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Static sliding mode control of systems with arbitrary relative degree by using artificial delay

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***Abstract*—Static output-feedback stabilization of systems with relative degree n with matched disturbances is considered. Assuming that the system is controllable, a static output-feedback sliding mode controller (SMC) is designed, where the output derivatives up to the order $(n-1)$ are approximated by using the current and the delayed values of the output. Numerical examples illustrate the efficiency of the method.**

***Index Terms*—Sliding mode control, static feedback, time-delay systems.**

I. Introduction

State-Of-Art. Sliding mode control(SMC) has attractive features for the theoretically exact compensation of the matched uncertainties and disturbances, and finite-time convergence of the system's trajectory to the sliding surface [1]. The static output-feedback SMC paradigm for systems with relative degree one was introduced in [2], where only the measured output (and not its derivatives) was used as SMC surface. For systems with state delays, state-feedback SMC and static output-feedback SMC were suggested [3] and [4] respectively by using the descriptor approach [6].

Note that the presence of input delays destroy the convergence to the sliding motions, or even lead to the instability of the closed-loop system [7]. Practical stabilization of systems with input delays by static output-feedback SMC was suggested in [5] by using a singular perturbation approach.

In [8] using of artificial delay for static output-feedback SMC of systems with relative degree one was introduced. LMI-based conditions were proposed for the stability analysis of the resulting delayed closed-loop system.

A new approach to stabilization of the wide class of systems introducing artificial delay and estimation of the upper bound of such delay was proposed in [9],[10],[11] using the Taylor expansion with the integral remainder and appropriate Lyapunov-Krasovskii funbctionals.

In this paper we propose a static output-feedback SMC design for systems with relative degree n with matched perturbations. For the design of sliding surface for such systems, the complete information about the system state is usually required. For the output-based sliding mode controllers, the estimation of the states has been based on Luenberger observers [12], addictive filters [13], and robust exact differentiators/observers with finite-time convergence [14],[15],[16],[22].

Paper Novelty. In this paper we consider stabilization of systems with relative degree n and matched disturbances. The objective is to design a static output-feedback sliding mode controller that for the estimation of the states employs the delayed values of the output. With this aim:

- the delayed sliding surface is proposed using estimation of the system states based on the artificial delay;
- finite-time attractivity of the proposed delayed sliding surface is proved (Theorem 2);
- Lyapunov-Krasovskii functional is employed to achieve the convergence of system states to the neighborhood of zero (Thereom 1);
- the design parameters are chosen for a good trade-off between the approximation accuracy and the reduction of controller gain.

Notations. For a real symmetric matrix X, $X \leq 0$ (respectively, $X < 0$) means that the matrix X is negative semidefinite (respectively, negative definite). $B(.,.)$ is Euler's beta function. Define a symmetric matrix as $\mathrm{He}(M) = M + M^T$, and the symmetric elements of a symmetric matrix is represented by $\star$. The notations $\|\cdot\|$ and $|\cdot|$ stand for the Euclidean norm and 1-norm of a vector, respectively. By using $O(h)$, a matrix/scalar function of $h \in \mathbb{R}_+$ is defined to satisfy $\lim_{h\to 0^+} |\frac{1}{h}O(h)| = m$, where $m > 0$.

II. Problem formulation and preliminaries

Consider the following n-th order relay system:

$$z^{(n)}(t) = \sum_{i=1}^{n} A_i z^{(i-1)}(t) + B_1[u(t) + d(z, \cdots, z^{(n-1)}, t)], \tag{1}$$

where $n \geq 2$, $z(t) = z^{(0)}(t) \in \mathbb{R}^k$ is the measurement, $z^{(i)}(t)$ is the ith derivative of $z(t)$, $u(t) \in \mathbb{R}^m$ is the control input, and $d(z, z^{(1)}, \cdots, z^{(n)}, t)$ is the matched perturbation. Model (1) represents the linearized nonlinear system of relative degree n with matched state-dependent uncertainties, which represents the local behavior of any Lipschitz system linearized in the vicinity of the equilibrium.

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For the convenience of representation, we define

$$x(t) = \text{col}\{z(t),\ z^{(1)}(t), \cdots,\ z^{(n-1)}(t)\} \triangleq \text{col}\{x_1(t),\ x_2(t), \cdots,\ x_n(t)\}.$$

System (1) is equivalent to

$$\dot{x}(t) = Ax(t) + B(u(t) + d(x,t)) \tag{2}$$

which arrives at

$$x_{n+1}(t) \triangleq \dot{x}_n(t) = \bar{A}x(t) + B_1[u(t) + d(x,t)] \tag{3}$$

where

$$A = \begin{bmatrix} 0 & I_k & 0 & \cdots & 0 \\ 0 & 0 & I_k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I_k \\ A_1 & A_2 & A_3 & \cdots & A_n \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ B_1 \end{bmatrix}$$

$$\bar{A} = [\ A_1 \quad A_2 \quad A_3 \quad \cdots \quad A_n\ ].$$

Inspired from [1], the following linear sliding motion is always used for system (1):

$$s^*(t) = \bar{C}x(t) \tag{4}$$

and the design matrix is given as

$$\bar{C} = [\ C_1 \quad C_2 \quad \cdots \quad C_n\ ]$$

with $C_l \in \mathbb{R}^{m\times k},\ l = 1, 2, \cdots, n-1$.

Since only $z(t)$ is accessible to the controller, we approximate the derivatives by a few past measurements:

$$\hat{x}(t,h) \approx N^{-1}(h)x(t) \tag{5}$$

which becomes

$$x(t) \approx N(h)\hat{x}(t,h) \tag{6}$$

where $N(h) = (MF(h))^{-1}$, $\hat{x}(t,h) = M\bar{x}(t,h)$, $\bar{x}(t,h) = F(h)x(t)$, and

$$\hat{x}(t,h) = \text{col}\{x_1(t),\ x_1(t-h), \cdots,\ x_1(t-(n-1)h)\}$$
$$F(h) = \text{diag}\{I_k,\ -hI_k, \cdots,\ (-h)^{n-1}I_k\}$$

$$M = \begin{bmatrix} I_k & 0 & 0 & \cdots & 0 \\ I_k & I_k & \frac{1}{2!}I_k & \cdots & \frac{1}{(n-1)!}I_k \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I_k & jI_k & \frac{j^2}{2!}I_k & \cdots & \frac{j^{n-1}}{(n-1)!}I_k \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I_k & (n-1)I_k & \frac{(n-1)^2}{2!}I_k & \cdots & \frac{(n-1)^{n-1}}{(n-1)!}I_k \end{bmatrix}.$$

Matrix M is a Vandermonde-type matrix. Equation (5) indicates that the states of system (5) can be estimated by using the past measurements at the time instant j ($j = 0, 1, \cdots, n-1$).

