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H_∞ Control of Linear Uncertain Time-Delay Systems—A Projection Approach

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***Abstract*—The issues of stability and H_∞ control of linear systems with time-varying delays are considered. Based on the Lyapunov–Krasovskii approach and on Finsler's projection lemma, delay-dependent sufficient conditions are obtained, in terms of linear matrix inequalities (LMIs), for the stability of these systems. These conditions generalize previous results that were derived using either the descriptor approach or the first and the third model transformations. The obtained criteria are extended to deal with: stabilizability, the bounded real lemma and the H_∞ state-feedback control.**

***Index Terms*—Lyapunov–Krasovskii approach, neutral systems, robust H_∞ control, time-delay systems.**

I. Introduction

During the last decade, a considerable amount of attention has been paid to stability and control of linear systems with uncertain delays (either constant or time-varying) lying in the given segment $[0, h]$ (see, e.g., [1]–[6] and the references therein). The so-called delay-dependent sufficient stability conditions in terms of linear matrix inequalities (LMIs) have been derived by using Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions (the latter is usually more conservative). Delay-dependent conditions via Lyapunov–Krasovskii functionals are based on different model transformations. Each model transformation leads to a corresponding form of Lyapunov–Krasovskii functional. The third model transformation (according to the classification of [2]), which was applied in [7] and [8], and the most recent and less conservative one, the descriptor representation of the system [4]–[6], lead to the same Lyapunov–Krasovskii functional depending on the derivatives of the state. The derivative of this functional is, however, different in the two approaches, where in the descriptor approach both the state vector and its derivative appear in the expression for the derivative of the Lyapunov–Krasovskii functional along the trajectories of the system.

For systems without delays the LMI stability conditions are obtained either by directly differentiating the quadratic Lyapunov function along the system trajectory [9] or by applying Finsler's lemma [10]. It turns out that the LMIs that are obtained by the latter two methods are equivalent in the case without uncertainty, but since the LMIs that are based on Finsler's lemma possess more degrees of freedom they provide better results in the case where parameter uncertainty is encountered [10].

It is shown in the present note that similar improvement is achieved when applying Finsler's lemma to the robust analysis and design of retarded and neutral systems. The sufficient conditions that are obtained for testing stability and for the bounded real lemma (BRL) are more general than the results achieved based on the descriptor approach or on the first and the third model transformations (see the classification of [2]). The latter results are obtained as a special case of the new conditions by taking few of the additional free matrix parameters to be zero. Moreover, for the first time, it is theoretically proved that the descriptor approach conditions are generalization of the conditions based on the first and the third model transformations and, thus, less conservative. Utilizing the geometric structure of the resulting inequalities, for these special cases, the results of [4]–[6] are obtained by solving LMIs with fewer decision variables, where there is no longer a need to find all the matrix blocks of the Lyapunov's kernel matrix explicitly.

A new effective method for state-feedback design is introduced. The merit of the new results lies not only in the fact that it provides another geometric approach to the analysis and the synthesis of retarded systems and that it reduces the complexity of the resulting LMIs. The main merit of the proposed method is the fact that it provides additional degrees of freedom which, similar to the case without delay, lead to less conservative results when uncertainty of the polytopic type is encountered. Some effort of applying Finsler's lemma to the case of systems with time delay has been recently made. A generalization of [8] was obtained in [11] where the elimination lemma was used to generalize the results of [8] that are based on the third model transformation. A delay-independent stability conditions via Finsler's lemma have been derived recently in [12].

II. Stability

Consider the following system (the system can be extended to include more delays):

$$\begin{aligned}\dot{x}(t) - F\dot{x}(t-g) &= A_0x(t) + A_1x(t-\tau(t)),\ t \geq t_0\\ x(\theta) &= \phi(\theta),\ \theta \in \mathcal{E}_{t_0}\end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the system state, A_0, A_1 and F are constant $n \times n$-matrices, t_0 is a given initial time, ϕ is a continuously differentiable initial function and $\mathcal{E}_{t_0} = \{\theta \in \mathcal{R} : \theta = \eta - \tau(\eta) \leq t_0,\ \eta \geq t_0\} \cup [t_0 - g,\ t_0]$. It is assumed that g is a known constant delay and that the delay $\tau(t)$ is a bounded differentiable function that satisfies

$$0 \leq \tau \leq h,\ \dot{\tau}(t) \leq d < 1. \tag{2}$$

Moreover, it is assumed that all the eigenvalues of F are inside the unit circle. The latter guarantees that the difference equation $x(t) - Fx(t-$

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$g) = 0$ is asymptotically stable for all g [13]. Similar to [3, Sec. 5.5.], results will be delay-independent in g and dependent in h and d. For $g = \tau$ (usually such models appear in the applications), one can apply the results with $d = 0$.

