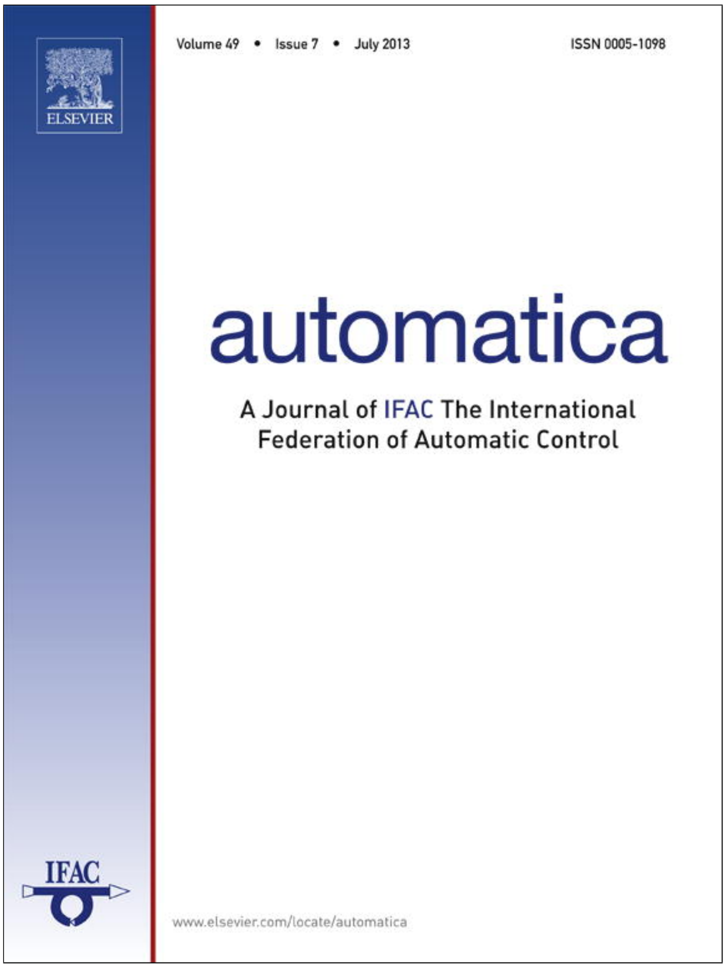
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Brief paper

Observers and initial state recovering for a class of hyperbolic systems via Lyapunov method ☆

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ABSTRACT

Recently the problem of estimating the initial state of some linear infinite-dimensional systems from measurements on a finite interval was solved by using the sequence of forward and backward observers Ramdani, Tucsnak, and Weiss (2010). In the present paper, we introduce a direct Lyapunov approach to the problem and extend the results to the class of semilinear systems governed by wave and beam equations with boundary measurements from a finite interval. We first design forward observers and derive Linear Matrix Inequalities (LMIs) for the exponential stability of the estimation errors. Further we obtain simple finite-dimensional conditions in terms of LMIs for an upper bound T^* on the minimal time, that guarantees the convergence of the sequence of forward and backward observers on $[0, T^*]$ for the initial state recovering. This T^* represents also an upper bound on the observability time. For observation times bigger than T^*, these LMIs give upper bounds on the convergence rate of the iterative algorithm in the norm defined by the Lyapunov functions. In our approach, T^* is found as the minimal dwelling time for the switched exponentially stable (forward and backward estimation error) systems with the different Lyapunov functions (Liberzon, 2003). The efficiency of the results is illustrated by numerical examples.

1. Introduction

Estimation of the initial state of a distributed parameter system from its input and output functions measured over some finite time interval is an important problem in engineering, oceanography, meteorology and medical imaging (see e.g. Ramdani et al., 2010, and the references therein). For the linear exactly observable distributed parameter system, the initial state can be recovered from the measured segment of the input and output functions by inverting the Gramian operator of the system (see, for instance Tucsnak & Weiss, 2009, Section 6.1), and this may be numerically very challenging. However, this is not applicable to nonlinear systems.

Recently the problem of estimating the initial state of some infinite-dimensional systems from measurements on a finite interval has been solved by using a sequence of forward and backward observers (Auroux & Nodet, 2012; Ramdani et al., 2010). For finite-dimensional systems this idea has appeared in Auroux and Blum (2005). In Ramdani et al. (2010) the condition on the convergence of the iterative procedure is given in terms of the bounds on the norms of the semigroups generated by the operators of the forward and backward estimation error equations. It is not easy to find the latter bounds. Moreover, the results of Ramdani et al. (2010) (and the convergence results of Auroux & Nodet, 2012) are confined to the linear time-invariant case.

It is of interest to develop consistent methods that are capable of utilizing nonlinear distributed parameter models and of providing simple conditions for the convergence of forward and backward observers. The LMI approach (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994) is definitely among such methods. For time-delay systems, this approach allowed to solve various control problems in terms of simple finite-dimensional conditions (see e.g. Fridman & Shaked, 2002; Gu, Kharitonov, & Chen, 2003; Richard, 2003, and the references therein). Its extension to distributed parameter systems has been started in Fridman and Orlov (2009a,b).

The LMI approach to observers and initial state recovering of distributed parameter systems is the primary concern of the present paper, where we consider semilinear 1-d wave and beam equations. We start with the design of forward observers and derive LMIs for the exponential stability of the estimation errors. Though the stability of the beam equation has been studied in the literature via direct Lyapunov method (see e.g. Guo & Yang, 2009; Krstic, Guo, Balogh, & Smyshlyaev, 2008), these are the first LMIs for the exponential stability. Their derivation is based on Wirtinger's inequality (Hardy, Littlewood, & Polya, 1934) and on the application of the S-procedure (Yakubovich, 1977).

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Further we find LMIs that give an upper bound T^* on the minimal time, that guarantees the convergence of the sequence of forward and backward observers on $[0, T^*]$ for the recovery of the initial state. This T^* represents also an upper bound on the exact observability time. The continuous dependence of the reconstructed initial state on the measurements follows from the integral input-to-state stability of the corresponding error system (see Angeli, Sontag, & Wang, 2000), which is guaranteed by the LMIs for the exponential stability. For observation times larger than T^*, these LMIs give upper bounds on the convergence rate of the iterative algorithm in the norm defined by the Lyapunov functions. Finding T^* is similar to finding the minimal dwelling time for the switched exponentially stable systems with different Lyapunov functions (Liberzon, 2003). It appears that the LMIs are not conservative for the linear homogeneous wave equation recovering the analytical value of the minimal observability time. Some preliminary results for wave equations were presented in Fridman (2013).

1.1. Notation and preliminaries

Throughout the paper $\mathbb{R}^n$ denotes the n dimensional Euclidean space with the norm $|\cdot|$, the notation $P > 0$ with $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. Functions, continuous (continuously differentiable) in all arguments, are referred to as of class C (of class C^1). $L^2(0,1)$ is the Hilbert space of square integrable functions $z(\xi)$, $\xi \in [0,1]$ with the corresponding norm $\|z\|_{L^2} = \sqrt{\int_0^1 z^2(\xi)d\xi}$. $\mathscr{H}^1(0,1)$ is the Sobolev space of absolutely continuous scalar functions $z : [0,1] \to R$ with $\frac{dz}{d\xi} \in L^2(0,1)$. $\mathscr{H}^2(0,1)$ is the Sobolev space of scalar functions $z : [0,1] \to R$ with absolutely continuous $\frac{dz}{d\xi}$ and with $\frac{d^2z}{d\xi^2} \in L^2(0,1)$.

The following inequalities will be useful:

Lemma 1.1. *Let $z \in \mathscr{H}^1(0,1)$ be a scalar function with $z(0) = 0$ or $z(1) = 0$. Then Wirtinger's inequality holds (Hardy et al., 1934)*

$$\int_0^1 z^2(x)dx \le \frac{4}{\pi^2}\int_0^1 z_x^2(x)dx. \tag{1.1}$$

Moreover,

$$\max_{x\in[0,1]} z^2(x) \le \int_0^1 z_x^2(x)dx. \tag{1.2}$$

2. Observers and initial state recovering: wave equation

2.1. Observers for semilinear wave equations

Consider the following one-dimensional semilinear wave equation

$$z_{tt}(x,t) = \frac{\partial}{\partial x}[a(x)z_x(x,t)] + f(z_x(x,t),x,t),\quad t \ge t_0,\ x \in (0,1), \tag{2.1}$$

under the boundary conditions

$$z(0,t) = 0, \qquad z_x(1,t) = 0. \tag{2.2}$$

Here subscripts denote the corresponding partial derivatives, f is a C^2 function with uniformly bounded first partial derivatives in the two first variables.

The initial conditions are given by

$$z(x,t_0) = z_1(x), \qquad z_1(0) = 0, \qquad z_{1x}(1) = 0,\quad z_t(x,t_0) = z_2(x). \tag{2.3}$$

The smooth function $a(x)$ satisfies the following inequalities:

$$0 < a(1) \le a(x), \qquad a_x(x) \le 0, \qquad \forall x \in (0,1). \tag{2.4}$$

Let $g_1 > 0$ be the known bound on the derivative of $f(\xi,x,t)$ with respect to the first argument:

$$|f_\xi(\xi,x,t)| \le g_1 \quad \forall(\xi,x,t) \in \mathbb{R}^3. \tag{2.5}$$

The boundary measurements are given by $y(t) = z_t(1,t)$, $t \ge t_0$.

