals $u(\xi_k)$ which relative change is greater than a constant $\sigma \in [0, 1)$ (see Fig. 3), namely, that violate the following event-triggering rule

$$\left(\hat u(\xi_{k-1}) - u(\xi_k)\right)^T \Omega \left(\hat u(\xi_{k-1}) - u(\xi_k)\right) \le \sigma u^T(\xi_k)\Omega u(\xi_k), \tag{13}$$

where a matrix $\Omega \ge 0$ and a scalar $\sigma \ge 0$ are event-triggering parameters and $\hat u(\xi_{k-1})$ is the last sent control value before the time instant ξ_k:

$$\hat u(\xi_k) = \begin{cases}\hat u(\xi_{k-1}), & \text{if (13) is true},\\ u(\xi_k), & \text{otherwise},\end{cases} \tag{14}$$

with $\hat u(\xi_{-1}) = 0$. Note that the sensor sends measurements at time instants s_k (such that $s_{k+1} - s_k \le h$) independent of the event-triggering events. Then (3) takes the form

$$\begin{aligned}\dot x(t) &= Ax(t), && t \in [0, t_0),\\ \dot x(t) &= Ax(t) + B\hat u(\xi_k), && t \in [t_k, t_{k+1}), k \in \mathbb{Z}_+.\end{aligned} \tag{15}$$

Consider the change of variable (5) with $v(\theta)$ to be defined. By substituting (15) for $\dot x(t)$, we obtain

$$\begin{aligned}&\dot z(t) = Az(t) + Bv(t + r_0) - e^{A(r_0+r_1)}Bv(t - r_1), \quad t \in [0, t_0),\\ &\dot z(t) = Az(t) + Bv(t + r_0) + e^{A(r_0+r_1)}B\left[\hat u(\xi_k) - v(t - r_1)\right], \\ &\quad t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+.\end{aligned} \tag{16}$$

We now show that for $\mu_M = 0$ and $\mu_M > 0$ one should pick different functions $v(\theta)$ in the predictor (5).

1. Let $\mu_M = 0$. To cancel the last term in (15) we take $v(t - r_1) = \hat u(\xi_k)$ for $t \in [t_k, t_{k+1})$ or, equivalently,

$$v(\xi) \triangleq \begin{cases}0, & \xi < \xi_0,\\ \hat u(\xi_k), & \xi \in [\xi_k, \xi_{k+1}),\ k \in \mathbb{Z}_+.\end{cases} \tag{17}$$

Then (5), (17) is the state prediction for the system (15), i.e. $z(t) = x(t + r_0 + r_1)$. The system (15) takes the form

$$\dot z(t) = Az(t) + Bv(t + r_0), \quad t \ge 0.$$

Let us define

$$e_0(t) \triangleq \begin{cases}0, & t < \xi_0,\\ \hat u(\xi_k) - u(\xi_k), & t \in [\xi_k, \xi_{k+1}),\ k \in \mathbb{Z}_+.\end{cases}$$

Then for $t \in [\xi_k - r_0, \xi_{k+1} - r_0)$ we have

$$\begin{aligned}v(t + r_0) &= \hat u(\xi_k) = u(\xi_k) + e_0(t + r_0) = Kz(s_k) + e_0(t + r_0)\\ &= Kz(t - \tau(t)) + e_0(t + r_0)\end{aligned}$$

with $\tau(t)$ defined in (11). For $t < \xi_0 - r_0$, $v(t + r_0) = 0 = Kz(t - \eta_0) + e_0(t + r_0)$. Therefore,

$$\dot z(t) = Az(t) + BKz(t - \tau(t)) + Be_0(t + r_0), \quad t \ge 0 \tag{18}$$

with (10), and (13), (14) yield

$$0 \le \sigma z^T(t - \tau(t))K^T\Omega Kz(t - \tau(t)) - e_0^T(t + r_0)\Omega e_0(t + r_0)$$

for $t \ge 0$. It may seem that (18) depends on the future, since $e_0(t + r_0)$ enters the system. This is not the case, since $e_0(\xi)$ for $\xi \in [\xi_k, \xi_{k+1})$ is fully defined by $z(s)$ with $s \le s_k = \xi_k - r_0 - \eta_k$.

2. Let $\mu_k \not\equiv 0$. Then the last term in (15) cannot be canceled, since this would require to take $v(\xi) = \hat u(\xi_k)$ for $\xi \in [\xi_k + \mu_k, \xi_{k+1} + \mu_{k+1})$ with unknown μ_k. If one defines $v(\xi)$ as in (17) and uses $v(\xi) = \hat u(\xi_k) = u(\xi_k) + e_0(\xi)$, the functions $v(t + r_0)$, $v(t - r_1)$, $\hat u(\xi_k)$ present in (15) will introduce three errors due to triggering e_0 with different arguments. To avoid additional triggering errors, we do not include them into the definition of $v(\xi)$, namely, we use (4) where $v(\xi) = u(\xi_k)$ or zero. Let us define

$$e_1(t) \triangleq \begin{cases}0, & t < t_0,\\ \hat u(\xi_k) - u(\xi_k), & t \in [t_k, t_{k+1}), k \in \mathbb{Z}_+.\end{cases}$$

Then we have

$$\begin{aligned}0 &= Kz(t - \tau_2(t)) + e_1(t), \quad t \in [0, t_0),\\ \hat u(\xi_k) &= u(\xi_k) + e_1(t) = Kz(s_k) + e_1(t)\\ &= Kz(t - \tau_2(t)) + e_1(t), \quad t \in [t_k, t_{k+1}), k \in \mathbb{Z}_+\end{aligned}$$

with $\tau_2(t)$ defined in (11). By arguments similar to those from Section 2 we obtain

$$\begin{aligned}\dot z(t) &= Az(t) + BKz(t - \tau(t)) + e^{A(r_0+r_1)}Be_1(t)\\ &\quad + e^{A(r_0+r_1)}BK[z(t - \tau_2(t)) - z(t - \tau_1(t))], \quad t \ge 0,\end{aligned} \tag{19}$$

with (10), where τ, τ_1, τ_2 are defined in (11) and due to (13), (14) for $t \ge 0$

$$0 \le \sigma z^T(t - \tau_2(t))K^T\Omega Kz(t - \tau_2(t)) - e_1^T(t)\Omega e_1(t). \tag{20}$$

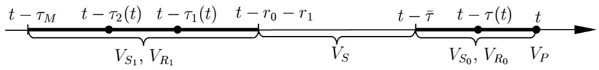

Fig. 4. Lyapunov–Krasovskii functional.