To this end, the following delay-dependent sliding variable is adopted:

$$s(t) = \bar{C}N(h)\hat{x}(t,h)$$

where h is the artificial time delay.

Our objective is to design the artificial time-delay estimator (5) in sliding mode control, which avoid introducing additional dynamics for estimation and enhancing the robustness to measurement noises. Then, we present the problem formulation of this work: right) a delay-dependent sliding surface $s(t) = 0$ and an SMC law $u(t)$ will be designed such that the sliding motion $s(t)$ is quadratically stable in finite time.

To this end, the following lemmas are necessary.

Lemma 1. (*Jensen's Inequality* [10]) *Define* $G = \int_a^b f(s)x(s)\mathrm{d}s$, *where* $a \le b$, $f : [a,b] \to [0,\infty)$, $x(s) \in \mathbb{R}^n$, *and the integration concerned is well defined. Then, for any* $n \times n$ *matrix* $R > 0$, *the following inequality holds:*

$$G^TRG \le \int_a^b f(\theta)\mathrm{d}\theta \int_a^b f(s)x^T(s)Rx(s)\mathrm{d}s. \tag{7}$$

Lemma 2. [20, 23] *Given a positive scalar* $\bar{\epsilon}$, *and symmetric matrices* M_1, M_2 *and* M_3 *with the same dimensions, the inequality holds for any* $\epsilon \in (0, \bar{\epsilon}]$: $M_1 + \epsilon M_2 + \epsilon^2 M_3 \le 0$, *if and only if* $M_1 \le 0,\ M_1 + \bar{\epsilon}M_2 \le 0,\ M_1 + \bar{\epsilon}M_2 + \bar{\epsilon}^2 M_3 \le 0$.

A. Taylor's formula with the integral remainder

For n-times continuously differentiable function $x_1(t)$ with absolutely continuous $x_n(t)$ over the time interval $[t - jh, t]$, the Taylor expansion is written as

$$\begin{aligned} x_1(t - jh) &= x_1(t) + j(-h)x_2(t) + \frac{j^2(-h)^2}{2!}x_3(t) \\ &\quad + \cdots + \frac{j^{n-1}(-h)^{n-1}}{(n-1)!}x_n(t) + \delta_j(t,h) \\ &= M(j)F(h)x(t) + \delta_j(t,h) \end{aligned} \tag{8}$$

where $j = 0,\ 1, \cdots,\ n-1$, and

$$M(j) = \left[\ I_k \quad jI_k \quad \tfrac{j^2}{2!}I_k \quad \cdots \quad \tfrac{j^{n-1}}{(n-1)!}I_k\ \right].$$

Wherein, the remainder $\delta_j(t,h)$ has two equivalent forms:

$$\delta_j(t,h) = \frac{(-1)^n}{(n-1)!}\int_{t-jh}^t (s - t + jh)^{n-1}x_{n+1}(s)\ \mathrm{d}s \tag{9}$$

$$\delta_j(t,h) = \frac{(-1)^{n-1}}{(n-2)!}\int_{t-jh}^t (s - t + jh)^{n-2}\mu(s,t)\ \mathrm{d}s \tag{10}$$

with $\mu(s,t) = x_n(s) - x_n(t)$.

Remark 1. The form of the integral terms of Lyapunov functionals is constructed based on (9) (as in [10]). Here, the representation of (10) is used for estimating the bound of the remainder $\delta_j(t,h)$.

Then, we have

$$\hat{x}(t,h) = N^{-1}(h)x(t) + \Delta_1(t,h) \tag{11}$$

where $\Delta_1(t,h) = \text{col}\{\delta_0(t,h),\ \delta_1(t,h),\ \cdots,\ \delta_{n-1}(t,h)\}$.

It is easy to verify that, if $\lim_{h\to 0} |\delta_i(t,h)| = 0$, $\lim_{h\to 0} \frac{1}{h^{n+1}}|\delta_i(t,h)| = 0$, then we have

$$\Delta_1(t) = O(h^n). \tag{12}$$

The approximation error is given by $\Delta_1(t)$, which is used to verify the approximation accuracy of the artificial time-delay method. Hence, we obtain $s(t) = s^*(t) + \bar{C}N(h)\Delta_1(t,h)$ $e_s(t) = s(t) - s^*(t) = \bar{C}N(h)\Delta_1(t,h) = O(h^n)$.

Remark 2. The sliding motion $s(t) = 0$ can imitate the behavior of $s^*(t) = 0$ with the accuracy $O(h^n)$ such that

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the delayed output-feedback controller contains more system information than the pure static one.

It is reasonable to assume that

$$|x_{n+1}(t)| \leq \beta \tag{13}$$

where β is the upper bound of $x_{n+1}(t)$, and β is a scalar defined by the domain in which model (1) is valid. The following assumptions will be used:

A1) The perturbation term $d(x,t)$ is bounded, i.e. $\mid d(x,t) \mid \leq d^*$, where d^* is a positive scalar.

A2) The dynamic $x_n(t)$ is β-Lipschizian on the small time interval $[t-jh-\delta, t+\delta]$:

$$|\mu(s,t)| \leq \beta|t-s|, \text{ for all } s \in [t-jh,\ t] \tag{14}$$

where δ is a small positive scalar.

Remark 3. All system states $x_i(t)$ $(i = 1, 2, \cdots, n)$ are still continuous, i.e. $x_i(t) = x_i(t_0) + \int_{t_0}^{t} x_{i+1}(s)\mathrm{d}s$, $\lim_{t\to t_0^-} x_i(t) = \lim_{t\to t_0^+} x_i(t)$, which reveals the validity of Assumption A2). For example, a mass-spring-damper system contains the states of position and velocity. The acceleration can change the directions by using forces with $\mathrm{sgn}(\cdot)$, but the velocity and position are still continuous.