The boundary-value problem (2.1), (2.2) can be represented as an abstract differential equation by defining the state $\zeta(t) = [\zeta_1(t)\ \zeta_2(t)]^T = [z(t)\ z_t(t)]^T$ and the operators

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ \frac{\partial}{\partial x}\left[a(x)\frac{\partial}{\partial x}\right] & 0 \end{bmatrix}, \qquad F(\zeta,t) = \begin{bmatrix} 0 \\ F_1(\zeta_1,t) \end{bmatrix},$$

where $F_1 : \mathscr{H}^1 \times R \to L^2(0,1)$ is defined as $F_1(\zeta_1,t) = f(\zeta_{1x}(x),x,t)$ so that it is continuous in t for each $\zeta_1 \in \mathscr{H}^1$. The differential equation is

$$\dot\zeta(t) = \mathscr{A}\zeta(t) + F(\zeta(t),t), \quad t \ge t_0 \tag{2.6}$$

in the Hilbert space $\mathscr{H} = \mathscr{H}_L^1(0,1) \times L^2(0,1)$, where

$$\mathscr{H}_L^1(0,1) = \left\{\zeta_1 \in \mathscr{H}^1(0,1) | \zeta_1(0) = 0\right\}$$

and $\|\zeta\|^2_{\mathscr{H}} = \|\zeta_{1x}\|^2_{L^2} + \|\zeta_2\|^2_{L^2}$. The operator $\mathscr{A}$ with the dense domain

$$\mathscr{D}(\mathscr{A}) = \left\{(\zeta_1,\zeta_2)^T \in \mathscr{H}^2(0,1) \bigcap \mathscr{H}_L^1(0,1) \times \mathscr{H}_L^1(0,1) \middle| \zeta_{1x}(1) = 0\right\}$$

is m-dissipative and hence it generates a strongly continuous contraction semigroup $\mathbb{T}$ (Pazy, 1983). Due to (2.5) the following Lipschitz condition holds:

$$\|F_1(\zeta_1,t) - F_1(\bar\zeta_1,t)\|_{L^2} \le g_1\|\zeta_{1x} - \bar\zeta_{1x}\|_{L^2} \tag{2.7}$$

where $\zeta_1, \bar\zeta_1 \in \mathscr{H}_L^1(0,1)$, $t \in \mathbb{R}$. Then by Theorem 6.1.2 of Pazy (1983), a unique continuous mild solution $\zeta(\cdot)$ of (2.6) in $\mathscr{H}$ initialized by

$$\zeta_1(t_0) = z_1 \in \mathscr{H}_L^1(0,1), \qquad \zeta_2(t_0) = z_2 \in L^2(0,1), \tag{2.8}$$

i.e. a unique solution of the integral equation

$$\zeta(t) = \mathbb{T}(t-t_0)\zeta(t_0) + \int_{t_0}^t \mathbb{T}(t-s)F(\zeta(s),s)ds \tag{2.9}$$

exists in $C([t_0,\infty),\mathscr{H})$. Moreover, this solution is locally Lipschitz in the initial state (i.e. for all $T > 0$ the mapping $(z_1,z_2) \to \zeta$ is Lipschitz from $\mathscr{H}$ to $C([t_0,T],\mathscr{H})$). Note that $F : \mathscr{H} \times [t_0,\infty) \to \mathscr{H}$ is continuously differentiable. If $\zeta(t_0) \in \mathscr{D}(\mathscr{A})$, then this mild solution is in $C^1([t_0,\infty),\mathscr{H})$ and it is a classical solution of (2.1), (2.2) with $\zeta(t) \in \mathscr{D}(\mathscr{A})$ (see Theorem 6.1.5 of Pazy, 1983).

We suggest a nonlinear Luenberger type observer of the form

$$\hat z_{tt}(x,t) = \frac{\partial}{\partial x}[a(x)\hat z_x(x,t)] + f(\hat z_x(x,t),x,t),\quad t \ge t_0,\ x \in (0,1) \tag{2.10}$$

under the boundary conditions

$$\hat z(0,t) = 0, \qquad \hat z_x(1,t) = k[y(t) - \hat z_t(1,t)], \tag{2.11}$$

and the initial conditions $[\hat z(\cdot,t_0), \hat z_t(\cdot,t_0)]^T \in \mathscr{H}$, where $k > 0$ is the injection gain. The well-posedness of (2.10), (2.11) will be established by showing the well-posedness of the estimation error $e = z - \hat{z}$, which satisfies the wave equation

$$e_{tt}(x,t) = \frac{\partial}{\partial x}[a(x)e_x(x,t)] + ge_x(x,t), \quad t \geq t_0,\ x \in (0,1). \tag{2.12}$$

Here $ge_x = f(z_x, x, t) - f(z_x - e_x, x, t)$ and

$$g = g(z_x, e_x, x, t) = \int_0^1 f_{z_x}(z_x + (\theta - 1)e_x, x, t)d\theta.$$

Note that (2.5) yields $|g| \leq g_1$. The boundary and the initial conditions are given by

$$e(0,t) = 0, \qquad e_x(1,t) = -ke_t(1,t) \tag{2.13}$$

and

$$e(x,t_0) = z_1(x) - \hat{z}(x,t_0), \qquad e_t(x,t_0) = z_2(x) - \hat{z}_t(x,t_0). \tag{2.14}$$

Let z be a mild solution of (2.1), (2.2). Then $z : [t_0, \infty) \to \mathscr{H}^1$ is continuous and, thus, the function $F_2 : \mathscr{H}^1 \times [t_0,\infty) \to L_2(0,1)$ defined as

$$F_2(\zeta_1, t) = f(z_x, x, t) - f(z_x - \zeta_{1x}, x, t)$$

satisfies the Lipschitz condition (2.7), where F_1 is replaced by F_2. By the above arguments, the error system (2.12)–(2.14) has a unique mild solution $\{e, e_t\} \in C([t_0,\infty), \mathscr{H})$ initialized by $[e(\cdot,t_0), e_t(\cdot,t_0)]^T \in \mathscr{H}$. Therefore there exists a unique mild solution $\{\hat{x}, \hat{x}_t\} \in C([t_0,\infty), \mathscr{H})$ to the observer system (2.10)–(2.11) with the initial conditions $[\hat{z}(\cdot,t_0), \hat{z}_t(\cdot,t_0)]^T \in \mathscr{H}$. If $[e(\cdot,t_0), e_t(\cdot,t_0)]^T \in \mathscr{D}(\mathscr{A})$, then $\{e, e_t\} \in C^1([t_0,\infty), \mathscr{H})$ is a classical solution of (2.12)–(2.14) with $[e(\cdot,t), e_t(\cdot,t)] \in \mathscr{D}(\mathscr{A})$ for $t \geq t_0$. Hence, if $[\hat{z}(\cdot,t), \hat{z}_t(\cdot,t)]^T \in \mathscr{D}(\mathscr{A})$ and $[z_1, z_2]^T \in \mathscr{D}(\mathscr{A})$, there exists a unique classical solution $\{\hat{z}, \hat{z}_t\} \in C^1([t_0,\infty), \mathscr{H})$ to the observer system (2.10)–(2.11) with $[\hat{z}(\cdot,t), \hat{z}_t(\cdot,t)]^T \in \mathscr{D}(\mathscr{A})$ for $t \geq t_0$.

We will derive further sufficient conditions for the exponential stability of the error wave equation (2.12) under the boundary conditions (2.13). Consider the Lyapunov function (see e.g. Nicaise & Pignotti, 2006)

$$V(t) = \int_0^1 [e_x(x,t)\ e_t(x,t)] \begin{bmatrix} a(x)p & \chi x \\ * & p \end{bmatrix} \begin{bmatrix} e_x(x,t) \\ e_t(x,t) \end{bmatrix} dx \tag{2.15}$$

with some constants $p > 0$, $\chi > 0$ defined on the mild solutions of (2.12). Assume that

$$\begin{bmatrix} a(1)p & \chi \\ * & p \end{bmatrix} > 0. \tag{2.16}$$

Since $a(1) \leq a(x) \leq a(0)$, the following holds

$$0 < \alpha I \leq \begin{bmatrix} a(1)p & \chi x \\ * & p \end{bmatrix} \leq \begin{bmatrix} a(x)p & \chi x \\ * & p \end{bmatrix} \leq \beta I$$

where

$$\alpha = \lambda_{\min}\left(\begin{bmatrix} a(1)p & \chi \\ * & p \end{bmatrix}\right), \qquad \beta = (\chi + \max\{a(0), 1\}p). \tag{2.17}$$

Then

$$\alpha \int_0^1 [e_x^2(x,t) + e_t^2(x,t)]dx \leq V(t) \leq \beta \int_0^1 [e_x^2(x,t) + e_t^2(x,t)]dx. \tag{2.18}$$

We consider first $[z_1, z_2]^T, [\hat{z}(\cdot,t_0), \hat{z}_t(\cdot,t_0)]^T \in \mathscr{D}(\mathscr{A})$. We are looking for conditions that guarantee $\frac{d}{dt}V(t) + 2\delta V(t) \leq 0$ along the classical solutions of the wave equation. Then $V(t) \leq e^{-2\delta(t-t_0)}V(t_0)$ and, thus, (2.18) yields

$$\int_0^1 [e_x^2(x,t) + e_t^2(x,t)]dx \leq \frac{\beta}{\alpha}e^{-2\delta(t-t_0)} \times \int_0^1 [(z_{0x}(x) - \hat{z}_x(x,t_0))^2 + (z_1(x) - \hat{z}_t(x,t_0))^2]dx. \tag{2.19}$$

Since $\mathscr{D}(\mathscr{A})$ is dense in $\mathscr{H}$ the same estimate (2.19) remains true (by continuous extension) for any initial conditions $[z_1, z_2]^T$, $[\hat{z}(\cdot,t_0), \hat{z}_t(\cdot,t_0)]^T \in \mathscr{H}$. For such initial conditions we have mild solutions of (2.1), (2.2) and of (2.10), (2.11). Similar to Fridman and Orlov (2009b) we arrive at the following conditions (see Appendix for the proof):

Proposition 2.1. *Given $k > 0$ and $\delta > 0$, assume that exist positive constants χ, p such that LMIs (2.16) and*

$$\begin{aligned} \psi_1 &\triangleq -2a(1)kp + (1 + a(1)k^2)\chi < 0, \\ \Psi_2 &\triangleq \begin{bmatrix} -a(1)\chi + 2\delta a(1)p + 2\chi g_1 & 2\chi\delta + pg_1 \\ * & -\chi + 2\delta p \end{bmatrix} < 0 \end{aligned} \tag{2.20}$$

are feasible. Then solutions of the boundary-value problem (2.12), (2.13) satisfy (2.19), where α and β are given by (2.17), i.e. the system governed by (2.12), (2.13) is exponentially stable with the decay rate $\delta > 0$.