Consider also the Lyapunov–Krasovski functional

$$V(t) = x^T(t)P_1x(t) + V_2 + V_3 + V_4 \quad (3)$$

where

$$V_2 = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta,\ V_3 = \int_{t-g}^{t} \dot{x}^T(s)U\dot{x}(s)ds$$
$$V_4 = \int_{t-\tau}^{t} x^T(s)Sx(s)ds$$

and where $P_1, R, U, andS$ are positive–definite matrics. Differentiating (3), with respect to t, we require

$$\dot{V} = [\,x^T(t) \quad \dot{x}^T(t)\,]^T \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 < 0 \quad (4)$$

for all $x(t)$ that satisfy (1). If (4) holds, then (1) is asymptotically stable (see [13, pp. 336–337]). We define: $\xi = col\{x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-g)\}$ and rewrite (4) in the following form:

$$\dot{V} = \xi^T \begin{bmatrix} 0 & P_1 & 0 & 0 \\ P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xi + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 < 0$$
$$\forall \xi \in R^{4n} \text{ s.t. } [A_0, -I, A_1, F]\,\xi = 0. \quad (5)$$

To (5), the following version of Finsler's lemma is applied.

Lemma 1: [9] The following statement holds:

$x^TQx + f(x) < 0, \forall \bar{B}x = 0, x \neq 0$, where Q is a symmetric matrix, $\bar{B} \in \mathcal{R}^{m\times n}$ and $f(x)$ is a scalar function, if there exists $X \in R^{n\times m}$ such that: $x^T[Q + X\bar{B} + \bar{B}^TX^T]x + f(x) < 0, \forall x \neq 0$.

In the sequel, we also use the following bounding result [8].

Lemma 2: For any $a \in \mathcal{R}^n, b \in \mathcal{R}^{2n}, \mathcal{N} \in \mathcal{R}^{2n\times n}, R \in \mathcal{R}^{n\times n}, Y \in \mathcal{R}^{n\times 2n}, Z \in \mathcal{R}^{2n\times 2n}$, the following holds:

$$-2b^T\mathcal{N}a \le \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} R & Y - \mathcal{N}^T \\ Y^T - \mathcal{N} & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix},\ \begin{bmatrix} R & Y \\ Y^T & Z \end{bmatrix} \ge 0. \quad (6)$$

Lemma 1 can be used with

$$x = \xi \quad Q = \begin{bmatrix} 0 & P_1 & 0 & 0 \\ P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\bar{B} = [A_0, -I, A_1, F] \quad X = [P_2, P_3, P_4, P_5]^T.$$

Carrying out the multiplications and substituting for $\dot{V}_2$, $\dot{V}_3$, $\dot{V}_4$ (7a), as shown at the bottom of the page, is obtained, where

$$\mu(t) \triangleq \xi^T \begin{bmatrix} P_2^T \\ P_3^T \\ 0 \\ 0 \end{bmatrix} [0 \quad 0 \quad A_1 \quad 0]\xi + \xi^T \begin{bmatrix} 0 \\ 0 \\ A_1^T \\ 0 \end{bmatrix} [\,P_2 \quad P_3 \quad 0 \quad 0\,]\,\xi - \int_{t-h}^{t} \dot{x}(s)^TR\dot{x}(s)ds. \quad (7b)$$

Since

$$\xi^T \begin{bmatrix} P_2^T \\ P_3^T \\ 0 \\ 0 \end{bmatrix} [0 \quad 0 \quad A_1 \quad 0]\xi = \xi^T \begin{bmatrix} P_2^T \\ P_3^T \\ 0 \\ 0 \end{bmatrix} A_1x(t) - \int_{t-\tau}^{t} \xi(t)^T \begin{bmatrix} P_2^T \\ P_3^T \\ 0 \\ 0 \end{bmatrix} A_1\dot{x}(s)ds$$

we have that

$$\mu(t) = 2\xi^T \begin{bmatrix} P_2^T \\ P_3^T \\ 0 \\ 0 \end{bmatrix} A_1x(t) - 2\int_{t-\tau}^{t} \xi(t)^T \begin{bmatrix} P_2^T \\ P_3^T \\ 0 \\ 0 \end{bmatrix} A_1\dot{x}(s)ds - \int_{t-h}^{t} \dot{x}(s)^TR\dot{x}(s)ds. \quad (8)$$

We apply Lemma 2 to the expression we have obtained above for $\mu(t)$. This is done by taking in (6) $\mathcal{N} = [\,P_2 \quad P_3\,]^T A_1$, $a = \dot{x}(s)$ and $b = \text{col}\{x(t), \dot{x}(t)\}$. We obtain

$$\mu(t) \le 2x^T(t)Y \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} - 2x^T(t-\tau)\left[Y - A_1^T\,[\,P_2 \quad P_3\,]\right] \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + h\,[\,x^T(t) \quad \dot{x}^T(t)\,]\,Z \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}. \quad (9)$$

Denoting $Y = [\,Y_1 \quad Y_2\,]$ and $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{bmatrix}$ the following is then obtained.

