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Exact controllability of a class of nonlinear distributed parameter systems using back-and-forth iterations

Vivek Natarajan[a], Hua-Cheng Zhou[b], George Weiss[b] and Emilia Fridman[b]

[a]Systems and Control Engineering Group, Indian Institute of Technology Bombay, Mumbai, India; [b]School of Electrical Engineering, Tel Aviv University, Ramat Aviv, Israel

ABSTRACT

We investigate the exact controllability of a nonlinear plant described by the equation $\dot{x}(t) = Ax(t) + Bu(t) + B_{\mathcal{N}}\mathcal{N}(x(t), t)$, where $t \geq 0$. Here A is the infinitesimal generator of a strongly continuous group $\mathbb{T}$ on a Hilbert space X, B and $B_{\mathcal{N}}$, defined on Hilbert spaces U and $U_{\mathcal{N}}$, respectively, are admissible control operators for $\mathbb{T}$ and the function $\mathcal{N} : X \times [0, \infty) \mapsto U_{\mathcal{N}}$ is continuous in t and Lipschitz in x, with Lipschitz constant $L_{\mathcal{N}}$ independent of t. Thus, B and $B_{\mathcal{N}}$ can be unbounded as operators from U and $U_{\mathcal{N}}$ to X, in which case the nonlinear term $B_{\mathcal{N}}\mathcal{N}(x(t), t)$ in the plant is in general not Lipschitz in x. We assume that there exist linear operators F and F_b such that the triples $(A, [B\ B_{\mathcal{N}}], F)$ and $(-A, [B\ B_{\mathcal{N}}], F_b)$ are regular and $A + BF_\Lambda$ and $-A + BF_{b,\Lambda}$ are generators of operator semigroups $\mathbb{T}^f$ and $\mathbb{T}^b$ on X such that $\|\mathbb{T}^f_t\| \cdot \|\mathbb{T}^b_t\|$ decays to zero exponentially. We prove that if $L_{\mathcal{N}}$ is sufficiently small, then the nonlinear plant is exactly controllable in some time $\tau > 0$. Our proof is constructive, i.e. given an initial state $x_0 \in X$ and a final state $x_\tau \in X$, we propose an approach for constructing a control signal u of class L^2 for the nonlinear plant which ensures that if $x(0) = x_0$, then $x(\tau) = x_\tau$. We illustrate our approach using two examples: a sine-Gordon equation and a nonlinear wave equation. Our main result can be regarded as an extension of Russell's principle on exact controllability to a class of nonlinear plants.

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1. Introduction

Let X, U and $U_{\mathcal{N}}$ be Hilbert spaces, let $\mathbb{T}$ be a strongly continuous group of operators on X with generator A and let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|$, where β is an arbitrary (but fixed) element in the resolvent set $\rho(A)$. In this paper we study the exact controllability of a nonlinear distributed parameter system described by the state equation

$$\dot{x}(t) = Ax(t) + Bu(t) + B_{\mathcal{N}}\mathcal{N}(x(t), t) + f(t), \qquad (1.1)$$

where $B \in \mathcal{L}(U, X_{-1})$ and $B_{\mathcal{N}} \in \mathcal{L}(U_{\mathcal{N}}, X_{-1})$ are admissible control operators for $\mathbb{T}$, the input signal u takes values in U and the nonlinear function $\mathcal{N} : X \times [0, \infty) \mapsto U_{\mathcal{N}}$ is continuous in the second argument t (the time) and Lipschitz in the first argument x with Lipschitz constant $L_{\mathcal{N}}$ independent of t, i.e.

$$\|\mathcal{N}(x_1, t) - \mathcal{N}(x_2, t)\| \leq L_{\mathcal{N}}\|x_1 - x_2\| \qquad \forall\, x_1, x_2 \in X, \qquad \forall\, t \geq 0. \qquad (1.2)$$

The function f (the drift term in (1.1)) is an X_{-1}-valued function such that

$$f \in H^1_{\text{loc}}((0, \infty); X_{-1}),$$

which means that the function $t \mapsto f(t) - f(0)$ can be written as the integral of a function in $L^2_{\text{loc}}([0, \infty); X_{-1})$. In the Abstract we took $f = 0$ for simplicity.

By *exact controllability* in time $\tau > 0$ of the plant (1.1), we mean the following: For any initial state $x(0) \in X$ we can steer the final state $x(\tau)$ of the plant to any desired point in X, by using an input function $u \in L^2([0, \tau]; U)$. By exact controllability we mean exact controllability in some time $\tau > 0$. Exact controllability of nonlinear partial differential equations is a classical research area, see for instance Balachandran and Dauer (2002), Chen (1979), Dehman and Lebeau (2009), Zuazua (1993) and the references therein. The plants considered in the present work differ from those in the above, and many other, works in that (i) B can be unbounded, so that the control signal may drive the PDE while acting on a measure zero subset of its domain, like in the case of boundary control, (ii) $B_{\mathcal{N}}$

CONTACT Vivek Natarajan vivekn@sc.iitb.ac.in

can be unbounded as well, so that the nonlinear perturbation term $B_{\mathcal{N}}\mathcal{N}(x(t), t)$ is not Lipschitz continuous on the state space X.

We assume that the pairs (A, B) and $(-A, B)$ are *jointly exponentially stabilisable* in a certain sense which is inspired by the definition of stabilisability in Weiss and Curtain (1997). Using terminology that will be recalled in Section 3, the requirements of joint exponential stabilisability are that there exist linear operators F and F_b such that the triples (A, B, F) and $(-A, B, F_b)$ are regular, I is an admissible feedback operator for both of the regular systems with generating operators $(A, B, F, 0)$ and $(-A, B, F_b, 0)$ and the corresponding closed-loop semigroups $\mathbb{T}^f$ (whose generator is $A + BF_\Lambda$) and $\mathbb{T}^b$ (whose generator is $-A + BF_{b,\Lambda}$) satisfy, for some $M \geq 1$ and $\mu > 0$,

$$\|\mathbb{T}_t^f\| \cdot \|\mathbb{T}_t^b\| \leq Me^{-\mu t} \qquad \forall\, t \geq 0. \tag{1.3}$$

Here, the subscript Λ denotes the Λ-extension of an operator, see Section 3. In addition, we assume that the triples $(A, B_{\mathcal{N}}, F)$ and $(-A, B_{\mathcal{N}}, F_b)$ are regular. These assumptions will be explained in more detail in Section 4.