Remark 4. Note that for $\mu_M = 0$ (19) transforms to

$$\dot{z}(t) = Az(t) + BKz(t-\tau(t)) + e^{A(r_0+r_1)}Be_1(t).$$

Since the triggering error $e_1(t)$ is multiplied by $e^{A(r_0+r_1)}$, to guarantee the stability of the system for unstable A and large r_0+r_1 one needs to retain $e_1(t)$ small enough. This problem does not appear in the system (18) for which the stability conditions are independent of r_0 and r_1 (see Proposition 1).

To avoid some technical complications, we assume that $\bar{\tau} = h + \eta_M \leq r_0 + r_1$. The stability conditions are derived using Lyapunov–Krasovskii functional (see Fig. 4)

$$V = V_P + V_{S_0} + V_{R_0} + V_S + V_{S_1} + V_{R_1},$$

where

$$V_P = z^T(t)Pz(t), \quad P > 0,$$
$$V_{S_0} = \int_{t-\bar{\tau}}^{t} e^{2\alpha(s-t)} z^T(s)S_0 z(s)\,ds, \quad S_0 \geq 0,$$
$$V_{R_0} = \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{z}^T(s)R_0\dot{z}(s)\,ds\,d\theta, \quad R_0 \geq 0,$$
$$V_S = \int_{t-r_0-r_1}^{t-\bar{\tau}} e^{2\alpha(s-t)} z^T(s)Sz(s)\,ds, \quad S \geq 0,$$
$$V_{S_1} = \int_{t-\tau_M}^{t-r_0-r_1} e^{2\alpha(s-t)} z^T(s)S_1 z(s)\,ds, \quad S_1 \geq 0,$$
$$V_{R_1} = (\tau_M - r_0 - r_1) \times \int_{-\tau_M}^{-r_0-r_1}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{z}^T(s)R_1\dot{z}(s)\,ds\,d\theta, \quad R_1 \geq 0.$$

Note that the delayed arguments of z in (19) belong to two bold regions in Fig. 4. To analyze these regions, we use standard delay-dependent terms in V (see, e.g., Fridman, 2014). To allow for large transport delays r_0 and r_1, we use only delay-independent term V_S for the interval $[t-r_0-r_1, t-\bar{\tau}]$.

Lemma 1. *For given $\mu_M \geq 0$, $\eta_M \geq 0$, and $\alpha > 0$ let there exist an $n\times n$ matrix $P > 0$, $n\times n$ non-negative matrices S, S_0, S_1, R_0, R_1, an $m\times m$ matrix $\Omega \geq 0$, and $n\times n$ matrices P_2, P_3, G_i $(i = 0, \ldots, 3)$ such that*

$$\Phi \leq 0, \quad \begin{bmatrix} R_0 & G_0 \\ * & R_0 \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_1 & G_i \\ * & R_1 \end{bmatrix} \geq 0, \quad i = 1, 2, 3,$$

where $\Phi = \{\Phi_{ij}\}$ is the symmetric matrix composed from

$$\Phi_{11} = 2\alpha P + S_0 - \bar{\rho}R_0 + P_2^T A + A^T P_2,$$
$$\Phi_{12} = P - P_2^T + A^T P_3,$$
$$\Phi_{13} = \bar{\rho}(R_0 - G_0) + P_2^T BK, \quad \Phi_{14} = \bar{\rho}G_0,$$
$$\Phi_{19} = P_2^T e^{A(r_0+r_1)}B, \quad \Phi_{17} = -\Phi_{16} = \Phi_{19}K,$$
$$\Phi_{22} = \bar{\tau}^2 R_0 + (\tau_M - r_0 - r_1)^2 R_1 - P_3 - P_3^T, \quad \Phi_{23} = P_3^T BK,$$
$$\Phi_{29} = P_3^T e^{A(r_0+r_1)}B, \quad \Phi_{27} = -\Phi_{26} = \Phi_{29}K,$$
$$\Phi_{34} = \bar{\rho}(R_0 - G_0), \quad \Phi_{33} = -\Phi_{34} - \Phi_{34}^T,$$
$$\Phi_{44} = \bar{\rho}(S - S_0 - R_0), \quad \Phi_{56} = \rho_M(R_1 - G_1),$$
$$\Phi_{55} = e^{-2\alpha(r_0+r_1)}(S_1 - S) - \rho_M R_1, \quad \Phi_{58} = \rho_M G_2,$$
$$\Phi_{57} = \rho_M(G_1 - G_2), \quad \Phi_{66} = -\Phi_{56} - \Phi_{56}^T,$$
$$\Phi_{67} = \rho_M(R_1 - G_1 + G_2 - G_3), \quad \Phi_{68} = \rho_M(G_3 - G_2),$$
$$\Phi_{78} = \rho_M(R_1 - G_3), \quad \Phi_{77} = -\Phi_{78} - \Phi_{78}^T + \sigma K^T\Omega K,$$
$$\Phi_{88} = -\rho_M(S_1 + R_1), \Phi_{99} = -\Omega, \quad \bar{\rho} = e^{-2\alpha\bar{\tau}},$$
$$\rho_M = e^{-2\alpha\tau_M},$$

other blocks are zero matrices. Then the system (10), (19) is exponentially stable with a decay rate α, i.e. for some $M > 0$ solutions of the system satisfy

$$|z(t)| \leq Me^{-\alpha t}|z(0)|, \quad t \geq 0. \tag{21}$$

Proof is given in Appendix A.