Theorem 1: System (1) is uniformly (with respect to t_0) asymptotically stable for a delay that satisfies (2) if there exist $n \times n$ matrices $0 < P_1, 0 < S, P_i, i = 2, \ldots, 5, Y_1, Y_2, Z_1, Z_2, Z_3, 0 < U$ and $0 < R$ that satisfy the LMIs shown in (10a) at the bottom of the page, and

$$\begin{bmatrix} R & Y_1 & Y_2 \\ * & Z_1 & Z_2 \\ * & * & Z_3 \end{bmatrix} > 0. \quad (10b)$$

$$\xi^T \begin{bmatrix} P_2^TA_0 + A_0^TP_2 + S & P_1 - P_2^T + A_0^TP_3 & A_0^TP_4 & P_2^TF + A_0^TP_5 \\ * & -P_3^T - P_3 + U + hR & -P_4 & P_3^TF - P_5 \\ * & * & -S(1-d) + A_1^TP_4 + P_4^TA_1 & P_4^TF + A_1^TP_5 \\ * & * & * & P_5^TF + F^TP_5 - U \end{bmatrix} \xi + \mu < 0 \quad (7a)$$

$$\begin{bmatrix} P_2^TA_0 + A_0^TP_2 + Y_1 + Y_1^T + S + hZ_1 & P_1 - P_2^T + A_0^TP_3 + Y_2 + hZ_2 & -Y_1^T + P_2^TA_1 + A_0^TP_4 & P_2^TF + A_0^TP_5 \\ * & -P_3^T - P_3 + U + h(R + Z_3) & -Y_2^T + P_3^TA_1 - P_4 & P_3^TF - P_5 \\ * & * & -S(1-d) + A_1^TP_4 + P_4^TA_1 & P_4^TF + A_1^TP_5 \\ * & * & * & P_5^TF + F^TP_5 - U \end{bmatrix} < 0 \quad (10a)$$

Remark 1: The case where $\tau(t)$ is a continuous arbitrarily time-varying function, satisfying for all $t \geq 0$, $0 \leq \tau(t) \leq h$,is solved by choosing, in Theorem 1, $S \rightarrow 0$.

Remark 2: The two stability conditions in Theorem 1 can be joined into one LMI by applying Schur complements formula to (10b), replacing the term hZ_1 in (10a) by $h[Y_1 \;\; Y_2]^T R^{-1}[Y_1 \;\; Y_2]$. The latter can then be transformed by Schur formula to additional column and row in (10a) thus producing a single LMI.

Inequality (10a) can be written as

$$\Xi + \begin{bmatrix} A_0^T \\ -I \\ A_1^T \\ F^T \end{bmatrix} [P_2 \quad P_3 \quad P_4 \quad P_5] + \begin{bmatrix} P_2^T \\ P_3^T \\ P_4^T \\ P_5^T \end{bmatrix} [A_0 \quad -I \quad A_1 \quad F] < 0 \quad (11)$$

where

$$\Xi = \begin{bmatrix} Y_1 + Y_1^T + S + hZ_1 & P_1 + Y_2 + hZ_2 & -Y_1^T & 0 \\ * & U + h(R + Z_3) & -Y_2^T & 0 \\ * & * & -S(1-d) & 0 \\ * & * & * & -U \end{bmatrix}.$$

Using Lemma 1 it is readily seen that there exist matrices P_2, P_3, P_4, P_5 that solve (10a) iff the following LMI has a solution:

$$\mathcal{N}_A^T \Xi \mathcal{N}_A < 0 \quad (12)$$

where we denote the full-rank matrix representations of the right annihilator of $[\, A_0 \quad -I \quad A_1 \quad F\,]$ by $\mathcal{N}_A$. Since

$$\mathcal{N}_A = \begin{bmatrix} I & 0 & 0 \\ A_0 & A_1 & F \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we obtain the following.

Theorem 2: The conditions of Theorem 1 are satisfied iff there exist $n \times n$ matrices $0 < P_1$, $0 < S$, Y_1, Y_2, Z_1, Z_3, X, $\bar{U}$ and R that satisfy the LMIs shown in (13a)–(13b) at the bottom of the page, where $\bar{U} \triangleq U + h(R + Z_3)$ and $X \triangleq P_1 + Y_2 + hZ_2$.