In the linear case (when $\mathcal{N} = 0$) our assumptions on A and B imply that the system (A, B) is exactly controllable: this is the well-known Russell's principle for controllability, for which we refer to Russell (1974, 1978), Chen (1979), Komornik (1992), Rebarber and Weiss (1997). This principle (which can be stated in many different versions, of varying generality) states that if the pairs (A, B) and $(-A, B)$ are jointly exponentially stabilisable (in the sense explained above), then (A, B) is exactly controllable. For the dual version (which concerns exact observability) we refer to Ramdani, Tucsnak, and Weiss (2010) (specifically, Proposition 3.3 and the comments after it). The latter paper studied exact observability via a sequence of forward and backward observers. This approach was extended to some nonlinear wave and beam equations in Fridman (2013), Fridman and Terushkin (2016). The most general statement of Russell's principle (the linear exact controllability question) known to us is in the conference paper (Natarajan & Weiss, 2013b), where it is stated informally in the comments after Theorem 4.1. In that reference $\mathbb{T}$ is not assumed to be invertible, the stabilisability concept is weaker than what we use in this work and the conclusion is the null-controllability of the system (A, B) in some time.

Our assumptions on A, B and $B_{\mathcal{N}}$ allow us to regard the nonlinear system (1.1) as a perturbation of the corresponding linear system (where $\mathcal{N} = 0$) if the Lipschitz constant $L_{\mathcal{N}}$ is small enough. The fact that exact controllability of a linear system is preserved under small bounded linear perturbations has been proved in Hadd (2005). A more general result (in dual form) is in Section 6.3 of Tucsnak and Weiss (2009). It concerns a class of unbounded perturbations, similar to what we have in this paper, but linear. A further generalisation (again in dual form) is in the recent work (Jiang, Liu, & Zhang, 2015) which considers nonlinear unbounded perturbations. However, our results are not a simple extension of the results in these works, because the approach we use here is entirely different. Our main result can be regarded as an extension of Russell's principle to a class of nonlinear plants. We remark that the idea of extending Russell's principle to nonlinear systems was raised (but not addressed) in Chen (1979).

We show that for each sufficiently small Lipschitz constant $L_{\mathcal{N}}$ there exists $\tau^* \geq 0$ such that the plant (1.1) is exactly controllable in each time $\tau > \tau^*$. Our proof is novel and constructive, i.e. we present an algorithm for constructing the required control signal u that takes x from a given $x(0)$ to a desired $x(\tau)$. Briefly, we define a sequence of forward and backward state equations that have the same structure as (1.1), but without drift terms and with different functions in place of $\mathcal{N}$. Using solutions of this sequence of systems on the time interval $[0, \tau]$, that satisfy suitable boundary conditions, we construct the required control signal u. When the plant is linear, our approach for determining u reduces to solving the plant with linear feedback, back and forth in time on the interval $[0, \tau]$, with different initial conditions. For simplicity, we explain our approach when X is finite-dimensional in Section 2. The details for infinite-dimensional X are in Section 4. Section 5 contains numerical examples.

We simplify our notation by adopting the following convention: if in a sum the summation index runs from $m \in \mathbb{Z}$ to $n \in \mathbb{Z}$ and $n < m$, then the sum is zero.

2. Discussion in finite dimensions

To make our ideas easily understood, we first present them in the simpler context of finite-dimensional control theory. Thus in this section $X = \mathbb{C}^p$, $U = \mathbb{C}^q$, $U_{\mathcal{N}} = \mathbb{C}^r$ and A, B, $B_{\mathcal{N}}$, F and F_b are matrices of suitable dimensions such that (1.3) holds (note that $\mathbb{T}_t^f = e^{(A+BF)t}$ and $\mathbb{T}_t^b = e^{(-A+BF_b)t}$). We shall see later (see Remark 2.2) that in this finite-dimensional context, the existence of F and F_b such that (1.3) holds is equivalent to the fact that (A, B) is controllable.

We assume that $\mathcal{N}$ is a nonlinear function as in the text before (1.2), $\mathcal{N}$ satisfies the Lipschitz assumption (1.2) and the plant is described by (1.1). The drift function $f : [0, \infty) \to X$ is now only assumed to be continuous.

In this setting, it follows from the standard theory for the existence and uniqueness of solutions to ordinary differential equations (ODEs) that for any input $u \in C([0, \infty); U)$ and every initial condition $x_0 \in X$, there exists a unique solution $x \in C^1([0, \infty); X)$ for (1.1) such that $x(0) = x_0$. In the theorem below we assume, for the sake of simplicity, that the desired final state is 0 (i.e. we study the null controllability of (1.1)). This simplifying assumption can be eliminated, see Remark 2.1.

Theorem 2.1: *Under the above assumptions, if $L_{\mathcal{N}}$ is sufficiently small, then there exists $\tau^* \geq 0$ such that the following holds: For any $\tau > \tau^*$ and any $x_0 \in X$ there exists an input function $u \in C([0, \tau]; U)$ such that the solution x of (1.1) corresponding to $x(0) = x_0$ and this u satisfies $x(\tau) = 0$.*

Proof: In the first step, we consider the systems

$$\dot{x}_f(t) = (A + BF)x_f(t) + B_{\mathcal{N}} h(x_f(t), t) \qquad \forall\, t \geq 0, \tag{2.1}$$

$$\dot{x}_b(t) = (-A + BF_b)x_b(t) + B_{\mathcal{N}} h_b(x_b(t), t) \qquad \forall\, t \geq 0, \tag{2.2}$$

and each ODE (except the first) in the sequence of ODEs that we will associate with the plant (1.1) in the next step resembles one of the above two systems. Here the functions h and h_b are continuous in their second argument t and Lipschitz in their first argument, with the same Lipschitz constant $L_{\mathcal{N}}$ as for $\mathcal{N}$. In other words, for all x_1, $x_2 \in X$ and each $t \geq 0$,

$$\|h(x_1, t) - h(x_2, t)\| \leq L_{\mathcal{N}} \|x_1 - x_2\|,$$
$$\|h_b(x_1, t) - h_b(x_2, t)\| \leq L_{\mathcal{N}} \|x_1 - x_2\|.$$

