



# $H_\infty$ -control of linear state-delay descriptor systems: an LMI approach<sup>☆</sup>

E. Fridman\*, U. Shaked

*Department of Electrical Engineering-Systems, Tel Aviv University, Ramat Aviv 69978, Tel Aviv, Israel*

Received 20 January 2001; accepted 30 October 2001

Submitted by P. Van Dooren

---

## Abstract

For continuous-time, linear descriptor system with state-delay a  $H_\infty$ -control problem is solved. Sufficient conditions for delay-dependent/delay-independent stability and  $L_2$ -gain analysis are obtained in terms of linear matrix inequalities (LMIs). A bounded real lemma and state-feedback solutions are derived for systems which may contain polytopic parameter uncertainties. The filtering problem is also solved and an output-feedback controller is then found by solving two LMIs. The first LMI is associated with a proportional-derivative state-feedback control. The second LMI is derived in two different forms, the first one corresponds to the adjoint of the system that describes the estimation error and the other stems from the original system. These two forms lead to different results. Numerical examples are given which illustrate the effectiveness of the new theory. © 2002 Elsevier Science Inc. All rights reserved.

*Keywords:* Time-delay systems; Descriptor systems;  $H_\infty$ -control; LMI; Delay-dependent criteria

---

## 1. Introduction

Delay differential–algebraic equations, which have both delay and algebraic constraints, often appear in various engineering systems, including aircraft stabilization, chemical engineering systems, lossless transition lines, etc. (see e.g. [4,13,14,22,28],

---

<sup>☆</sup> This work was supported by the Ministry of Absorption of Israel and by C&M Maus Chair at Tel Aviv University.

\* Corresponding author. Tel.: +972-3640-5313; fax: +972-3640-7095.

*E-mail address:* emilia@eng.tau.ac.il (E. Fridman).

and the references therein). Depending on the area of application, these models are called singular or implicit or descriptor systems with delay. As has been pointed out in [6,7], descriptor systems with delay may in fact be systems of advanced type. Descriptor systems may be destabilized by small delay in the feedback [20].

There are only few papers on descriptor systems with delay [6,7,10,15,22,23,28]. A particular case of these systems (the so-called lossless propagation models), described by

$$\dot{x}_1(t) = Ax_1(t) + Bx_2(t-h), \quad x_2(t) = Cx_1(t) + Dx_2(t-h), \quad (1)$$

has been treated as a special class of neutral systems either by letting  $x_2(t) = \dot{y}_2(t)$  [23] or by writing the second equation as [15,22]

$$\frac{d}{dt}[x_2(t) - Cx_1(t) - Dx_2(t-h)] = 0. \quad (2)$$

The stability of a general neutral type descriptor system with a single delay described by

$$E\dot{x}(t) + Ax(t) + B\dot{x}(t-h) + Cx(t-h) = 0 \quad (3)$$

with a singular matrix  $E$  has been studied in [28] by analyzing its characteristic equation

$$\det[sE + A + (sB + C)\exp(-hs)] = 0$$

and finding frequency domain conditions which guarantee that all roots of the latter equation have negative real parts bounded away from 0. A Lyapunov-based approach to stability analysis of descriptor system with delay has been introduced in [10], where delay-independent and delay-dependent linear matrix inequalities (LMIs) conditions have been derived. For information on LMI approach to control, see [3].

All the above-mentioned results only analyze the existence and the stability of solutions of descriptor systems with delay. To the best of our knowledge no control problem solution has been derived for this class of systems. For descriptor systems without delay,  $H_\infty$ -control problems have been treated in the frequency domain [19,27] and in the time-domain [21,26,29]. In [21,26] an LMI approach has been proposed. For nondescriptor systems with state-delay, LMI delay-dependent and delay-independent  $H_\infty$ -controllers were derived in [12,16,24] (see also the references therein). These finite-dimensional LMIs provide sufficient conditions only for infinite-dimensional systems with state-delay. Unlike infinite-dimensional methods (see e.g. [1,11]) they lead to effective numerical algorithms and may be applied for systems with polytopic uncertainties.

In the present paper, we adopt the finite-dimensional LMI approach to  $H_\infty$ -control of descriptor system with delay. Our objective is to obtain delay-dependent solutions which are less conservative than the delay-independent ones. We apply the descriptor model transformation that has been introduced recently for delay-dependent stability and  $H_\infty$ -control of nondescriptor systems [9,12]. We derive bounded real lemmas (BRLs) and find solutions to the  $H_\infty$  filtering, the state-feedback and

the output-feedback  $H_\infty$ -control problems. The solutions are delay-dependent with respect to the ‘slow’ variable and delay-independent with respect to the ‘fast’ one. The latter guarantees the robustness of the system behavior with respect to the small changes in the delay.

*Notation.* Throughout the paper the superscript ‘T’ stands for matrix transposition,  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space with vector norm  $|\cdot|$ ,  $\mathbb{R}^{n \times m}$  is the set of all  $n \times m$  real matrices, and the notation  $P > 0$  for  $P \in \mathbb{R}^{n \times n}$  means that  $P$  is symmetric and positive definite. The space of functions in  $\mathbb{R}^q$  that are square integrable over  $[0, \infty)$  is denoted by  $\mathcal{L}_2^q[0, \infty)$  with norm  $\|\cdot\|_{L_2}$ . Let  $C_n[a, b]$  denote the space of continuous functions  $\phi : [a, b] \rightarrow \mathbb{R}^n$  with the supremum norm  $|\cdot|$ . We also denote  $x_t(\theta) = x(t + \theta)$  ( $\theta \in [-h, 0]$ ) and  $j = \sqrt{-1}$ .

## 2. Problem formulation

Given the following system:

$$E\dot{x}(t) = \sum_{i=0}^2 A_i x(t - h_i) + B_1 w(t) + B_2 u(t), \quad x(t) = 0, \quad \forall t \leq 0, \quad (4)$$

$$\bar{y}(t) = \bar{C}_2 x(t) + D_{21} w(t), \quad (5a)$$

$$z(t) = \text{col}\{\bar{C}_1 x(t); D_{12} u(t)\}, \quad (5b)$$

where  $x(t) = \text{col}\{x_1(t), x_2(t)\}$ ,  $x_1(t) \in \mathbb{R}^{n_1}$ ,  $x_2(t) \in \mathbb{R}^{n_2}$  is the system state vector,  $u(t) \in \mathbb{R}^\ell$  is the control input,  $w(t) \in \mathcal{L}_2^q[0, \infty)$  is the exogenous disturbance signal,  $\bar{y}(T) \in \mathbb{R}^r$  is the measurement vector and  $z(t) \in \mathbb{R}^p$  is the state combination (objective function signal) to be attenuated. The time delays  $h_0 = 0$ ,  $h_i > 0$ ,  $i = 1, 2$ , are assumed to be known. We took for simplicity two delays, but all the results are easily generalized for the case of any finite number of delays  $h_1, \dots, h_m$ . The singular matrix  $E$  and the matrices  $A_i, B_i$  are constant matrices of appropriate dimensions. Denote  $n \triangleq n_1 + n_2$ .

Following [21,26], we assume for simplicity that

$$E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}. \quad (6)$$

Every descriptor system can be represented in a form satisfying this assumption. Note that in [21] there is  $\Sigma = \Sigma^T > 0$  instead of  $I_{n_1}$ , but from such a system, the system with  $E$  of (6) follows immediately. The matrices in (4), (5a) and (5b) have the following structure:

$$A_i = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, \quad i = 0, 1, 2, \quad (7)$$

$$B_i = \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}, \quad \bar{C}_i = [\bar{C}_{i1} \quad \bar{C}_{i2}], \quad i = 1, 2.$$

A descriptor system without delay

$$E\dot{x}(t) = A_0x(t) + B_1w(t) \tag{8}$$

is *regular* if the characteristic polynomial  $\det(sE - A_0)$  does not vanish identically in  $s \in C$ . It is well known that descriptor system may have impulsive solutions. The existence of the latter solutions is usually studied in terms of the Weierstrass canonical form and the *index* of the system which are defined as follows [5,8,20]: there exist nonsingular matrices  $P, Q \in \mathbb{R}^{n \times n}$  such that

$$QEP = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad QA_0P = \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{9}$$

where  $N \in \mathbb{R}^{n_2 \times n_2}$  and  $J \in \mathbb{R}^{n_1 \times n_1}$  are in Jordan form. The matrix  $N$  is nilpotent of index  $\nu$ , i.e.  $N^\nu = 0, N^{\nu-1} \neq 0$ . The index of (8) is the index of nilpotence  $\nu$  of  $N$ . The index of the system with delay

$$E\dot{x}(t) = \sum_{i=0}^2 A_i x(t - h_i) + B_1 w(t) \tag{10}$$

is defined in [10] to be equal to the index of (8). The descriptor system (10) admits impulsive solutions iff  $\nu > 1$  [10].

We do not require  $A_{04}$  in (4) to be nonsingular. If  $A_{04}$  is singular, then (8) has index greater than 1 (see e.g. [5,8]). Hence, the index of the open loop system (4) with delay is higher than one. Such a system may have an impulsive solution. The nonsingularity of  $A_{04}$  guarantees the existence and the uniqueness of solution to (4) with  $u = 0$  (see Proposition 3.1).

The following class of neutral descriptor systems

$$\left[ \dot{x}_1(t) - \sum_{i=1}^2 F_i \dot{x}_1(t - h_i) \right] = \sum_{i=0}^2 A_i x(t - h_i) + B_1 w(t) + B_2 u(t), \tag{11}$$

can be reduced to the form of (4) and (6). This follows from the fact that the augmented system

$$\begin{aligned} \dot{x}_1(t) &= y, \\ \left[ y(t) - \sum_{i=1}^2 F_i y(t - h_i) \right] &= \sum_{i=0}^2 A_i x(t - h_i) + B_1 w(t) + B_2 u(t), \end{aligned} \tag{12}$$

is a particular case of (4) and (6).

For a prescribed scalar  $\gamma > 0$ , we define the performance index

$$J(w) = \int_0^\infty (z^T z - \gamma^2 w^T w) d\tau. \tag{13}$$

The problem is to find a controller such that the resulting closed loop system has index at most one is internally stable (i.e. asymptotically stable for  $w = 0$ ) and  $J(w) < 0$  for all disturbances  $w(t) \in \mathcal{L}_2^q[0, \infty]$ .

### 3. Stability and $L_2$ -gain analysis of a descriptor system with delay

BRLs will be obtained for systems with discrete and distributed delays. Given the following system:

$$E\dot{x}(t) = \sum_{i=0}^2 A_i x(t - h_i) + \int_{-d}^0 A_d(s)x(t + s) ds + B_1 w(t), \tag{14a}$$

$$x(t) = 0 \quad \forall t \leq 0, \tag{14b}$$

$$z(t) = \text{col}\{C_0 x(t), C_1 x(t - h_1), C_2 x(t - h_2)\}, \tag{15}$$

where  $E$  is defined in (6),  $x(t) = \text{col}\{x_1(t), x_2(t)\}$ ,  $x_1(t) \in \mathbb{R}^{n_1}$ ,  $x_2(t) \in \mathbb{R}^{n_2}$ , is the system state vector,  $w(t) \in \mathcal{L}_2^q[0, \infty]$  is the exogenous disturbance signal and  $z(t) \in \mathbb{R}^p$  is objective function signal,  $A_d(s)$  is a piecewise-continuous and uniformly bounded  $(n_1 + n_2) \times (n_1 + n_2)$ -matrix-function. We assume that the matrices in (14a) and (14b) have the structure of (7) and

$$C_i = [C_{i1} \quad C_{i2}], \quad i = 0, 1, 2, \quad A_d = \begin{bmatrix} A_{d1} & A_{d2} \\ A_{d3} & A_{d4} \end{bmatrix}.$$

Denote  $h = \max\{h_1, h_2, d\}$ . By solution of (14a) and (14b) on the segment  $[0, t_1]$  ( $t_1 > 0$ ) we understand a pair of functions  $\{x_1(t), x_2(t)\}$ , such that  $x_1$  is absolutely continuous and  $x_2$  is integrable on  $[0, t_1]$ , these functions satisfy system (14a) almost for all  $t \in [0, t_1]$  and the initial conditions (14b).