Remark 3: The LMIs of the latter theorem are independent of P_i, $i = 2, \ldots, 5$. They thus involve a smaller number of decision variables.

A. Comparison With Other Methods

The above approach generalizes the main existing methods for delay-dependent stability such as: model transformation 1 [1], model transformation 3 [7], [8] and the descriptor approach [4]–[6]. In the sequel we show that the results of the latter three approaches are special cases of Theorems 1 and 2. We also discuss what are the extra degrees of freedom offered by the new approach.

The Descriptor Model Transformation: It is readily seen that by choosing $P_4 = P_5 = 0$, the LMIs of Theorem 1 are identical to those obtained in [4]. The question arises, however, to what an extent the introduction of the additional decision variables P_4 and P_5 in Theorem 1 lead to results that are less conservative than those obtained by the descriptor approach (where P_4 and P_5 are zero). Rewriting the stability condition of [4] in the following way:

$$\Xi + \begin{bmatrix} A_0^T \\ -I \\ A_1^T \\ F^T \end{bmatrix} [\, P_2 \quad P_3 \quad P_4 \quad P_5\,] \, \mathrm{diag}\{I, I, 0, 0\} + \mathrm{diag}\{I, I, 0, 0\} \begin{bmatrix} P_2^T \\ P_3^T \\ P_4^T \\ P_5^T \end{bmatrix} [\, A_0 \quad -I \quad A_1 \quad F\,] < 0 \quad (14)$$

where Ξ is as in (11), we use Lemma 1 and find that there exist matrices P_2, P_3, P_4, P_5 that solve (14) iff (10b) and the following LMIs in Y_1, Y_2, Z_i, $i = 1, \ldots, 3$, U, and R have a solution

$$\mathcal{N}_A^T \Xi \mathcal{N}_A < 0 \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}^T \Xi \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} < 0. \quad (15ab)$$

Carrying out multiplication in (15b), we obtain

$$\begin{bmatrix} -(1-d)S & 0 \\ 0 & -U \end{bmatrix} < 0.$$

This inequality is redundant because it is a part of the LMI in (15a). Since (15a) is equivalent to the LMI in Theorem 1 no improvement can thus be obtained by the new method in the case where the parameters of the system are all known. It is shown, however, in Example 1 below that the new approach has a considerable advantage in the case where polytopic type parameter uncertainty is encountered.

Model Transformations 1 and 3: We choose [8] to be one of the recent publication that is based on the third model transformation. In order to allow comparison with the results of [8], we consider retarded system with constant unknown delay (namely, we take $F = 0$ and $d = 0$ in (1)). Choosing $Y_2 = 0$, $Z_2 = 0$, $X = P_1$, $U = hR$ and $Z_3 = 0$ in (13), we obtain the following LMI condition for stability that is identical to the one that appears in [8, Th. 1]:

$$\begin{bmatrix} A_0^T P_1 + P_1 A_0 + Y_1 + Y_1^T + S & -Y_1 + P_1 A_1 & hA_0^T R & hY_1^T \\ * & -S & hA_1^T R & 0 \\ * & * & -hR & 0 \\ * & * & * & -hR \end{bmatrix} < 0. \quad (16)$$

It is shown in [18] that the results of the first model transformation, as expressed in [1], are a special case of the third model transformation results of [7]. Since [8] is a generalization of [7], we conclude that Theorem 1 and Theorem 2 of the present note are generalizations of the results obtained by the first and the third model transformations.

III. The BRL

We consider the following system:

$$\begin{aligned} \dot{x} - F\dot{x}(t-g) &= A_0 x(t) + A_1 x(t - \tau(t)) + Bu(t) + B_1 w(t) \\ x(t) &= 0, \; t \leq 0 \\ z(t) &= Cx(t) \end{aligned} \quad (17ab)$$

$$\begin{bmatrix} Y_1 + Y_1^T + S + hZ_1 + XA_0 + A_0^T X^T & -Y_1^T + XA_1 - A_0^T Y_2^T & XF & A_0^T \bar{U} \\ * & -S(1-d) - Y_2 A_1 - A_1^T Y_2^T & -Y_2 F & A_1^T \bar{U} \\ * & * & -\bar{U} + h(R + Z_3) & F^T \bar{U} \\ * & * & * & -\bar{U} \end{bmatrix} < 0 \quad (13a)$$

$$\begin{bmatrix} R & Y_1 & hY_2 \\ * & Z_1 & X - P_1 - Y_2 \\ * & * & h^2 Z_3 \end{bmatrix} > 0 \quad (13b)$$