Theorem 1 (*Sampled Event-Triggering*)**.** *Under the conditions of Lemma 1 the system (7), (13)–(15) with $\upsilon(\theta)$ given by (4) is exponentially stable with a decay rate α, i.e. for some $M > 0$ solutions of the system satisfy*

$$|x(t)| \leq Me^{-\alpha t}|x(0)|. \tag{22}$$

Proof is given in Appendix B.

Remark 5. If $A + BK$ is Hurwitz and $\alpha = \tau_M = 0$ the LMIs of Lemma 1 are always feasible by the standard arguments for delay-dependent conditions (Fridman, 2014). That is, LMIs of Lemma 1 establish relation between the decay rate, sampling period, and time-delays that preserve exponential stability of the system (4), (7), (13)–(15).

Corollary 1. *If conditions of Lemma 1 are satisfied with $\sigma = 0$, $\Omega > 0$, the system (3) under the control law (7) with $\upsilon(\theta)$ given by (4) is exponentially stable with a decay rate α.*

Proof. For $\sigma = 0$, $\Omega > 0$ event-triggering mechanism (13), (14) implies $\hat{u}(\xi_k) = u(\xi_k)$ and $e_1(t) \equiv 0$, therefore, (19) coincides with (9). Then under conditions of Lemma 1 (9) is exponentially stable. This implies exponential stability of (3), (4), (7) due to the change of variable (4), (5). □

For the case of $\mu_M = 0$ the next proposition gives stability conditions independent of r_0 and r_1.

Proposition 1. *For $\mu_M = 0$ and given $\eta_M \geq 0$, $\alpha > 0$, if there exist an $n\times n$ matrix $P > 0$, $n\times n$ non-negative matrices S, R, an $m\times m$ matrix $\Omega \geq 0$, and $n\times n$ matrices P_2, P_3, G such that*

$$\Psi \leq 0, \quad \begin{bmatrix} R & G \\ * & R \end{bmatrix} \geq 0,$$

where $\Psi = \{\Psi_{ij}\}$ is the symmetric matrix composed from

$$\Psi_{11} = 2\alpha P + S - \bar{\rho}R + P_2^T A + A^T P_2, \quad \Psi_{12} = P - P_2^T + A^T P_3,$$
$$\Psi_{13} = \bar{\rho}(R - G) + P_2^T BK, \quad \Psi_{14} = \bar{\rho}G, \quad \Psi_{15} = P_2^T B,$$
$$\Psi_{55} = -\Omega, \quad \Psi_{22} = \bar{\tau}^2 R - P_3 - P_3^T,$$
$$\Psi_{25} = P_3^T B, \quad \Psi_{23} = \Psi_{25}K, \quad \Psi_{34} = \bar{\rho}(R - G),$$
$$\Psi_{33} = -\Psi_{34} - \Psi_{34}^T + \sigma K^T\Omega K, \quad \Psi_{44} = -\bar{\rho}(S + R), \bar{\rho} = e^{-2\alpha\bar{\tau}},$$

other blocks are zero matrices, then (7), (13)–(15) with $\upsilon(\theta)$ given by (17) is exponentially stable with a decay rate α.

Proof is based on the representation (18) and is very similar to the proof of Lemma 1.

4. Event-triggering with continuous measurements

In Section 2 the control signals are sent at $\xi_k = s_k + r_0 + \eta_k$, where $r_0 + \eta_k$ are sensors-to-controller delays and s_k are measurement sampling instants. In this section we consider the system (3) without a sensors-to-controller network ($r_0 = \eta_k = 0$) and with measurements continuously available to the controller. The control law is given by

$$u(\xi) = Kz(\xi), \quad \xi \geq 0, \tag{23}$$

where $z(\xi)$ is given by (5) with $v(\theta)$ to be defined. To obtain the time instants $\{\xi_k\}$ when a continuously changing control signal $u(\xi)$ is sampled and sent through a controller-to-actuators network, we use a switching approach to event-triggered control (Selivanov & Fridman, in press). Namely, we choose $\xi_0 = 0$,

$$\begin{aligned}\xi_{k+1} = \min\{\xi \geq \xi_k + h \mid (u(\xi_k) - u(\xi))^T \Omega (u(\xi_k) - u(\xi)) \\ \geq \sigma u^T(\xi)\Omega u(\xi)\},\end{aligned} \tag{24}$$

where a matrix $\Omega \geq 0$ and scalars $h > 0$, $\sigma \geq 0$ are event-triggering parameters. According to (24), after the controller sends out the control signal $u(\xi_k)$, it waits for at least h seconds. Then it starts to continuously check the event-triggering rule and sends the next control signal when the event-triggering condition is violated. The idea of a switching approach to event-triggered control is to present the closed-loop system as a switching between a system with sampling h and a system with event-triggering mechanism. This allows to ensure large inter-event times and reduce the amount of sent signals (Selivanov & Fridman, in press).

Calculating $\dot{z}$ given by (5) in view of (3) we obtain (5) (with $r_0 = \eta_k = 0$). Similar to Section 3 depending on the value of μ_M one should choose different functions $v(\theta)$.

1. Let $\mu_M = 0$. For $v(\theta)$ given in (4), Eq. (5) takes the form

$$\dot{z}(t) = Az(t) + Bu(\xi_k), \quad t \in [\xi_k, \xi_{k+1}).$$

Following (Selivanov & Fridman, in press) we present the latter system as a switching between two systems:

$$\begin{aligned}\dot{z}(t) &= Az(t) + BKz(t - \tau_3(t)), \quad t \in [\xi_k, \xi_k + h), \\ \dot{z}(t) &= (A + BK)z(t) + Be_2(t), \quad t \in [\xi_k + h, \xi_{k+1}),\end{aligned} \tag{25}$$

where the initial conditions are given by (10), and

$$\begin{aligned}\tau_3(t) &\triangleq t - \xi_k \leq h, & t \in [\xi_k, \xi_k + h), \\ e_2(t) &\triangleq Kz(\xi_k) - Kz(t), & t \in [\xi_k + h, \xi_{k+1})\end{aligned}$$

and (24) implies

$$0 \leq \sigma z^T(t)K^T\Omega Kz(t) - e_2^T(t)\Omega e_2(t), \quad t \in [\xi_k + h, \xi_{k+1}).$$