**Proposition 3.1.** *Assume that  $A_{04}$  is nonsingular. For  $w(t) \in \mathcal{L}_2^q[0, \infty)$  the solution to (14a) and (14b) exists and is unique on  $[0, t_1]$  for all  $t_1 > 0$ .*

**Proof.** By denoting  $\dot{y}_2 = x_2$  we obtain from (14a) a neutral type system with the zero initial conditions  $x_1(t) = 0, \dot{y}_2(t) = 0 \forall t \leq 0$ . Hence,  $y_2(t) = c \forall t \leq 0, c \in \mathbb{R}^{n_2}$ . This initial value problem for neutral system has a piecewise absolutely continuous solution  $x_1(t), y_2(t)$  on  $[0, t_1]$  such that  $x_1(t)$  is absolutely continuous on  $[0, t_1]$  (see [17, p. 143]). Therefore,  $x_2(t)$  is integrable and solution of (14a) and (14b) exists.

To prove the uniqueness we assume that there are two solutions of (14a) and (14b). Then their difference satisfies the homogeneous equations (14a) and (14b) with  $w = 0$ , which has a unique solution  $x \equiv 0$  [10]. Hence, solution of (14a) and (14b) is unique.  $\square$

#### 3.1. Stability of the difference operator and of the descriptor system

We assume:

**A1.** The matrix  $A_{04}$  is nonsingular and the difference operator  $\mathcal{D} : C_{n_2}[-h, 0] \rightarrow \mathbb{R}^n$  given by

$$\mathcal{D}(x_{2t}) = x_2(t) + \sum_{i=1}^2 A_{04}^{-1} A_{i4} x_2(t - h_i) + \int_{-d}^0 A_{04}^{-1} A_{d4}(s) x_2(t + s) ds$$

is stable for all delays  $h_1$  and  $h_2$  (i.e. equation  $\mathcal{D}x_{2t} = 0$  is asymptotically stable for all  $h_1$  and  $h_2$ ).

A sufficient condition for **A1** is the following inequality:

$$\sum_{i=1}^2 |A_{04}^{-1} A_{i4}| + \int_{-d}^0 |A_{04}^{-1} A_{d4}(s)| ds < 1,$$

where  $|\cdot|$  is any matrix norm.

In the case of single delay (e.g.  $h_1$ ) in the fast variable  $x_2$  we assume instead of **A1** the following:

**A1'**. All the eigenvalues of  $A_{04}^{-1} A_{14}$  are inside of the unit circle.

In the case of multiple discrete delays in  $\mathcal{D}$ , where  $A_{d4} = 0$ , **A1** is equivalent to the following one (see [14, Theorem 6.1, p. 286]):

**A1''**. If  $\sigma(B)$  is the spectral radius of matrix  $B$ , then  $\sigma_0 < 1$ , where

$$\sigma_0 \triangleq \sup \left\{ \sigma \left( \sum_{k=1}^2 A_{04}^{-1} A_{i4} e^{i\theta_k} \right) : \theta_k \in [0, 2\pi], k = 1, 2 \right\}. \tag{16}$$

Evidently **A1'** is equivalent to **A1''** in the case of single delay  $h_1$ . A sufficient LMI condition for **A1''** is given by the following:

**Lemma 3.2** [10]. *If there exist  $n_2 \times n_2$ -matrices  $P_f, U_{1f}, U_{2f}$  that satisfy the following LMI:*

$$\begin{bmatrix} P_f^T A_{04} + A_{04}^T P_f + \sum_{i=1}^2 U_{if} & P_f^T A_{14} & P_f^T A_{24} \\ * & -U_{1f} & 0 \\ * & * & -U_{2f} \end{bmatrix} < 0, \tag{17}$$

then  $A_{04}$  is nonsingular and

- (i) **A1''** holds;
- (ii) the difference operator

$$\mathcal{D}(x_{2t}) = x_2(t) + \sum_{i=1}^2 A_{04}^{-1} A_{i4} x_2(t - h_i)$$

is stable for all  $h_1, h_2$ ;

- (iii) under additional assumption that  $P_f > 0$  the “fast system”

$$\dot{x}_2(t) = A_{04} x_2(t) + \sum_{i=1}^2 A_{i4} x_2(t - h_i) \tag{18}$$

is asymptotically stable for all  $h_1, h_2$ .

The following result on stability of (14a) and (14b) with  $w = 0$  has been obtained recently [10].

**Lemma 3.3.** *Under A1 if there exist positive numbers  $\alpha, \beta, \gamma$  and a continuous functional  $V : C_n[-h, 0] \rightarrow \mathbb{R}$  such that*

$$\beta|\phi_1(0)|^2 \leq V(\phi) \leq \gamma|\phi|^2, \tag{19a}$$

$$\dot{V}(\phi) \leq -\alpha|\phi(0)|^2, \tag{19b}$$

and the function  $\bar{V}(t) = V(x_t)$  is absolutely continuous for  $x_t$  satisfying (14a) and (14b) with  $w = 0$ , then (14a), (14b) with  $w = 0$  is asymptotically stable.

### 3.2. Delay-independent BRL (with respect to discrete delays)

Descriptor type Lyapunov–Krasovskii functional for system (14a), (14b) has the following form:

$$V(x_t) = x^T(t)EPx(t) + V_1 + V_2, \tag{20}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 = P_1^T > 0, \tag{21}$$

$$V_1 = \sum_{i=1}^2 \int_{t-h_i}^t x^T(s)U_i x(s) ds, \quad U_i > 0, \tag{22}$$

and

$$V_2 = \int_{-d}^0 \int_{t+\theta}^t x^T(s)A_d^T(\theta)RA_d(\theta)x(s) ds d\theta, \quad R > 0. \tag{23}$$

The first term of (20) corresponds to the descriptor system,  $V_1$  corresponds to the delay-independent stability with respect to the discrete delays and  $V_2$ —to delay-dependent stability with respect to the distributed delays [18]. The functional (20) is *degenerated* (i.e. nonpositive-definite) as it is usual for descriptor systems.

We obtain analogously to [10] the following:

**Theorem 3.4.** *Under A1 (14a) and (14b), (15) is internally asymptotically stable and for a prechosen  $\gamma > 0$   $J(w) < 0$  for all nonzero  $w(t) \in \mathcal{L}_2^q[0, \infty]$  and for all  $h_1 \geq 0, h_2 \geq 0$  if there exist  $n \times n$ -matrix  $P$  of (21) with  $n_1 \times n_1$ -matrix  $P_1$  and  $n_2 \times n_2$ -matrix  $P_3$  and  $n \times n$  matrices  $U_i = U_i^T, i = 1, 2, R = R^T$  that satisfy the following LMI:*

$$\begin{bmatrix} P^T A_0 + A_0^T P + \sum_{i=1}^2 U_i + \int_{-d}^0 A_d^T(s) R A_d(s) ds & P^T B_1 & P^T A_1 & P^T A_2 & dP^T \\ * & -\gamma^2 I_q & 0 & 0 & 0 \\ * & * & -U_1 & 0 & 0 \\ * & * & * & -U_2 & 0 \\ * & * & * & * & -dR \end{bmatrix} < 0. \tag{24}$$

**Remark 1.** From Theorem 3.4 it follows that the system

$$E\dot{x}(t) = A_0 x(t) + A_1 x(t - h) \tag{25}$$

is asymptotically stable for all  $h \geq 0$  if the following LMI is feasible:

$$\begin{bmatrix} P^T A_0 + A_0^T P & P^T A_1 \\ A_1^T P & 0 \end{bmatrix} + \begin{bmatrix} U_1 & 0 \\ 0 & -U_1 \end{bmatrix} < 0. \tag{26}$$

Multiplying (26) by  $\Omega$  from the left and by  $\Omega^*$  from the right, where

$$\Omega = \begin{bmatrix} A_1^T (-j\omega E - A_0^T)^{-1} & I \end{bmatrix}, \quad \Omega^* = \begin{bmatrix} (j\omega E - A_0)^{-1} A_1 \\ I \end{bmatrix}, \quad \omega \in \mathbb{R},$$

the following frequency domain inequality is readily obtained

$$\Omega \operatorname{diag}\{U_1, -U_1\} \Omega^* < 0$$

or

$$A_1^T (-j\omega E - A_0^T)^{-1} U_1 (j\omega E - A_0)^{-1} A_1 < U_1. \tag{27}$$

Therefore, if LMI (24) is feasible (and thus, (26) is feasible), then for all  $\omega \in \mathbb{R}$  the frequency domain inequality (27) holds. Hence the  $H_\infty$ -norm of  $U_1^{1/2} (j\omega E - A_0)^{-1} A_1 U_1^{-1/2}$  is less than 1. This is a counterpart of the Kalman–Yakubovich–Popov lemma for descriptor systems.

### 3.3. Delay-dependent BRL

We are looking for delay-dependent conditions with respect to slow variable  $x_1$ . With respect to discrete delays in the fast variables the results will be delay-independent. The latter guarantees robust stability with respect to small changes of delay [10]. Following [9,10] we represent (14a) and (14b) in the equivalent form:

$$\begin{aligned} \dot{x}_1(t) &= y(t), \\ \begin{bmatrix} y(t) \\ 0 \end{bmatrix} &= \begin{bmatrix} \sum_{i=0}^2 A_{i1} & A_{02} \\ \sum_{i=0}^2 A_{i3} & A_{04} \end{bmatrix} x(t) + \sum_{i=1}^2 \begin{bmatrix} A_{i2} \\ A_{i4} \end{bmatrix} x_2(t - h_i) \\ &\quad - \sum_{i=1}^2 \begin{bmatrix} A_{i1} \\ A_{i3} \end{bmatrix} \int_{-h_i}^0 y(t+s) ds + \int_{-d}^0 A_d(s) x(t+s) ds + B_1 w(t). \end{aligned} \tag{28}$$



The latter system can be represented in the form:

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) = & \sum_{i=0}^2 \bar{A}_i \bar{x}(t - h_i) + \sum_{i=1}^2 h_i H_i \int_{-h_i}^0 y(t + s) ds \\ & + \int_{-d}^0 \bar{A}_d(s) \bar{x}(t + s) ds + \bar{B}_1 w(t), \end{aligned} \tag{29}$$

where

$$\begin{aligned} \bar{x} = \begin{bmatrix} x_1 \\ y \\ x_2 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0_{n_1 \times n_1} & 0 \\ 0 & 0 & 0_{n_2 \times n_2} \end{bmatrix}, \\ \bar{A}_0 = \begin{bmatrix} 0 & I & 0 \\ \sum_{i=0}^2 A_{i1} & -I_{n_1} & A_{02} \\ \sum_{i=0}^2 A_{i3} & 0 & A_{04} \end{bmatrix}, \quad \bar{A}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{i2} \\ 0 & 0 & A_{i4} \end{bmatrix}, \quad i = 1, 2, \tag{30} \\ H_i = \begin{bmatrix} 0 \\ A_{i1} \\ A_{i3} \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} 0 & 0 & 0 \\ A_{d1} & 0 & A_{d2} \\ A_{d3} & 0 & A_{d4} \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}. \end{aligned}$$