III. Main results

A. Analysis of sliding motions

From (10), we have

$$\dot{\Delta}_1(t,h) = \Delta_2(t,h) + Y(h)x_{n+1}(t) \tag{15}$$

where $Y(h) = \frac{(-1)^{n-2}}{(n-1)!}\mathrm{col}\{0, h^{n-1}I_k, \cdots, [(n-1)h]^{n-1}I_k\}$. The form of $\Delta_2(t,h)$ is written as

$$\Delta_2(t,h) = \mathrm{col}\{\rho_0(t), \rho_1(t,h), \cdots, \rho_{n-1}(t,h)\}$$

where $\rho_j(t,h)$ takes the following two forms:

$$\rho_j(t,h) = \frac{(-1)^{n+1}}{(n-2)!}\int_{t-jh}^{t}(s-t+jh)^{n-2}x_{n+1}(s)\mathrm{d}s \tag{16}$$

$$\rho_j(t,h) = \begin{cases} \dfrac{(-1)^n}{(n-3)!}\displaystyle\int_{t-jh}^{t}(s-t+jh)^{n-3}\mu(s,t)\mathrm{d}s, n \geq 3 \\ x_n(t) - x_n(t-jh),\ n=2. \end{cases} \tag{17}$$

From (2) and (15), the derivative of $s(t)$ is given as

$$\dot{s}(t) = \bar{C}\dot{x}(t) + \bar{C}H(h)x_{n+1}(t) + \bar{C}N(h)\Delta_2(t,h) \tag{18}$$

where $H(h) = N(h)Y(h)$. By virtue of (1) and (2), we can rewrite (18) as

$$\dot{s}(t) = J_1(h)x(t) + J_2(h)(u(t) + d(x,t)) + \bar{C}N(h)\Delta_2(t,h)$$

where $J_1(h) = \bar{C}A + \bar{C}H(h)\bar{A},\ J_2(h) = \bar{C}B + \bar{C}H(h)B_1$.

The equivalent control law is formulated as

$$u_{eq}^*(t) = -d(x,t) - J_2^{-1}(h)(J_1(h)x(t) + \bar{C}N(h)\Delta_2(t,h)) \tag{19}$$

which requires the full state information of $x(t)$. By using the estimation (6) in (19), we arrive at

$$\begin{aligned} u_{eq}(t) = -d(x,t) - J_2^{-1}(h)[J_1(h)N(h)\hat{x}(t,h) \\ + \bar{C}N(h)\Delta_2(t,h)] \end{aligned} \tag{20}$$

which is equivalent to

$$\begin{aligned} u_{eq}(t) = -d(x,t) - J_2^{-1}(h)[J_1(h)x(t) \\ + J_1(h)N(h)\Delta_1(t) + \bar{C}N(h)\Delta_2(t,h)]. \end{aligned} \tag{21}$$

Remark 4. In (20), the equivalent control law is implemented without using the derivatives of the measurement output.

B. Stabilization of the closed-loop system

Based on the descriptor model transformation in [3, 5], substituting (21) into system (2) yields

$$E\dot{\eta}(t) = \mathcal{A}_c\eta(t) + \mathcal{A}_{d1}\Delta_1(t) + \mathcal{A}_{d2}\Delta_2(t) \tag{22}$$

where $\eta(t) = \mathrm{col}\{x(t), x_{n+1}\}$, and

$$E = \begin{bmatrix} I_{\bar{n}} & 0 \\ 0 & 0 \end{bmatrix},\ \mathcal{A} = \begin{bmatrix} \Gamma_1 & 0 \\ \bar{A} & -I_k \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 0 \\ B_1 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} 0 & I_{\bar{n}-k} \end{bmatrix},\ \bar{n} = nk,$$

$$\mathcal{A}_c = \mathcal{A} + \mathcal{B}\Gamma_2(h),\ \Gamma_2(h) = \begin{bmatrix} -J_2^{-1}(h)J_1(h) & 0 \end{bmatrix}$$

$$\mathcal{A}_{d1} = -\mathcal{B}J_2^{-1}(h)J_1(h)N(h),\ \mathcal{A}_{d2} = -\mathcal{B}J_2^{-1}(h)\bar{C}N(h).$$

Define $A_m = \begin{bmatrix} \bar{A} - B_1J_2^{-1}(h)J_1(h) & 0 \end{bmatrix}$, and

$$A_{n1} = -B_1J_2^{-1}(h)J_1(h)N(h),\ A_{n2} = -B_1J_2^{-1}(h)\bar{C}N(h).$$

With $u_{eq}(t)$, equation (1) can be further represented as

$$x_{n+1}(t) = A_m\eta(t) + A_{n1}\Delta_1(t) + A_{n2}\Delta_2(t). \tag{23}$$

Then, we will derive the delay-dependent LMI conditions for the stabilization of the closed-loop system (22).

Theorem 1. *For the given tuning scalar $\epsilon^* > 0$ and the prescribed matrix $\bar{C}$, the descriptor system (22) is asymptotically stable, if there exist symmetric matrices $P_1 \in \mathbb{R}^{(\bar{n}-k)\times(\bar{n}-k)}$, $P_3 \in \mathbb{R}^{k\times k}$, $X \in \mathbb{R}^{k\times k}$, and $W \in \mathbb{R}^{k\times k}$, and the matrix $P_2 \in \mathbb{R}^{k\times(\bar{n}-k)}$ such that the following inequalities hold:*

$$\begin{bmatrix} \mathrm{He}(P^T\mathcal{A}_c) & P^T\mathcal{A}_{d1} & P^T\mathcal{A}_{d2} & 0 \\ \star & -\mathcal{W} & 0 & 0 \\ \star & \star & -\mathcal{X} & 0 \\ \star & \star & \star & -\bar{Q} \end{bmatrix} \leq 0 \tag{24}$$

$$\begin{bmatrix} \mathrm{He}(P^T\mathcal{A}_c) & P^T\mathcal{A}_{d1} & P^T\mathcal{A}_{d2} & \sqrt{\epsilon^*}A_m^T\bar{X} \\ \star & -\mathcal{W} & 0 & \sqrt{\epsilon^*}A_{n1}^T\bar{X} \\ \star & \star & -\mathcal{X} & \sqrt{\epsilon^*}A_{n2}^T\bar{X} \\ \star & \star & \star & -\bar{X} \end{bmatrix} \leq 0 \tag{25}$$

$$\begin{bmatrix} \mathrm{He}(P^T\mathcal{A}_c) & P^T\mathcal{A}_{d1} & P^T\mathcal{A}_{d2} & \sqrt{\epsilon^*}A_m^T\bar{Q}(h^*) \\ \star & -\mathcal{W} & 0 & \sqrt{\epsilon^*}A_{n1}^T\bar{Q}(h^*) \\ \star & \star & -\mathcal{X} & \sqrt{\epsilon^*}A_{n2}^T\bar{Q}(h^*) \\ \star & \star & \star & -\bar{Q}(h^*) \end{bmatrix} \leq 0 \tag{26}$$

where $h^* = \sqrt[2(n-1)]{\epsilon^*}$, and

$$\bar{X} = \sum_{j=0}^{n-1}\bar{X}_{j+1},\ \bar{X}_{j+1} = j^{2n-1}X_{j+1}$$

$$\bar{Q} = \sum_{j=0}^{n-1}(j)^{2(n-1)}X_{j+1},\ \bar{Q}(h^*) = \sum_{j=0}^{n-1}Q_{j+1}(h^*)$$