2. Let $\mu_k \not\equiv 0$. As it has been explained in Section 3, in this case it is reasonable not to include the error due to triggering in the definition of $v(\theta)$. Therefore, we take

$$v(\xi) \triangleq u(\xi) = Kz(\xi), \quad \xi \geq 0. \tag{26}$$

Then by calculating $\dot{z}$ we obtain

$$\begin{aligned}&\dot{z}(t) = Az(t) + BKz(t) - e^{Ar_1}BKz(t - r_1), \quad t \in [0, t_0), \\ &\dot{z}(t) = Az(t) + BKz(t) + e^{Ar_1}BK[z(\xi_k) - z(t - r_1)], \\ &\quad t \in [t_k, t_{k+1}), k \in \mathbb{Z}_+.\end{aligned} \tag{27}$$

Further analysis of the system (26) is based on a switching approach to event-triggered control (Selivanov & Fridman, in press). Define

$$t_{-1}^* \triangleq \min\{h, t_0\}, \qquad t_k^* \triangleq \min\{t_k + h, t_{k+1}\} \quad \text{for } k \in \mathbb{Z}_+.$$

We have $z(t - r_1 - \tau_4(t)) = 0$ for $t \in [0, t_{-1}^*)$ and $z(\xi_k) = z(t - r_1 - \tau_4(t))$ for $t \in [t_k, t_k^*)$, where

$$\tau_4(t) \triangleq \begin{cases} \mu_0, & t \in [0, t_{-1}^*), \\ t - \xi_k - r_1, & t \in [t_k, t_k^*). \end{cases}$$

Note that $\tau_4(t) \leq \tilde{\tau} \triangleq h + \mu_M$. Further, $Kz(t - r_1 - \mu(t)) + e_3(t) = 0$ for $t \in [t_{-1}^*, t_0)$ and $Kz(\xi_k) = Kz(t - r_1 - \mu(t)) + e_3(t)$ for $t \in [t_k^*, t_{k+1})$, where

$$\mu(t) \triangleq \begin{cases} \mu_0, & t \in [t_{-1}^*, t_0), \\ \mu_k + (t - t_k - h)\dfrac{\mu_{k+1} - \mu_k}{t_{k+1} - t_k - h}, & t \in [t_k^*, t_{k+1}), \end{cases}$$

$$e_3(t) \triangleq \begin{cases} 0, & t \in [t_{-1}^*, t_0), \\ Kz(\xi_k) - Kz(t - r_1 - \mu(t)), & t \in [t_k^*, t_{k+1}). \end{cases}$$

The function $\mu(t)$ is chosen so that $t - r_1 - \mu(t) \in [\xi_k + h, \xi_{k+1})$ for $t \in [t_k^*, t_{k+1})$, therefore, (24) implies

$$\begin{aligned}0 \leq\ & \sigma z^T(t - r_1 - \mu(t))K^T\Omega Kz(t - r_1 - \mu(t)) \\ & - e_3^T(t)\Omega e_3(t)\end{aligned} \tag{28}$$

for $t \in [t_{k-1}^*, t_k)$ with $k \in \mathbb{Z}_+$.

Finally, the system (26) is presented in the form

$$\begin{aligned}\dot{z}(t) \ =\ & (A + BK)z(t) + e^{Ar_1}BK[z(t - r_1 - \tau_4(t)) \\ & - z(t - r_1)], \quad t \in [0, t_{-1}^*) \cup [t_k, t_k^*),\end{aligned} \tag{29}$$

$$\begin{aligned}\dot{z}(t) \ =\ & (A + BK)z(t) + e^{Ar_1}BK[z(t - r_1 - \mu(t)) \\ & - z(t - r_1)] + e^{Ar_1}Be_3(t), \quad t \in [t_{-1}^*, t_0) \cup [t_k^*, t_{k+1})\end{aligned} \tag{30}$$

with (10) and $0 \leq \tau_4(t) \leq \tilde{\tau} = h + \mu_M$, $0 \leq \mu(t) \leq \mu_M$.

Lemma 2. *For given $\mu_M \geq 0$ and $\alpha > 0$ let there exist an $n \times n$ matrix $P > 0$, $n \times n$ non-negative matrices S, S_0, S_1, R_0, R_1, an $m \times m$ matrix $\Omega \geq 0$, and $n \times n$ matrices P_2, P_3, G_0, G_1 such that*

$$\Sigma \leq 0, \qquad \Xi \leq 0, \qquad \begin{bmatrix} R_0 & G_0 \\ * & R_0 \end{bmatrix} \geq 0, \qquad \begin{bmatrix} R_1 & G_1 \\ * & R_1 \end{bmatrix} \geq 0,$$

where $\Sigma = \{\Sigma_{ij}\}$ and $\Xi = \{\Xi_{ij}\}$ are the symmetric matrices composed from the matrices