with τ that satisfies (2), where $u(t) \in \mathcal{R}^p$ is the control input, $w(t) \in \mathcal{R}^q$ is an arbitrary disturbance vector in $L_2[0\ \infty)$, $z(t) \in \mathcal{R}^m$ is the objective vector. We take $B = 0$. For a prescribed scalar $\gamma > 0$, we define the performance index

$$J = \int_0^\infty (z^T(s)z(s) - \gamma^2 w^T(s)w(s))ds. \tag{18}$$

Using the arguments of the previous section we apply the Lyapunov–Krasovskii functional of (3) for $\xi = col\{x, \dot{x}, x(t-\tau), \dot{x}(t-g), w(t)\}$ and require

$$\begin{aligned} &\dot{V} + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\ &= \xi^T \begin{bmatrix} C^TC & P_1 & 0 & 0 & 0 \\ P_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma^2 I_q \end{bmatrix} \xi + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 < 0 \\ &\forall \xi \text{ s.t. } [A_0, -I, A_1, F, B_1]\,\xi = 0. \end{aligned} \tag{19}$$

Similarly to the derivation of Theorem 1, we thus obtain the following.

Theorem 3: System (17) with $B = 0$ is asymptotically stable for all delays that satisfy (2) and, for a prescribed scalar γ, $J < 0\ \forall\ w(t) \in L_2[0\ \infty)$ if there exist $n \times n$ matrices $0 < P_1$, $0 < S$, $P_i, i = 2, \ldots, 5$, Y_1, Y_2, Z_1, Z_2, Z_3, $0 < U$, R and P_6 of the appropriate dimensions that satisfy (10b) and (20), as shown at the bottom of the page.

IV. H_∞ State-Feedback Control

The aforementioned results are most suitable for stability analysis. In order to apply these results to synthesis problems where a state-feedback is sought that stabilizes the system and achieves a prescribed performance bound we consider the control problem for $B \neq 0$ where the objective vector z is defined as

$$z(t) = Cx(t) + D_{12}u(t). \tag{21}$$

We apply the latter BRL, taking $P_i = \epsilon_{i-2}P_2$, $i = 3, \ldots, 5$, $P_6 = \epsilon_4 B_1^T$, where ϵ_i, $i = 1, 2, 3$ are tuning scalar parameters. Note that P_2 is nonsingular due to the fact that the only matrix which can be negative definite in the second block on the diagonal of (20) is $-\epsilon_1(P_2 + P_2^T)$. Defining

$$\begin{aligned} \bar{P} = P_2^{-1}, \quad [\,\bar{P}_1\ \ \bar{Y}_1\ \ \bar{Y}_2\ \ \bar{S}\ \ \bar{U}\ \ \bar{R}\ \ \bar{Z}_1\ \ \bar{Z}_2\ \ \bar{Z}_3\,] = \\ \bar{P}^T[P_1\bar{P}\ \ Y_1\bar{P}\ \ Y_2\bar{P}\ \ S\bar{P}\ \ U\bar{P}\ \ R\bar{P}\ \ Z_1\bar{P}\ \ Z_2\bar{P}\ \ Z_3\bar{P}] \end{aligned}$$

and $\hat{Y} = K\bar{P}$, and multiplying (20) by $\mathrm{diag}\{\bar{P}, \bar{P}, \bar{P}, \bar{P}, I_q, I_m\}$ and its transpose, from the right and the left, respectively, the following is obtained:

Theorem 4: Under the feedback law $u(t) = Kx(t)$, the system of (17) and (21) is asymptotically stable for all delays that satisfy (2) and for a prescribed scalar γ, $J < 0\ \forall\ w(t) \in L_2[0\ \infty)$ if for some tuning scalar parameters ϵ_i, $i = 1, \ldots, 4$ there exist $n \times n$ matrices $0 < \bar{P}_1$, $0 < \bar{S}$, $\bar{P}, \bar{Y}_1, \bar{Y}_2, \bar{Z}_1, \bar{Z}_2, \bar{Z}_3$, $0 < \bar{U}$, $\bar{R}$, and $\hat{Y} \in \mathcal{R}^{p\times n}$ that satisfy

$$\Psi + \Psi^T < 0 \quad \text{and} \quad \begin{bmatrix} \bar{R} & \bar{Y}_1 & \bar{Y}_2 \\ * & \bar{Z}_1 & \bar{Z}_2 \\ * & * & \bar{Z}_3 \end{bmatrix} > 0 \tag{22ab}$$

where (22c), as shown at the bottom of the page, holds. The state-feedback gain matrix is then given by

$$K = \hat{Y}\bar{P}^{-1}. \tag{22d}$$

In Theorem 4, the solution was obtained by applying the BRL to the system (17). Another solution to the problem is obtained using the adjoint system of (17), in the case where the delay does not vary in time. We consider the following "forward adjoint" of (17) with $B = 0$ (see, e.g., [14] for the case of constant delays):

$$\begin{aligned} \dot{\zeta} - F^T\dot{\zeta}(t-g) &= A_0^T\zeta(t) + A_1^T\zeta(t-\tau) + C^Tv(t),\ t \geq 0 \\ x(t) &= 0,\ t \leq 0 \\ \mu(t) &= B_1^T\zeta(t). \end{aligned} \tag{23}$$

By applying standard Laplace transform arguments, it is easily verified that the H_∞ norms of (17), with $B = 0$, and of (23) are identical. Theorem 3 can thus be applied to (23) to produce an alternative BRL.