2.2. Iterative forward and backward observer design

Our next objective is to recover (if possible) the unique initial state (2.3) of the solution to (2.1)–(2.3) from the measurements on the finite time interval

$$y(t) = z_t(1,t), \quad t \in [t_0, t_0 + T],\ T > 0. \tag{2.21}$$

Definition 2.1. The system (2.1), (2.2) with the measurements (2.21) is called exactly observable in time T, if

(i) for any initial state $\zeta(t_0) \in \mathscr{H}$, it is possible to find a sequence of $\zeta_0^n \in \mathscr{H} (n = 1, 2, \ldots)$ from the measurements (2.21) such that $\lim_{n\to\infty} \|\zeta_0^n - \zeta(t_0)\|_{\mathscr{H}} = 0$ (i.e. it is possible to recover the unique initial state as $\zeta(t_0) = \lim_{n\to\infty} \zeta_0^n$);
(ii) there exists a constant $C > 0$ such that for any initial states $\zeta(t_0) \in \mathscr{H}$ and $\bar{\zeta}(t_0) \in \mathscr{H}$ leading to the measurements $y(t)$ and $\bar{y}(t)$ and to the corresponding sequences ζ_0^n and $\bar{\zeta}_0^n$, the following holds:

$$\|\lim_{n\to\infty} \zeta_0^n - \lim_{n\to\infty} \bar{\zeta}_0^n\|_{\mathscr{H}}^2 \leq C \int_{t_0}^{t_0+T} |y(s) - \bar{y}(s)|^2 ds. \tag{2.22}$$

The time T is called the observability time.

Note that (2.22) means the continuous in the measurements recovery of the initial state. In order to recover the initial state we use the iterative procedure as in Ramdani et al. (2010). Define the sequences of forward $z^{(n)}$ and backward observers $z^{b(n)}$, $n = 1, 2, \ldots$ with the injection gain k:

$$\begin{aligned} z_{tt}^{(n)}(x,t) &= \frac{\partial}{\partial x}[a(x)z_x^{(n)}(x,t)] + f(z_x^{(n)}(x,t), x, t), \\ z^{(n)}(0,t) &= 0, \qquad z_x^{(n)}(1,t) = k[y(t) - z_t^{(n)}(1,t)], \\ &\quad t \in [t_0, t_0+T], \\ z^{(n)}(x,t_0) &= z^{b(n-1)}(x,t_0), \qquad z_t^{(n)}(x,t_0) = z_t^{b(n-1)}(x,t_0), \end{aligned} \tag{2.23}$$

where $z^{b(0)}(x,t_0)=z_t^{b(0)}(x,t_0)\equiv 0$, and

$$\begin{aligned}
&z_{tt}^{b(n)}(x,t)=\frac{\partial}{\partial x}[a(x)z_x^{b(n)}(x,t)]+f(z_x^{b(n)}(x,t),x,t),\\
&z^{b(n)}(0,t)=0,\qquad z_x^{b(n)}(1,t)=-k[y(t)-z_t^{b(n)}(1,t)],\\
&\quad t\in[t_0,t_0+T],\\
&z^{b(n)}(x,t_0+T)=z^{(n)}(x,t_0+T),\\
&z_t^{b(n)}(x,t_0+T)=z_t^{(n)}(x,t_0+T).
\end{aligned}\tag{2.24}$$

This results in the sequence of the forward $e^{(n)}=z-z^{(n)}$ and the backward $e^{b(n)}=z-z^{b(n)},\ n=1,2,\ldots$ errors satisfying

$$\begin{aligned}
&e_{tt}^{(n)}(x,t)=\frac{\partial}{\partial x}[a(x)e_x^{(n)}(x,t)]+g^{(n)}e_x^{(n)}(x,t),\\
&e^{(n)}(0,t)=0,\qquad e_x^{(n)}(1,t)=-ke_t^{(n)}(1,t),\\
&\quad t\in[t_0,t_0+T],\\
&e^{(n)}(x,t_0)=e^{b(n-1)}(x,t_0),\qquad e_t^{(n)}(x,t_0)=e_t^{b(n-1)}(x,t_0),
\end{aligned}\tag{2.25}$$

where $e^{b(0)}(x,t_0)=z_1(x)-z^{b(0)}(\cdot,t_0)$, $e_t^{b(0)}(x,t_0)=z_2(x)-z_t^{b(0)}(\cdot,t_0)$ and

$$\begin{aligned}
&e_{tt}^{b(n)}(x,t)=\frac{\partial}{\partial x}[a(x)e_x^{b(n)}(x,t)]+g^{b(n)}e_x^{b(n)}(x,t),\\
&e^{b(n)}(0,t)=0,\qquad e_x^{b(n)}(1,t)=ke_t^{b(n)}(1,t),\\
&\quad t\in[t_0,t_0+T],\\
&e^{b(n)}(x,t_0+T)=e^{(n)}(x,t_0+T),\\
&e_t^{b(n)}(x,t_0+T)=e_t^{(n)}(x,t_0+T).
\end{aligned}\tag{2.26}$$

Here

$$g^{(n)}=g(z_x^{(n)},e_x^{(n)},x,t)=\int_0^1 f_{z_x}(z_x^{(n)}+\theta e_x^{(n)},x,t)d\theta,$$

$$g^{b(n)}=g(z_x^{b(n)},e_x^{b(n)},x,t)=\int_0^1 f_{z_x}(z_x^{b(n)}+\theta e_x^{b(n)},x,t)d\theta.$$

2.3. Observability time and convergence rate

For (2.25) and (2.26) we consider for $t\in[t_0,t_0+T]$ the Lyapunov functions

$$\begin{aligned}
V^{(n)}(t)=&\int_0^1\Big[a(x)p[e_x^{(n)}(x,t)]^2+p[e_t^{(n)}(x,t)]^2\\
&+2\chi xe_x^{(n)}(x,t)e_t^{(n)}(x,t)\Big]dx
\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}
V^{b(n)}(t)=&\int_0^1\Big[a(x)p[e_x^{b(n)}(x,t)]^2+p[e_t^{b(n)}(x,t)]^2\\
&-2\chi xe_x^{b(n)}(x,t)e_t^{b(n)}(x,t)\Big]dx
\end{aligned}\tag{2.28}$$

with constants $p>0$ and $\chi>0$, satisfying (2.16). Then $\forall t\geq t_0$ (cf. (2.19))

$$\begin{aligned}
&\beta\int_0^1\Big[[e_x^{(n)}(x,t)]^2+[e_t^{(n)}(x,t)]^2\Big]dx\geq V^{(n)}(t)\\
&\geq\alpha\int_0^1\Big[[e_x^{(n)}(x,t)]^2+[e_t^{(n)}(x,t)]^2\Big]dx,\\
&\beta\int_0^1\Big[[e_x^{b(n)}(x,t)]^2+[e_t^{(bn)}(x,t)]^2\Big]dx\\
&\geq V^{b(n)}(t)\geq\alpha\int_0^1\Big[[e_x^{b(n)}(x,t)]^2+[e_t^{b(n)}(x,t)]^2\Big]dx.
\end{aligned}\tag{2.29}$$

Lemma 2.1. *Consider $V^{(n)}$ and $V^{b(n)}$ given by (2.27) and (2.28) respectively with $p>0$ and $\chi>0$ satisfying (2.16). Assume there exist $\delta>0$ and $T>0$ such that for all $n=1,2,\ldots$ and for all $t\in[t_0,t_0+T]$ the inequalities*

$$\dot V^{(n)}(t)+2\delta V^{(n)}(t)\leq 0\tag{2.30}$$

and

$$\dot V^{b(n)}(t)-2\delta V^{b(n)}(t)\geq 0\tag{2.31}$$

hold along (2.25) and (2.26) respectively. Assume additionally that for some $T^*\in(0,T)$

$$\begin{aligned}
&V^{(n)}(t_0)e^{-2\delta T^*}\leq V^{b(n-1)}(t_0),\\
&V^{b(n)}(t_0+T)e^{-2\delta T^*}\leq V^{(n)}(t_0+T).
\end{aligned}\tag{2.32}$$

Then the iterative algorithm converges on $[t_0,t_0+T]$:

$$V^{b(n)}(t_0)\leq qV^{b(n-1)}(t_0)\leq q^nV^{b(0)}(t_0),\tag{2.33}$$

where $q=e^{-4\delta(T-T^*)}$ is the convergence rate.

Proof. Inequalities (2.30), (2.31) yield

$$V^{b(n)}(t_0)\leq V^{b(n)}(t_0+T)e^{-2\delta T},\qquad V^{(n)}(t_0+T)\leq V^{(n)}(t_0)e^{-2\delta T}.$$

Hence, (2.32) implies

$$\begin{aligned}
V^{b(n)}(t_0)&\leq V^{b(n)}(t_0+T)e^{-2\delta T}\leq V^{(n)}(t_0+T)e^{-2\delta(T-T^*)}\\
&\leq V^{(n)}(t_0)e^{-2\delta(T-T^*)}e^{-2\delta T}\leq V^{b(n-1)}(t_0)e^{-4\delta(T-T^*)}.\quad\square
\end{aligned}$$

Remark 2.1. The forward and backward error estimation systems (2.25) and (2.26) can be considered as the switched exponentially stable systems with the dwelling time T and with the (different) Lyapunov functions V^n and $V^{b(n)}$ respectively. Then the inequalities (2.32) represent the minimal dwelling time condition that preserves the stability of the switched systems (Liberzon, 2003).