Furthermore, these functions satisfy $h(0, t) = h_b(0, t) = 0$ for all $t \geq 0$. It follows from the standard theory on the solutions to ODEs that for each $x_0 \in X$ there exist unique C^1 solutions x_f for (2.1) and x_b for (2.2), on the time interval $[0, \infty)$, such that $x_f(0) = x_0$ and $x_b(0) = x_0$. In addition, since (1.3) holds, if $L_{\mathcal{N}}$ is sufficiently small (as discussed in more detail in Section 4), then there exist constants M_f, $M_b \geq 1$ and $\omega_f, \omega_b \in \mathbb{R}$ (independent of x_0 and the specific forms of h and h_b) such that

$$\|x_f(t)\| \leq M_f e^{-\omega_f t} \|x_0\|,$$
$$\|x_b(t)\| \leq M_b e^{-\omega_b t} \|x_0\| \qquad \forall\, t \geq 0 \tag{2.3}$$

and $\omega_f + \omega_b > 0$. The above estimates are established in Section 4 when X is infinite-dimensional (see Lemma 4.1 and Lemma 4.2). Let $\tau^* \geq 0$ be such that $M_f M_b e^{-(\omega_f + \omega_b)\tau^*} = 1$. Fix $\tau > \tau^*$, then clearly

$$\Gamma := M_f M_b e^{-(\omega_f + \omega_b)\tau} < 1. \tag{2.4}$$

The second step is to introduce a sequence of ODEs defined recursively, meaning that each ODE in the sequence (except the first) is defined using the solutions of the previous ODEs in the sequence. Fix $x_0 \in X$. The first ODE is

$$\dot{x}^0(t) = Ax^0(t) + B_{\mathcal{N}} \mathcal{N}(x^0(t), t) + f(t) \quad \forall\, t \in [0, \tau],$$
$$x^0(0) = x_0. \tag{2.5}$$

For each $n \in \mathbb{N}$ and all $t \in [0, \tau]$ we consider the 'backward' ODE

$$\dot{x}_b^n(t) = (A - BF_b)x_b^n(t) + B_{\mathcal{N}} h_b^n(x_b^n(t), t),$$
$$x_b^n(\tau) = x^{n-1}(\tau), \tag{2.6}$$

where h_b^n is defined using the solutions of the previous ODEs: for all $x \in X$ and $t \in [0, \tau]$,

$$h_b^n(x, t) = \mathcal{N}\left(\sum_{k=0}^{n-1} x^k(t) - \sum_{k=1}^{n-1} x_b^k(t), t\right) - \mathcal{N}\left(\sum_{k=0}^{n-1} x^k(t) - \sum_{k=1}^{n-1} x_b^k(t) - x, t\right). \tag{2.7}$$

For each $n \in \mathbb{N}$ and all $t \in [0, \tau]$ we consider the 'forward' ODE

$$\dot{x}^n(t) = (A + BF)x^n(t) + B_{\mathcal{N}} h^n(x^n(t), t),$$
$$x^n(0) = x_b^n(0), \tag{2.8}$$

where h^n is defined using the solutions of the previous ODEs: $\forall\, x \in X$ and $\forall\, t \in [0, \tau]$

$$h^n(x, t) = \mathcal{N}\left(x + \sum_{k=0}^{n-1} x^k(t) - \sum_{k=1}^{n} x_b^k(t), t\right) - \mathcal{N}\left(\sum_{k=0}^{n-1} x^k(t) - \sum_{k=1}^{n} x_b^k(t), t\right). \tag{2.9}$$

The structure of h_b^n and h^n implies that these ODEs must be solved sequentially on $[0, \tau]$, i.e. we must find x^0, x_b^1, x^1, x_b^2, x^2 and so on, in this order. For example,

$$\dot{x}_b^1(t) = (A - BF_b)x_b^1(t) + B_{\mathcal{N}}\big[\mathcal{N}(x^0(t), t) - \mathcal{N}(x^0(t) - x_b^1(t), t)\big],$$

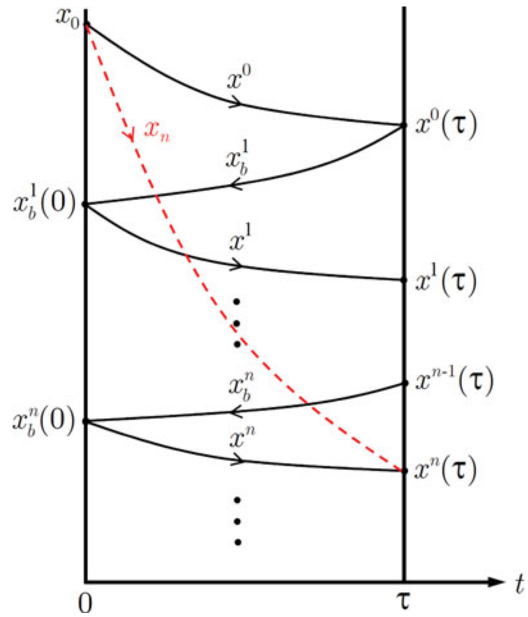

Figure 1. Illustration of the back-and-forth algorithm used to construct a null control signal u for the nonlinear plant (1.1) with initial state $x(0) = x_0$. While x^0 is a state trajectory of (1.1), x_b^n and x^n (for $n \geq 1$) are state trajectories of (2.6) and (2.8). The dashed curve x_n (see (2.14)) is a state trajectory of (1.1) when $u = u_n$. We have $\lim_{n\to\infty} x_n(\tau) = 0$.

which is solved backwards on $[0, \tau]$ starting from $x_b^1(\tau) = x^0(\tau)$, and

$$\dot{x}^1(t) = (A + BF)x^1(t) + B_{\mathcal{N}}\big[\mathcal{N}(x^1(t) - x_b^1(t) + x^0(t), t) - \mathcal{N}(x^0(t) - x_b^1(t), t)\big],$$

which is solved forwards on $[0, \tau]$ starting from $x^1(0) = x_b^1(0)$. This sequence of back-and-forth state trajectories is illustrated in Figure 1.