A Lyapunov–Krasovskii functional for system (28) has the form:

$$\begin{aligned} V(t) = & \bar{x}^T(t) \bar{E} P \bar{x}(t) + \sum_{i=1}^2 \int_{t-h_i}^t x_1^T(\tau) S_i x_1(\tau) d\tau \\ & + \sum_{i=1}^2 \int_{t-h_i}^t x_2^T(\tau) U_i x_2(\tau) d\tau \\ & + \sum_{i=1}^2 \int_{-h_i}^0 \int_{t+\theta}^t y^T(s) \begin{bmatrix} A_{i1}^T & A_{i3}^T \end{bmatrix} R_{i3} \begin{bmatrix} A_{i1} \\ A_{i3} \end{bmatrix} y(s) ds d\theta \\ & + \int_{-d}^0 \int_{t+\theta}^t \bar{x}^T(s) \bar{A}_d^T R_d \bar{A}_d \bar{x}(s) ds d\theta, \end{aligned} \tag{31}$$

where  $P$  has the structure of (21) with  $P_1 \in \mathbb{R}^{n_1 \times n_1}$ ,  $P_3 \in \mathbb{R}^{n \times n}$  and

$$\begin{aligned} 0 < S_i \in \mathbb{R}^{n_1 \times n_1}, \quad 0 < U_i \in \mathbb{R}^{n_2 \times n_2}, \\ 0 < R_{i3} \in \mathbb{R}^{n \times n}, \quad R_d \in \mathbb{R}^{(n_1+n) \times (n_1+n)}. \end{aligned}$$

The first term of (31) corresponds to the descriptor system, the second and the fourth terms—to the delay-dependent conditions with respect to  $x_1$  and the third—to the delay-independent conditions with respect to  $x_2$ , the fifth term corresponds to delay-dependent with respect to distributed delay.

We obtain the following:

**Theorem 3.5.** Under **A1** the system of (14a), (14b) and (15) is internally asymptotically stable and for a prechosen  $\gamma > 0$ ,  $J(w) < 0$  for all nonzero  $w(t) \in \mathcal{L}_2^q[0, \infty]$  if there exist matrices  $P \in \mathbb{R}^{(n_1+n) \times (n_1+n)}$  of (21),  $0 < P_1 \in \mathbb{R}^{n_1 \times n_1}$ ,  $P_2, P_3 \in \mathbb{R}^{n \times n}$ ,  $S_i = S_i^T \in \mathbb{R}^{n_1 \times n_1}$ ,  $U_i = U_i^T \in \mathbb{R}^{n_2 \times n_2}$ ,  $W_i \in \mathbb{R}^{(n_1+n) \times (n_1+n)}$  and  $R_i = R_i^T \in \mathbb{R}^{(n_1+n) \times (n_1+n)}$ ,  $i = 1, 2$  that satisfy the following LMI:

$$\begin{bmatrix} \bar{\Psi} & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_1 X_1 & h_2 X_2 & -W_1^T \begin{bmatrix} 0 \\ A_{11} \\ A_{13} \end{bmatrix} & -W_2^T \begin{bmatrix} 0 \\ A_{21} \\ A_{23} \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_{12} \\ A_{14} \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_{22} \\ A_{24} \end{bmatrix} & dP^T \\ * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -h_1 R_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h_2 R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -S_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & * & * & * & -U_2 & 0 \\ * & * & * & * & * & * & * & * & -dR_d \end{bmatrix} < 0, \tag{32}$$

where

$$\begin{aligned} X_i &= W_i^T + P^T, \quad i = 1, 2, \\ \bar{\Psi} &= \Psi + \sum_{i=0}^2 \begin{bmatrix} C_{i1}^T C_{i1} & 0_{n_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_{i2}^T C_{i2} \end{bmatrix} + \sum_{i=1}^2 W_i^T \begin{bmatrix} 0 & 0 & 0 \\ A_{i1} & 0 & 0 \\ A_{i3} & 0 & 0 \end{bmatrix} \\ &\quad + \sum_{i=1}^2 \begin{bmatrix} 0 & A_{i1}^T & A_{i3}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W_i \end{aligned}$$

and

$$\begin{aligned} \Psi &\triangleq P^T \begin{bmatrix} 0 & I_{n_1} & 0 \\ \sum_{i=0}^2 A_{i1} & -I_{n_1} & A_{02} \\ \sum_{i=0}^2 A_{i3} & 0 & A_{04} \end{bmatrix} + \begin{bmatrix} 0 & I_{n_1} & 0 \\ \sum_{i=0}^2 A_{i1} & -I_{n_1} & A_{02} \\ \sum_{i=0}^2 A_{i3} & 0 & A_{04} \end{bmatrix}^T P \\ &\quad + \sum_{i=1}^2 \begin{bmatrix} S_i & 0 & 0 \\ 0 & h_i [A_{i1}^T & A_{i3}^T] R_{i3} \begin{bmatrix} A_{i1} \\ A_{i3} \end{bmatrix} \\ 0 & 0 & U_i \end{bmatrix} \\ &\quad + \int_{-d}^0 \bar{A}_d^T(s) R_d \bar{A}_d(s) ds \end{aligned} \tag{33}$$

and where  $R_{i3} \in \mathbb{R}^{n \times n}$  is the (2, 2) block of  $R_i$ .

**Proof.** Since

$$\bar{x}^T(t)\bar{E}P\bar{x}(t) = x_1^T(t)P_1x_1(t)$$

differentiating the first term of (31) with respect to  $t$  we have

$$\frac{d}{dt}\bar{x}^T(t)\bar{E}P\bar{x}(t) = 2x_1^T(t)P_1\dot{x}_1(t) = 2\bar{x}^T(t)P^T \begin{bmatrix} \dot{x}_1(t) \\ 0 \\ 0 \end{bmatrix}. \tag{34}$$

Substituting (28) into (34) we obtain

$$\begin{aligned} & \frac{dV(x_t)}{dt} + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\ &= \xi^T \begin{bmatrix} \Psi & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_{12} \\ A_{14} \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_{22} \\ A_{24} \end{bmatrix} \\ * & -\gamma^2 I_q & 0 & 0 \\ * & * & -U_1 & 0 \\ * & * & * & -U_2 \end{bmatrix} \xi + z^T z + \sum_{i=0}^2 \eta_i \\ & - \sum_{i=1}^2 \left[ x_1^T(t-h_i)S_i x_1(t-h_i) + x_2^T(t-h_i)U_i x_2(t-h_i) \right. \\ & \quad \left. + \int_{t-h_i}^t y^T(s) \begin{bmatrix} A_{i1}^T & A_{i3}^T \end{bmatrix} R_{i3} \begin{bmatrix} A_{i1} \\ A_{i3} \end{bmatrix} y(s) ds \right] \\ & - \int_{-d}^0 \bar{x}^T(t+\theta)\bar{A}_d^T(\theta)R_d A_d(\theta)\bar{x}(t+\theta) d\theta, \tag{35} \end{aligned}$$

where

$$\xi \triangleq \text{col}\{\bar{x}(t), w(t), x_2(t-h_1), x_2(t-h_2)\},$$

$\Psi$  is defined by (33) and

$$\begin{aligned} \eta_i(t) &\triangleq -2 \int_{t-h_i}^t \bar{x}^T(t)P^T \begin{bmatrix} 0 \\ A_{i1} \\ A_{i3} \end{bmatrix} y(s) ds, \quad i = 1, 2, \\ \eta_0(t) &\triangleq -2 \int_{t-d}^t \bar{x}^T(t)P^T \bar{A}_d(s)\bar{x}(t+s) ds. \end{aligned}$$

For any  $(n_1 + n) \times (n_1 + n)$ -matrices  $R_i > 0$  and  $M_i$  the following inequality holds [25]:

$$-2 \int_{t-h_i}^t b^T(s)a(s) ds \leq \int_{t-h_i}^t \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R_i & R_i M_i \\ M_i^T R_i & (2, 2) \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \tag{36}$$

for  $a(s) \in \mathbb{R}^{n_1+n}$ ,  $b(s) \in \mathbb{R}^{n_1+n}$  given for  $s \in [t - h_i, t]$ . Here

$$(2, 2) = (M_i^T R_i + I) R_i^{-1} (R_i M_i + I).$$

Denoting  $W_i = R_i M_i P$  and using this inequality for  $a(s) = \text{col}\{0 \ A_{i1} \ A_{i3}\}y(s)$  and  $b = P\bar{x}(t)$  we obtain for  $i = 1, 2$

$$\begin{aligned} \eta_i(t) &\leq h_i \bar{x}^T(t) (W_i^T + P^T) R_i^{-1} (W_i + P) \bar{x}(t) \\ &\quad + 2(x_1^T(t) - x_1^T(t - h_i)) \begin{bmatrix} 0 & A_{i1}^T & A_{i3}^T \end{bmatrix} W_i \bar{x}(t) \\ &\quad + \int_{t-h_i}^t y^T(s) \begin{bmatrix} A_{i1}^T & A_{i3}^T \end{bmatrix} R_{i3} \begin{bmatrix} A_{i1} \\ A_{i3} \end{bmatrix} y(s) \, ds. \end{aligned} \tag{37}$$

Similarly

$$\begin{aligned} \eta_0(t) &\leq d \bar{x}^T(t) P^T R_d^{-1} P \bar{x}(t) \\ &\quad + \int_{t-d}^t \bar{x}^T(t + s) \bar{A}_d^T(s) R_d \bar{A}_d(s) \bar{x}(t + s) \, ds. \end{aligned} \tag{38}$$

We substitute (37), (38) into (35) and integrate the resulting inequality in  $t$  from 0 to  $\infty$ . Because  $V(x_0) = 0$ ,  $V(x_\infty) \geq 0$  and

$$\begin{aligned} \int_0^\infty z^T z \, dt &= \sum_{i=0}^2 \int_0^\infty x^T(t - h_i) C_i^T C_i x(t - h_i) \, dt \\ &= \sum_{i=0}^2 \int_0^\infty x^T(t) C_i^T C_i x(t) \, dt, \end{aligned}$$

we obtain (by Schur complements) that

$$\|z\|_{L_2}^2 - \gamma^2 \|w\|_{L_2}^2 \leq \bar{\xi}^T \Gamma \bar{\xi} < -\alpha \|x\|_{L_2}^2, \quad \alpha > 0,$$

where  $\Gamma$  is the matrix in the left-hand side of (32) and

$$\bar{\xi} \triangleq \text{col}\{\bar{x}(t), w(t), x_2(t - h_1), x_2(t - h_2), \bar{\eta}\},$$

where  $\bar{\eta}$  is vector of fictitious states. Hence, for  $w(t) \in L_2[0, \infty]$  we have  $x(t) \in L_2[0, \infty]$  and  $J(w) < 0$  if (32) holds. Moreover,  $V$  of (31) satisfies (19a), (19b) and hence (14a) and (14b) are internally stable.  $\square$

### 3.4. Another delay-independent BRL (with respect to discrete delays)

For

$$W_i = -P, \quad R_i = \frac{\varepsilon I_{2n}}{h_i}, \quad i = 1, \dots, m, \tag{39}$$