2.4. LMIs for the observability time and the convergence rate

Taking into account that $e^{(n)}(x,t_0+T)=e^{b(n)}(x,t_0+T)$ and $e_t^{(n)}(x,t_0+T)=e_t^{b(n)}(x,t_0+T)$, we have

$$\begin{aligned}
&V^{b(n)}(t_0+T)e^{-2\delta T^*}-V^{(n)}(t_0+T)\\
&\leq\int_0^1\begin{bmatrix}e_x^{(n)}(x,t_0+T)\\ e_t^{(n)}(x,t_0+T)\end{bmatrix}^T\Phi(\chi)\begin{bmatrix}e_x^{(n)}(x,t_0+T)\\ e_t^{(n)}(x,t_0+T)\end{bmatrix}dx\leq 0,
\end{aligned}$$

where $\Phi(\chi)=\begin{bmatrix}a(x)p(e^{-2\delta T^*}-1) & -\chi(e^{-2\delta T^*}+1)x\\ * & p(e^{-2\delta T^*}-1)\end{bmatrix}$, if

$$\begin{bmatrix}a(1)p(e^{-2\delta T^*}-1) & -\chi(e^{-2\delta T^*}+1)\\ * & p(e^{-2\delta T^*}-1)\end{bmatrix}<0\tag{2.34}$$

Therefore, (2.34) implies the second inequality of (2.32)

By the same arguments, (2.34) implies the first inequality of (2.32), where $e^{(n)}(x,t_0)=e^{b(n-1)}(x,t_0)$ and $e_t^{(n)}(x,t_0)=e_t^{b(n-1)}(x,t_0)$, since

$$\begin{aligned}
V^{(n)}(t_0)e^{-2\delta T^*}-V^{b(n-1)}(t_0)\leq&\int_0^1\begin{bmatrix}e_x^{(n)}(x,t_0)\\ e_t^{(n)}(x,t_0)\end{bmatrix}^T\Phi(-\chi)\\
&\times\begin{bmatrix}e_x^{(n)}(x,t_0)\\ e_t^{(n)}(x,t_0)\end{bmatrix}dx\leq 0.
\end{aligned}$$

Note that by Schur complements (2.34) yields (2.16). We arrive to the following:

Theorem 2.1. *Given positive tuning parameters T^* and δ, assume positive constants p and χ exist such that LMIs (2.20) and (2.34) are feasible. Then*

(i) *the iterative algorithm with $T = T^*$ converges and the system (2.1), (2.2) is exactly observable in time T^*;*
(ii) *for all $\Delta T > 0$ the iterative algorithm with $T = T^* + \Delta T$ converges:*

$$\int_0^1 \Big[[e_x^{b(n)}(x,t_0)]^2 + [e_t^{b(n)}(x,t_0)]^2\Big]dx \le \frac{\beta}{\alpha} q^n \int_0^1 \Big[z_{1x}^2(x) + z_2^2(x)\Big]dx, \tag{2.35}$$

where $q = e^{-4\delta\Delta T}$ and where α and β are given by (2.17).

Proof. (i) Given $\delta > 0$ and $T^* > 0$, if the strong LMIs (2.20) and (2.34) are feasible for some $p > 0$ and $\chi > 0$, then for small enough $\delta_0 > 0$ the LMIs (2.20), where δ is changed by $\delta + \delta_0$, are satisfied with the same p and χ. Hence, LMIs (2.20) with δ changed by $\delta+\delta_0$ and (2.34) lead to

$$\begin{aligned} V^{b(n)}(t_0) &\le e^{-2(\delta+\delta_0)T^*} V^{b(n)}(t_0+T^*) \le e^{-2\delta_0 T^*} V^{(n)}(t_0+T^*) \\ &\le e^{-2(\delta+\delta_0)T^*} e^{-2\delta_0 T^*} V^{(n)}(t_0) \le e^{-4\delta_0 T^*} V^{b(n-1)}(t_0), \end{aligned}$$

which yields (2.33) with $q = e^{-4\delta_0 T^*}$ and $T = T^*$.

To prove the exact observability in time T^*, consider initial states $\zeta(t_0) \in \mathscr{H}$ and $\bar\zeta(t_0) \in \mathscr{H}$ of (2.1), (2.2) that lead to the measurements $y(t)$ and $\bar y(t)$ and to the corresponding forward and backward observers $z^n, z^{b(n)}$ and $\bar z^n, \bar z^{b(n)}$. Note that $\bar z^n, \bar z^{b(n)}$ satisfy (2.23) and (2.24), where $z^n, z^{b(n)}$ and y are replaced by $\bar z^n, \bar z^{b(n)}$ and $\bar y$. The resulting $e^n = z^n - \bar z^n$, $e^{b(n)} = z^{b(n)} - \bar z^{b(n)}$ satisfy (2.25), (2.26) with the perturbed boundary conditions at $x = 1$:

$$\begin{aligned} &e_x^{(n)}(1,t) = -ke_t^{(n)}(1,t) + w(t), \quad w(t) \triangleq k[y(t) - \bar y(t)], \\ &e_x^{b(n)}(1,t) = ke_t^{b(n)}(1,t) - w(t). \end{aligned} \tag{2.36}$$

Let $V^{(n)}$ and $V^{b(n)}$ be defined by (2.27) and (2.28). LMI (2.34) implies inequalities (2.32). By arguments of Proposition 2.1 (see also Fridman, Mondie, & Saldivar, 2010), we find that

$$\dot V^{(n)}(t) + 2\delta V^{(n)}(t) - \gamma |w(t)|^2 \le 0 \tag{2.37}$$

for some $\gamma > 0$ if $\psi_1 < 0$ and

$$\left[\begin{array}{c|c} \Psi_2 & \begin{matrix} -a(1)\chi k + a(1)p \\ 0 \end{matrix} \\ \hline * & -\gamma + \chi a(1) \end{array}\right] \le 0. \tag{2.38}$$

By Schur complements, the latter inequality is feasible for large enough γ if $\Psi_2 < 0$, i.e. if LMIs (2.20) are satisfied. Then, by the comparison principle (see e.g. Khalil, 1992),

$$V^{(n)}(t) \le e^{-2\delta(t-t_0)} V^{(n)}(t_0) + \gamma \int_{t_0}^t |w(s)|^2 ds. \tag{2.39}$$

Similarly, LMIs (2.20) guarantee that $\dot V^{b(n)}(t) - 2\delta V^{b(n)}(t) + \gamma|w(t)|^2 \ge 0$ for large enough $\gamma > 0$, i.e.

$$V^{b(n)}(t) \ge e^{2\delta(t-t_0)} V^{b(n)}(t_0) - \gamma \int_{t_0}^t e^{2\delta(t-s)} |w(s)|^2 ds$$

and, thus,

$$V^{b(n)}(t_0) \le e^{-2\delta(t-t_0)} V^{b(n)}(t) + \gamma \int_{t_0}^t |w(s)|^2 ds. \tag{2.40}$$

Therefore,

$$\begin{aligned} V^{b(n)}(t_0) &\le e^{-2(\delta+\delta_0)T^*} V^{b(n)}(t_0+T^*) + \gamma \int_{t_0}^{t_0+T^*} |w(s)|^2 ds \\ &\le e^{-2\delta_0 T^*} V^{(n)}(t_0+T^*) + \gamma \int_{t_0}^{t_0+T^*} |w(s)|^2 ds \\ &\le e^{-2(\delta+2\delta_0)T^*} V^{(n)}(t_0) \\ &\quad + (e^{-2\delta_0 T^*} + 1)\gamma \int_{t_0}^{t_0+T^*} |w(s)|^2 ds \\ &\le e^{-4\delta_0 T^*} V^{b(n-1)}(t_0) \\ &\quad + (e^{-2\delta_0 T^*} + 1)\gamma \int_{t_0}^{t_0+T^*} |w(s)|^2 ds. \end{aligned}$$

We arrive at

$$\begin{aligned} &\alpha \int_0^1 \Big[[e_x^{b(n)}(x,t_0)]^2 + [e_t^{b(n)}(x,t_0)]^2\Big]dx \\ &\le V^{b(n)}(t_0) \le e^{-4\delta_0 T^*} e^{-4\delta_0 T^*} V^{b(n-2)}(t_0) \\ &\quad + (e^{-6\delta_0 T^*} + e^{-4\delta_0 T^*} + e^{-2\delta_0 T^*} + 1)\gamma \int_{t_0}^t |w(s)|^2 ds \\ &\le (e^{-4\delta_0 T^*})^n V^{b(0)}(t_0) + \frac{\gamma}{1 - e^{-2\delta_0 T^*}} \int_{t_0}^{t_0+T^*} |w(s)|^2 ds \end{aligned}$$

which implies (2.22), where $\| \lim_{n\to\infty} \zeta_0^n - \lim_{n\to\infty} \bar\zeta_0^n \|_{\mathscr{H}} = \lim_{n\to\infty} \|\zeta_0^n - \bar\zeta_0^n\|_{\mathscr{H}}$ and $C = \frac{\gamma}{\alpha[1-e^{-2\delta_0 T^*}]}$.

(ii) From Proposition 2.1 it follows that LMIs (2.20) yield (2.30). By the similar derivations, LMIs (2.20) imply (2.31) for the backward system. Moreover, (2.34) guarantees (2.32) and (2.16). Then (ii) follows from Lemma 2.1 and (2.29). □

Remark 2.2. The proof of the exact observability is based on the integral input-to-state stability of (2.25) with the perturbed boundary condition (2.36) at $x = 1$:

$$\begin{aligned} &\int_0^1 \Big[[e_x^{(n)}(x,t)]^2 + [e_t^{(n)}(x,t)]^2\Big]dx \\ &\le \frac{\beta}{\alpha} e^{-2\delta(t-t_0)} \int_0^1 \Big[[e_x^{(n)}(x,t_0)]^2 + [e_t^{(n)}(x,t_0)]^2\Big]dx \\ &\quad + \frac{\gamma}{\alpha} \int_{t_0}^t |w(s)|^2 ds, \quad t \ge t_0. \end{aligned}$$

The latter property is guaranteed by LMIs $\psi_1 < 0$ and (2.38).

Remark 2.3. If f is linear in the state (e.g. $f = g(x,t)z_x + h(x,t)$) subject to (2.5) the presented results are still new: they give constructive finite-dimensional conditions for finding the observability time and the convergence rate, which is a non-trivial problem. Therefore, Theorem 2.1 *completes the existing results* (Auroux & Nodet, 2012; Ramdani et al., 2010) (even in the linear time-invariant case) and *extend them to time-varying/semilinear systems*. We assume that the nonlinearities are globally Lipschitz, which may be restrictive. For locally Lipschitz f, by the standard arguments for the nonlinear systems (see e.g. Baroun, Jacob, Maniar, & Schnaubelt, 2013; Khalil, 1992), the presented results hold locally.