$$\begin{aligned}
&\Sigma_{11} = \Xi_{11} = 2\alpha P + S + P_2^T(A + BK) + (A + BK)^T P_2, \\
&\Sigma_{12} = \Xi_{12} = P - P_2^T + (A + BK)^T P_3, \qquad \Sigma_{34} = \rho_M R_0, \\
&\Sigma_{46} = \tilde{\rho}G_1, \qquad \Sigma_{15} = \Xi_{14} = -\Sigma_{13} = -\Xi_{13} = P_2^T e^{Ar_1}BK, \\
&\Sigma_{45} = \tilde{\rho}(R_1 - G_1), \\
&\Sigma_{22} = \Xi_{22} = \mu_M^2 R_0 + h^2 R_1 - P_3 - P_3^T, \qquad \Sigma_{55} = -\Sigma_{45} - \Sigma_{45}^T, \\
&\Sigma_{25} = \Xi_{24} = -\Sigma_{23} = -\Xi_{23} = P_3^T e^{Ar_1}BK, \qquad \Sigma_{56} = \tilde{\rho}(R_1 - G_1), \\
&\Sigma_{33} = \Xi_{33} = e^{-2\alpha r_1}(S_0 - S) - \rho_M R_0, \qquad \Sigma_{66} = -\tilde{\rho}(S_1 + R_1), \\
&\Sigma_{44} = -\rho_M(R_0 + S_0 - S_1) - \tilde{\rho}R_1, \qquad \Xi_{17} = P_2^T e^{Ar_1}B, \\
&\Xi_{77} = -\Omega, \qquad \Xi_{27} = P_3^T e^{Ar_1}B, \\
&\Xi_{34} = \Xi_{45} = \rho_M(R_0 - G_0), \qquad \Xi_{35} = \rho_M G_0, \\
&\Xi_{44} = -\Xi_{34} - \Xi_{34}^T + \sigma K^T\Omega K, \\
&\Xi_{55} = \rho_M(S_1 - S_0 - R_0) - \tilde{\rho}R_1, \qquad \Xi_{56} = \tilde{\rho}R_1, \\
&\Xi_{66} = -\tilde{\rho}(S_1 + R_1), \; \tilde{\rho} = e^{-2\alpha(r_1 + \tilde{\tau})}, \\
&\rho_M = e^{-2\alpha(r_1 + \mu_M)},
\end{aligned}$$

other blocks are zero matrices. Then the system (10), (29), (30) with ξ_k given by (24) is exponentially stable with a decay rate α (i.e. (21) holds).

Proof is given in Appendix C.

Theorem 2 (*Continuous Event-Triggering*)**.** *Under the conditions of Lemma 2 the system (3), (5), (23), (24) with $v(\theta)$ given by (26) is exponentially stable with a decay rate α (i.e. (22) holds).*

Proof is similar to the proof of Theorem 1.

Remark 6. The control law (5), (23) with $v(\theta)$ given by (26) requires the knowledge of $z(t)$ for any $t \geq 0$. To obtain $z(t)$ during the evolution of the system (3), (5), (24), (23), (26) one has to solve the differential equation

$$\dot{z}(t) = (A+BK)z(t) - e^{Ar_1}BKz(t-r_1), \quad t \in [0, t_0),$$
$$\dot{z}(t) = (A+BK)z(t) + e^{Ar_1}BK[z(\xi_k) - z(t-r_1)], \quad t \in [t_k, t_{k+1})$$

with $z(0) = e^{Ar_1}x(0)$ and $z(t) = 0$ for $t < 0$.

Proposition 2. *For $\mu_M = 0$ and a given $\alpha > 0$, if there exist $n \times n$ matrices $P > 0$, $S \geq 0$, $R \geq 0$, an $m \times m$ matrix $\Omega \geq 0$, and $n \times n$ matrices P_2, P_3, G such that*

$$M \leq 0, \qquad N \leq 0, \qquad \begin{bmatrix} R & G \\ * & R \end{bmatrix} \geq 0,$$

where $M = \{M_{ij}\}$ and $N = \{N_{ij}\}$ are the symmetric matrices composed from the matrices

$$M_{11} = 2\alpha P + S - \rho_h R + P_2^T A + A^T P_2, \qquad M_{12} = P - P_2^T + A^T P_3,$$
$$M_{13} = \rho_h(R-G) + P_2^T BK, \qquad M_{14} = \rho_h G,$$
$$M_{22} = h^2 R - P_3 - P_3^T,$$
$$M_{23} = P_3^T BK, \qquad M_{34} = \rho_h(R-G), \qquad M_{33} = -M_{34} - M_{34},$$
$$M_{44} = -\rho_h(S+R), \qquad N_{12} = P - P_2^T + (A+BK)^T P_3,$$
$$N_{11} = 2\alpha P + S - \rho_h R + \sigma K^T \Omega K + P_2^T(A+BK) + (A+BK)^T P_2,$$
$$N_{13} = \rho_h R, \qquad N_{14} = P_2^T B, \qquad N_{22} = h^2 R - P_3 - P_3^T,$$
$$N_{24} = P_3^T B,$$
$$N_{33} = -\rho_h(S+R), \qquad N_{44} = -\Omega, \qquad \rho_h = e^{-2\alpha h},$$

other blocks are zero matrices, then the system (3), (5), (24), (23) with $v(\theta)$ given by (26) is exponentially stable with a decay rate α.

Proof is based on the representation (25) and is very similar to the proof of Lemma 2.

Remark 7. Let us set $P_3 = \varepsilon_1 P_2$, $\Omega = \varepsilon_2 I_m$ and multiply LMIs of Lemmas 1 and 2, Propositions 1 and 2 by $I \otimes P_2^{-1}$ and its transpose from the right and the left, respectively. By denoting $\bar{P}_2 = P_2^{-1}$, $Y = K\bar{P}_2$ and applying Schur complement to $\sigma Y^T \Omega Y$, we obtain LMIs with tuning parameters ε_1, ε_2 that allow to find controller gain $K = Y\bar{P}_2^{-1}$. Since requirements $P_3 = \varepsilon_1 P_2$, $\Omega = \varepsilon_2 I_m$ may be restrictive, after obtaining K one should use Lemmas 1 and 2 or Propositions 1, 2 to obtain larger bound for time-delays and a decay rate. For the details on the LMI-based design see Fridman (2014) and Suplin, Fridman, and Shaked (2007).

5. Example: inverted pendulum on a cart

Following (Wang & Lemmon, 2009) we consider an inverted pendulum on a cart controlled through a network described by (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mgM^{-1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ M^{-1} \\ 0 \\ -(Ml)^{-1} \end{bmatrix}, \qquad (31)$$

where $M = 10$ kg is the cart mass, $m = 1$ kg is the bob mass, $l = 3$ m is the arm length and $g = 10$ m/s^2 is the gravitational acceleration. The state $x = (y, \dot{y}, \theta, \dot{\theta})^T$ is combined of cart's position y, cart's velocity $\dot{y}$, bob's angle θ and bob's angular velocity $\dot{\theta}$. For such parameters the open-loop system is unstable and can be stabilized by the control law $u(t) = Kx(t)$ with $K =$ [2, 12, 378, 210]. In what follows we compare different control strategies proposed in this paper.