LMI (32) implies for  $\varepsilon \rightarrow 0^+$  the following delay-independent LMI:

$$\begin{bmatrix}
 \Phi & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_{11} \\ A_{13} \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_{21} \\ A_{23} \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_{12} \\ A_{14} \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_{22} \\ A_{24} \end{bmatrix} & dP^T \\
 * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 \\
 * & * & -S_1 & 0 & 0 & 0 & 0 \\
 * & * & * & -S_2 & 0 & 0 & 0 \\
 * & * & * & * & -U_1 & 0 & 0 \\
 * & * & * & * & * & -U_2 & 0 \\
 * & * & * & * & * & * & -dR_d
 \end{bmatrix} < 0, \tag{40}$$

where

$$\begin{aligned}
 \Phi = & P^T \begin{bmatrix} 0 & I & 0 \\ A_{01} & -I_{n_1} & A_{02} \\ A_{03} & 0 & A_{04} \end{bmatrix} + \begin{bmatrix} 0 & I & 0 \\ A_{01} & -I_{n_1} & A_{02} \\ A_{03} & 0 & A_{04} \end{bmatrix}^T P \\
 & + \sum_{i=1}^2 \begin{bmatrix} S_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U_i \end{bmatrix} + \int_{-d}^0 \bar{A}_d^T(s) R_d \bar{A}_d(s) ds.
 \end{aligned}$$

If LMI (40) is feasible, then (32) is feasible for a small enough  $\varepsilon > 0$  and for  $R_i$  and  $W_i$  that are given by (39). Thus, from Theorem 2.1 the following corollary holds:

**Corollary 3.6.** *Under A1 the system of (14a), (14b), and (15) is stable for all  $h_i > 0, i = 1, 2$  and  $J < 0$  if there exist  $0 < P_1 = P_1^T, P_2, P_3, U_i = U_i^T$  and  $S_i = S_i^T, i = 1, 2$  that satisfy (40).*

**Remark 2.** As we have seen above, the delay-dependent BRL of Theorem 2.1 is most powerful in the sense that it provides sufficient conditions for both the delay-dependent and the delay-independent cases (where (40) holds). In the latter case, (32) is feasible for  $h_i \rightarrow \infty, i = 1, 2$ .

### 3.5. Delay-dependent BRL for systems with polytopic uncertainties

The BRL of Theorem 2.1 was derived for system (14a), (14b) where the system matrices  $A_i, C_i, i = 1, 2, B_1, A_d$  are all known. However, since the LMI of (32) is affine in the system matrices, the theorem can be used to derive a criterion that will guarantee stability and the required attenuation level in the case where the system matrices are not exactly known and they reside within a given polytope.

Denoting

$$\Omega = [A_i \quad A_d \quad B_1 \quad C_i \quad i = 0, 1, 2]$$

we assume that  $\Omega \in \mathcal{CO}\{\Omega_j, j = 1, \dots, N\}$ , namely,

$$\Omega = \sum_{j=1}^N f_j \Omega_j \quad \text{for some } 0 \leq f_j \leq 1, \quad \sum_{j=1}^N f_j = 1,$$

where the  $N$  vertices of the polytope are described by

$$\Omega_j = [A_i^{(j)} \quad A_d^{(j)} \quad B_1^{(j)} \quad C_i^{(j)} \quad i = 0, 1, 2].$$

We readily obtain the following:

**Corollary 3.7.** *Assume that for all  $j = 1, \dots, N$ , **A1** holds. Consider the system of (14a), (14b), where the system matrices reside within the polytope  $\Omega$ . For a prescribed  $\gamma > 0$ , the cost function (13) achieves  $J(w) < 0$  over  $\Omega$  for all nonzero  $w \in \mathcal{L}_2^q[0, \infty)$  if there exist  $n \times n$ -matrices  $0 < P_1^{(j)}, W_{i1}^{(j)}, W_{i2}^{(j)}, W_{i3}, W_{i4}, j = 1, \dots, N, P_2, P_3$ , and  $0 < R_i^{(j)}, 0 < R_d^{(j)}, 0 < U_i^{(j)}, 0 < S_i^{(j)}, i = 1, 2, j = 1, \dots, N$  that satisfy (32) for  $j = 1, \dots, N$ , where the matrices*

$$A_i, A_d, B_1, C_i, P_1, W_1, W_2, R_1, R_2, R_d, S_1, S_2, \quad i = 0, 1, 2,$$

are taken with the upper index  $j$ .

### 3.6. On LMI conditions in the case of discrete delays

We consider  $A_d = 0$ . Even in this simpler case condition **A1** is not easily verifiable. That is why instead of **A1** one can assume that the fast LMI (17) is feasible for some  $P_f, U_{kf}, k = 1, 2$ . Another possibility is to look for  $P_3$  in Theorem 3.4 in the diagonal form:

$$P_3 = \text{diag}\{P_{31}, P_{32}\}, \quad P_{32} \in \mathbb{R}^{n_2 \times n_2}. \tag{41}$$

In the latter case if the full-order LMI (24) holds for  $P_3$  of (41), then (17) holds for  $P_f = P_{32}$ , where  $U_{kf}$  are (2, 2) blocks of  $U_k$ .

Consider now a difference continuous system

$$0 = A_{04}x_2(t) + \sum_{i=1}^2 A_{i4}x_2(t - h_i) + B_{12}w(t), \tag{42}$$

$$z(t) = \text{col}\{C_{02}x_2, C_{12}x_2(t - h_1), C_{22}x_2(t - h_2)\}.$$

From Theorem 3.4, the following (delay-independent) BRL follows:

**Corollary 3.8.** Given  $\gamma > 0$ , if there exist  $n_2 \times n_2$ -matrices  $P_f, U_{1f}, U_{2f}$  that satisfy the following LMI:

$$\begin{bmatrix} P_f^T A_{04} + A_{04}^T P_f + \sum_{i=1}^2 U_{if} + \sum_{i=0}^2 C_{i2}^T C_{i2} & P_f^T A_{14} & P_f^T A_{24} & P_f^T B_{12} \\ * & -U_{1f} & 0 & 0 \\ * & * & -U_{2f} & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{43}$$

then for all  $h_1 > 0, h_2 > 0$  the difference system (42) is internally stable and  $J(w) < 0$ .

### 3.7. $H_\infty$ -norm of the ‘adjoint’ system

We begin by noting that the  $H_\infty$ -norm of the system  $\Sigma_1$  of (14a) and (14b), where

$$z = C_0 x(t), \quad A_d = 0, \tag{44}$$

is given by (see e.g. [2, vol. 2, p. 32]):

$$\|\Sigma_1\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma} \left\{ C_0 \left( j\omega E - A_0 - \sum_{i=1}^2 A_i e^{-j\omega h_i} \right)^{-1} B_1 \right\}, \tag{45}$$

where  $\bar{\sigma}\{D\}$  denotes the largest singular value of  $D$ . Since

$$\bar{\sigma}\{H(j\omega)\} = \bar{\sigma}\{H^T(-j\omega)\}$$

for all the transfer function matrices  $H(s)$  with real coefficients, it follows that the  $H_\infty$ -norm of  $\Sigma_1$  is equal to the  $H_\infty$ -norm of the following system:

$$\begin{aligned} -E\dot{\xi}(t) &= \sum_{i=0}^2 A_i^T \xi(t + h_i) + C_0^T \tilde{z}(t), & \tilde{w}(t) &= B_1^T \xi(t), \\ \xi &= 0 \quad \forall t \in [0 \ h] \end{aligned} \tag{46}$$

where  $\xi(t) \in \mathbb{R}^n, \tilde{z}(t) \in \mathbb{R}^p$  and  $\tilde{w}(t) \in \mathbb{R}^q$ . Note that the latter system represents the backward adjoint of  $\Sigma_1$  (as defined for nondescrptor case in [2, vol. 1]). Its forward representation,  $\Sigma_2$ , is described by

$$\begin{aligned} E\dot{\xi}(\tau) &= \sum_{i=0}^2 A_i^T \xi(\tau - h_i) + C_0^T \tilde{z}(\tau), & \tilde{w}(\tau) &= B_1^T \xi(\tau), \\ \xi &= 0 \quad \forall \tau \in [-h \ 0]. \end{aligned} \tag{47}$$

Since the characteristic equations of  $\Sigma_2$  and  $\Sigma_1$  are identical, the former system is asymptotically stable iff  $\Sigma_1$  is.

Sufficient conditions of Theorem 3.5 for (14a), (14b), (44) and for its ‘adjoint’ may lead to different results. Therefore, one can apply Theorem 3.5 for the original system and for its ‘adjoint’ and then choose the less conservative result.

**Example 1.** We consider the following system

$$\begin{bmatrix} \dot{x}_1(t) \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 1 \\ 0 & 0.5 \end{bmatrix} x(t-h) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} w(t), \tag{48}$$

$$z(t) = [0.5 \quad 1]x(t),$$

where  $x(t) = \text{col}\{x_1(t), x_2(t)\} \in \mathbb{R}^2$ . Applying the LMI condition of Theorem 3.5 to (48) and its ‘adjoint’ we obtain in both cases that the system is internally stable for  $h \leq 1.15$ . The minimum achievable value of  $\gamma$  is however different in the two cases. For  $h = .1$  we obtain for both systems  $\gamma = 2.3$ , while for  $h = 1$  we obtain for (48)  $\gamma_0 = 9$  and for its ‘adjoint’  $\gamma_t = 6$ . For  $h = 1.12$  the corresponding results are  $\gamma_0 = 40$  and  $\gamma_t = 28$ , respectively. We see that in this example the conditions of Theorem 3.5 for the ‘adjoint’ system are less conservative than those obtained for the original system. Note that the same results are obtained by choosing block-diagonal  $P_3$  with  $P_{32} > 0$ .

#### 4. Delay-dependent state-feedback control

We apply the results of the previous section to the infinite-horizon  $H_\infty$ -control problem. Given system (4), (6) with the objective vector (5b). For a prescribed scalar  $\gamma > 0$ , we consider the performance index of (13). We look for the state-feedback gain matrix  $K$  which, via the control law

$$u(t) = Kx(t), \quad K = [K_1, K_2] \tag{49}$$

achieves  $J(w) < 0$  for all nonzero  $w \in \mathcal{L}_2^q[0, \infty)$ . Substituting (49) into (4), we obtain the structure of (14a) and (14b) with  $A_0 + B_2K$  instead of  $A_0$  and

$$C_0^T C_0 = \bar{C}_1^T \bar{C}_1 + K^T D_{12}^T D_{12} K. \tag{50}$$

Applying the BRL of Section 3 to the above matrices, results in a nonlinear matrix inequality because of the terms  $P_2^T B_2 K$  and  $P_3^T B_2 K$ . We therefore consider another version of the BRL which is derived from (32).