Once the BRL is derived for the system (17) with $B = 0$ we consider the control problem for $B \neq 0$ where the objective vector z is defined in (21). We apply the latter BRL, taking $P_i = \epsilon_{i-2}P_2$, $i = 3, \ldots, 5$, $P_6 = \epsilon_4 P_2 C^T$, where ϵ_i, $i = 1, 2, 3$ are tuning scalar parameters, and defining $\bar{Y} = KP_2$. We obtain the following.

$$\begin{bmatrix} P_2^TA_0 + A_0^TP_2 + Y_1 + Y_1^T + S + hZ_1 & P_1 - P_2^T + A_0^TP_3 + Y_2 + hZ_2 & -Y_1^T + P_2^TA_1 + A_0^TP_4 & P_2^TF + A_0^TP_5 & P_2^TB_1 + A_0^TP_6 & C^T \\ * & -P_3^T - P_3 + U + h(R+Z_3) & -Y_2^T + P_3^TA_1 - P_4 & P_3^TF - P_5 & P_3^TB_1 - P_6 & 0 \\ * & * & -S(1-d) + A_1^TP_4 + P_4^TA_1 & P_4^TF + A_1^TP_5 & P_4^TB_1 + A_1^TP_6 & 0 \\ * & * & * & P_5^TF + F^TP_5 - U & P_5^TB_1 + F^TP_6 & 0 \\ * & * & * & * & -\gamma^2 I_q + P_6^TB_1 + B_1^TP_6 & 0 \\ * & * & * & * & * & -I_m \end{bmatrix} < 0 \tag{20}$$

$$\Psi = \begin{bmatrix} A_0\bar{P} + B\hat{Y} + \bar{Y}_1 + \frac{1}{2}\bar{S} + \frac{h}{2}\bar{Z}_1 & \bar{P}_1 - \bar{P} + \bar{Y}_2 + h\bar{Z}_2 & A_1\bar{P} & F\bar{P} & B_1 & 0 \\ \epsilon_1(A_0\bar{P} + B\hat{Y}) & -\epsilon_1\bar{P} + \frac{1}{2}\bar{U} + \frac{h}{2}(\bar{R} + \bar{Z}_3) & \epsilon_1A_1\bar{P} & \epsilon_1F\bar{P} & \epsilon_1B_1 & 0 \\ -\bar{Y}_1 + \epsilon_2(A_0\bar{P} + B\hat{Y}) & -\epsilon_2\bar{P} - \bar{Y}_2 & -\frac{1}{2}S(1-d) + \epsilon_2A_1\bar{P} & \epsilon_2F\bar{P} & \epsilon_2B_1 & 0 \\ \epsilon_3(A_0\bar{P} + B\hat{Y}) & -\epsilon_3\bar{P} & \epsilon_3A_1\bar{P} & \epsilon_3F\bar{P} - \frac{1}{2}\bar{U} & \epsilon_3B_1 & 0 \\ \epsilon_4B_1^T(A_0\bar{P} + B\hat{Y}) & -\epsilon_4B_1^T\bar{P} & \epsilon_4B_1^TA_1\bar{P} & \epsilon_4B_1^TF\bar{P} & -\frac{1}{2}\gamma^2I_q + \epsilon_4B_1^TB_1 & 0 \\ C\bar{P} + D_{12}\hat{Y} & 0 & C\bar{P} + D_{12}\hat{Y} & 0 & 0 & -\frac{1}{2}I_m \end{bmatrix} \tag{22c}$$

Theorem 5: Under the feedback law $u(t) = Kx(t)$, the system (17) is asymptotically stable for all constant $0 \le \tau \le h$ and for a prescribed scalar γ, $J_\infty < 0 \ \forall \ w(t) \in L_2[0 \ \infty)$ if for some tuning scalar parameters ϵ_i, $i = 1, \ldots, 4$ there exist $n \times n$ matrices $0 < P_1$, $0 < S, P_2, Y_1, Y_2, Z_1, Z_2, Z_3, 0 < U$, R, *and* $\bar{Y} \in \mathcal{R}^{p \times n}$ that satisfy (10b) and

$$\Psi + \Psi^T < 0 \tag{24a}$$

where (24b), as shown at the bottom of the page, holds. The state-feedback gain matrix is then given by:

$$K = \bar{Y} P_2^{-1}. \tag{24c}$$

Remark 4: The matrix P_2 is nonsingular due to the fact that the only matrix which can be negative definite in the second block on the diagonal of (24a) is $-\epsilon_1(P_2 + P_2^T)$.