We start by considering a system with both sensors-to-controller and controller-to-actuators networks. The numerical simulations show that the system (3), (31) under the controller $u(t) = Kx(t)$ (without a predictor) is not stable for $r_0 = r_1 = 0.1$, $h = 0.0369$, and $\eta_M = \mu_M = 0$. The conditions of Corollary 1 are satisfied for the same h and larger $r_0 = r_1 = 0.2$, $\eta_M = \mu_M = 0.01$, whereas the decay rate is $\alpha = 0.01$. That is, the predictor-based control admits significantly larger network delays. Furthermore, this implies that within 20 s of simulation $\lfloor 20/h \rfloor + 1 = 543$ signals are sent through each network in the system (3), (31) under the predictor-based controller (4), (7) ($\lfloor \cdot \rfloor$ stands for the integer part). For the system (15), (31) under the event-triggered controller (4), (7), (13), (14) with $\sigma = 0.01$ Theorem 1 gives $h = 0.0315$. This bound is smaller than the one given by Corollary 1, which means that the event-triggered control requires the measurements $x(s_k)$ to be sent more often but allows to reduce the amount of sent control values $u(\xi_k)$. To obtain the amount of sent signals under the event-triggered control, we perform numerical simulations with $x(0) = [0.98, 0, 0.2, 0]$ and random η_k, μ_k satisfying (2). The results are given in Table 1. As one can see event-triggering allows to reduce the workload of the controller-to-actuators network by more than 75%. The total amount of signals sent through both sensors-to-controller and controller-to-actuators networks is $543 \cdot 2 = 1086$ for the predictor-based controller (4), (7) and $\lfloor 20/h \rfloor + 1 + 116 = 751$ for the event-triggered controller (4), (7), (13), (14).

Now we consider a system with only a controller-to-actuators network ($r_0 = \eta_M = 0$) and continuous measurements. For this case one can apply sampled predictor-based controller (4), (7) or sampled event-triggered controller (4), (7), (13), (14) (with $s_k = \xi_k$). The sampled approach simplifies the calculation of the integral term in (5) but does not take advantage of the continuously available measurements. Indeed, as one can see from Table 1 continuous predictor (5), (26) without event-triggering ($\sigma = 0$ in (24)) reduces the network workload compared to the sampled predictor by almost 40%.

To compare the sampled event-triggering mechanism (4), (5), (13), (14) and the switching event-triggering mechanism (5), (24), (26), for $\alpha = 0.01$ and each value of $\sigma = 0.01, 0.02, \ldots, 1$ we apply Theorems 1 and 2 to find the maximum allowable h. Then we perform numerical simulations for each pair of (σ, h) with μ_k subject to (2) ($r_1 = 0.2$, $\mu_M = 0.01$) and choose the pair (σ, h) that leads to the smallest amount of sent control signals. In Table 1 one can see that both event-triggering mechanisms significantly reduce the amount of sent control signals. The switching event-triggering reduces the network workload by almost 15% compared to the sampled event-triggering.

6. Conclusions

We considered predictor-based control of NCSs with uncertain network delays. For the event-triggered control we showed that one should use different predictor models depending on the value of the controller-to-actuators delay uncertainty. To take advantage of the continuously available measurements in the systems with only a controller-to-actuators network, we considered a continuous-time predictor with a switching event-triggering mechanism. For the proposed control strategies we obtained LMI-based stability conditions that guaranty the desired exponential decay rate of convergence and allow to find appropriate controller gains. An example of inverted pendulum on a cart demonstrates that event-triggering mechanism allows to reduce the network workload and in those cases where the continuous-time predictor can be applied it has some advantages over the sampled one.

Table 1
Sent control signals (SCS) for different control strategies ($\alpha = 0.01, r_1 = 0.2, \mu_M = 0.01$).

	$r_0 = 0.2, \eta_M = 0.01$			$r_0 = \eta_M = 0$		
	σ	h	SCS	σ	h	SCS
Sampled predictor (4), (7)	0	0.0369	543	0	0.0646	310
Sampled event-triggering (13), (14)	0.01	0.0315	116	0.07	0.046	56
Continuous predictor (5), (26)	–	–	–	0	0.105	191
Switching event-triggering (24)	–	–	–	0.13	0.105	48

Appendix A. Proof of Lemma 1

For $t \geq \tau_M$ we have

$$\begin{aligned}
&\dot V_P + 2\alpha V_P = 2z^T(t)P\dot z(t) + 2\alpha z^T(t)Pz(t),\\
&\dot V_{S_0} + 2\alpha V_{S_0} = z^T(t)S_0 z(t) - e^{-2\alpha\bar\tau}z^T(t-\bar\tau)S_0 z(t-\bar\tau),\\
&\dot V_S + 2\alpha V_S = e^{-2\alpha\bar\tau}z^T(t-\bar\tau)Sz(t-\bar\tau)\\
&\qquad - e^{-2\alpha(r_0+r_1)}z^T(t-r_0-r_1)Sz(t-r_0-r_1),\\
&\dot V_{S_1} + 2\alpha V_{S_1} = e^{-2\alpha(r_0+r_1)}z^T(t-r_0-r_1)S_1\\
&\quad \times z(t-r_0-r_1) - e^{-2\alpha\tau_M}z^T(t-\tau_M)S_1 z(t-\tau_M).
\end{aligned} \tag{A.1}$$