$$Q_{j+1}(h^*) = (j)^{2(n-1)}[X_{j+1} + (jh^*)^2W_{j+1}].$$

Proof. The following Lyapunov functional is used:

$$V(\eta, y, t) = V_1(\eta, t) + V_2(x_{n+1}) + V_3(x_{n+1})$$

where

$$V_1(\eta, t) = \eta^T(t)EP\eta(t)$$
$$V_2(x_{n+1}) = \sum_{j=0}^{n-1}(jh)^n \int_{t-jh}^{t}(s-t+jh)^n \nu_{j+1}(s)\mathrm{d}s$$
$$V_3(x_{n+1}) = \sum_{j=0}^{n-1}(jh)^{n-1} \int_{t-jh}^{t}(s-t+jh)^{n-1} \varphi_{j+1}(s)\mathrm{d}s$$
$$\nu_{j+1}(s) = x_{n+1}^T(s)W_{j+1}x_{n+1}(s),\ W_{j+1} = W_{j+1}^T > 0$$
$$\varphi_{j+1}(s) = x_{n+1}^T(s)X_{j+1}x_{n+1},\ X_{j+1} = X_{j+1}^T > 0.$$

Matrix P is specified as

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix},\ P_1 = P_1^T > 0.$$

Differentiating $V_1(\eta, t)$ with respect to time t yields

$$\begin{aligned}\dot{V}_1(\eta, t) = \eta^T(t)\mathrm{He}(P^T\mathcal{A}_c)\eta(t) + 2\eta^T(t)P^T\mathcal{A}_{d1}\Delta_1(t,h) \\ + 2\eta^T(t)P^T\mathcal{A}_{d2}\Delta_2(t,h).\end{aligned} \tag{27}$$

To deal with the term with $\Delta_1(t)$ in (27), the functional $V_2(x_{n+1})$ is taken into account, with its derivative given as

$$\begin{aligned}\dot{V}_2(x_{n+1}) = \sum_{j=0}^{n-1}(jh)^{2n}x_{n+1}^T(t)W_{j+1}x_{n+1}(t) \\ - n\sum_{j=0}^{n-1}(jh)^n \int_{t-jh}^{t}(s-t+jh)^{n-1}\nu_{j+1}(s)\mathrm{d}s.\end{aligned} \tag{28}$$

It follows from Lemma 1 that

$$\begin{aligned}- n\sum_{j=0}^{n-1}(jh)^n \int_{t-jh}^{t}(s-t+jh)^{n-1}\nu_{j+1}(s)\ \mathrm{d}s \\ \le -\Delta_1^T(t,h)\mathcal{W}\Delta_1(t,h),\end{aligned} \tag{29}$$

where $\mathcal{W} = (n!)^2\mathrm{diag}\{W_0,\ W_1,\ \cdots,\ W_{n-1}\}$.

From (28) and (29), we have

$$\begin{aligned}\dot{V}_2(x_{n+1}) \le \zeta^T(t)\mathcal{V}^T\{\sum_{j=0}^{n-1}(jh)^{2n}W_{j+1}\}\mathcal{V}\zeta(t) \\ - \Delta_1^T(t,h)\mathcal{W}\Delta_1(t,h),\end{aligned} \tag{30}$$

where

$$\mathcal{V} = \begin{bmatrix} A_m & A_{n1} & A_{n2} \end{bmatrix},\ \zeta(t) = \mathrm{col}\{\eta(t),\ \Delta_1(t),\ \Delta_2(t)\}.$$

Moreover, the time derivative of the delay-dependent function $V_3(x_{n+1})$ along the solution of (22) is written as

$$\begin{aligned}\dot{V}_3(x_{n+1}) = \sum_{j=0}^{n-1}(jh)^{2(n-1)}x_{n+1}^T(t)X_{j+1}x_{n+1}(t) \\ - (n-1)\sum_{j=0}^{n-1}(jh)^{n-1}\int_{t-jh}^{t}(s-t+jh)^{n-2}\varphi_{j+1}(s)\mathrm{d}s.\end{aligned}$$

By using the representation (16) and applying Lemma 1, we find

$$\begin{aligned}- (n-1)\sum_{j=0}^{n-1}(jh)^{n-1}\int_{t-jh}^{t}(s-t+jh)^{n-2}\varphi_{j+1}(s)\ \mathrm{d}s \\ \le -\Delta_2^T(t,h)\mathcal{X}\Delta_2(t,h)\end{aligned} \tag{31}$$

where

$$\mathcal{X} = (n-1)!^2\mathrm{diag}\{X_0,\ X_1,\ \cdots,\ X_{n-1}\}.$$

Taking (31) into consideration, it is easy to obtain that

$$\begin{aligned}\dot{V}_3(x_{n+1}) \le \zeta^T(t)\mathcal{V}^T\sum_{j=0}^{n-1}[(jh)^{2(n-1)}X_{j+1}]\mathcal{V}\zeta(t) \\ - \Delta_2^T(t,h)\mathcal{X}\Delta_2(t,h).\end{aligned} \tag{32}$$

Adding (27), (30) to (32), we have

$$\begin{aligned}\dot{V}(t) \le \zeta^T(t)\Xi\zeta(t) \\ + \zeta^T(t)\mathcal{V}^T\sum_{j=0}^{n-1}[(jh)^{2(n-1)}X_{j+1} + (jh)^{2n}W_{j+1}]\mathcal{V}\zeta(t)\end{aligned}$$

where

$$\Xi = \begin{bmatrix} \mathrm{He}(P^T\mathcal{A}_c) & P^T\mathcal{A}_{d1} & P^T\mathcal{A}_{d2} \\ \star & -\mathcal{W} & 0 \\ \star & \star & -\mathcal{X} \end{bmatrix}.$$

After some manipulation using the Schur Complement Lemma, the inequality $\dot{V}(t) \le 0$ is equivalently represented as

$$\Gamma(\epsilon, h) \le 0 \tag{33}$$

where

$$\epsilon = h^{2(n-1)},\ \ \bar{Q}(h) = \sum_{j=0}^{n-1}(j)^{2(n-1)}[X_{j+1} + (jh)^2W_{j+1}]$$

$$\Gamma(\epsilon, h) = \begin{bmatrix} \mathrm{He}(P^T\mathcal{A}_c) & P^T\mathcal{A}_{d1} & P^T\mathcal{A}_{d2} & \epsilon A_m^TQ(h) \\ \star & -\mathcal{W} & 0 & \epsilon A_{n1}^TQ(h) \\ \star & \star & -\mathcal{X} & \epsilon A_{n2}^TQ(h) \\ \star & \star & \star & -\epsilon Q(h) \end{bmatrix}.$$