In Example 2 bellow it is found that the adjoint based approach provided a significantly less conservative result. This approach is, however, limited to the case of constant delays.

Remark 5: Theorem 2 led us to new simplified and more efficient (see Example 2) synthesis procedures of Theorems 4 and 5. The understanding that P_2 and P_3 are less important made it possible to assume that these matrices are proportional and to apply in the design procedure only $\bar{P}_1$ and $\bar{P} = P_2^{-1}$ (see Theorem 4), or P_1 and P_2 in the adjoint based results (Theorem 5). An alternative approach was suggested in [5], where the LMIs that were derived from (20), were multiplied by $\mathrm{diag}\{P^{-1}, \ I\}$ and its transpose, from the right and the left, respectively, where $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$.

Remark 6: The results of Theorems 4 and Theorem 5 apply the tuning scalar parameters ε_i, $i = 1, \ldots, 4$. The question arises how to find the optimal combination of these parameters. One way to address the tuning issue is to choose for a cost function the parameter γ and to apply a numerical optimization algorithm, such as the program **fminsearch** in the optimization toolbox of Matlab, to the latter cost function. A locally convergent solution to the tuning problem is thus obtained.

Remark 7: The above results assume that the parameters of the system are all known. They are easily applicable, however, also to the case where the system encounters a parameter uncertainty of the polytopic type [9]. Since the LMIs of (22) and (24) are affine in the system matrices, the results of Theorems 4 and 5 can be used to derive a criterion that will guarantee the required attenuation level in the case where the system matrices are not exactly known and they reside within a given polytope.

Denoting: $\Omega = \begin{bmatrix} A_0 & A_1 & F \\ B & B_1 & C \end{bmatrix}$ we assume that

$$\Omega \in Co\{\Omega_j, \quad j = 1, \ldots N\}$$

namely

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j, \quad \text{for some } \ 0 \le f_j \le 1, \sum_{j=1}^{N} f_j = 1$$

where the N vertices of polytope are described by

$$\Omega_j = \begin{bmatrix} A_0^j & A_1^j & F^j \\ B^j & B_1 & C^j \end{bmatrix}.$$

We readily obtain that considering (17), where the system matrices reside within the polytope Ω, the cost function (18) achieves $J < 0$ over Ω for all nonzero $\omega \in L_2[0 \ \infty)$ and for all positive delays τ if there exist $n \times n$ $0 < \bar{P}_1^j, 0 < \bar{S}^j, \bar{P}, \bar{Y}_1^j, \bar{Y}_2^j, \bar{Z}_1^j, \bar{Z}_2^j, \bar{Z}_3^j, 0 < \bar{U}^j, \bar{R}^j$, and $\hat{Y} \in \mathcal{R}^{p \times n}$-matrices that satisfy (10b) and (22) for $j = 1, \ldots N$, where the matrices $0 < \bar{P}_1$, $0 < \bar{S}, \bar{Y}_1, \bar{Y}_2, \bar{Z}_1, \bar{Z}_2, \bar{Z}_3, \bar{U}, \bar{R}, A_0, A_1, F, C$, are taken with the superscript j.

In the case where also B_1 in (17b) is uncertain, a problem may arise in the LMI (22a) because of the special choice we made for P_6. In such a case, one may take $P_6 = \epsilon_4 \bar{B}_1$ where $\bar{B}_1$ is any constant matrix of the dimensions of B_1. Preferably, one may take $\bar{B}_1$ to be one of the matrices B_1 in the uncertainty polytope.

A similar result can be readily derived that corresponds to the result of Theorem 5.

V. Examples

We bring two examples. The first considers the robust stability analysis problem that was treated in [17]. The second example solves the robust H_∞ state-feedback control problem. In both example a comparison is made with the results that are obtained using the descriptor approach ([5]).

A. *Example 1 [17]*

Consider (1), where

$$A_0 = \begin{bmatrix} 0 & -0.12 + 12\rho \\ 1 & -0.465 - \rho \end{bmatrix} \quad A_1 = \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix}$$

and where $|\rho| \le 0.035$. We consider first the case where $F = 0$ and the time-delay is constant bounded by h. In this case, applying Theorem 1, and following the arguments of Section V, a maximum value of $h = 0.863$ is obtained for which the asymptotic stability of the system is assured over the uncertainty interval for ρ. Applying the method of [5], the corresponding bound on the time-delay is found to be $h = 0.782$. The improvement achieved by the results of this note stem from the additional decision variable P_5.