Remark 2.4. In this section, in order to present the new method in a simpler form, we have not considered f that may depend on z as well. However, our results can be extended to a more general z, z_x, x, t-dependent f, which is Lipschitz in z and z_x. Modified LMIs for this case can be derived by applying Wirtinger's inequality and S-procedure (as it is done in the next section for the beam equation).

Table 1
Minimum T_m^* and $q = e^{-4\delta(T-T^*)}$ for $T \geq T_m^*$.

g_1	T_m^*	δ_m	T	T^*	δ	q
0	2.0000001	10^{-8}	2.5	2.24	0.05	0.9493
0.1	4.67	0.1	5.5	4.67	0.1	0.7175
0.2	8.65	0.08	9.5	8.65	0.08	0.7619

2.5. Example

Consider (2.1)–(2.3) with $a(x) \geq 1, a_x \leq 0, k = 1$ and with the values of g_1 as given in Table 1. By applying the LMI Toolbox of Matlab and verifying the feasibility of (2.20) and (2.34), we find first the minimal values of T_m^* and the corresponding δ_m for the convergence of the iterative algorithm and, thus, for the exact observability (see Table 1). Note that for $g_1 = 0$ the observability time is $T_m^* = 2.0000001$, which is very close to the exact analytical value 2 (for the constant $a \equiv 1$). Thus, our results are not conservative for this case.

Further given $T > T_m^*$, we try to minimize the upper bound on the convergence of the iterative algorithm with T that is guaranteed by (ii) of Theorem 2.1. For this purpose we choose $T^* \in [T_m^*, T)$ and maximize δ which preserves the feasibility of (2.20) and (2.34). The resulting convergence rate $q = e^{-4\delta(T-T^*)}$ for different g_1 is given in Table 1. It is found that for $g_1 = 0$ and $T = 2.5 > T_m^* = 2$, the fastest convergence rate with minimal achievable $q = 0.94933$ corresponds to $T^* = 2.24$. For positive g_1 the choice of $T^* = T_m^*$ leads to smaller q. The observer gain k can also be optimized/tuned from LMIs so as to minimize the resulting observability time and the convergence rate. Here $k = 1$ leads to better results.

3. Observers and initial state recovery: beam equation

3.1. Problem formulation

We consider the following one-dimensional semilinear beam equation

$$z_{tt}(x,t) + \frac{\partial^2}{\partial x^2}[a(x)z_{xx}(x,t)] + f(z_{xx}(x,t), z_x(x,t), x, t) = 0,$$
$$t \geq t_0,\ x \in (0,1), \tag{3.1}$$

under the boundary conditions

$$\begin{aligned} &z(0,t) = z_x(0,t) = 0,\\ &z_{xx}(1,t) = 0, \qquad z_{xxx}(1,t) = 0,\end{aligned} \tag{3.2}$$

where f is a smooth function with uniformly bounded partial derivatives in the two first variables. The smooth function $a(x)$ satisfies (2.4). The initial conditions are given by

$$\begin{aligned} &z(x,t_0) = z_1(x), \qquad z_{1xx}(1) = z_{1xxx}(1) = 0,\\ &z_t(x,t_0) = z_2(x).\end{aligned} \tag{3.3}$$

Define

$$\mathscr{H}_L^2(0,1) = \{z_1 \in \mathscr{H}^2(0,1) : z_1(0) = z_{1x}(0) = 0\}.$$

The boundary-value problem (3.1), (3.2) can be represented as the differential equation (2.6) in the Hilbert space $\mathscr{H} = \mathscr{H}_L^2(0,1) \times L^2(0,1)$. In the above equation, the infinitesimal operator $\mathscr{A} = \begin{bmatrix} 0 & I \\ -\frac{\partial^2}{\partial x^2}[a(x)\frac{\partial^2}{\partial x^2}] & 0\end{bmatrix}$ has the dense domain

$$\mathscr{D}(\mathscr{A}) = \Big\{(\zeta_1,\zeta_2)^T \in \mathscr{H}^4(0,1) \bigcap \mathscr{H}_L^2(0,1) \times \mathscr{H}_L^2(0,1) : \zeta_{1xx}(1) = 0,\ \zeta_{1xxx}(1) = 0\Big\}$$

and generates a strongly continuous contraction semigroup (see e.g. Li & Xu, 2011).

The second component $F_1(\zeta_1, t) : \mathscr{H}^2(0,1) \times \mathscr{H}^1(0,1) \times R \to L_2(0,1)$ of the nonlinear term $F = (0, F_1)^T$ is defined as $F_1(\zeta_1, t) = f(\zeta_{1xx}(x,t), \zeta_{1x}(x,t), x, t)$. Since f has bounded derivatives, the following Lipschitz condition

$$\|F_1(\zeta_1,t) - F_1(\bar\zeta_1,t)\|_{L^2} \leq L\|\zeta_{1xx} - \bar\zeta_{1xx}\|_{L^2}$$

holds for $\zeta_1, \bar\zeta_1 \in \mathscr{H}^2(0,1), t \in [t_0, t_0+T]$ with some constants $L > 0$. Then a unique mild solution of (2.6), initialized with $\zeta(t_0) \in \mathscr{H}$ exists in $\mathscr{H}$ (see, Theorem 6.1.2 of Pazy, 1983). Moreover, this solution is locally Lipschitz in the initial condition (i.e. for all $T > 0$ the mapping $\zeta(t_0) \to \zeta$ is Lipschitz from $\mathscr{H}$ to $C([t_0,T],\mathscr{H})$). Note that $F : \mathscr{H} \times [t_0,\infty) \to \mathscr{H}$ is continuously differentiable. If $\zeta(t_0) \in \mathscr{D}(\mathscr{A})$, then this mild solution is in $C^1([t_0,\infty),\mathscr{H})$ and it is a classical solution of (2.1), (2.2) with $\zeta(t) \in \mathscr{D}(\mathscr{A})$ (see Theorem 6.1.5 of Pazy, 1983).

The boundary measurements are given by $y(t) = z_t(1,t),\ t \geq t_0$. We suggest a nonlinear observer of the form

$$\hat z_{tt}(x,t) + \frac{\partial^2}{\partial x^2}[a(x)\hat z_{xx}(x,t)] + f(\hat z_{xx}(x,t), \hat z_x(x,t), x, t) = 0,$$
$$t \geq t_0,\ x \in (0,1) \tag{3.4}$$

under the boundary conditions

$$\begin{aligned} &\hat z(0,t) = \hat z_x(0,t) = 0,\\ &\hat z_{xx}(1,t) = 0, \qquad \hat z_{xxx}(1,t) = -k[y(t) - \hat z_t(1,t)]\end{aligned} \tag{3.5}$$

and the initial condition $[\hat z(\cdot,t_0), \hat z_t(\cdot,t_0)]^T \in \mathscr{H}$, where $k > 0$ is the injection gain.

Then the estimation error $e = z - \hat z$ satisfies the beam equation

$$e_{tt}(x,t) + \frac{\partial^2}{\partial x^2}[a(x)e_{xx}(x,t)] + ge_{xx}(x,t) + ce_x(x,t) = 0,$$
$$t \geq t_0,\ x \in (0,1), \tag{3.6}$$

where

$$\begin{aligned} g &= g(\hat z_{xx}, e_{xx}, \hat z_x, e_x, x, t)\\ &= \int_0^1 f_{z_{xx}}(\hat z_{xx} + \theta e_{xx}, \hat z_x + \theta e_x, x, t)d\theta,\\ c &= c(\hat z_{xx}, e_{xx}, \hat z_x, e_x, x, t)\\ &= \int_0^1 f_{z_x}(\hat z_{xx} + \theta e_{xx}, \hat z_x + \theta e_x, x, t)d\theta,\end{aligned}$$

and the boundary conditions

$$\begin{aligned} &e(0,t) = e_x(0,t) = 0, \qquad e_{xx}(1,t) = 0,\\ &e_{xxx}(1,t) = ke_t(1,t),\end{aligned} \tag{3.7}$$

whereas the initial conditions are given by $e(x,t_0) = z_1(x) - \hat z(x,t_0), e_t(x,t_0) = z_2(x) - \hat z_t(x,t_0),\ x \in (0,1)$.

We further assume that $|f_{z_{xx}}| \leq g_1, |f_{z_x}| \leq c_1$ and, thus,

$$|g| \leq g_1, \qquad |c| \leq c_1. \tag{3.8}$$

The existence and uniqueness of the mild/classical solutions to the error and the observer equations can be proved similar to the wave equations.

3.2. Exponential stability of the beam equation

We will derive further sufficient conditions for the exponential stability of the error beam equation (3.6), (3.7). Consider the Lyapunov function

$$V(t) = \int_0^1 [a(x)pe_{xx}^2(x,t) + pe_t^2(x,t) + 2\chi xe_x(x,t)e_t(x,t)]dx \tag{3.9}$$

with some constants $p > 0$, $\chi > 0$. We will first derive the condition which guarantees that for some $\alpha > 0$

$$V(t) \geq \alpha \int_0^1 [e_{xx}^2(x,t) + e_t^2(x,t)]dx. \tag{3.10}$$

Since $e_x(0,t) = 0$, we have by Wirtinger's inequality (1.1)

$$\int_0^1 \left[e_x^2(x,t) - \frac{4}{\pi^2} e_{xx}^2(x,t)\right] dx \leq 0. \tag{3.11}$$

By applying to (3.10) and (3.11) the S-procedure with $\lambda_0 > 0$ (Yakubovich, 1977), we conclude that for some $\lambda_0 > 0$ and $\alpha > 0$ the following holds

$$\begin{aligned} V(t) &\geq V(t) + \lambda_0 \int_0^1 [e_x^2(x,t) - 4/\pi^2 e_{xx}^2(x,t)]dx \\ &\geq (a(1)p - 4\lambda_0/\pi^2) \int_0^1 e_{xx}^2(x,t)dx \\ &\quad + \int_0^1 [e_x(x,t)\; e_t(x,t)] \begin{bmatrix} \lambda_0 & \chi x \\ * & p \end{bmatrix} \begin{bmatrix} e_x(x,t) \\ e_t(x,t) \end{bmatrix} dx \\ &\geq \alpha \int_0^1 [e_{xx}^2(x,t) + e_t^2(x,t)]dx \end{aligned} \tag{3.12}$$

if

$$a(1)p - 4\lambda_0/\pi^2 > 0, \qquad \begin{bmatrix} \lambda_0 & \chi \\ * & p \end{bmatrix} > 0. \tag{3.13}$$