The third step is to obtain estimates for the solutions of this sequence of ODEs. For this, notice that for $t \in [0, \tau]$, (2.8) is an equation of the form (2.1), with $h = h^n$. It is clear that (2.6) can be rewritten, using the notation $z(t) = x_b^n(\tau - t)$, as

$$\dot{z}(t) = (-A + BF_b)z(t) - B_{\mathcal{N}}h_b^n(z(t), \tau - t),$$
$$z(0) = x^{n-1}(\tau),$$

which for $t \in [0, \tau]$ is an equation of the form (2.2), with

$$h_b(x, t) = -h_b^n(x, \tau - t) \qquad \forall\, x \in X.$$

It is easy to see that the nonlinear functions h and h_b just defined possess all the properties assumed after Equations (2.1) and (2.2). Therefore, the estimates (2.3) hold, meaning that for every $n \in \mathbb{N}$ and $t \in [0, \tau]$,

$$\|x_b^n(\tau - t)\| \leq M_b e^{-\omega_b t}\|x^{n-1}(\tau)\|,$$

$$\|x^n(t)\| \leq M_f e^{-\omega_f t}\|x_b^n(0)\|. \tag{2.10}$$

Combining the above estimates for $n = 1$ and $t = \tau$ and using the notation Γ from (2.4), we obtain $\|x^1(\tau)\| \leq \Gamma\|x^0(\tau)\|$. By iterating this process, we get

$$\|x^n(\tau)\| \leq \Gamma^n\|x^0(\tau)\| \qquad \forall\, n \in \mathbb{N} \tag{2.11}$$

which, along with the first inequality in (2.10), implies that

$$\|x_b^n(t)\| \leq M_b e^{-\omega_b(\tau - t)}\Gamma^{n-1}\|x^0(\tau)\| \quad \forall\, t \in [0, \tau],\;\; n \in \mathbb{N}. \tag{2.12}$$

From here and (2.10) it follows that

$$\|x^n(t)\| \leq M_f e^{-\omega_f t} M_b e^{-\omega_b \tau}\Gamma^{n-1}\|x^0(\tau)\| \quad \forall\, t \in [0, \tau],\;\; n \in \mathbb{N}. \tag{2.13}$$

The fourth step is to find the desired input function u for the plant (1.1). For each $n \in \mathbb{N}$ and $t \in [0, \tau]$, we define

$$x_n(t) = \sum_{k=0}^{n} x^k(t) - \sum_{k=1}^{n} x_b^k(t),$$
$$u_n(t) = F\sum_{k=1}^{n} x^k(t) + F_b\sum_{k=1}^{n} x_b^k(t). \tag{2.14}$$

Then x_n (shown in Figure 1 as a dashed line) is the unique solution of (1.1) on the time interval $[0, \tau]$ with $x_n(0) = x_0$ and input $u = u_n$, i.e.

$$\dot{x}_n(t) = Ax_n(t) + Bu_n(t) + B_{\mathcal{N}}\mathcal{N}(x_n(t), t) + f(t) \qquad \forall\, t \in [0, \tau], \tag{2.15}$$

and moreover $x_n(\tau) = x^n(\tau)$. The functions h_b^n and h^n have been chosen precisely so that x_n satisfies (2.15). This can be shown by a simple computation based on (2.5)–(2.9) (which involves many cancellations). We define $u : [0, \tau] \to U$ by

$$u(t) = F\sum_{k=1}^{\infty} x^k(t) + F_b\sum_{k=1}^{\infty} x_b^k(t) \qquad \forall\, t \in [0, \tau]. \tag{2.16}$$

It follows using (2.12) and (2.13) that both series in (2.16) converge to a finite sum for each $t \in [0, \tau]$, and moreover $\lim_{n\to\infty}\|u - u_n\|_{L^\infty[0,\tau]} = 0$. The function u, being the uniform limit of a sequence of continuous functions, is continuous on $[0, \tau]$.

Let x be the unique solution of (1.1) on $[0, \tau]$ when $x(0) = x_0$ and u is as in (2.16). From the continuous dependence of the state trajectories of systems described by ODEs on their inputs (see, for instance, Khalil, 2002, Theorem 3.4), we get that

$$x(t) = \lim_{n\to\infty} x_n(t) \qquad \forall\, t \in [0, \tau].$$

This, together with the fact that $x_n(\tau) = x^n(\tau)$ and the estimate in (2.11), implies that $x(\tau) = 0$ as desired. ■

We remark that when $\mathcal{N} = 0$ and $f = 0$, if we replace A with $A + BF$ in (2.5) and add $Fx^0(t)$ to the right side of (2.16), then the above construction of the null control signal u reduces to the one described in Natarajan and Weiss (2013b).

As promised at the beginning of this section, in the next remark we explain how the null controllability in Theorem 2.1 can be replaced with controllability.

Remark 2.1: Suppose that for the plant (1.1), with $\mathcal{N}$ as in (1.2), the initial state is $x(0) = x_0$ and the desired final state is $x(\tau) = x_\tau \in X$. Define $\mathcal{M}(z, t) = \mathcal{N}(z + x_\tau, t)$ for each $z \in X$ and $t \geq 0$. Consider the modified nonlinear plant

$$\dot z(t) = Az(t) + Bu(t) + B_{\mathcal{N}}\mathcal{M}(z(t), t) + f(t) + Ax_\tau \qquad \forall\, t \geq 0. \tag{2.17}$$

It is easy to verify that if $z(0) = x_0 - x_\tau$ and $u \in C([0, \tau]; U)$ is a null control signal for (2.17) in time τ, so that the corresponding solution $z \in C^1([0, \tau]; X)$ of (2.17) satisfies $z(\tau) = 0$, then x defined by $x(t) = z(t) + x_\tau$ for each $t \in [0, \tau]$ solves (1.1) for the same u and satisfies $x(0) = x_0$ and $x(\tau) = x_\tau$. Hence, this control signal u takes (1.1) from x_0 to x_τ in time τ (which is our control objective). If we introduce the new drift function $g(t) = f(t) + Ax_\tau$, then the problem of finding a null control signal for (2.17) for any given initial state fits into the framework of Theorem 2.1 (with $\mathcal{M}$ in place of $\mathcal{N}$ and g in place of f) and hence it can be solved under the assumptions of that theorem. Thus, Theorem 2.1 can be strengthened by replacing the conclusion of null controllability in time τ with controllability in time τ.