In order to obtain an LMI we have to restrict ourselves to the case of the diagonal matrix  $P_3$  of (41) and (as well as in the nondescriptor problem) to the case of  $W_i = \varepsilon_i P$ ,  $i = 1, 2$ , where  $\varepsilon_i \in \mathbb{R}$  is a scalar parameter. Note that for  $\varepsilon_i = -1$  (32) yields the delay-independent condition of Corollary 3.6. It is obvious from the requirement of  $0 < P_1$ , and the fact that in (32)  $(P_3 A_{04}^T + A_{04} P_3^T)$  must be negative definite, that  $P$  is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}, \tag{51a}$$



$$Q_2 = \begin{bmatrix} Q_{21} \\ Q_{22} \end{bmatrix}, \tag{51b}$$

$$Q_3 = \text{diag}\{Q_{31}, Q_{32}\}, \quad Q_{32} \in \mathbb{R}^{n_2 \times n_2}, \tag{51c}$$

and  $\Delta = \text{diag}\{Q, I_{q+4n+2n_1}\}$  we multiply (32) by  $\Delta^T$  and  $\Delta$ , on the left and on the right, respectively. Applying the Schur formula to the quadratic term in  $Q$ , we obtain the following inequality:

$$\begin{bmatrix} \Xi_1 + \Xi_2 & \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_1(\varepsilon_1 + 1)I_{n+n_1} & h_2(\varepsilon_2 + 1)I_{n+n_1} & \varepsilon_1 \begin{bmatrix} 0 \\ A_{11} \\ A_{13} \end{bmatrix} & \varepsilon_2 \begin{bmatrix} 0 \\ A_{21} \\ A_{23} \end{bmatrix} & \begin{bmatrix} 0 \\ A_{12} \\ A_{14} \end{bmatrix} & \begin{bmatrix} 0 \\ A_{22} \\ A_{24} \end{bmatrix} & Q^T \begin{bmatrix} C_{01}^T \\ 0 \\ C_{02}^T \end{bmatrix} \\ * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -h_1 R_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h_2 R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -S_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & * & * & * & -U_2 & 0 \\ * & * & * & * & * & * & * & * & -I_p \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{bmatrix} \\ \\ Q^T \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} & Q^T \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} & h_1 Q^T \begin{bmatrix} 0_{n_1 \times n_1} & 0 & 0 \\ 0 & A_{11}^T & A_{13}^T \\ 0 & 0 & 0 \end{bmatrix} & h_2 Q^T \begin{bmatrix} 0_{n_1 \times n_1} & 0 & 0 \\ 0 & A_{21}^T & A_{23}^T \\ 0 & 0 & 0 \end{bmatrix} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -S_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -S_2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -U_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -U_2^{-1} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -h_1 R_1^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -h_2 R_2^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & * & -h_2 R_2^{-1} & 0 & 0 \end{bmatrix} < 0, \tag{52}$$

where  $C_0 = [C_{01} \ C_{02}]$ ,

$$\begin{aligned} \Xi_1 = & \begin{bmatrix} 0 & I_{n_1} & 0 \\ \sum_{i=0}^2 A_{i1} & -I_{n_1} & A_{02} \\ \sum_{i=0}^2 A_{i3} & 0 & A_{04} \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & I_{n_1} & 0 \\ \sum_{i=0}^2 A_{i1} & -I_{n_1} & A_{02} \\ \sum_{i=0}^2 A_{i3} & 0 & A_{04} \end{bmatrix}^T \\ & + \sum_{i=1}^2 \varepsilon_i \begin{bmatrix} 0 & 0 & 0 \\ A_{i1} & 0 & 0 \\ A_{i3} & 0 & 0 \end{bmatrix} Q + \sum_{i=1}^2 \varepsilon_i Q^T \begin{bmatrix} 0 & A_{i1}^T & A_{i3}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{53} \end{aligned}$$



where

$$\bar{E} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \begin{bmatrix} Y_1 & 0_{n_1} & Y_2 \end{bmatrix} + \begin{bmatrix} Y_1^T \\ 0_{n_1} \\ Y_2^T \end{bmatrix} \begin{bmatrix} 0 & B_2^T \end{bmatrix}.$$

The state-feedback gain is then given by

$$K_2 = Y_2 Q_{32}^{-1}, \quad K_1 = (Y_1 - K_2 Q_{22}) Q_1^{-1}. \tag{56}$$

**Example 2.** We consider the system

$$\begin{aligned} E\dot{x}(t) &= A_1 x(t - h) + B_1 w(t) + B_2 u(t), \quad x_0 = 0, \\ z(t) &= \bar{C}_1 x(t) + D_{12} u(t), \end{aligned} \tag{57}$$

where

$$\begin{aligned} E &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \\ C_1 &= [1 \quad 0.2], \quad D_{12} = [0.1]. \end{aligned}$$

Note that in this example  $A_{04} = 0$ . We first find the state-feedback solution. We obtained a near minimum value of  $\gamma = 21$  for  $h = 1.2$  and  $\varepsilon_1 = -0.255$ . The state-feedback control law that achieves the later bound on the  $H_\infty$ -norm of the closed loop is  $u = Kx$ , where  $K = [175.62 \quad -430680]$ .

The LMI in Theorem 4.1 is affine in the system matrices. It can thus be applied also to the case where these matrices are uncertain and are known to reside within a given polytope. Considering the system of (4) and denoting

$$\Omega = \begin{bmatrix} E & A_0 & A_1 & A_2 \\ B_1 & B_2 & \bar{C}_1 & D_{12} \end{bmatrix},$$

we assume that  $\Omega \in \mathcal{CO}\{\Omega_j, j = 1, \dots, N\}$ , where the  $N$  vertices of the polytope are described by

$$\Omega^{(j)} = \begin{bmatrix} E & A_0^{(j)} & A_1^{(j)} & A_2^{(j)} \\ B_1^{(j)} & B_2^{(j)} & \bar{C}_1^{(j)} & D_{12}^{(j)} \end{bmatrix}.$$

We obtain the following:

**Theorem 4.2.** Consider the system of (4), (6), (5b), where the system matrices reside within the polytope  $\Omega$  and the cost function of (13). For a prescribed  $0 < \gamma$ , the state-feedback law of (49) achieves,  $J(w) < 0$  for all nonzero  $w \in \mathcal{L}_2^q[0, \infty)$

and for all the matrices in  $\Omega$  if for some prescribed scalars  $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$  there exist  $0 < Q_1 \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{S}_k = S_k^{-1} \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{U}_k = U_k^{-1} \in \mathbb{R}^{n_2 \times n_2}, k = 1, 2, Q_2 \in \mathbb{R}^{n \times n_1}$  and  $Q_3 \in \mathbb{R}^{n \times n}$  of (51a)–(51c),  $0 < \bar{R}_1 = R_1^{-1}, 0 < \bar{R}_2 = R_2^{-1} \in \mathbb{R}^{(n+n_1) \times (n+n_1)}, Y_1 \in \mathbb{R}^{\ell \times n_1}$  and  $Y_2 \in \mathbb{R}^{\ell \times n_2}$  that satisfy LMIs (55) for  $j = 1, \dots, N$ , where the matrices

$$A_i, i = 0, 1, 2, B_1, B_2, \bar{C}_1, D_{12}, \bar{R}_1, \bar{R}_2$$

are taken with the upper index  $j$ . The state-feedback gain is then given by (56).

### 5. Delay-dependent filtering

We consider system (4) with the measurement law of (5a). We seek a filter of the following observer form:

$$E \dot{\hat{x}}(t) = \sum_{i=0}^2 A_i \hat{x}(t - h_i) + K_f (\bar{y}(t) - \bar{C}_2 x(t)) \tag{58}$$

such that the  $H_\infty$ -norm of the resulting transference between the exogenous signal  $w$  and the estimation error  $z$  is less than a prescribed value  $\gamma$ , where

$$z(t) \triangleq L(x(t) - \hat{x}(t)). \tag{59}$$

From (4), (5a) and (58) it follows that the estimation error  $e(t) = x(t) - \hat{x}(t)$  is described by the following model:

$$E \dot{e}(t) = (A_0 - K_f \bar{C}_2) e(t) + \sum_{i=1}^2 A_i \bar{e}(t - h_i) + (B_1 - K_f D_{21}) w, \tag{60}$$

$$z(t) = L e(t).$$

The problem then becomes one of finding the filter gain  $K_f$  such that  $J(w) < 0$ .

We consider the ‘adjoint’ to (60) system described by

$$E \dot{\xi}(\tau) = (A_0^T - \bar{C}_2^T K_f^T) \xi(\tau) + \sum_{i=1}^2 A_i^T \xi(\tau - h_i) + L^T \bar{z}(\tau), \tag{61}$$

$$\bar{w}(\tau) = (B_1^T - D_{21}^T K_f^T) \xi(\tau), \quad \xi = 0 \quad \forall \tau \in [-h \quad 0].$$

Analogously to Theorem 4.1 (by applying BRL of Theorem 3.5 to (61)) we obtain:

**Theorem 5.1.** Consider the system of (4), (5a) and the cost function  $J(w)$ . For a prescribed  $0 < \gamma$ , the filter gain achieves,  $J(w) < 0$  for all nonzero  $w \in \mathcal{L}_2^q[0, \infty)$  if for some prescribed scalars  $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ , there exist  $0 < Q_1 \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{S}_k \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{U}_k \in \mathbb{R}^{n_2 \times n_2}, k = 1, 2, Q_2 \in \mathbb{R}^{n \times n_1}$  and  $Q_3 \in \mathbb{R}^{n \times n}$  of (51a)–(51c),  $0 < \bar{R}_1, 0 < \bar{R}_2 \in \mathbb{R}^{(n+n_1) \times (n+n_1)}, Y_1 \in \mathbb{R}^{\ell \times n_1}$  and  $Y_2 \in \mathbb{R}^{\ell \times n_2}$  that satisfy the following LMI:



$$\bar{E} = - \begin{bmatrix} 0 \\ \bar{C}_2^T \end{bmatrix} \begin{bmatrix} Y_1 & 0_{n_1} & Y_2 \end{bmatrix} - \begin{bmatrix} Y_1^T \\ 0_{n_1} \\ Y_2^T \end{bmatrix} \begin{bmatrix} 0 & \bar{C}_2 \end{bmatrix},$$

$$\mathcal{A}_i = \begin{bmatrix} Q_{21}^T \\ Q_{31}^T \\ 0 \end{bmatrix} \begin{bmatrix} 0 & A_{i1} & A_{i2} \end{bmatrix}, \quad i = 1, 2.$$

The filter gain is then given by

$$K_2^T = Y_2 Q_{32}^{-1}, \quad K_1^T = (Y_1 - K_2^T Q_{22}) Q_1^{-1}, \quad K_f = \text{col}\{K_1, K_2\}. \quad (63)$$

The LMI in Theorem 5.1 is affine in the system matrices. Similarly to Theorem 4.2, it can thus be reformulated also to the case of matrices with polytopic uncertainties.

### 6. Delay-dependent output-feedback control

We adopt in this section the dissipation approach to the solution of the output-feedback problem. It applies a controller of a state-feedback—observer structure and requires a solution of two LMIs. We assume:

**A2.** The matrices  $B_1$  and  $D_{21}$  are orthogonal, i.e.  $B_1 D_{21}^T = 0$ , and  $\tilde{R} = D_{12}^T D_{12}$  is not singular.