Note that (3.13) is feasible for some $\lambda_0 > 0$ if

$$\begin{bmatrix} a(1)p\pi^2/4 & \chi \\ * & p \end{bmatrix} > 0. \tag{3.14}$$

By the Cauchy–Schwarz inequality

$$\begin{aligned} \left|\int_0^1 2\chi x e_x(x,t)e_t(x,t)dx\right| &\leq 2\chi \int_0^1 |e_x(x,t)e_t(x,t)|dx \\ &\leq \chi \int_0^1 [e_x^2(x,t) + e_t^2(x,t)]dx \\ &\leq \chi \int_0^1 \left[\frac{4}{\pi^2} e_{xx}^2(x,t) + e_t^2(x,t)\right] dx, \end{aligned}$$

where we applied Wirtinger's inequality (1.1). Hence,

$$\begin{aligned} \alpha \int_0^1 [e_{xx}^2(x,t) + e_t^2(x,t)]dx &\leq V(t) \\ &\leq \beta \int_0^1 [e_{xx}^2(x,t) + e_t^2(x,t)]dx, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} &\alpha = \min\left\{a(1)p - 4\lambda_0/\pi^2,\ \lambda_{\min}\left(\begin{bmatrix} \lambda_0 & \chi \\ * & p \end{bmatrix}\right)\right\}, \\ &\beta = \chi + \max\{1, a(0)\}p. \end{aligned} \tag{3.16}$$

We are looking for conditions that guarantee $\frac{d}{dt}V(t) + 2\delta V(t) \leq 0$ along the beam equation and, thus, imply

$$\int_0^1 [e_{xx}^2(x,t) + e_t^2(x,t)]dx \leq \frac{\beta}{\alpha} e^{-2\delta(t-t_0)} \int_0^1 [e_{xx}^2(x,t_0) + e_t^2(x,t_0)]dx. \tag{3.17}$$

Differentiating V along (3.1) we have

$$\begin{aligned} \frac{d}{dt}V(t) + 2\delta V(t) &= 2\int_0^1 a(x)pe_{xx}(x,t)e_{xxt}(x,t)dx \\ &\quad + 2\int_0^1 pe_t(x,t)e_{tt}(x,t)dx \\ &\quad + 2\chi \frac{d}{dt}\int_0^1 [xe_x(x,t)e_t(x,t)]dx \\ &\quad + \int_0^1 2\delta[a(x)pe_{xx}^2(x,t) + pe_t^2(x,t) \\ &\quad + 2\chi xe_x(x,t)e_t(x,t)]dx \end{aligned} \tag{3.18}$$

Integrating by parts twice and taking into account the boundary conditions (3.7) with $e_{xx}(1,t) = e_{xt}(0,t) = e_t(0,t) = 0$, $e_{xxx}(1,t) = ke_t(1,t)$, we have

$$\begin{aligned} &2\int_0^1 a(x)pe_{xx}(x,t)e_{xxt}(x,t)dx = 2a(x)pe_{xx}(x,t)e_{xt}(x,t)\,|_0^1 \\ &\quad - 2\int_0^1 p\frac{\partial}{\partial x}[a(x)e_{xx}(x,t)]e_{xt}(x,t)dx \\ &\quad = -2a(1)pke_t^2(1,t) + 2\int_0^1 p\frac{\partial^2}{\partial x^2}[a(x)e_{xx}(x,t)]e_t(x,t)dx. \end{aligned}$$

Therefore, substituting the right-hand side of (3.6) for e_{tt} we arrive at

$$\begin{aligned} &2\int_0^1 p[a(x)e_{xx}(x,t)e_{xxt}(x,t) + e_t(x,t)e_{tt}(x,t)]dx \\ &\quad = -2a(1)pke_t^2(1,t) - 2\int_0^1 pe_t(x,t) \\ &\qquad \times [ge_{xx}(x,t) + ce_x(x,t)]dx \\ &\quad \leq -2a(1)pke_t^2(1,t) + pg_1\int_0^1 [re_t^2(x,t) \\ &\qquad + r^{-1}e_{xx}^2(x,t)]dx - 2\int_0^1 pce_t(x,t)e_x(x,t)dx, \end{aligned} \tag{3.19}$$

where we used Young's inequality with $r > 0$.

Integration by parts leads to

$$\begin{aligned} &2\chi\frac{d}{dt}\int_0^1 [xe_x(x,t)e_t(x,t)]dx = 2\chi\int_0^1 \Big[xe_{xt}(x,t)e_t(x,t)\Big]dx \\ &\quad - 2\chi\int_0^1 xe_x(x,t)\Bigg[\frac{\partial^2}{\partial x^2}[a(x)e_{xx}(x,t)] \\ &\quad + ge_{xx}(x,t) + ce_x(x,t)\Bigg]dx, \end{aligned}$$

where

$$\begin{aligned} 2\chi\int_0^1 xe_{xt}(x,t)e_t(x,t)dx &= \chi\int_0^1 x\frac{\partial}{\partial x}[e_t^2(x,t)]dx \\ &= \chi e_t^2(1,t) - \chi\int_0^1 e_t^2(x,t)dx \end{aligned}$$

and

$$\begin{aligned} &2\chi\int_0^1 xe_x(x,t)\frac{\partial^2}{\partial x^2}[a(x)e_{xx}(x,t)]dx = 2\chi a(1)e_x(1,t)e_{xxx}(1,t) \\ &\quad - 2\chi\int_0^1 \Big[[xe_{xx}(x,t) + e_x(x,t)]\frac{\partial}{\partial x}[a(x)e_{xx}(x,t)]\Big]dx. \end{aligned}$$

Further integration by parts and substitution of the boundary conditions (3.7) yield

$$2\chi\int_0^1 e_x(x,t)\frac{\partial}{\partial x}[a(x)e_{xx}(x,t)]dx = 2\chi a(x)e_{xx}(x,t)e_x(x,t)\mid_0^1$$
$$-2\chi\int_0^1 a(x)e_{xx}^2(x,t)dx \le -2\chi a(1)\int_0^1 e_{xx}^2(x,t)dx$$

and due to $a_x(x)\le 0$

$$\begin{aligned}&2\chi\int_0^1 xe_{xx}(x,t)\frac{\partial}{\partial x}[a(x)e_{xx}(x,t)]dx\\ &=\chi\int_0^1\frac{x}{a}\frac{\partial}{\partial x}[(ae_{xx}(x,t))^2]dx\\ &=-\chi\int_0^1\frac{a(x)-a_x(x)x}{a^2}(ae_{xx}(x,t))^2dx\\ &\le -\chi a(1)\int_0^1 e_{xx}^2(x,t)dx.\end{aligned}\tag{3.20}$$

By using the boundary conditions, we find

$$\begin{aligned}&2\frac{d}{dt}\int_0^1[\chi xe_x(x,t)e_t(x,t)]dx \le -\chi\int_0^1 e_t^2(x,t)dx\\ &+\chi e_t^2(1,t)-3\chi a(1)\int_0^1 e_{xx}^2(x,t)dx\\ &-2\chi a(1)ke_x(1,t)e_t(1,t)\\ &-2\chi\int_0^1 xe_x(x,t)\Big[ge_{xx}(x,t)+ce_x(x,t)\Big]dx.\end{aligned}\tag{3.21}$$

Since $e_x(0,t)=0$, by (1.2) we have

$$\int_0^1 e_{xx}^2(x,t)dx - e_x^2(1,t)\ge 0.\tag{3.22}$$

Inequalities (3.18)–(3.22) and (3.11) yield for $\lambda_1>0,\lambda_2>0$ the following

$$\begin{aligned}&\frac{d}{dt}V(t)+2\delta V(t)\le pg_1\int_0^1[re_t^2(x,t)+r^{-1}e_{xx}^2(x,t)]dx\\ &-2\int_0^1 pce_t(x,t)e_x(x,t)dx-\chi\int_0^1 e_t^2(x,t)dx\\ &+(\chi-2a(1)kp)e_t^2(1,t)-3\chi a(1)\int_0^1 e_{xx}^2(x,t)dx\\ &-2\chi a(1)ke_x(1,t)e_t(1,t)-2\chi\int_0^1 gxe_x(x,t)e_{xx}(x,t)dx\\ &+2c_1\chi\int_0^1 e_x^2(x,t)dx+\int_0^1 2\delta[a(1)pe_{xx}^2(x,t)\\ &+pe_t^2(x,t)+2\chi xe_x(x,t)e_t(x,t)]dx\\ &+\lambda_1\left[\int_0^1 e_{xx}^2(x,t)dx-e_x^2(1,t)\right]\\ &+\lambda_2\int_0^1\left[\frac{4}{\pi^2}e_{xx}^2(x,t)-e_x^2(x,t)\right]dx.\end{aligned}$$

Set $\eta_1^T=[e_t(1,t)e_x(1,t)]$, $\eta_2^T(t)=[e_t(x,t)e_x(x,t)e_{xx}(x,t)]$. Then

$$\frac{d}{dt}V(t)+2\delta V(t)\le \eta_1^T\Psi_1\eta_1+\int_0^1\eta_2^T\Psi_2\eta_2dx\le 0,$$

if the LMIs (3.23) (see Box I) are feasible.

By applying the Schur complements to the 2nd and the 3rd columns and rows of Ψ_2 and to pg_1r^{-1} we find that $\Psi_2<0$ if the LMI (3.24) (see Box II) is feasible.