Remark 2.2: If we take $\mathcal{N} = 0$ in Theorem 2.1, then we get that the joint stabilisability of (A, B) and $(-A, B)$ (i.e. the estimate (1.3)) implies that (A, B) is controllable. The converse statement is clearly also true. This equivalence is the finite-dimensional version of Russell's principle.

3. Background on regular systems

This section is a brief introduction to regular linear systems based on Staffans (2004), Staffans and Weiss (2004), Weiss (1994a), Weiss (1994b), Weiss and Curtain (1997) and Tucsnak and Weiss (2009). For a Hilbert space Y and $\alpha \in \mathbb{R}$, we define the Hilbert space

$$L^2_\alpha([0,\infty); Y) = \left\{\phi \in L^2_{\text{loc}}([0,\infty); Y) \,\middle|\, \int_0^\infty e^{-2\alpha t}\|\phi(t)\|^2\,\mathrm{d}t < \infty\right\},$$

with the norm being the square-root of the integral appearing above. For each $\tau \geq 0$, let $\mathbf{P}_\tau$ be the projection of $L^2([0, \infty); Y)$ onto $L^2([0, \tau]; Y)$ by truncation.

Let A be the generator of a strongly continuous operator semigroup $\mathbb{T}$ on a Hilbert space X. The *growth bound* of $\mathbb{T}$ is denoted by $\omega_{\mathbb{T}}$. This means that $\omega_{\mathbb{T}}$ is the smallest real number with the following property: For each $\omega > \omega_{\mathbb{T}}$ there exists a $M_\omega > 0$ such that

$$\|\mathbb{T}_t\| \leq M_\omega e^{\omega t} \qquad \forall\, t \geq 0.$$

We call $\mathbb{T}$ (or A) *exponentially stable* if $\omega_{\mathbb{T}} < 0$. For $\beta \in \rho(A)$, we define

$$X_1 = \mathcal{D}(A) \text{ with the norm } \|x\|_1 = \|(\beta I - A)x\|.$$

The space X_{-1} has been defined before (1.1). These spaces are independent of the choice of β. The operators $\mathbb{T}_t$ extend to X_{-1} and form an operator semigroup on it. The generator of the extended operator semigroup is an extension of A to an operator in $\mathcal{L}(X, X_{-1})$. We use the same notation $\mathbb{T}_t$ and A for these extended operators. If $\mathbb{T}$ is a strongly continuous group on X, then so is its extension to X_{-1}. The space X_1^d is defined similarly to X_1, but with A^* in place of A. Then X_{-1} can be regarded as the dual space of X_1^d with respect to the pivot space X.

Let U be a Hilbert space. An operator $B \in \mathcal{L}(U, X_{-1})$ is an *admissible control operator* for $\mathbb{T}$ if for some (hence, for each) $\tau > 0$ and for every $u \in L^2([0, \infty); U)$,

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma}Bu(\sigma)\,\mathrm{d}\sigma \in X \tag{3.1}$$

(the integral is computed in X_{-1}). Then, this integral gives the strong solution of $\dot z(t) = Az(t) + Bu(t)$ in the space X_{-1}, corresponding to $z(0) = 0$, evaluated at the time τ. B is called *bounded* if $B \in \mathcal{L}(U, X)$, and *unbounded* otherwise. If B is admissible and $\alpha > \omega_{\mathbb{T}}$, then there exists $M_\alpha \geq 0$ such that

$$\|(sI - A)^{-1}B\|_{\mathcal{L}(U,X)} \leq \frac{M_\alpha}{\sqrt{\operatorname{Re} s - \alpha}} \quad \text{for } \operatorname{Re} s > \alpha.$$

If the semigroup $\mathbb{T}$ is normal, or invertible, or contractive, then the above estimate is sufficient for the admissibility of B. For A and B as above (so that in particular, B is admissible), for each $\tau \geq 0$ we define the *input map* Φ_τ: $L^2([0, \infty); U) \to X$ by (3.1). It can be shown that these operators are bounded. Since $\Phi_\tau u = \Phi_\tau \mathbf{P}_\tau u$, the operators Φ_τ have an obvious extension to $L^2_{\text{loc}}([0, \infty); U)$. For any $u \in L^2_{\text{loc}}([0, \infty); U)$, $\Phi_t u$ is a continuous X-valued function of the time t. If A is exponentially stable, then the operators Φ_t are uniformly bounded. If we denote their uniform bound by $\|\Phi_\infty\|$, then obviously

$$\|\Phi_t u\| \leq \|\Phi_\infty\| \cdot \|u\|_{L^2} \quad \forall\, u \in L^2([0, \infty); U), \quad \forall\, t \geq 0. \tag{3.2}$$

For A and B as above, if $\mathbb{T}_t$ is invertible for some $t > 0$, then it is invertible for every $t > 0$ and the inverses form a new operator semigroup with generator $-A$. In this case $\mathbb{T}$ can be extended to a strongly continuous group of operators via $\mathbb{T}_{-t} = \mathbb{T}_t^{-1}$ and B is admissible also for the inverse semigroup.

Let Y be a Hilbert space. An operator $C \in \mathcal{L}(X_1, Y)$ is an *admissible observation operator* for $\mathbb{T}$ if for some (hence, for every) $\tau > 0$ there exists $m_\tau > 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t z\|^2 \,\mathrm{d}t \leq m_\tau \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A). \tag{3.3}$$

If $B \in \mathcal{L}(U, X_{-1})$, then $B^* \in \mathcal{L}(X_1^d, U)$. B is an admissible control operator for $\mathbb{T}$ iff B^* is an admissible observation operator for the adjoint operator semigroup $\mathbb{T}^*$. The Λ*-extension* of an operator $C \in \mathcal{L}(X_1, Y)$ (with respect to A), denoted C_Λ, is defined as follows:

$$C_\Lambda x = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} x \tag{3.4}$$

and its domain $\mathcal{D}(C_\Lambda)$ consists of those $x \in X$ for which the above limit exists. If C is admissible for $\mathbb{T}$, then for every $x \in X$ the formula $y(t) = C_\Lambda \mathbb{T}_t x$ makes sense for almost every $t \geq 0$ and it defines a function $y \in L^2_\alpha([0, \infty); Y)$ for every $\alpha > \omega_{\mathbb{T}}$. We call C *bounded* if it can be extended so that $C \in \mathcal{L}(X, Y)$ and *unbounded* otherwise.