#### 6.1. The first phase: a state-feedback controller design

**Lemma 6.1.** Assume **A2**. Consider system (4), (5b). For a prescribed  $\gamma > 0$ , the feedback law

$$u(t) = - \begin{bmatrix} 0 & \tilde{R}^{-1} B_2^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ y(t) \\ x_2(t) \end{bmatrix}, \quad (64a)$$

$$\begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}^{-1} \quad (64b)$$

achieves  $J(w) < 0$  for all nonzero  $w \in \mathcal{L}_2^q[0, \infty)$  if for some prescribed scalars  $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ , there exist  $0 < Q_1 \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{S}_k = S_k^{-1} \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{U}_k = U_k^{-1} \in \mathbb{R}^{n_2 \times n_2}, k = 1, 2, Q_2 \in \mathbb{R}^{n \times n_1}$  and  $Q_3 \in \mathbb{R}^{n \times n}$  of (51a)–(51c),  $0 < \bar{R}_1 = R_1^{-1}$  and  $0 < \bar{R}_2 = R_2^{-1} \in \mathbb{R}^{(n+n_1) \times (n+n_1)}$  that satisfy the following LMI:

$$\begin{bmatrix} \Xi_1 + \bar{\Xi} & \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_1(\varepsilon_1 + 1)\bar{R}_1 & h_2(\varepsilon_2 + 1)\bar{R}_2 & \varepsilon_1 \begin{bmatrix} 0 \\ A_{11} \\ A_{13} \end{bmatrix} & \bar{S}_1 & \varepsilon_2 \begin{bmatrix} 0 \\ A_{21} \\ A_{23} \end{bmatrix} & \bar{S}_2 & \begin{bmatrix} 0 \\ A_{12} \\ A_{14} \end{bmatrix} & \bar{U}_1 & \begin{bmatrix} 0 \\ A_{22} \\ A_{24} \end{bmatrix} & \bar{U}_2 & Q^T \begin{bmatrix} \bar{C}_{11}^T \\ 0 \\ \bar{C}_{12}^T \end{bmatrix} \\ * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -h_1 \bar{R}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h_2 \bar{R}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{U}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\bar{U}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & -I_p \\ * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} & \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} & \begin{bmatrix} Q_{22}^T \\ 0 \\ Q_{32}^T \end{bmatrix} & \begin{bmatrix} Q_{22}^T \\ 0 \\ Q_{32}^T \end{bmatrix} & h_1 \begin{bmatrix} Q_{21}^T \\ Q_{31}^T \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ A_{11}^T \\ A_{13}^T \end{bmatrix} & A_{11}^T & A_{13}^T & h_2 \begin{bmatrix} Q_{21}^T \\ Q_{31}^T \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ A_{21}^T \\ A_{23}^T \end{bmatrix} & \begin{bmatrix} 0 \\ A_{21}^T \\ A_{23}^T \end{bmatrix} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\bar{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\bar{U}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{U}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -h_1 \bar{R}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -h_2 \bar{R}_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \tag{65}$$

where  $\Xi_1$  is given by (53) and where

$$\hat{\Xi} = - \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tilde{R}^{-1} [0 \quad B_2^T].$$

**Proof.** The proof readily follows by choosing  $V$  as in (31) and applying (36) for  $R_i M_i = \varepsilon_i I, i = 1, 2$ . Denote by  $\Gamma_u$  the matrix in the left-hand side of (65). We obtain by integrating  $dV(t)/dt$  that

$$\begin{aligned} J &\leq \int_0^\infty \bar{\xi}^T \Gamma_p \bar{\xi} dt + \int_0^\infty (u^T - u^{*T}) \tilde{R} (u - u^*) dt \\ &\quad - \gamma^2 \int_0^\infty (w^T - w^{*T})(w - w^*) dt, \end{aligned} \tag{66}$$

where

$$\begin{aligned} \Gamma_p &= \text{diag}\{P^T, I_{8n+4n_1+p+q}\} \Gamma_u \text{diag}\{P, I_{8n+4n_1+p+q}\}, \\ w^* &= \gamma^{-2} \begin{bmatrix} 0 & B_1^T \end{bmatrix} P \begin{bmatrix} x_1 \\ y \\ x_2 \end{bmatrix}, \\ u^* &= -\tilde{R}^{-1} \begin{bmatrix} 0 & B_2^T \end{bmatrix} P \begin{bmatrix} x_1 \\ y \\ x_2 \end{bmatrix} \end{aligned} \tag{67}$$

and where the relation between  $P$  and  $Q_i, i = 1, \dots, 3$ , is given in (51a)–(51c) and  $\tilde{\xi} = \text{col}\{x_1, y, x_2, w, \eta_1\}$ , with  $\eta_1$  representing the fictitious states that emerge when applying Schur formula to construct  $\Gamma_p$ .  $\square$

Unfortunately, the feedback law of (64a) and (64b) cannot be implemented even when there exists a solution to (65), namely when the first term in the right-hand side of (66) is negative for all  $\tilde{\xi} \in \mathbb{R}^{8n+4n_1+p+q}$ .

### 6.2. The second phase: filtering via the adjoint system

Denoting  $\bar{r} = w - w^*$  we represent (4) and (5a) in the form:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ 0 \\ 0 \end{bmatrix} &= \sum_{i=0}^2 \hat{A}_i \begin{bmatrix} x_1(t - h_i) \\ y(t - h_i) \\ x_2(t - h_i) \end{bmatrix} + \hat{B}_1 \bar{r}(t) + \begin{bmatrix} B_{21} \\ B_{21} \\ B_{22} \end{bmatrix} u(t), \\ \bar{y}(t) &= \hat{C}_2 \begin{bmatrix} x_1(t) \\ y(t) \\ x_2(t) \end{bmatrix} + D_{21} \bar{r}(t), \end{aligned} \tag{68}$$

where

$$\begin{aligned} \hat{A}_0 &= \begin{bmatrix} A_{01} & 0 & A_{02} \\ A_{01} & -I_{n_1} & A_{02} \\ A_{03} & 0 & A_{04} \end{bmatrix} + \gamma^{-2} \hat{B}_1 \begin{bmatrix} 0_{q \times n_1} & B_1^T \end{bmatrix} P, \\ \hat{A}_i &= \begin{bmatrix} A_{i1} & 0 & A_{i2} \\ A_{i1} & 0 & A_{i2} \\ A_{i3} & 0 & A_{i4} \end{bmatrix}, \quad i = 1, 2, \quad \hat{B}_1 = \begin{bmatrix} B_{11} \\ B_{11} \\ B_{12} \end{bmatrix}, \\ \hat{C}_2 &= [\bar{C}_{21} \quad 0_{r \times n_1} \quad \bar{C}_{22}]. \end{aligned} \tag{69}$$

and where  $P$  solves (65). The objective function of (13) and (66) will then be negative if there exist  $\hat{x}(t)$  and  $\hat{y}(t)$  in  $\mathbb{R}^n$  that satisfy



$$J_a = \int_0^\infty (\bar{z}^T \bar{z} - \gamma^2 \bar{r}^T \bar{r}) dt < 0 \quad \forall \bar{r} \in \mathcal{L}_2^q[0, \infty),$$

$$\bar{z}(t) = \tilde{R}^{-1/2} \begin{bmatrix} 0 & B_2^T \end{bmatrix} P \begin{bmatrix} x_1(t) - \hat{x}_1(t) \\ y(t) - \hat{y}(t) \\ x_2(t) - \hat{x}_2(t) \end{bmatrix}. \tag{70}$$

The problem of finding  $\hat{x}(t)$  and  $\hat{y}(t)$  is, in fact, a  $H_\infty$  filtering problem for the descriptor system (68).

Consider the following ‘innovation’ filter

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ 0 \\ 0 \end{bmatrix} = \sum_{i=0}^2 \hat{A}_i \begin{bmatrix} \hat{x}_1(t - h_i) \\ \hat{y}(t - h_i) \\ \hat{x}_2(t - h_i) \end{bmatrix} + K_f \bar{y}(t) - K_f \hat{C}_2 \begin{bmatrix} \hat{x}_1(t) \\ \hat{y}(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{21} \\ B_{22} \end{bmatrix} u(t). \tag{71}$$

Denoting

$$e = \begin{bmatrix} e_1 \\ e_0 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y \\ x_2 \end{bmatrix} - \begin{bmatrix} \hat{x}_1 \\ \hat{y} \\ \hat{x}_2 \end{bmatrix} \tag{72}$$

and using the assumption on  $D_{21}$  and the definition of  $w^*$  in (67) we find

$$\begin{bmatrix} \dot{e}_1(t) \\ 0 \\ 0 \end{bmatrix} = (\hat{A}_0 - K_f \hat{C}_2) e(t) + \sum_{i=1}^2 \hat{A}_i e(t - h_i) + (\hat{B}_1 - K_f D_{21}) \bar{r}(t), \tag{73}$$

$$\bar{z}(t) = \tilde{R}^{-1/2} \begin{bmatrix} 0 & B_2^T \end{bmatrix} P e(t).$$

The problem now becomes one of finding the gain matrix  $K_f$  that will ensure the stability of system (73) and that the  $H_\infty$ -norm of the transference from  $\bar{r}$  to  $\bar{z}$  is less than  $\gamma$ . This problem was solved in Section 5. By applying Theorem 5.1 we obtain the following result:

**Theorem 6.2.** *Assume A2. Consider the system of (4), (5a), (5b) and the cost function of (13). For a prescribed  $0 < \gamma$ , there exists an output-feedback controller that achieves,  $J(w) < 0$  for all nonzero  $w \in \mathcal{L}_2^q[0, \infty)$  if for some prescribed scalars  $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ , there exist  $0 < Q_1 \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{S}_k \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{U}_k \in \mathbb{R}^{n_2 \times n_2}, k = 1, 2, Q_2 \in \mathbb{R}^{n \times n_1}$  and  $Q_3 \in \mathbb{R}^{n \times n}$  of (51a)–(51c),  $0 < \bar{R}_1$  and  $0 < \bar{R}_2 \in \mathbb{R}^{(n+n_1) \times (n+n_1)}$  that satisfy (65) and for some prescribed scalars  $\hat{\varepsilon}_1, \hat{\varepsilon}_2 \in \mathbb{R}$ , there exist  $0 < \hat{Q}_1 \in \mathbb{R}^{n_1 \times n_1}, 0 < \hat{S}_k \in \mathbb{R}^{n_1 \times n_1}, 0 < \hat{U}_k \in \mathbb{R}^{n \times n}, k = 1, 2, \hat{Q}_2 \in \mathbb{R}^{(n+n_1) \times n_1}$  and  $\hat{Q}_3 \in \mathbb{R}^{(n+n_1) \times (n+n_1)}$  of the form*

$$\hat{Q}_2 = \begin{bmatrix} \hat{Q}_{21} \\ \hat{Q}_{22} \end{bmatrix}, \quad \hat{Q}_{22} \in \mathbb{R}^{n \times n_1},$$

$$\hat{Q}_3 = \text{diag}\{\hat{Q}_{31}, \hat{Q}_{32}\}, \quad \hat{Q}_{31} \in \mathbb{R}^{n_1 \times n_1}, \quad \hat{Q}_{32} \in \mathbb{R}^{n \times n},$$

$$0 < \hat{R}_1, \quad 0 < \hat{R}_2 \in \mathbb{R}^{(2n_1+n) \times (2n_1+n)}, \quad Y_1 \in \mathbb{R}^{r \times n_1}, \quad \text{and} \quad Y_2 \in \mathbb{R}^{r \times n}$$

that satisfy the following LMI:

$$\begin{bmatrix} \Xi_1 + \Xi & \begin{bmatrix} 0_{n_1 \times r} \\ P \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \hat{R}^{-1/2} \end{bmatrix} & h_1(\hat{\varepsilon}_1 + 1)\hat{R}_1 & h_2(\hat{\varepsilon}_2 + 1)\hat{R}_2 & \hat{\varepsilon}_1 \begin{bmatrix} 0 \\ \hat{A}_{11}^T \\ \hat{A}_{12}^T \end{bmatrix} & \hat{\varepsilon}_1 \begin{bmatrix} 0 \\ \hat{A}_{21}^T \\ \hat{A}_{22}^T \end{bmatrix} & \hat{\varepsilon}_2 \begin{bmatrix} 0 \\ \hat{A}_{13}^T \\ \hat{A}_{14}^T \end{bmatrix} & \hat{\varepsilon}_2 \begin{bmatrix} 0 \\ \hat{A}_{23}^T \\ \hat{A}_{24}^T \end{bmatrix} & \hat{U}_1 & \hat{U}_2 \\ * & -\gamma^2 I_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -h_1 \hat{R}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h_2 \hat{R}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\hat{S}_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\hat{S}_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\hat{U}_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\hat{U}_2 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