We have proved the following sufficient conditions for the exponential stability of (3.6), (3.7) with the decay rate $\delta>0$:

Proposition 3.1. *Given $\delta>0$, assume that exist positive constants $p,\chi,\bar r,\lambda_0,\lambda_1,\lambda_2$ such that LMIs (3.13), (3.23) and (3.24) are feasible. Then solutions of the boundary-value problem (3.6), (3.7) satisfy (3.17), where α and β are given by (3.15).*

3.3. Iterative forward and backward observer design

In order to determine the initial state $z(x,t_0)$, $z_t(x,t_0)$ of (3.1) from the boundary measurements on the finite time interval (2.21), we apply the iterative procedure of Ramdani et al. (2010). Define on the finite interval $t\in[t_0,t_0+T]$ the sequences of forward $z^{(n)}$ and backward $z^{b(n)}$, $n=1,2,\dots$ observers with the injection gain $k>0$:

$$\begin{aligned}&z_{tt}^{(n)}(x,t)+\frac{\partial^2}{\partial x^2}[a(x)z_{xx}^{(n)}(x,t)]\\ &\quad+f(z_{xx}^{(n)}(x,t),z_x^{(n)}(x,t),x,t)=0,\\ &z^{(n)}(0,t)=z_x^{(n)}(0,t)=0,\quad t\in[t_0,t_0+T],\\ &z_{xx}^{(n)}(1,t)=0,\quad z_{xxx}^{(n)}(1,t)=k[y(t)-z_t^{(n)}(1,t)],\\ &z^{(n)}(x,t_0)=z^{b(n-1)}(x,t_0),\quad z_t^{(n)}(x,t_0)=z_t^{b(n-1)}(x,t_0),\end{aligned}\tag{3.25}$$

where $z^{b(0)}(x,t_0)=z_t^{b(0)}(x,t_0)\equiv 0$ and

$$\begin{aligned}&z_{tt}^{b(n)}(x,t)+\frac{\partial^2}{\partial x^2}[a(x)z_{xx}^{b(n)}(x,t)]\\ &\quad+f(z_{xx}^{b(n)}(x,t),z_x^{b(n)}(x,t),x,t)=0,\quad t\in[t_0,t_0+T],\\ &z^{b(n)}(0,t)=z_x^{b(n)}(0,t)=0,\\ &z_{xx}^{b(n)}(1,t)=0,\quad z_{xxx}^{b(n)}(1,t)=-k[y(t)-z_t^{b(n)}(1,t)]\\ &z^{b(n)}(x,t_0+T)=z^{(n)}(x,t_0+T),\\ &z_t^{b(n)}(x,t_0+T)=z_t^{(n)}(x,t_0+T).\end{aligned}\tag{3.26}$$

This results in the sequence of the forward $e^{(n)}=z-z^{(n)}$ and the backward $e^{b(n)}=z-z^{b(n)}$, $n=1,2,\dots$ errors satisfying

$$\begin{aligned}&e_{tt}^{(n)}(x,t)+\frac{\partial^2}{\partial x^2}[a(x)e_{xx}^{(n)}(x,t)]\\ &\quad+g^{(n)}e_{xx}^{(n)}(x,t)+c^{(n)}e_x^{(n)}(x,t)=0\quad t\in[t_0,t_0+T],\\ &e^{(n)}(0,t)=e_x^{(n)}(0,t)=0,\\ &e_{xx}^{(n)}(1,t)=0,\quad e_{xxx}^{(n)}(1,t)=-ke_t^{(n)}(1,t),\\ &e^{(n)}(x,t_0)=e^{b(n-1)}(x,t_0),\quad e_t^{(n)}(x,t_0)=e_t^{b(n-1)}(x,t_0),\end{aligned}\tag{3.27}$$

where $e^{b(0)}(x,t_0)=z_1(x)$, $e_t^{b(0)}(x,t_0)=z_2(x)$, $x\in(0,1)$ and

$$\begin{aligned}&e_{tt}^{b(n)}(x,t)+\frac{\partial^2}{\partial x^2}[a(x)e_{xx}^{b(n)}(x,t)]+g^{b(n)}e_{xx}^{b(n)}(x,t)\\ &\quad+c^{b(n)}e_x^{b(n)}(x,t)=0,\quad t\in[t_0,t_0+T],\\ &e^{b(n)}(0,t)=e_x^{b(n)}(0,t)=0,\\ &e_{xx}^{b(n)}(1,t)=0,\quad e_{xxx}^{b(n)}(1,t)=ke_t^{b(n)}(1,t),\\ &e^{b(n)}(x,t_0+T)=e^{(n)}(x,t_0+T),\\ &e_t^{b(n)}(x,t_0+T)=e_t^{(n)}(x,t_0+T).\end{aligned}\tag{3.28}$$

Here

$$\begin{aligned}g^{(n)}&=g(\hat z_{xx}^{(n)},e_{xx}^{(n)},\hat z_x^{(n)},e_x^{(n)},x,t)\\ &=\int_0^1 f_{z_{xx}}(\hat z_{xx}^{(n)}+\theta e_{xx}^{(n)},\hat z_x^{(n)}+\theta e_x^{(n)},x,t)d\theta,\\ c^{(n)}&=c(\hat z_{xx}^{(n)},e_{xx}^{(n)},\hat z_x^{(n)},e_x^{(n)},x,t)\\ &=\int_0^1 f_{z_x}(\hat z_{xx}^{(n)}+\theta e_{xx}^{(n)},\hat z_x^{(n)}+\theta e_x^{(n)},x,t)d\theta\end{aligned}$$

and $g^{b(n)}$, $c^{b(n)}$ are defined similarly.

$$\Psi_1 \triangleq \begin{bmatrix} \chi - 2a(1)kp & -a(1)k\chi \\ * & -\lambda_1 \end{bmatrix} < 0, \tag{3.23}$$
$$\Psi_2 \triangleq \begin{bmatrix} 2\delta p - \chi + pg_1 r & 2\delta\chi x - cp & 0 \\ * & -\lambda_2 + 2c_1\chi & -xg\chi \\ * & * & 2\delta a(1)p + pg_1 r^{-1} - 3a(1)\chi + \frac{4}{\pi^2}\lambda_2 + \lambda_1 \end{bmatrix} < 0$$

Box I.

$$\begin{bmatrix} 2\delta p - \chi + g_1\bar{r} & 2\delta\chi + c_1 p & 0 & 0 \\ * & -\lambda_2 + 2c_1\chi & g_1\chi & 0 \\ * & * & 2\delta a(1)p - 3a(1)\chi + \frac{4}{\pi^2}\lambda_2 + \lambda_1 & g_1 p \\ * & * & * & -g_1\bar{r} \end{bmatrix} < 0, \tag{3.24}$$
where $\bar{r} = pr$

Box II.

3.4. Observability time and convergence rate via LMIs

For (3.27) and (3.28) we consider the Lyapunov functions

$$V^{(n)}(t) = \int_0^1 \Big[a(x)p[e_{xx}^{(n)}(x,t)]^2 + p[e_t^{(n)}(x,t)]^2 + 2\chi x e_x^{(n)}(x,t)e_t^{(n)}(x,t)\Big]dx \tag{3.29}$$

and

$$V^{b(n)}(t) = \int_0^1 \Big[a(x)p[e_{xx}^{b(n)}(x,t)]^2 + p[e_t^{b(n)}(x,t)]^2 - 2\chi x e_x^{b(n)}(x,t)e_t^{b(n)}(x,t)\Big]dx \tag{3.30}$$

respectively with positive constants p and χ satisfying (3.13) for some $\lambda_0 > 0$. Then for all $t \ge t_0$

$$\begin{aligned} &\beta\int_0^1 \Big[[e_{xx}^{(n)}(x,t)]^2 + [e_t^{(n)}(x,t)]^2\Big]dx \ge V^{(n)}(t) \\ &\quad\ge \alpha\int_0^1 \Big[[e_{xx}^{(n)}(x,t)]^2 + [e_t^{(n)}(x,t)]^2\Big]dx, \\ &\beta\int_0^1 \Big[[e_{xx}^{b(n)}(x,t)]^2 + [e_t^{b(n)}(x,t)]^2\Big]dx \ge V^{b(n)}(t) \\ &\quad\ge \alpha\int_0^1 \Big[[e_{xx}^{b(n)}(x,t)]^2 + [e_t^{b(n)}(x,t)]^2\Big]dx \end{aligned} \tag{3.31}$$

where α and β are given by (3.16). Similarly for Lemma 2.1, the following can be proved:

Lemma 3.1. *Consider $V^{(n)}$ and $V^{b(n)}$ given by (3.29) and (3.30) respectively with $p > 0$ and $\chi > 0$ satisfying (3.13). Assume $\delta > 0$ and $T > 0$ exist such that for all $n = 1, 2, \dots$ and $t \in [t_0, t_0 + T]$ the inequalities (2.30) and (2.31) hold along (3.27) and (3.28) respectively. Assume additionally that (2.32) is valid for some $T^* \in (0, T)$. Then the iterative algorithm with $T = T^* + \Delta T$ converges in the sense of (2.33), where $q = e^{-4\delta(T-T^*)}$ is the convergence rate.*

By using arguments of Section 3.2 we find that (3.13), (3.23) and (3.24) guarantee (3.31), (2.30) and (2.31). We are now looking for conditions to satisfy (2.32). We start with the second inequality of (2.32), where we take into account that $e^{(n)}(x, t_0+T) = e^{b(n)}(x, t_0+T)$ and $e_t^{(n)}(x, t_0+T) = e_t^{b(n)}(x, t_0+T)$. Since $e_x^{(n)}(0, t_0+T) = 0$, we have by Wirtinger's inequality (1.1)

$$\int_0^1 \Big[[e_x^{(n)}(x,t_0+T)]^2 - \frac{4}{\pi^2}[e_{xx}^{(n)}(x,t_0+T)]^2\Big]dx \le 0.$$