Performing the congruent transformation given as

$$\mathcal{T}_s = \mathrm{diag}\{I, I, I, \frac{1}{\sqrt{\epsilon}}I\}$$

to inequality (33) yields

$$\Gamma^*(\epsilon, h) \le 0 \tag{34}$$

where

$$\Gamma^*(\epsilon, h) = \begin{bmatrix} \mathrm{He}(P^T\mathcal{A}_c) & P^T\mathcal{A}_{d1} & P^T\mathcal{A}_{d2} & \sqrt{\epsilon}A_m^TQ(h) \\ \star & -\mathcal{W} & 0 & \sqrt{\epsilon}A_{n1}^TQ(h) \\ \star & \star & -\mathcal{X} & \sqrt{\epsilon}A_{n2}^TQ(h) \\ \star & \star & \star & -Q(h) \end{bmatrix}.$$

It follows from Lemma 2 that the sufficient conditions for achieving (34) are given as

$$\exists\ h^* > 0,\ \text{s.t.}\ \Gamma(0,0) \le 0,\ \Gamma(\epsilon^*, 0) \le 0,\ \Gamma(\epsilon^*, h^*) \le 0,$$

for all $h \in (0, h^*]$, with $\epsilon^* = (h^*)^{2(n-1)}$, which can be transformed into LMIs (24)–(26). This completes the proof. □

Remark 5. Theorem 1 can be used for investigating the upper bound of the delay h. We can set any h satisfying $0 < h \le h^*$ in the formulation of $s(t)$ and $u(t)$.

Remark 6. Consider $n(t)$ as the measurement noises with a frequency f_n. The robustness to the high-frequency $n(t)$ can be concluded from the averaging theory, which requires

$$f_n \ll 1/h,\ 0 < h \le h^* \tag{35}$$

For low-frequency noise $n(t)$, the artificial time-delay estimator will not work as a filter. Thus, using the results in Theorem 1, it is possible to choose the suitable time delay for the finite-time convergence to the vicinity of origin.

C. Attractivity of the sliding surface

In this section, the attractivity of the delay-dependent sliding surface $s(t) = 0$ will be analyzed. The physical SMC law is

$$u(t) = -kJ_2^{-1}(h)\text{sgn}(s(t)) - J_2^{-1}(h)J_1(h)N(h)\hat{x}(t,h) \tag{36}$$

where k is a positive scalar. Moreover, an equivalent form of (36) can be represented as

$$u(t) = -kJ_2^{-1}(h)\text{sgn}(s(t)) - J_2^{-1}(h)J_1(h)x(t) - J_2^{-1}(h)J_1(h)N(h)\Delta_1(t,h). \tag{37}$$

Next, we will estimate the upper bound of $|\Delta_1(t,h)|$ and $|\Delta_2(t,h)|$. With Assumption A2), it is easy to verify that

$$\begin{aligned}|\Delta_1(t,h)| &= \sum_{j=0}^{n-1}|\delta_j(t,h)| \\ &= \frac{1}{(n-2)!}\sum_{j=0}^{n-1}\int_{t-jh}^{t}|(s-t+jh)^{n-2}\cdot\mu(s,t)|\text{d}s \\ &\le \frac{1}{(n-2)!}\sum_{j=0}^{n-1}\int_{t-jh}^{t}|(s-t+jh)^{n-2}|\cdot|\mu(s,t)|\text{d}s \\ &\le b_{r1}(\beta)\end{aligned}$$

where

$$b_{r1}(\beta) = \frac{\beta}{(n-2)!}\sum_{j=0}^{n-1}\int_{t-jh}^{t}|(s-t+jh)^{n-2}|\cdot|t-s|\text{d}s.$$

Similarly, we obtain that $|\Delta_2(t,h)| \le b_{r2}(\beta)$, where

$$b_{r2}(\beta) = \begin{cases} g(\beta), \ n \ge 3 \\ |x_n(t) - x_n(t-jh)|, \ n = 2, \end{cases}$$
$$g(\beta) = \frac{\beta}{(n-3)!}\sum_{j=0}^{n-1}\int_{t-jh}^{t}|(s-t+jh)^{n-3}|\cdot|t-s|\text{d}s.$$

Using (37) in the representation of $\dot{s}(t)$ yields

$$\dot{s}(t) = -k\text{sgn}(s(t)) - J_1N(h)\Delta_1(t,h) + J_2d(x,t) + \bar{C}N(h)\Delta_2(t,h). \tag{38}$$

The switching term $-k\text{sgn}(s(t))$ is used to compensate for $d(x,t)$, $\Delta_1(t,h)$ and $\Delta_2(t,h)$ to force the state trajectories to be attractive to the sliding surface $s(t) = 0$.

The following theorem investigates the reachability of the sliding motion in finite time.

Theorem 2. *Under Assumptions* A1)–A2)*, the control input (36) makes the sliding surface stable and globally attractive in finite time, if the following condition holds:*

$$k \ge \gamma_1\beta + \|J_2\|d^* + k_0 \tag{39}$$
$$\gamma_2\beta \ge k\|B_1J_1^{-1}\| + \|B_1\|d^* + k_1 \tag{40}$$

where k_0 and k_1 are positive scalars, and

$$\begin{aligned}\gamma_1 &= \|J_1N(h)\|b_1^* + \|\bar{C}N(h)\|b_2^*, \\ \gamma_2 &= 1 - \|A_{n1}\|b_1^* - \|A_{n2}\|b_2^*, \\ b_1^* &= \sum_{j=0}^{n-1}\frac{1}{(n-2)!}(jh)^nB(2,n-1), \\ b_2^* &= \begin{cases}\sum_{j=0}^{n-1}\frac{1}{(n-3)!}(jh)^{n-1}B(2,n-2), \ n \ge 3 \\ h, \ n = 2\end{cases}\end{aligned} \tag{41}$$

and the reaching time is given as $t_r = \sqrt{2V(0)}/k_0$.