Using Jensen's inequality (Gu, Kharitonov, & Chen, 2003), Park's theorem (Park, Ko, & Jeong, 2011) and taking into account that $\tau_1(t) \leq \tau_2(t)$ (Liu, Fridman, & Hetel, 2012) we obtain

$$\begin{aligned}
\dot V_{R_0} + 2\alpha V_{R_0} &= \bar\tau^2 \dot z^T(t)R_0\dot z(t)\\
&\quad - \bar\tau\int_{t-\bar\tau}^{t} e^{2\alpha(s-t)}\dot z^T(s)R_0\dot z(s)\,ds \leq \bar\tau^2\dot z^T(t)R_0\dot z(t)\\
&\quad - e^{-2\alpha\bar\tau}\begin{bmatrix} z(t)-z(t-\tau(t))\\ z(t-\tau(t))-z(t-\bar\tau)\end{bmatrix}^T \begin{bmatrix} R_0 & G_0\\ G_0^T & R_0\end{bmatrix}\\
&\quad \times \begin{bmatrix} z(t)-z(t-\tau(t))\\ z(t-\tau(t))-z(t-\bar\tau)\end{bmatrix},
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\dot V_{R_1} + 2\alpha V_{R_1} = (\tau_M - r_0 - r_1)^2 \dot z^T(t)R_1\dot z(t)\\
&\quad -(\tau_M - r_0 - r_1)\int_{t-\tau_M}^{t-r_0-r_1} e^{2\alpha(s-t)}\dot z^T(s)R_1\dot z(s)\,ds\\
&\leq (\tau_M - r_0 - r_1)^2\dot z^T(t)R_1\dot z(t) - e^{-2\alpha\tau_M}\\
&\quad \times \begin{bmatrix} z(t-r_0-r_1)-z(t-\tau_1(t))\\ z(t-\tau_1(t))-z(t-\tau_2(t))\\ z(t-\tau_2(t))-z(t-\tau_M)\end{bmatrix}^T \begin{bmatrix} R_1 & G_1 & G_2\\ * & R_1 & G_3\\ * & * & R_1\end{bmatrix}\\
&\quad \times \begin{bmatrix} z(t-r_0-r_1)-z(t-\tau_1(t))\\ z(t-\tau_1(t))-z(t-\tau_2(t))\\ z(t-\tau_2(t))-z(t-\tau_M)\end{bmatrix}.
\end{aligned} \tag{A.3}$$

We use the following descriptor representation of (19)

$$\begin{aligned}
0 &= 2[z^T(t)P_2^T + \dot z^T(t)P_3^T][-\dot z(t) + Az + BKz(t-\tau(t))\\
&\quad + e^{A(r_0+r_1)}B\big(e_1(t) + Kz(t-\tau_2(t)) - Kz(t-\tau_1(t))\big)].
\end{aligned} \tag{A.4}$$

By summing up (20), (A.1)–(A.4) we obtain

$$\dot V + 2\alpha V \leq \varphi^T\Phi\varphi \leq 0,$$

where $\varphi = \mathrm{col}\{z(t), \dot z(t), z(t-\tau(t)), z(t-\bar\tau), z(t-r_0-r_1), z(t-\tau_1(t)), z(t-\tau_2(t)), z(t-\tau_M), e_1(t)\}$. This implies $\dot V \leq -2\alpha V$ and, therefore,

$$V(t) \leq e^{-2\alpha(t-\tau_M)}V(\tau_M), \quad t \geq \tau_M. \tag{A.5}$$

Define $z_t = z(t+\theta)$, $\theta \in [-\tau_M, 0]$ and $\|z_t\|_{PC} = \max_{\theta\in[-\tau_M,0]} |z(t+\theta)|$. For $t \geq 0$ function $\|z_t\|_{PC}$ is continuous in t and (19), (20) imply $|\dot z(t)| \leq m\|z_t\|_{PC}$ for some $m > 0$. Therefore,

$$\|z_t\|_{PC} \leq |z(0)| + \int_0^t m\|z_s\|_{PC}\,ds, \quad t \geq 0.$$

By the Gronwall–Bellman Lemma this implies

$$\|z_t\|_{PC} \leq |z(0)|e^{mt}, \quad t \geq 0. \tag{A.6}$$

Since $|\dot z(t)| \leq m\|z_t\|_{PC}$, there exists c_1 such that $V(\tau_M) \leq c_1\|z_{\tau_M}\|_{PC}^2 \leq c_1|z(0)|^2 e^{2m\tau_M}$. Since $|z(t)|^2\lambda_{\min}(P) \leq V(t)$, (A.5) and (A.6) imply (21) for some $M > 0$.

Appendix B. Proof of Theorem 1

From (4), (5), (8) we have

$$\begin{aligned}
x(t) &= e^{-A(r_0+r_1)}z(t)\\
&\quad - \int_{t-r_1}^{t+r_0} e^{A(t-r_1-\theta)}BKz(\theta - r_0 - \tau(\theta - r_0))\,d\theta, \quad t \geq 0,
\end{aligned}$$

where z satisfies (10), (19). By Lemma 1 (21) holds, thus

$$|x(t)| \leq Ce^{-\alpha t}|z(0)| \leq Ce^{-\alpha t}\left\|e^{A(r_0+r_1)}\right\| |x(0)|.$$

Appendix C. Proof of Lemma 2

For $t \geq r_1 + \tilde\tau$ ($\tilde\tau = h + \mu_M$) consider the functional

$$\begin{aligned}
V &= V_P + V_S + V_{S_0} + V_{R_0} + V_{S_1} + V_{R_1},\\
V_P &= z^T(t)Pz(t),\\
V_S &= \int_{t-r_1}^{t} e^{2\alpha(s-t)}z^T(s)Sz(s)\,ds,\\
V_{S_0} &= \int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}z^T(s)S_0 z(s)\,ds,\\
V_{R_0} &= \mu_M\int_{-r_1-\mu_M}^{-r_1}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot z^T(s)R_0\dot z(s)\,ds\,d\theta,\\
V_{S_1} &= \int_{t-r_1-\tilde\tau}^{t-r_1-\mu_M} e^{2\alpha(s-t)}z^T(s)S_1 z(s)\,ds,\\
V_{R_1} &= h\int_{-r_1-\tilde\tau}^{-r_1-\mu_M}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot z^T(s)R_1\dot z(s)\,ds\,d\theta.
\end{aligned}$$