Definition 3.1: Let U, X and Y be Hilbert spaces. Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X and let $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$ be admissible control and observation operators, respectively, for $\mathbb{T}$. Then the triple (A, B, C) is called *regular* if for some (hence, for every) $s \in \rho(A)$, $C_\Lambda(sI - A)^{-1}B$ exists and the map $s \mapsto C_\Lambda(sI - A)^{-1}B$ is bounded on some right half-plane.

For any regular triple (A, B, C) and any $D \in \mathcal{L}(U, Y)$, the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = C_\Lambda x(t) + Du(t), \tag{3.5}$$

define a *regular linear system* Σ with *input space* U, *state space* X and *output space* Y. For any initial state $x(0) = x_0 \in X$ and any *input* $u \in L^2_\alpha([0, \infty); U)$ (where $\alpha \in \mathbb{R}$) there exists a unique *state trajectory* $x \in C([0, \infty); X)$ and *output* y for Σ such that $y \in L^2_\gamma([0, \infty); Y)$ for all $\gamma \geq \alpha$ with $\gamma > \omega_{\mathbb{T}}$ and both equations in (3.5) hold for almost every $t \geq 0$. The state trajectory x can be written as

$$x(t) = \mathbb{T}_t x_0 + \Phi_t u \qquad \forall\, t \geq 0,$$

where Φ_t is as in (3.1). The operators (A, B, C, D) are called the *generating operators* of Σ. The *transfer function* of Σ is

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D,$$

which is an $\mathcal{L}(U, Y)$-valued analytic function defined on the right half-plane where $\operatorname{Re} s > \omega_{\mathbb{T}}$. Denoting the Laplace transformation by a hat, for u, x_0, y and γ as in the text after (3.5), we have

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + \mathbf{G}(s)\hat{u}(s) \quad \text{for } \operatorname{Re} s > \gamma$$

and $\lim_{s \to \infty} \mathbf{G}(s)v = Dv$ for any $v \in U$, where s is positive.

For a system Σ as above, there exist linear maps $\Psi_\infty : X \to L^2_{\text{loc}}([0, \infty); Y)$ (called the *extended output map*) and $\mathbb{F}_\infty : L^2_{\text{loc}}([0, \infty); U) \to L^2_{\text{loc}}([0, \infty); Y)$ (called the *extended input-output map*) such that the output y from (3.5) can be expressed as $y = \Psi_\infty x_0 + \mathbb{F}_\infty u$. The extended input-output map is causal, i.e. $\mathbf{P}_\tau \mathbb{F}_\infty = \mathbf{P}_\tau \mathbb{F}_\infty \mathbf{P}_\tau$ for all $\tau \geq 0$. This and the above expression for the output y give

$$\mathbf{P}_\tau y = \mathbf{P}_\tau \Psi_\infty x_0 + \mathbf{P}_\tau \mathbb{F}_\infty \mathbf{P}_\tau u \qquad \forall\, \tau \geq 0. \tag{3.6}$$

For every $\omega > \omega_{\mathbb{T}}$, Ψ_∞ is bounded from X to $L^2_\omega([0, \infty); Y)$ with norm $\|\Psi_\infty\|_\omega$ and $\mathbb{F}_\infty$ is bounded from $L^2_\omega([0, \infty); U)$ to $L^2_\omega([0, \infty); Y)$ with norm $\|\mathbb{F}_\infty\|_\omega$. Moreover

$$\|\mathbb{F}_\infty\|_\omega = \sup_{\operatorname{Re} s > \omega} \|\mathbf{G}(s)\|.$$

Definition 3.2: With the notation from Definition 3.1, let $\mathbf{G}$ be the transfer function of Σ and let $K \in \mathcal{L}(Y, U)$. We say that K is an *admissible feedback operator* for Σ if $[I - K\mathbf{G}(s)]^{-1}$ exists and is uniformly bounded on some right half-plane (equivalently, $[I - \mathbf{G}(s)K]^{-1}$ exists and is uniformly bounded on some right half-plane).

Proposition 3.1: *We work under the assumptions of the above definition, and moreover we assume that $I - KD$ is invertible (equivalently, $I - DK$ is invertible). Then there exists a unique regular linear system Σ^K with generating operators (A^K, B^K, C^K, D^K) and transfer function $\mathbf{G}^K$ determined as follows:*

$$A^K x = [A + BK(I - DK)^{-1}C_\Lambda]x \qquad \forall\, x \in \mathcal{D}(A^K),$$

where $\mathcal{D}(A^K)$ consists precisely of those $x \in \mathcal{D}(C_\Lambda)$ for which the above expression for $A^K x$ is in X. For any $x \in \mathcal{D}(A^K)$ we have

$$B^K = B(I - KD)^{-1}, \qquad C^K x = (I - DK)^{-1}C_\Lambda x,$$

$$D^K = D(I - KD)^{-1} = (I - DK)^{-1}D,$$

$$\mathbf{G}^K = \mathbf{G}(I - K\mathbf{G})^{-1} = (I - \mathbf{G}K)^{-1}\mathbf{G}.$$

With the notation and the assumptions of the above proposition, let $\tilde{U}$ be a Hilbert space and $\tilde{B} \in \mathcal{L}(\tilde{U}, X_{-1})$. If $(A, \tilde{B}, C)$ is a regular triple, then $\tilde{B}$ is an admissible control operator for the semigroup generated by A^K. (A similar statement holds for admissible observation operators, we omit the details.) Furthermore, if $D = 0$, then $(A^K, \tilde{B}, \tilde{C})$ is a regular triple, where $\tilde{C}$ is the restriction of C_Λ from (3.4) to $\mathcal{D}(A^K)$. The Λ-extension of $\tilde{C}$ with respect to A^K is C_Λ (the same as for C).

The following definition uses the above proposition with $D = 0$.

Definition 3.3: Suppose that A and B are as in (3.1). Then (A, B) is *exponentially stabilisable* if there exists $F \in \mathcal{L}(X_1, U)$ such that:

(1) (A, B, F) is a regular triple,
(2) $[I - F_\Lambda(sI - A)^{-1}B]^{-1}$ exists and is bounded on some right half-plane,
(3) The operator semigroup generated by $A + BF_\Lambda$ is exponentially stable.