Proof. Consider the Lyapunov function as $V(t) = 0.5s^T(t)s(t)$. By differentiating $V(t)$ with respect to time t, we have

$$\begin{aligned}\dot{V}(t) &= s(t)\dot{s}(t) \\ &= s(t)[-k\text{sgn}(s(t)) - J_1(h)N(h)\Delta_1(t,h) \\ &\quad + J_2(h)d(x,t) + \bar{C}N(h)\Delta_2(t,h)], \\ &\le -k|s(t)| + \|s(t)\|\cdot\|J_2(h)\|\cdot\|d(x,t)\| \\ &\quad + \|s(t)\|\cdot\|J_1(h)N(h)\|\cdot\|\Delta_1(t,h)\| \\ &\quad + \|s(t)\|\cdot\|\bar{C}N(h)\|\cdot\|\Delta_2(t,h)\|.\end{aligned} \quad N(h) \tag{42}$$

Note that $\|s(t)\| \le |s(t)|$, $\|\Delta_1(t,h)\| \le |\Delta_1(t,h)|$, and $\|\Delta_2(t,h)\| \le |\Delta_2(t,h)|$. With Assumption A1) and A2), inequality (42) becomes

$$\begin{aligned}\dot{V}(t) \le -|s(t)|\cdot[&k - \|J_1(h)N(h)\|\cdot b_{r1}(\beta) \\ &- \|J_2(h)\|d^* - \|\bar{C}N(h)\|\cdot b_{r2}(\beta)]\end{aligned}$$

which indicates that (39). Finally, we obtain that $\dot{V}(t) \le -k_0|s| < 0$, which guarantees the convergence of system (1) towards the surface $s(t) = 0$, and

$$\begin{aligned}b_{r1}(\beta) &= \frac{\beta}{(n-2)!}\sum_{j=0}^{n-1}\int_{t-jh}^{t}|(s-t+jh)^{n-2}|\cdot|t-s|\text{d}s \\ &= \beta\sum_{j=0}^{n-1}(jh)^n\int_0^1|s^{n-1}(1-s)|\text{d}s = b_1^*\beta, \\ b_{r2}(\beta) &= \frac{\beta}{(n-3)!}\sum_{j=0}^{n-1}\int_{t-jh}^{t}|(s-t+jh)^{n-3}|\cdot|t-s|\text{d}s \\ &= \sum_{j=0}^{n-1}(jh)^{n-1}\int_0^1|s^{n-2}(1-s)|\text{d}s\cdot\beta = b_2^*\beta, \ n \ge 3, \\ b_{r2}(\beta) &= h\beta = b_2^*\beta, \ n = 2.\end{aligned}$$

Here, b_1^* and b_2^* are the linear coefficients.

Hence, the state trajectory is capable to reach the sliding surface $s(t) = 0$ in finite time, and the reaching time given below can be adjusted by changing k_0:

$$t_r = |s(0)|/k_0 = \sqrt{2V(0)}/k_0.$$

Moreover, with control law in (36), we obtain

$$\begin{aligned}x_{n+1}(t) &= A_m\eta(t) + A_{n1}\Delta_1(t,h) + A_{n2}\Delta_2(t,h) \\ &\quad - kB_1J_2^{-1}(h)\text{sgn}(s(t)) + B_1d(x,t)\end{aligned}$$

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which implies that

$$\|x_{n+1}(t)\| \leq \|A_m\eta(t)\| + \beta\|A_{n1}\|b_1^* + \beta\|A_{n2}\|b_2^* + k\|B_1J_2^{-1}(h)\| + \|B_1\|d^*.$$

Considering that $\|x_{n+1}(t)\| < |x_{n+1}(t)| < \beta$, we arrive at

$$k_1+\beta\|A_{n1}\|b_1^*+\beta\|A_{n2}\|b_2^*+k\|B_1J_1^{-1}(h)\|+\|B_1\|d^* < \beta. \quad (43)$$

Note that in (40), the item k_1 is added to (43) in order to compensate the bounded item $\|A_m\eta(t)\|$, i.e. $\|A_m\eta(t)\| \leq |(\bar{A}-B_1J_2^{-1}(h)J_1(h))x| < k_1$. This completes the proof. □

Remark 7. The upper bounds of estimation errors are $b_1^*\beta = O(h^n)$ and $b_2^*\beta = O(h^n)$, which are small scalars for small enough delay h. Thus, $\gamma_1\beta \approx 0,\ 1 \geq \gamma_2 > 0$. First, we approximately choose k satisfying $k \geq \|J_2\|d^*+k_0$. Then, the value of β is determined via $\beta \geq k\|B_1J_1^{-1}\|/\gamma_2+\|B_1\|d^*/\gamma_2+k_1/\gamma_2$.

Design Steps. The model (1) is derived based on the linearization around the equilibrium. Then, the matrices A_i and B_1 are obtained, where the domain for model (2) is defined as $|x(t)| \leq \psi$. Here ψ is a known scalar. The upper bound d^* is known. With Assumptions A1)–A2), we will follow

1) Select k_1 that satisfies
$$|(\bar{A}-B_1J_2^{-1}J_1)x| \leq |(\bar{A}-B_1J_2^{-1}J_1)|\cdot\psi \leq k_1.$$
2) Choose $\bar{C}$, and formulate the ideal sliding surface $s^*(t)$ and the delay-dependent sliding surface $s(t)$.
3) Find $h^* = \max h$, if the solution to (24)–(26) exists:
$$\max h,\ \text{s.t. LMIs (24)–(26) hold;}$$
4) Choose h, k, k_0 and β to satisfy the constraints (39)–(40), and $0 < h \leq h^*$;
5) Design the output SMC controller in the form of (36) by using the values of $\bar{C}$, k and h.

IV. Simulation Example

In this section, a simulation example of a Magnetic Levitation System (MLS) is used. The following nonlinear model in [21] is considered:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{k}{M}x_2(t) + \dfrac{aL_0}{2M}\dfrac{x_3^2(t)}{(a+x_1(t))^2} - g \\ \dot{x}_3(t) = \dfrac{1}{L(x_1(t))}(-R_0x_3(t) - aL_0\dfrac{x_2(t)x_3(t)}{(a+x_1(t))^2} + v(t)) \end{cases} \quad (44)$$

where $x_1(t)$, x_2 and $x_3(t)$ are, respectively, the plate's position in [m], velocity in [m/s] and coil current in [A], and $v(t)$ is the control input (voltage applied to coil). Moreover, M is the mass of the plate, g is the gravity acceleration, k is a viscous friction coefficient, R_0 is the electric resistance, and $L(x_1(t)) = L_1 + \frac{aL_0}{a+x_1(t)}$ is the coil inductance, and a, L_0 and L_1 are positive constants. In the simulation setup, these parameters are given as $M = 0.1203\text{kg}$, $g = 9.815\text{m/s}^2$, $k = 0.01\text{N}\cdot\text{m/s}$, $L_1 = 0.1\text{H}$, $L_0 = 0.245\text{H}$, $a = 0.0088\text{m}$, and $R_0 = 1.75\Omega$. By diffeomorphism, the following variables are defined:

$$\begin{cases} \sigma_1(t) = x_1(t),\ \sigma_2(t) = x_2(t), \\ \sigma_3(t) = -\dfrac{k}{M}x_2(t) + \dfrac{aL_0}{2M}\dfrac{x_3^2(t)}{(a+x_1(t))^2} - g. \end{cases}$$