$$\hat{Q}^T \begin{bmatrix} 0 \\ B_{11} \\ B_1 \end{bmatrix} \begin{bmatrix} Y_1^T \\ 0 \\ Y_2^T \end{bmatrix} D_{21} \begin{bmatrix} \hat{Q}_1 \\ 0 \end{bmatrix} \begin{bmatrix} \hat{Q}_1 \\ 0 \end{bmatrix} \begin{bmatrix} \hat{Q}_{22}^T \\ 0 \\ \hat{Q}_{32}^T \end{bmatrix} \begin{bmatrix} \hat{Q}_{22}^T \\ 0 \\ \hat{Q}_{32}^T \end{bmatrix} \begin{bmatrix} h_1 \hat{\mathcal{A}}_1 & h_2 \hat{\mathcal{A}}_2 \end{bmatrix} < 0, \tag{74}$$

where

$$\begin{aligned} P_1 &= Q_1^{-1}, \quad P_2 = -Q_3^{-1} Q_2 Q_1^{-1}, \\ P_3 &= Q_3^{-1} = \text{diag}\{Q_{31}^{-1}, Q_{32}^{-1}\} = \text{diag}\{P_{31}, P_{32}\}, \\ P &= \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \end{aligned} \tag{75}$$

$$\begin{aligned} \Xi_1 = & \begin{bmatrix} 0 & I & 0 \\ \sum_{i=0}^2 \hat{A}_{i1}^T & -I_{n_1} & \hat{A}_{03}^T \\ \sum_{i=0}^2 \hat{A}_{i2}^T & 0 & \hat{A}_{04}^T \end{bmatrix} \hat{Q} + \hat{Q}^T \begin{bmatrix} 0 & I & 0 \\ \sum_{i=0}^2 \hat{A}_{i1}^T & -I_{n_1} & \hat{A}_{03}^T \\ \sum_{i=0}^2 \hat{A}_{i2}^T & 0 & \hat{A}_{04}^T \end{bmatrix}^T \\ & + \sum_{i=1}^2 \hat{\varepsilon}_i \begin{bmatrix} 0 & 0 & 0 \\ \hat{A}_{i1}^T & 0 & 0 \\ \hat{A}_{i2}^T & 0 & 0 \end{bmatrix} \hat{Q} + \sum_{i=1}^2 \hat{\varepsilon}_i \hat{Q}^T \begin{bmatrix} 0 & \hat{A}_{i1} & \hat{A}_{i2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\bar{\Xi} = - \begin{bmatrix} 0 \\ \hat{C}_2^T \end{bmatrix} [Y_1 \quad 0_{n_1} \quad Y_2] - \begin{bmatrix} Y_1^T \\ 0_{n_1} \\ Y_2^T \end{bmatrix} [0 \quad \hat{C}_2],$$

$$\hat{A}_{i1} = A_{i1}, \quad \hat{A}_{i2} = [0 \quad A_{i2}], \quad \hat{A}_{i3} = \begin{bmatrix} A_{i1} \\ A_{i3} \end{bmatrix}, \quad \hat{A}_{i4} = \begin{bmatrix} 0 & A_{i2} \\ 0 & A_{i4} \end{bmatrix}, \quad i = 1, 2,$$

$$\hat{A}_{01} = A_{01} + \gamma^{-2} B_{11} B_1^T P_2,$$

$$\hat{A}_{02} = [0 \quad A_{02}] + \gamma^{-2} [B_{11} B_{11}^T P_{31} \quad B_{11} B_{12}^T P_{32}],$$

$$\hat{A}_{03} = \begin{bmatrix} A_{01} \\ A_{03} \end{bmatrix} + \gamma^{-2} \begin{bmatrix} B_{11} B_1^T P_2 \\ B_{12} B_1^T P_2 \end{bmatrix},$$

$$\hat{A}_{04} = \begin{bmatrix} -I_{n_1} & A_{02} \\ 0 & A_{04} \end{bmatrix} + \gamma^{-2} \begin{bmatrix} B_{11} B_{11}^T P_{31} & B_{11} B_{12}^T P_{32} \\ B_{12} B_{11}^T P_{31} & B_{12} B_{12}^T P_{32} \end{bmatrix},$$

$$\hat{\mathcal{A}}_i = \begin{bmatrix} \hat{Q}_{21}^T \\ \hat{Q}_{31}^T \\ 0 \end{bmatrix} [0 \quad A_{i1} \quad 0 \quad A_{i2}], \quad i = 1, 2.$$

The filter gain is then given by

$$K_2^T = Y_2 \hat{Q}_{32}^{-1}, \quad K_1^T = (Y_1 - K_2^T \hat{Q}_{22}) \hat{Q}_1^{-1}, \quad K_f = \text{col}\{K_1, K_2\}. \quad (76)$$

If a solution to (65) and (74) exists, then the output-feedback controller is obtained by:

$$u(t) = -\tilde{R}^{-1} B_2^T [P_2 \hat{x}_1(t) + P_3 \text{col}\{\hat{y}(t) \quad \hat{x}_2(t)\}], \quad (77)$$

where  $\hat{x}$  and  $\hat{y}$  are obtained by (71).

**Example 3.** We consider the following system:

$$\begin{aligned} E \dot{x}(t) &= \sum_{i=0}^1 A_1 x(t - h_i) + B_1 w(t) + B_2 u(t), \quad x_0 = 0, \\ z(t) &= \bar{C}_1 x(t) + D_{12} u(t), \\ y(t) &= \bar{C}_2 x(t) + D_{21} w(t), \end{aligned} \quad (78)$$

where

$$\begin{aligned}
 E &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \\
 B_1 &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = [1 \quad 0.2], \quad C_2 = [1 \quad 0], \\
 D_{12} &= 0.1, \quad D_{21} = [0 \quad 0.1].
 \end{aligned}$$

We first find the state-feedback solution. We obtained a near minimum value of  $\gamma = 11$  for  $h_1 = 1.2$  and  $\varepsilon_1 = -0.3$ . The state-feedback control law that achieves the later bound on the  $H_\infty$ -norm of the closed loop is  $u = Kx$ , where  $K = [42.6854 \quad -0.3426]$ . Using (65) we find that the feedback law  $u^*$  does not apply  $y$  because  $B_{21} = 0$  and  $P_3$  is diagonal. We thus obtained the same near minimum value of  $\gamma = 11$  for  $h = 1.2$  and  $\varepsilon_1 = -0.3$ . The feedback control law that achieves the later bound on the  $H_\infty$ -norm of the closed loop is

$$u^* = 9.72x_1 + 3.22x_2.$$

The output-feedback control is derived for the same values of  $h$ ,  $\varepsilon_1$  and  $\hat{\varepsilon}_1 = -1$ . A minimum value of  $\gamma = 2.4$  is obtained. The resulting output-feedback has the form  $9.72\hat{x}_1 + 3.22\hat{x}_2$ , where  $\hat{x}$  is obtained by (71) with

$$K_f = [6.1814 \quad 1.1882 \quad -3.1710]^T.$$

### 6.3. The second phase: direct filtering

The filtering of the previous section suffers from an additional overdesign that stems from the use of the adjoint system which must be stable independently of the delays in the variable  $e_0$ . The advantage of the approach of Section 6.2 in comparison with [12], where nondescriptor systems were considered, lies in the fact that it applies the efficient bounds introduced by Park [25]. For smaller values of  $h$  the method of [12] may lead to less conservative results (see Example 4 below). In the present section, we generalize the method of [12] to the case of descriptor systems with delay.

Denoting  $\bar{r} = w - w^*$  we represent (4) in the form:

$$\begin{aligned}
 \begin{bmatrix} \dot{\hat{x}}_1(t) \\ 0 \\ 0 \end{bmatrix} &= \hat{A} \begin{bmatrix} x_1(t) \\ y(t) \\ x_2(t) \end{bmatrix} - \sum_{i=1}^2 \begin{bmatrix} 0 \\ A_{i1} \\ A_{i3} \end{bmatrix} \int_{t-h_i}^t y(s) ds \\
 &+ \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \bar{r}(t) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(t),
 \end{aligned} \tag{79}$$

where

$$\hat{A} = \begin{bmatrix} 0 & I & 0 \\ \sum_{i=0}^2 A_{i1} & -I & A_{02} \\ \sum_{i=0}^2 A_{i3} & 0 & A_{04} \end{bmatrix} + \gamma^{-2} \hat{B}_1 \hat{B}_1^T P, \quad \hat{B}_1 = \begin{bmatrix} 0 \\ B_1 \end{bmatrix},$$

$$\hat{C}_2 = [\bar{C}_{21} \quad 0 \quad \bar{C}_{22}]$$

and where  $P$  solves (65). The objective function of (13) and (66) will then be negative if there exist  $\hat{x}(t)$  and  $\hat{y}(t)$  in  $\mathbb{R}^n$  that satisfy (70).

Consider the following ‘innovation’ filter

$$\begin{aligned} \begin{bmatrix} \hat{\dot{x}}_1(t) \\ 0 \\ 0 \end{bmatrix} &= \hat{A} \begin{bmatrix} \hat{x}_1(t) \\ \hat{y}(t) \\ \hat{x}_2(t) \end{bmatrix} - \sum_{i=1}^2 \begin{bmatrix} 0 \\ A_{i1} \\ A_{i3} \end{bmatrix} \int_{t-h_i}^t \hat{y}(s) \, ds \\ &+ K_f [\bar{y}(t) - \hat{C}_2 \hat{x}(t)] + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(t). \end{aligned} \tag{80}$$

Using the notation of (72), assumption **A2** and the definition of  $w^*$  in (67) we find

$$\begin{aligned} \begin{bmatrix} \dot{e}_1(t) \\ 0 \\ 0 \end{bmatrix} &= [\hat{A} - K_f \hat{C}_2] e(t) - \sum_{i=1}^2 \begin{bmatrix} 0 \\ A_{i1} \\ A_{i3} \end{bmatrix} \int_{t-h_i}^t \hat{e}_0(s) \, ds \\ &+ (\hat{B}_1 - K_f D_{21}) \bar{r}(t), \\ \bar{z}(t) &= \tilde{R}^{-1/2} \begin{bmatrix} 0 & B_2^T \end{bmatrix} P e(t). \end{aligned} \tag{81}$$

The problem now becomes one of finding the gain matrix  $K_f$  that will ensure the stability of system (81) and that the  $H_\infty$ -norm of the transference from  $\bar{r}$  to  $\bar{z}$  is less than  $\gamma$ . Similarly to [12] we obtain the following result:

**Theorem 6.3.** *Consider the system of (4), (5a), (5b) and the cost function of (13). For a prescribed  $0 < \gamma$ , there exists an output-feedback controller that achieves,  $J(w) < 0$  for all nonzero  $w \in \mathcal{L}_2^q[0, \infty)$  if for some prescribed scalars  $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ , there exist  $0 < Q_1 \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{S}_k \in \mathbb{R}^{n_1 \times n_1}, 0 < \bar{U}_k \in \mathbb{R}^{n_2 \times n_2}, k = 1, 2, Q_2 \in \mathbb{R}^{n \times n_1}$  and  $Q_3 \in \mathbb{R}^{n \times n}$  of (51a)–(51c),  $0 < \bar{R}_1$  and  $0 < \bar{R}_2 \in \mathbb{R}^{(n+n_1) \times (n+n_1)}$  that satisfy (65) and there exist  $0 < \hat{P}_1 \in \mathbb{R}^{n_1 \times n_1}, 0 < \hat{U}_k \in \mathbb{R}^{n_2 \times n_2}, k = 1, 2, \hat{P}_2 \in \mathbb{R}^{n \times n_1}$  and  $\hat{P}_3 \in \mathbb{R}^{n \times n}$  of the form*

$$\begin{aligned} \hat{P}_2 &= \begin{bmatrix} \hat{P}_{21} \\ \hat{P}_{22} \end{bmatrix}, \quad \hat{P}_{22} \in \mathbb{R}^{n_2 \times n_1}, \\ \hat{P}_3 &= \text{diag}\{\hat{P}_{31}, \hat{P}_{32}\}, \quad \hat{P}_{31} \in \mathbb{R}^{n_1 \times n_1}, \quad \hat{P}_{32} \in \mathbb{R}^{n_2 \times n_2}, \\ 0 < \hat{R}_1, \quad 0 < \hat{R}_2 &\in \mathbb{R}^{n_1 \times n_1}, \quad Y \in \mathbb{R}^{(n+n_1) \times r} \end{aligned}$$

that satisfy the following LMI:

$$\begin{bmatrix}
 \bar{\Psi}_1 & \hat{P}^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} - Y D_{21} & h_1 \hat{P}^T \begin{bmatrix} 0 \\ A_{11} \\ A_{13} \end{bmatrix} & h_2 \hat{P}^T \begin{bmatrix} 0 \\ A_{11} \\ A_{13} \end{bmatrix} & \begin{bmatrix} P_2^T \\ P_3^T \end{bmatrix} B_2 & \hat{P}^T \begin{bmatrix} 0 \\ A_{12} \\ A_{14} \end{bmatrix} & \hat{P}^T \begin{bmatrix} 0 \\ A_{22} \\ A_{24} \end{bmatrix} \\
 * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 \\
 * & * & -h_1 \hat{R}_1 & 0 & 0 & 0 & 0 \\
 * & * & 0 & -h_2 \hat{R}_2 & 0 & 0 & 0 \\
 * & * & * & * & -\bar{R} & 0 & 0 \\
 * & * & * & * & * & -\hat{U}_1 & 0 \\
 * & * & * & * & * & * & -\hat{U}_2
 \end{bmatrix} < 0, \tag{82}$$

where  $P, P_1, P_2, P_3$  are defined by (75) and

$$\bar{\Psi}_1 = \hat{P}^T \hat{A} + \hat{A}^T \hat{P} - Y \hat{C}_2 - \hat{C}_2^T Y^T + \begin{bmatrix} 0_{n_1 \times n_1} & 0 & 0 \\ 0 & \sum_{i=1}^2 h_i \hat{R}_i & 0 \\ 0 & 0 & \sum_{i=1}^2 \hat{U}_i \end{bmatrix}.$$

If a solution to (65) and (82) exists, then the output-feedback controller is obtained by (77), where  $\hat{x}$  and  $\hat{y}$  are obtained by (80) with  $K_f = \hat{P}^{-T} Y$ .

**Example 4.** We consider the nondescriptor system of [12]

$$\begin{aligned}
 \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B_1 w(t) + B_2 u(t), \\
 z(t) &= \text{col}\{C_1 x(t), D_{12} u(t)\}, \\
 \bar{y}(t) &= C_2 x(t) + D_{21} w(t),
 \end{aligned} \tag{83}$$

where

$$\begin{aligned}
 A_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
 C_1 &= [0 \quad 1], \quad D_{12} = 0.1, \quad C_2 = [0 \quad 1], \quad D_{21} = [0 \quad 0.1].
 \end{aligned} \tag{84}$$

We compare the output-feedback controller designs achieved by the two methods presented in Sections 6.2 and 6.3. By the counterpart of Theorem 6.3 for the non-descriptor case, the output-feedback control was derived in [12] for  $h = 0.999$  and  $\varepsilon_1 = -0.29$ . A minimum value of  $\gamma = 0.86$  was obtained. By Theorem 6.2 a greater value of  $\gamma = 11$  is found for the same value of  $h$  and  $\varepsilon_1 = \varepsilon_2 = -0.3$ .

For  $h \geq 1$  the LMI of Theorem 6.3 is not feasible for any  $\gamma > 0$ . By Theorem 6.2, the output-feedback control is obtained for  $h = 1.28, \varepsilon_1 = \hat{\varepsilon}_1 = -.3$ . A minimum value of  $\gamma = 20$  is then achieved.

This example shows that for greater values of  $h$  Theorem 6.2 is less conservative due to Park’s inequality [25] that is used for bounding the cross terms. For smaller values of  $h$  Theorem 6.2 leads to more conservative results owing to the fact that the adjoint system for  $e$  contains  $e_0$  with delays and that the results are delay-independent with respect to delays in  $e_0$ .

## 7. Conclusions

An LMI solution is proposed for the problem of stability and  $H_\infty$ -control of linear time-invariant descriptor systems. This solution is based on the Lyapunov function approach to descriptor systems with delay introduced in [10] and on the LMI approach to  $H_\infty$  control of nondescriptor systems of [12]. The LMI sufficient conditions that are obtained allow solutions to the  $H_\infty$ -control problem in the uncertain case where the system parameters lie within an uncertainty polytope. As a byproduct, new LMI conditions for stability and  $H_\infty$ -control of difference continuous time equations are obtained.

The design of the output-feedback controller is achieved by two methods: one is based on the BRL for the adjoint of the system that describes the estimation error; the other applies the BRL directly to the system of the estimation error (and thus generalizes the result of [12] to the descriptor case). Both methods suffer from an additional overdesign that stems from the need to estimate the state and its derivative. These methods lead to complementary results: for greater values of the delay the first method is less conservative, while for smaller values of the delay—the second one provides less conservative results. In the special case where a result is sought which is delay-independent with respect to the process and delay-dependent with respect to observer, the latter overdesign can be removed since the estimate of the state (and not of its derivative) is needed.

One question that often arises when solving control and estimation problems for systems with time-delay is whether the solution obtained for certain delays  $h_i$  will satisfy the design requirements for all delays  $\bar{h}_i \leq h_i$ . In the problems of state-feedback and filtering the answer is the affirmative since the LMIs in Theorems 4.1 and 5.1 are convex in the time delays. The situation in the output-feedback control case is however different, in spite of the seemingly convexity of the LMI of Theorems 6.2 and 6.3 in the delay parameters. The fact that the  $P_2$  and  $P_3$  depend nonlinearly on the delay implies that the output-feedback controller that is derived for a certain delay will not necessarily satisfy the design specifications for smaller delays.

## References

- [1] T. Azuma, K. Ikeda, K. Uchida, Infinite-dimensional LMI approach to  $H_\infty$ -control synthesis for linear systems with time delay, in: Proceedings of the 5th European Control Conference, Karlsruhe, Germany, September 1999.
- [2] A. Bensoussan, G. DoPrato, M.C. Delfour, S.K. Mitter, Representation and control of infinite dimensional systems, vol. 1–2, Birkhäuser, Basel, 1992.
- [3] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear matrix inequality in systems and control theory, SIAM Frontier Series (April 1994).
- [4] R. Brayton, Small signal stability criterion for electrical networks containing lossless transmission lines, IBM J. Res. Develop. 12 (1968) 431–440.
- [5] A. Bunse-Gerstner, R. Byers, V. Mehrmann, N. Nickols, Feedback design for regularizing descriptor systems, Linear Algebra Appl. 299 (1999) 119–151.

- [6] S. Campbell, Singular linear systems of differential equations with delays, *Appl. Anal.* 2 (1980) 129–136.
- [7] S. Campbell, 2-D (differential-delay) implicit systems, in: *Proceedings of IMACS World Congress on Scientific Computation*, Dublin, 1991, pp. 1828–1829.
- [8] L. Dai, *Singular Control Systems*, Springer, Berlin, 1989.
- [9] E. Fridman, New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems, *Systems Control Lett.* 43 (4) (2001) 309–319.
- [10] E. Fridman, Stability of linear descriptor systems with delay: a Lyapunov-based approach, *J. Math. Anal. Appl.* (under review).
- [11] E. Fridman, U. Shaked,  $H_\infty$ -state-feedback control of linear systems with small state-delay, *Systems Control Lett.* 33 (1998) 141–150.
- [12] E. Fridman, U. Shaked, A descriptor system approach to  $H_\infty$  control of linear time-delay systems, *IEEE Trans. Automat. Control* 47 (2002).
- [13] A. Halanay, V. Rasvan, Stability radii for some propagation models, *IMA J. Math. Control Inform.* 14 (1997) 95–107.
- [14] J. Hale, S. Lunel, *Introduction to Functional Differential Equations*, Springer, New York, 1993.
- [15] J. Hale, P. Martinez Amores, Stability in neutral equations, *J. Nonlinear Anal. Theory, Meth. Appl.* 1 (1977) 161–172.
- [16] D. Ivanescu, J.-M. Dion, L. Dugard, S.I. Niculescu, Dynamical compensation for time-delay systems: an LMI approach, *Internat. J. Robust Nonlinear Control* 10 (2000) 611–628.
- [17] V. Kolmanovskii, A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.
- [18] V. Kolmanovskii, S.I. Niculescu, J.P. Richard, On the Liapunov–Krasovskii functionals for stability analysis of linear delay systems, *Internat. J. Control* 72 (1999) 374–384.
- [19] H. Kwakernaak, Frequency domain solution of the  $H_\infty$  problem for descriptor systems, in: S. Hara, Y. Yamamoto (Eds.), *Learning Control and Hybrid Systems*, Lecture Notes in Control and Information Sciences, vol. 241, Springer, New York, 1999, pp. 317–336.
- [20] H. Logemann, Destabilizing effects of small time delays on feedback-controlled descriptor systems, *Linear Algebra Appl.* 272 (1998) 131–153.
- [21] I. Masubuchi, Y. Kamitane, A. Ohara, N. Suda,  $H_\infty$  control for descriptor systems: a matrix inequalities approach, *Automatica* 33 (1997) 669–673.
- [22] S.-I. Niculescu, V. Rasvan, Delay-independent stability in lossless propagation models with applications, in: *Proceedings of MTNS 2000*, Perpignan, France, 2000.
- [23] V. Rasvan, *Absolute Stability of Time Lag Control Systems*, Ed. Academiei, Bucharest, 1975 (in Romanian).
- [24] C.E. de Souza, X. Li, Delay-dependent robust  $H_\infty$  control of uncertain linear state-delayed systems, *Automatica* 35 (1999) 1313–1321.
- [25] P. Park, A Delay-dependent stability criterion for systems with uncertain time-invariant delays, *IEEE Trans. Automat. Control* 44 (1999) 876–877.
- [26] A. Rehm, F. Allgower, Descriptor and nondescriptor controllers in  $H_\infty$  control of descriptor systems, in: *Proceedings of ROCOND*, Prague, 2000.
- [27] K. Takaba, N. Morihira, T. Katayama,  $H_\infty$  control for descriptor systems—a J-spectral factorization approach, *Proceedings of the 33rd IEEE Conf. Decision and Control*, 1994, pp. 2251–2256.
- [28] W. Zhu, L. Pitzold, Asymptotic stability of linear delay differential–algebraic equations and numerical methods, *Appl. Numer. Math.* 24 (1997) 247–264.
- [29] H. Xu, K. Mizukami, Upper and lower bounds of  $H_\infty$ -optimal performance for a class of continuous type descriptor systems, *Proceedings of the 38th Conf. Decision and Control*, Kobe, Japan, vol. 1, 1996, pp. 1021–1026.