We have

$$\begin{aligned}
&\dot V_P + 2\alpha V_P = 2z^T(t)P\dot z(t) + 2\alpha z^T(t)Pz(t),\\
&\dot V_S + 2\alpha V_S = z^T(t)Sz(t) - e^{-2\alpha r_1}z^T(t-r_1)Sz(t-r_1),\\
&\dot V_{S_0} + 2\alpha V_{S_0} = e^{-2\alpha r_1}z^T(t-r_1)S_0 z(t-r_1)\\
&\quad - e^{-2\alpha(r_1+\mu_M)}z^T(t-r_1-\mu_M)S_0 z(t-r_1-\mu_M),
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
&\dot V_{S_1} + 2\alpha V_{S_1} = e^{-2\alpha(r_1+\mu_M)}z^T(t-r_1-\mu_M)S_1\\
&\quad \times z(t-r_1-\mu_M) - e^{-2\alpha(r_1+\tilde\tau)}z^T(t-r_1-\tilde\tau)S_1 z(t-r_1-\tilde\tau),\\
&\dot V_{R_0} + 2\alpha V_{R_0} = \mu_M^2\dot z^T(t)R_0\dot z(t)\\
&\quad - \mu_M\int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}\dot z^T(s)R_0\dot z(s)\,ds,\\
&\dot V_{R_1} + 2\alpha V_{R_1} = h^2\dot z^T(t)R_1\dot z(t)\\
&\quad - h\int_{t-r_1-\tilde\tau}^{t-r_1-\mu_M} e^{2\alpha(s-t)}\dot z^T(s)R_1\dot z(s)\,ds.
\end{aligned} \tag{C.2}$$

I. For $t \in [t_k^*, t_{k+1})$ we have

$$0 = 2[z^T(t)P_2^T + \dot{z}^T(t)P_3^T][-\dot{z}(t) + (A+BK)z(t) + e^{Ar_1}B(Kz(t-r_1-\mu(t)) - Kz(t-r_1) + e_3(t))]. \quad (C.3)$$

To compensate the term $z(t-r_1-\mu(t))$ using Jensen's inequality and Park's theorem we derive

$$\begin{aligned} -\mu_M \int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}\dot{z}^T(s)R_0\dot{z}(s)\,ds &\le e^{-2\alpha(r_1+\mu_M)} \\ \times \begin{bmatrix} z(t-r_1)-z(t-r_1-\mu(t)) \\ z(t-r_1-\mu(t))-z(t-r_1-\mu_M)\end{bmatrix}^T &\begin{bmatrix} R_0 & G_0 \\ G_0^T & R_0 \end{bmatrix} \\ \times \begin{bmatrix} z(t-r_1)-z(t-r_1-\mu(t)) \\ z(t-r_1-\mu(t))-z(t-r_1-\mu_M)\end{bmatrix}&, \end{aligned} \quad (C.4)$$

$$\begin{aligned} -\mu_M \int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}\dot{z}^T(s)R_0\dot{z}(s)\,ds \\ \le e^{-2\alpha(r_1+\tilde{\tau})}\left[z(t-r_1-\mu_M)-z(t-r_1-\tilde{\tau})\right]^T R_1 \\ \times [z(t-r_1-\mu_M)-z(t-r_1-\tilde{\tau})]. \end{aligned} \quad (C.5)$$

By summing up (28) and (C.1)–(C.3) in view of (C.4), (C.5) we obtain $\dot{V} + \alpha V \le \xi^T \Xi \xi \le 0$, where $\xi = \mathrm{col}\{z(t), \dot{z}(t), z(t-r_1), z(t-r_1-\mu(t)), z(t-r_1-\mu_M), z(t-r_1-\tilde{\tau}), e_3(t)\}$.

For $t \in [t_k, t_k^*)$ the system (29) with $\tau_4(t) \in [0, \mu_M)$ is described by (30) with $e_3(t) = 0$ satisfying (28).

II. For $t \in [t_k, t_k^*)$, $\tau_4(t) \in [\mu_M, \mu_M + h)$ we have

$$0 = 2\left[z^T(t)P_2^T + \dot{z}^T(t)P_3^T\right]\left[-\dot{z}(t) + (A+BK)z(t) + e^{Ar_1}BKz(t-r_1-\tau_4(t)) - e^{Ar_1}BKz(t-r_1)\right]. \quad (C.6)$$

To compensate the term $z(t-r_1-\tau_4(t))$ using Jensen's inequality and Park's theorem we derive

$$\begin{aligned} -\mu_M \int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}\dot{z}^T(s)R_0\dot{z}(s)\,ds \\ \le -e^{-2\alpha(r_1+\mu_M)}[z(t-r_1)-z(t-r_1-\mu_M)]^T R_0 \\ \times [z(t-r_1)-z(t-r_1-\mu_M)], \end{aligned} \quad (C.7)$$

$$\begin{aligned} -h \int_{t-r_1-\tilde{\tau}}^{t-r_1-\mu_M} e^{2\alpha(s-t)}\dot{z}^T(s)R_1\dot{z}(s)\,ds &\le -e^{-2\alpha(r_1+\tilde{\tau})} \\ \times \begin{bmatrix} z(t-r_1-\mu_M)-z(t-r_1-\tau_4(t)) \\ z(t-r_1-\tau_4(t))-z(t-r_1-\tilde{\tau})\end{bmatrix}^T &\begin{bmatrix} R_1 & G_1 \\ G_1^T & R_1 \end{bmatrix} \\ \times \begin{bmatrix} z(t-r_1-\mu_M)-z(t-r_1-\tau_4(t)) \\ z(t-r_1-\tau_4(t))-z(t-r_1-\tilde{\tau})\end{bmatrix}&. \end{aligned} \quad (C.8)$$

By summing up (C.1), (C.2), (C.6) in view of (C.7), (C.8) we obtain $\dot{V} + 2\alpha V \le \eta^T \Sigma \eta \le 0$, where $\eta = \mathrm{col}\{z(t), \dot{z}(t), z(t-r_1), z(t-r_1-\mu_M), z(t-r_1-\tau_4(t)), z(t-r_1-\tilde{\tau})\}$.

Therefore, we obtain $\dot{V} \le -2\alpha V$ for $t \ge r_1 + \tilde{\tau}$. The end of the proof is similar to that of Lemma 1.

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