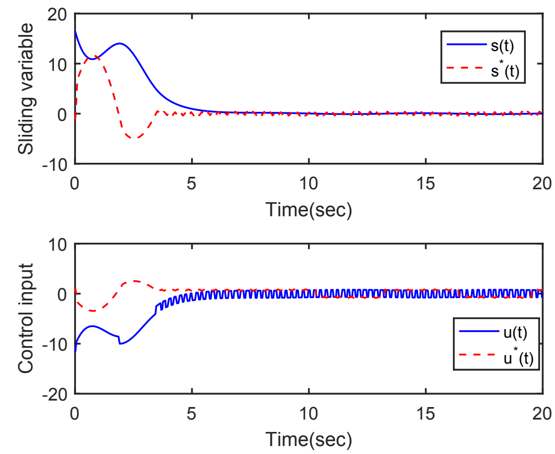

Fig. 1. Comparison of the proposed method with the full-state-information sliding mode controller($n(t) = 0$).

Then, system (44) is equivalently represented as

$$\dot{\sigma}_1(t) = \sigma_2(t),\ \dot{\sigma}_2(t) = \sigma_3(t),\ \dot{\sigma}_3(t) = f(\sigma) + g(\sigma)v(t) \quad (45)$$

where $\sigma(t) = \text{col}\{\sigma_1(t),\ \sigma_2(t),\ \sigma_3(t)\}$, and

$$f(\sigma) = \frac{k^2}{M^2}\sigma_1(t) + \frac{kg}{M} - 2(\frac{R}{L(\sigma_1(t))} + \frac{k}{2M} + \frac{\sigma_2(t)}{a+\sigma_1(t)} - \frac{aL_0\sigma_2(t)}{L(\sigma_1(t))(a+\sigma_1(t))})\cdot(\sigma_3(t) + \frac{k}{M}\sigma_2(t) + g),$$

$$g(\sigma) = \frac{2aL_0(\sigma_3(t) + \frac{k}{M}\sigma_2(t) + g)}{M^{\frac{1}{2}}L(\sigma_1(t))(a+\sigma_1(t))}.$$

Applying $v(t) = -g^{-1}(\sigma(t))[f(\sigma(t)) + u(t)]$ to system (46) yields the following perturbed triple integrator:

$$\dot{\sigma}_1(t) = \sigma_2(t),\ \dot{\sigma}_2(t) = \sigma_3(t),\ \dot{\sigma}_3(t) = u(t) + d(x,t) \quad (46)$$

where $v(t)$ is the virtual control input to be designed, and $d(x,t)$ is a perturbation caused by external signals or parameter or model uncertainties. The considered domain is defined as $|x(t)| \leq |x_1(t)|+|x_2(t)|+|x_3(t)| \leq \psi$, with $\psi = 20$. Then, $d(x,t) = 0.5\cos(t) + 0.5\sin(t)$, and $d^* = 1$. Only the noisy measurement of $\sigma_1(t)$ is available for control purpose, which is denoted as $f(t) = \sigma_1(t) + n(t)$, where $n(t)$ is the noises satisfying $|n(t)| \leq \varepsilon$, and ε is a positive scalar.

A. Design of output sliding mode controller via artificial time-delay estimation

We characterize the initial values as $x_1(0) = 3,\ x_2(0) = -3,\ x_3(0) = 3$. The design steps are listed:

1) Based on the interested domain, we select $k_1 = 13.2$ to satisfy that $|(A-B_1J_2^{-1}J_1)x| \leq |0.51|\cdot\psi \leq k_1$.
2) Choose $\bar{C} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$ with $c_1 = 1.3$, $c_2 = 2.45$, $c_3 = 1$. Then, sliding variables $s^*(t)$ and $s(t)$ are
$$s^*(t) = 1.3x_1(t) + 2.45x_2(t) + x_3(t),\ \ s(t) = \bar{C}_d\hat{x}(t,h),$$
where $\hat{x}(t,h) = \text{col}\{x_1(t),\ x_1(t-h),\ x_1(t-2h)\}$, $c_{d1} = c_1 + 1.5c_2/h + c_3/h^2$, and
$$\bar{C}_d = \begin{bmatrix} c_{d1} & -2c_2/h - 2c_3/h^2 & 0.5c_2/h + c_3/h^2 \end{bmatrix}.$$

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3) By solving LMIs of Theorem 1, the feasible solutions are obtained as $P_3 = 0.976,\ X = 0.568,\ W = 3.124$, and

$$P_1 = \begin{bmatrix} 0.876 & 0.622 \\ * & 1.456 \end{bmatrix}, \quad P_2 = \begin{bmatrix} -0.234 & 1.723 \end{bmatrix}.$$

Meanwhile, we find that $h^* = 0.87$.

4) Choose $k_0 = 0.5$, $k = 2$ and $\beta = 9$ such that the constraints (39) and (40) simultaneously hold.
5) From (36), the output sliding mode controller via artificial time-delay estimation is written as

$$u(t) = -2\mathrm{sgn}(s(t)) + g_1 x_1(t) + \sum_{i=2}^{3} g_i x_1(t-(i-1)h)$$

and $g_1 = (-10 - 10/h)$, $g_2 = 10/h + 10/h^2$, $g_3 = -5/h^2$, and the value of h should satisfy condition (35).

Inspired from [1], the control law for the case of the differentiator or sliding mode observer is written as

$$u^*(t) = -\frac{k_c}{2.7879}\mathrm{sgn}(s^*(t)) - C_c\hat{x}(t)$$

where $k_c = 6$, $C_c = \begin{bmatrix} 0.7609 & 1.8913 & 1 \end{bmatrix}$, and $\hat{x}(t) = \mathrm{col}\{f(t), \hat{x}_2(t), \hat{x}_3(t)\}$. States $x_2(t)$ and $x_3(t)$ are estimated via a robust exact differentiator/sliding mode observer. Fig.1 reveals the trajectories of $s^*(t)$, $s(t)$, $u(t)$ and $u^*(t)$, which shows that the artificial time-delay sliding surface can approximate the linear sliding surface with the acceptable precision.