Then for some $\lambda_0 > 0$ the following holds:

$$\begin{aligned} &V^{b(n)}(t_0+T)e^{-2\delta T^*} - V^{(n)}(t_0+T) \le V^{b(n)}(t_0+T)e^{-2\delta T^*} \\ &\quad - V^{(n)}(t_0+T) + \lambda_0\int_0^1 \Big[\frac{4}{\pi^2}[e_{xx}^{(n)}(x,t_0+T)]^2 \\ &\quad - [e_x^{(n)}(x,t_0+T)]^2\Big]dx \\ &\le [a_1 p(e^{-2\delta T^*} - 1) + 4\lambda_0/\pi^2]\int_0^1 [e_{xx}^{(n)}(x,t_0+T)]^2 dx \\ &\quad + \int_0^1 \begin{bmatrix} e_x^{(n)}(x,t_0+T) \\ e_t^{(n)}(x,t_0+T) \end{bmatrix}^T \Xi(T^*) \begin{bmatrix} e_x^{(n)}(x,t_0+T) \\ e_t^{(n)}(x,t_0+T) \end{bmatrix} dx \le 0 \end{aligned}$$

if

$$\begin{aligned} &\Xi(T^*) \triangleq \begin{bmatrix} -\lambda_0 & \chi(e^{-2\delta T^*} + 1) \\ * & p(e^{-2\delta T^*} - 1) \end{bmatrix} < 0, \\ &a(1)p(e^{-2\delta T^*} - 1) + 4\lambda_0/\pi^2 < 0. \end{aligned} \tag{3.32}$$

By the same arguments, inequalities (3.32) imply the first inequality of (2.32), where $e^{(n)}(x,t_0) = e^{b(n-1)}(x,t_0)$ and $e_t^{(n)}(x,t_0) = e_t^{b(n-1)}(x,t_0)$, because

$$\begin{aligned} &V^{(n)}(t_0)e^{-2\delta T^*} - V^{b(n-1)}(t_0) \\ &\quad\le [a(1)p(e^{-2\delta T^*} - 1) + 4\lambda_0/\pi^2]\int_0^1 [e_{xx}^{(n)}(x,t_0)]^2 dx \\ &\quad + \int_0^1 [e_x^{(n)}(x,t_0)\ e_t^{(n)}(x,t_0)]\,\Xi(T^*)\begin{bmatrix} e_x^{(n)}(x,t_0) \\ e_t^{(n)}(x,t_0) \end{bmatrix} dx \le 0 \end{aligned}$$

Since $e^{-2\delta T} < e^{-2\delta T^*}$ for $T > T^*$, the feasibility of (3.32) with some p, χ, λ_0 implies the feasibility of $a(1)p(e^{-2\delta T} - 1) + 4\lambda_0/\pi^2 < 0$ and, by Schur complements, of $\Xi(T) < 0$ for all $T \ge T^*$ with the same p, χ, λ_0. Particularly (for $T = \infty$), (3.32) yields (3.10). Similarly to Theorem 2.1 we arrive at the following:

Theorem 3.1. *Given positive tuning parameters T^* and δ, assume that positive constants $p, \chi, \bar{r}$ and $\lambda_i (i = 0, 1, 2)$ exist such that the LMIs (3.23), (3.24) and (3.32) are feasible. Then*

(i) *the iterative algorithm with $T = T^*$ converges and the system is exactly observable in T^*;*
(ii) *for all $\Delta T > 0$ the iterative algorithm with $T = T^* + \Delta T$ converges:*

$$\begin{aligned} &\int_0^1 \Big[[e_{xx}^{b(n)}(x,t_0)]^2 + [e_t^{b(n)}(x,t_0)]^2\Big]dx \\ &\quad\le \frac{\beta}{\alpha}(e^{-4\delta\Delta T})^n \int_0^1 \Big[z_{1xx}^2(x) + z_2^2(x)\Big]dx, \end{aligned} \tag{3.33}$$

where α and β are given by (3.15).

Table 2
Minimum T_m^* and $q = e^{-4\delta(T-T^*)}$ for $T^* \geq T_m^*$.

g_1	c_1	T_m^*	δ_m	T	T^*	δ	q
0	0	1.28	0.06	1.6	1.4	0.24	0.8253
0	0.1	1.6	0.29	1.8	1.6	0.29	0.7929
0.1	0.1	1.8	0.32	2.2	1.8	0.32	0.7741
0.2	0.2	2.5	0.26	2.8	2.5	0.26	0.7320

3.5. Example

Consider (3.1)–(3.3) with $a(x) \geq 1, a_x \leq 0, k = 1$ and with the values of g_1, c_1 as given in Table 2. By verifying the feasibility of (3.23), (3.24) and (3.32), we find first the minimal values of T_m^* and the corresponding δ_m for the convergence of the iterative algorithm. Further given $T > T_m^*$, we try to minimize the upper bound on the convergence of the iterative algorithm with T that is guaranteed by (ii) of Theorem 3.1. For this purpose we choose $T^* \in [T_m^*, T)$ and maximize δ which preserves the feasibility of (2.20) and (2.34). The resulting convergence rate $q = e^{-4\delta(T-T^*)}$ for different g_1 is given in Table 2. It is found that for $g_1 = c_1 = 0$ and $T = 1.6 > T_m^* = 1.28$, the fastest convergence rate with the minimal achievable $q = 0.8253$ corresponds to $T^* = 1.4$. For the positive $g_1 + c_1$ the choice of $T^* = T_m^*$ leads to the fastest convergence rate q.

4. Conclusions

In the present paper an LMI approach is introduced for the observer design and for the initial state recovering by iterative forward and backward observers for a class of distributed parameter systems. These are semilinear systems governed by 1-d wave or beam equations with boundary measurements from a finite interval. For the beam equation, these are the first LMI conditions for the exponential stability (of the estimation errors). We have derived LMIs for an upper bound on the observability time and on the convergence rate of the iterative algorithm in the norm defined by the Lyapunov functions. The continuous dependence of the reconstructed initial state on the measurements follows from the integral input-to-state stability of the corresponding error system, which is guaranteed by the LMIs for the exponential stability.

Extension of the method to various classes of distributed parameter systems and its improvement may be topics for future research. As it happened with time-delay systems, LMIs are expected to provide effective constructive tools for analysis and control of distributed parameter systems.

Appendix

Proof of Proposition 2.1. Note that

$$2\frac{d}{dt}\left(\int_0^1 xe_t(x,t)e_x(x,t)dx\right) = 2\int_0^1 xe_{tt}(x,t)e_x(x,t)dx$$
$$+2\int_0^1 xe_t(x,t)e_{xt}(x,t)dx = 2\int_0^1 x\frac{\partial}{\partial x}[a(x)e_x(x,t)]e_x(x,t)dx$$
$$+2\int_0^1 xe_t(x,t)e_{xt}(x,t)dx + 2\int_0^1 xge_x^2(x,t)dx.$$

Integration by parts gives

$$2\int_0^1 xe_t(x,t)e_{xt}(x,t)dx = 2e_t^2(1,\ t) - 2\int_0^1 xe_{xt}e_t(x,t)dx$$
$$-2\int_0^1 e_t^2(x,t)dx$$

i.e. $2\int_0^1 xe_t(x,t)e_{xt}(x,t)dx = -\int_0^1 e_t^2(x,t)dx + e_t^2(1,\ t)$. Similarly

$$2\int_0^1 x\frac{\partial}{\partial x}[a(x)e_x(x,t)]e_x(x,t)dx = 2a(1)e_x^2(1,t)$$
$$-2\int_0^1 [xe_x(x,t)]_x a(x)e_x(x,t)dx = 2a(1)e_x^2(1,t)$$
$$-2\int_0^1 a(x)e_x^2(x,t)dx - 2\int_0^1 x\frac{\partial}{\partial x}[a(x)e_x(x,t)]e_x(x,t)dx$$
$$+2\int_0^1 xa_x(x)e_x^2(x,t)dx,$$

where the last term is not positive due to the assumption $a_x \leq 0$. Then

$$2\int_0^1 x\frac{\partial}{\partial x}[a(x)e_x(x,t)]e_x(x,t)dx \leq a(1)e_x^2(1,t)$$
$$-\int_0^1 a(x)e_x^2(x,t)dx.$$

Therefore, under (2.5)

$$2\frac{d}{dt}\left(\int_0^1 xe_t(x,t)e_x(x,t)dx\right)$$
$$\leq -\int_0^1 [e_t^2(x,t) + a(x)e_x^2(x,t)]dx$$
$$+e_t^2(1,\ t) + a(1)e_x^2(1,\ t) + 2g_1\int_0^1 e_x^2(x,t)dx. \tag{A.1}$$

Differentiating V along (2.12), we obtain

$$\frac{d}{dt}V = 2p\int_0^1 a(x)e_x(x,t)e_{tx}(x,t)dx + 2p\int_0^1 e_t(x,t)$$
$$\times e_{tt}(x,t)dx + 2\chi\frac{d}{dt}\left(\int_0^1 xe_t(x,t)e_x(x,t)dx\right)$$
$$\leq 2p\int_0^1 \frac{\partial}{\partial x}[a(x)e_x(x,t)e_t(x,t)]dx + 2p\int_0^1 e_t(x,t)$$
$$\times ge_x(x,t)dx + 2\chi\frac{d}{dt}\left(\int_0^1 xe_t(x,t)e_x(x,t)dx\right).$$

Then due to (A.1) and (2.13) the following holds:

$$\frac{d}{dt}V + 2\delta V \leq -2a(1)kpe_t^2(1,t) + 2p\int_0^1 e_t(x,t)ge_x(x,t)dx$$
$$-\chi\left[\int_0^1 (e_t^2(x,t) + a(x)e_x^2(x,t))dx\right.$$
$$\left.-(1+a(1)k^2)e_t^2(1,t) - 2g_1\int_0^1 e_x^2(x,t)dx\right]$$
$$+\int_0^1 2\delta[a(x)pe_x^2(x,t)$$
$$+2\chi xe_x(x,t)e_t(x,t) + pe_t^2(x,t)]dx.$$

Setting $\eta^T = [e_t(1,t)\ e_x(x,t)\ e_t(x,t)]$ and using $a \geq a(1)$, we conclude that $\frac{d}{dt}V + 2\delta V \leq \int_0^1 \eta^T\Psi\eta dx \leq 0$, if

$$\Psi = \begin{bmatrix} \psi_1 & 0 & 0 \\ * & -a(1)\chi + 2\delta a(1)p + 2\chi xg & 2\chi\delta x + pg \\ * & * & -\chi + 2\delta p \end{bmatrix} \leq 0.$$

By Schur complements LMIs (2.20) yield $\Psi|_{x=0,g=\pm g_1} \leq 0$, $\Psi|_{x=1,g=\pm g_1} \leq 0$ and, thus, imply $\Psi \leq 0$. □

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