We consider next the case where $F = 0.05 I_2$. For a constant time delay, bounded by h, a maximum value of $\mathrm{h} = 0.462$ is obtained for which the asymptotic stability of the system is assured over the uncertainty interval. For a time-varying delay satisfying (2), maximum

$$\Psi = \begin{bmatrix} A_0P_2 + B\bar{Y} + Y_1 + \frac{1}{2}S + \frac{h}{2}Z_1 & P_1 + \epsilon_1(A_0P_2 + B\bar{Y}) + Y_2 + hZ_2 & \epsilon_2(A_0P_2 + B\bar{Y}) & \epsilon_3(A_0P_2 + B\bar{Y}) & \epsilon_4(A_0P_2 + B\bar{Y})C^T & B_1 \\ -P_2 & -\epsilon_1 P_2 + \frac{1}{2}U + \frac{h}{2}(R + Z_3) & -\epsilon_2 P_2 & -\epsilon_3 P_2 & -\epsilon_4 P_2 C^T & 0 \\ -Y_1 + A_1 P_2 & -Y_2 + \epsilon_1 A_1 P_2 & -\frac{1}{2}S(1-d) + \epsilon_2 A_1 P_2 & \epsilon_3 A_1 P_2 & \epsilon_4 A_1 P_2 C^T & 0 \\ FP_2 & \epsilon_1 F P_2 & \epsilon_2 F P_2 & \epsilon_3 F P_2 - \frac{1}{2}U & \epsilon_4 F P_2 C^T & 0 \\ CP_2 + D_{12}\bar{Y} & \epsilon_1(CP_2 + D_{12}\bar{Y}) & \epsilon_2(CP_2 + D_{12}\bar{Y}) & \epsilon_3(CP_2 + D_{12}\bar{Y}) & -\frac{\gamma^2}{2}I_m + \epsilon_4(CP_2 + D_{12}\bar{Y})C^T & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}I_q \end{bmatrix} \tag{24b}$$

values of $h = 0.408$ and $h = 0.263$ are obtained for $d = 0.1$ and $d = 0.5$, respectively.

B. Example 2

Given two systems that are described by (17). The matrices of the first system are

$$A_0 = \begin{bmatrix} -1.3 & 0.2 \\ 0.2 & -1 \end{bmatrix} \quad A_1 = \begin{bmatrix} -0.6 & -0.5 \\ -0.5 & -0.6 \end{bmatrix} \quad F = 0$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

The matrices of the second system are identical to the first, except for A_1 that is given by

$$A_1 = \begin{bmatrix} -2.3 & 0 \\ 0 & -0.8 \end{bmatrix}.$$

It is required to find a state-feedback controller $u = Kx(t)$ that minimizes an upper bound on the disturbance attenuation γ over all convex combinations of the two systems. Applying the descriptor method of [5] to the above system a minimum value of $\gamma = 33.861$ is obtained for $\epsilon = -0.9923$. Using Theorem 4 with $\epsilon_2 = \epsilon_4 = 0$ (i.e., descriptor method with a design procedure of Theorem 4, where $P_3 = \epsilon_1 P_2$), a minimum bound of $\gamma = 8.2641$ is obtained. On the other hand, applying Theorem 4 with no restrictions on ϵ_2 and ϵ_4, a better bound of $\gamma = 6.87$ is obtained for $\epsilon_1 = 0.4153$, $\epsilon_2 = 0.6068$ and $\epsilon_4 = 0.0021$. The corresponding state-feedback gain matrix is $K = [-754.41 \quad -236.15]$.

Applying Theorem 5, for $\epsilon_1 = 0.3442$, $\epsilon_2 = -0.1047$ and $\epsilon_4 = 0.1788$ a minimum value of $\gamma = 4.5157$ is obtained with a corresponding state-feedback gain matrix $K = [-49.8490 \quad -15.6345]$. A clear advantage of the method that is based on the adjoint system is evident.

VI. CONCLUSION

A generalization of the results obtained by using either the descriptor approach or the conditions found by the first or the third model transformation to the analysis and synthesis of time-delay systems is presented. Extra degrees of freedom are introduced which allow solutions to problems with polytopic uncertainty that are less conservative than those obtained in the past. The new approach also simplifies the inequalities that have to be solved by the descriptor approach in cases with no uncertainty. It leads to new effective design procedures.

The LMIs that have to be solved by the new approach are of larger size and their solution may require longer computation time. The benefit of these LMIs will thus be in the robust control case where the improvement they introduce compensate for the larger computation complexity.

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