B. Comparison of the artificial time-delay estimation with the robust differentiator/sliding mode observer

We use a robust exact differentiator (RED) [15]:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) - 3L_d^{\frac{1}{3}} e_d^{\frac{2}{3}}(t)\mathrm{sgn}(e_d(t)) \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) - 1.5L_d^{\frac{1}{2}} e_d^{\frac{1}{2}}(t)\mathrm{sgn}(e_d(t)) \\ \dot{\hat{x}}_3(t) = -1.1L_d\mathrm{sgn}(e_d(t)) \end{cases}$$

where $e_d(t) = \hat{x}_1(t) - f(t)$, and $L_d \geq d^* + \sup|u|$ ([15]). Moreover, a higher order sliding mode observer (HOSMO) is used:

$$\begin{cases} \dot{\hat{z}}_1(t) = \hat{z}_2(t) + 3L_o^{\frac{1}{3}} e_o^{\frac{2}{3}}(t)\mathrm{sgn}(e_o(t)) \\ \dot{\hat{z}}_2(t) = \hat{z}_3(t) + 1.5L_o^{\frac{1}{2}} e_o^{\frac{1}{2}}(t)\mathrm{sgn}(e_o(t)) \\ \dot{\hat{z}}_3(t) = 1.1L_o\mathrm{sgn}(e_o(t)) + u^*(t) \end{cases}$$

where $e_o(t) = \hat{z}_1(t) - f(t)$, and $L_o \geq d^*$ (see [22]). Here, we choose $L_d = 250$ ([15]), and $L_o = 50$, because HOSMO takes into account the known part of MLS dynamics ([22]). When $n(t) = 0$, Fig. 2 reveals that RED and achieve the better tracking precision of system dynamics than the artificial time-delay estimator without measurement noises. Apparently, RED/HOSMO have faster response rate than the proposed one, because RED and HOSMO adopt the dynamic output-feedback structure, while the artificial time-delay estimator is of static structure.

Next, sinuous measurement noises will be imposed in $x_1(t)$ for the the comparative simulations. They are assumed to be in the form of $n(t) = A_n \sin(f_n t)$, where A_n is the amplitude of noises and f_n is the noise frequency. Our method is very flexible, because h will be chosen with respect to

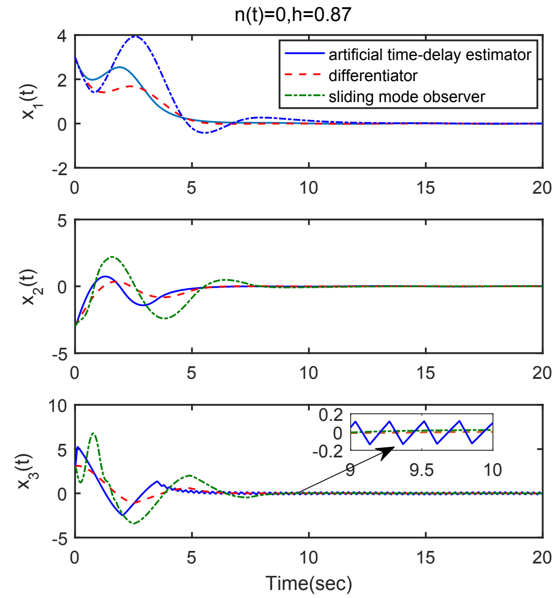

Fig. 2. Comparison of the proposed method with the RED and HOSMO ($n(t) = 0$).

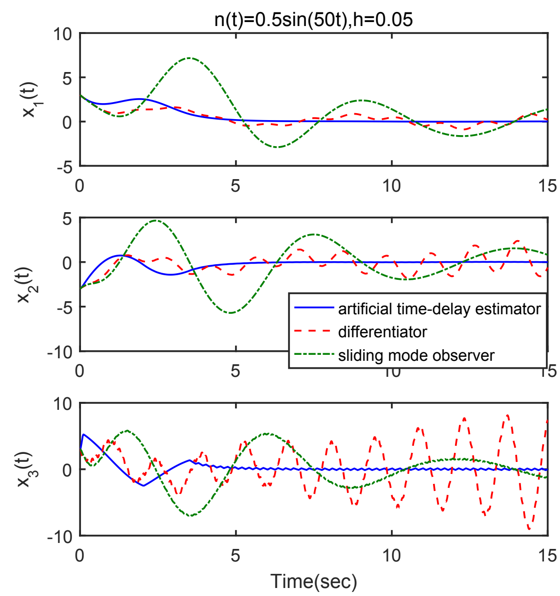

Fig. 3. Comparison of the proposed method with the RED and HOSMO ($n(t) = 0.5\sin(50t)$).

f_n. Here, noises with different frequencies are, respectively, imposed on the measurement $x_1(t)$: $n(t) = 0.5\cos(50t)$ and $n(t) = 0.1\cos(1000t)$. For the selection of h, the trade-off will be made between the approximation accuracy and filtration against noises. According to condition (35), we choose $h = 0.05$ and $h = 0.01$, respectively, for $n(t) = 0.5\cos(50t)$ and $n(t) = 0.5\cos(1000t)$. Figs. 3–4 illustrate that the artificial time-delay estimator has certain filtering quantities against the measurement noises and can achieve a better robustness than the RED/HOSMO. Moreover, the proposed artificial time-delay estimator is of static output-feedback structure, which

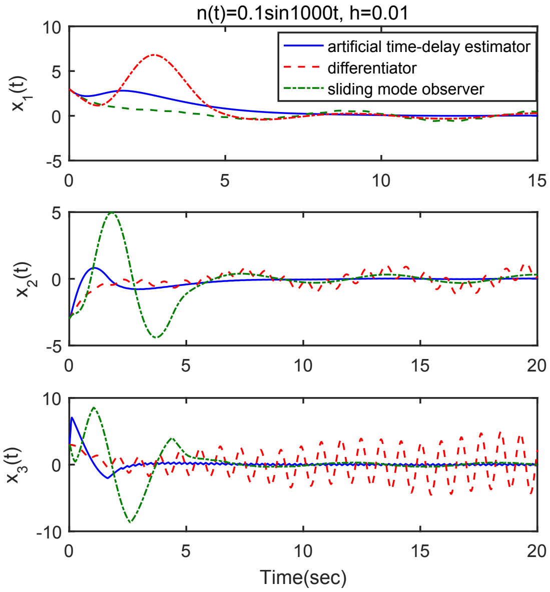

Fig. 4. Comparison of the proposed method with the RED and HOSMO ($n(t) = 0.1\sin(1000t)$).

does not require introducing additional dynamics for state estimation.

V. Conclusion

In this paper, the static output-feedback sliding mode controller for systems with relative degree n has been introduced, where the output derivatives up to the order $n-1$ are approximated by using the current and the delayed values of the output. First we design a delayed sliding surface, and then prove its finite-time attractivity. Lyapunov-Krasovskii functional is suggested to achieve the practical stabilization. The design parameters are chosen for a good trade-off between the approximation accuracy in the presence of measurement noises and the reduction of controller gain. A simulation example is given to show the merits of the proposed design method.

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