In this case, we call F a *stabilising state feedback operator* for (A, B).

4. Exact controllability of nonlinear DPS

In this section we study the exact controllability of the possibly infinite-dimensional nonlinear plant (1.1) using the approach described for finite-dimensional systems in Section 2. In this section X, U, $U_\mathcal{N}$ and Y denote Hilbert spaces.

First, we define a solution concept for the nonlinear systems encountered in this section, following Definition 4.1.1 in Tucsnak and Weiss (2009).

Definition 4.1: Consider the nonlinear differential equation

$$\dot{x}(t) = Ax(t) + B_\mathcal{G}\mathcal{G}(x(t), t) + f(t) \quad \forall\, t \geq 0, \quad (4.1)$$

where A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X, $B_\mathcal{G} \in \mathcal{L}(U, X_{-1})$, $\mathcal{G} : X \times [0, \infty) \to U$ is a nonlinear function and $f \in H^1_{\text{loc}}((0, \infty); X_{-1})$. A *strong solution of* (4.1) *in* X_{-1} is a function

$$x \in L^1_{\text{loc}}([0, \infty); X) \cap C([0, \infty); X_{-1})$$

with the following properties:

(a) $\mathcal{G}x \in L^2_{\text{loc}}([0, \infty); U)$, where $(\mathcal{G}x)(t) = \mathcal{G}(x(t), t)$ for almost all $t \geq 0$.

(b) The following equality holds for every $t \geq 0$:

$$x(t) - x(0) = \int_0^t \big[Ax(\sigma) + B_\mathcal{G}(\mathcal{G}x)(\sigma) + f(\sigma)\big]\, \mathrm{d}\sigma.$$

It is easy to see that the strong solution x in the above definition (if it exists) satisfies (4.1) pointwise in X_{-1} for almost every $t \geq 0$.

Proposition 4.1: *Suppose that $\mathbb{T}$ is a strongly continuous semigroup of operators on X with infinitesimal generator A, $B \in \mathcal{L}(U, X_{-1})$ and $B_\mathcal{N} \in \mathcal{L}(U_\mathcal{N}, X_{-1})$ are admissible control operators for $\mathbb{T}$ and $\mathcal{N} : X \times [0, \infty) \to U_\mathcal{N}$ is continuous in its second argument and satisfies the uniform Lipschitz condition (1.2) for some $L_\mathcal{N} > 0$. Assume that $f \in H^1_{\text{loc}}((0, \infty); X_{-1})$ and define*

$$\mathcal{F}(t) = \int_0^t \mathbb{T}_{t-\sigma} f(\sigma)\, d\sigma \qquad \forall\, t \geq 0.$$

For each $t \geq 0$, recall the operator Φ_t from (3.1) and define a similar operator $\Phi^\mathcal{N}_t : L^2([0, \infty); U_\mathcal{N}) \to X$ as follows:

$$\Phi^\mathcal{N}_t z = \int_0^t \mathbb{T}_{t-\sigma} B_\mathcal{N} z(\sigma)\, d\sigma \qquad \forall\, z \in L^2([0, \infty); U_\mathcal{N}).$$

Then for every $u \in L^2_{\text{loc}}([0, \infty); U)$ and each $x_0 \in X$ there exists a unique function $x \in C([0, \infty); X)$ such that

$$x(t) = \mathbb{T}_t x_0 + \Phi_t u + \Phi^\mathcal{N}_t(\mathcal{N}x)(t) + \mathcal{F}(t) \qquad \forall\, t \geq 0, \quad (4.2)$$

where $(\mathcal{N}x)(t) = \mathcal{N}(x(t), t)$ for all $t \geq 0$. For every $t > 0$, $x(t)$ depends continuously on x_0 and on $\mathbf{P}_\tau u$ (the truncation of u to $[0, \tau]$). Moreover, the function x defined above *is the unique strong solution of* (1.1) *in* X_{-1} *that satisfies* $x(0) = x_0$.

Outline of the proof. First we notice that from Theorem 4.1.6 in Tucsnak and Weiss (2009) we have $\mathcal{F} \in C([0, \infty); X)$.

The existence of x and its continuous dependence on x_0 and on $\mathbf{P}_\tau u$ can be shown by a slight adaptation of the proof of Theorem 3.1 in Natarajan and Weiss (2013a). Indeed, in the cited reference, $\mathcal{N}$ is independent of time, while here $\mathcal{N}$ may depend on t, but this does not change anything. In the cited reference, for our purpose, we must take the output y of the well-posed system Σ^P to be equal to its state x and replace $\Phi_t^1 u_1$ with $\Phi_t u + \mathcal{F}(t)$ and $\mathbb{F}_t^1 u_1$ with $\mathbf{P}_t(\Phi u + \mathcal{F})$ in Section 3 there. Here $(\Phi u)(t) = \Phi_t u$ for all $t \geq 0$. It follows from Remark 3.2 in the cited reference that this proposition holds without any restriction on the Lipschitz constant $L_\mathcal{N}$. This is because our observation operator is the identity operator on X, which is bounded. Theorem 3.1 from Natarajan and Weiss (2013a) is presented as Theorem 7.2 in the survey paper Tucsnak and Weiss (2014).

To use the concept of strong solution for (1.1), we have to fit (1.1) into the framework of (4.1). For this, we define $B_\mathcal{G} = [B \ B_\mathcal{N}]$ and $\mathcal{G}(x, t) = [u(t) \ \mathcal{N}(x, t)]^\top$. Adapting the proof of Proposition 4.2.5 in Tucsnak and Weiss (2009) we can check that x is the unique strong solution for (1.1) in X_{-1} that satisfies $x(0) = x_0$. □

The function x in the above proposition is called the *state trajectory* of (1.1) corresponding to the initial state x_0 and the input u.

Let A, B and $\mathbb{T}$ be as in Proposition 4.1. Then clearly $A - \gamma I$ is the generator of an operator semigroup $\mathbb{S}^\gamma$ for each $\gamma \in \mathbb{R}$, with $\mathbb{S}_t^\gamma = e^{-\gamma t}\mathbb{T}_t$ for all $t \geq 0$, and B is an admissible control operator for $\mathbb{S}^\gamma$. Define Φ_t^γ analogously to Φ_t by replacing $\mathbb{T}$ with $\mathbb{S}^\gamma$ in (3.1). If $\gamma > \omega_\mathbb{T}$, then $\mathbb{S}^\gamma$ is exponentially stable and from the discussion above (3.2) we get that the operators Φ_t^γ are uniformly bounded by a constant $\|\Phi_\infty^\gamma\|$.

Lemma 4.1: *Suppose that* A *is the generator of a strongly continuous semigroup* $\mathbb{T}$ *on* X *with growth bound* $\omega_\mathbb{T}$. *Let* $B \in \mathcal{L}(U, X_{-1})$ *be an admissible control operator for* $\mathbb{T}$ *and let* $g\colon X \times [0, \infty) \to U$ *be a function which is continuous in its second argument,* $g(0, t) = 0$ *for all* $t \geq 0$ *and* g *satisfies the Lipschitz condition*

$$\|g(x_1, t) - g(x_2, t)\| \leq L\|x_1 - x_2\| \quad \forall x_1, x_2 \in X, \quad \forall t \geq 0, \tag{4.3}$$

for some $L > 0$. *Then for every* $\gamma > \omega_\mathbb{T}$ *there exists* $M_\gamma > 1$ *such that for each* $x_0 \in X$, *the state trajectory* x *of the nonlinear system*

$$\dot{x}(t) = Ax(t) + Bg(x(t), t) \qquad \forall t \geq 0, \qquad x(0) = x_0 \tag{4.4}$$

satisfies $\|x(t)\| \leq M_\gamma \|x_0\| e^{-\omega_\gamma t}$ *for all* $t \geq 0$. *Here* $-\omega_\gamma = \gamma + \|\Phi_\infty^\gamma\|^2 L^2$, *where* $\|\Phi_\infty^\gamma\|$ *is as introduced just before the lemma.*

Furthermore, if $C \in \mathcal{L}(X_1, Y)$ *is an admissible observation operator for* $\mathbb{T}$ *and* (A, B, C) *is a regular triple, then for every trajectory* x *of* (4.4) $x(t) \in \mathcal{D}(C_\Lambda)$ *for almost all* $t \geq 0$ *and for each* $\gamma > \omega_\mathbb{T}$ *and* $\alpha > -\omega_\gamma$ *there exist constants* $c_\alpha, d_\alpha > 0$ *such that*

$$\|\mathbf{P}_\tau C_\Lambda x\|_{L^2_\alpha} \leq c_\alpha \|x_0\| + d_\alpha \|\mathbf{P}_\tau x\|_{L^2_\alpha} \qquad \forall \tau \geq 0. \tag{4.5}$$

Proof: For each $x_0 \in X$, it follows from Proposition 4.1 that there exists a unique state trajectory $x \in C([0, \infty); X)$ for (4.4) which satisfies

$$x(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-\sigma} Bg(x(\sigma), \sigma)\,\mathrm{d}\sigma \qquad \forall t \geq 0.$$

Fix $\gamma > \omega_\mathbb{T}$ and define $z(t) = e^{-\gamma t}x(t)$. Recall the notation $\mathbb{S}^\gamma$ and Φ_t^γ introduced just before this lemma. It is easy to verify using the last equation that

$$z(t) = \mathbb{S}_t^\gamma x_0 + \int_0^t \mathbb{S}_{t-\sigma}^\gamma B e^{-\gamma\sigma} g(e^{\gamma\sigma} z(\sigma), \sigma)\,\mathrm{d}\sigma \quad \forall t \geq 0.$$

For any fixed $t \geq 0$, if we define $u(\sigma) = e^{-\gamma\sigma} g(e^{\gamma\sigma} z(\sigma), \sigma)$ for $\sigma \in [0, t]$ and $u(\sigma) = 0$ for $\sigma > t$, then the above equation gives us that $z(t) = \mathbb{S}_t^\gamma x_0 + \Phi_t^\gamma u$. Using this and (4.3) we get that for some $M \geq 1$

$$\|z(t)\| \leq M\|x_0\| + \|\Phi_\infty^\gamma\| \cdot \|u\|_{L^2}, \qquad \|u\|_{L^2}^2 \leq L^2 \int_0^t \|z(\sigma)\|^2\,\mathrm{d}\sigma.$$

Therefore, for all $t \geq 0$,

$$\|z(t)\|^2 \leq 2M^2\|x_0\|^2 + 2\|\Phi_\infty^\gamma\|^2 L^2 \int_0^t \|z(\sigma)\|^2\,\mathrm{d}\sigma.$$

Applying Gronwall's inequality (see, for instance, (Coddington & Levinson, 1955), (Khalil, 2002)) to the above expression, we get that

$$\|z(t)\|^2 \leq 2M^2\|x_0\|^2 e^{2\|\Phi_\infty^\gamma\|^2 L^2 t} \qquad \forall t \geq 0.$$

This and the definitions $z(t) = e^{-\gamma t}x(t)$, $M_\gamma = \sqrt{2}M$ imply the estimate for $\|x(t)\|$.

Let C be as in the theorem. Then we can regard (4.4) along with the equation $y(t) = C_\Lambda x(t)$ as a regular linear system with input u given by $u(t) = g(x(t), t)$ for all $t \geq 0$ and output y. The existence of c_α and d_α such that (4.5) holds follows from (3.6) and the discussion below it, (4.3) and the estimate for $\|x(t)\|$. ■

Remark 4.1: The constants M_γ, ω_γ, c_α and d_α in the above lemma do not depend on the specific form of the nonlinear function g, as long as g satisfies (4.3). This is evident from the proof of Lemma 4.1.

Assumption 4.1: *The operators in (1.1) are such that (A, B) and $(-A, B)$ are jointly exponentially stabilisable, as defined in Section 1. Thus, there exist $F, F_b \in \mathcal{L}(X_1, U)$ such that the triples (A, B, F) and $(-A, B, F_b)$ are regular, I is an admissible feedback operator for both of the regular systems with generating operators $(A, B, F, 0)$ and $(-A, B, F_b, 0)$ and the corresponding closed-loop generators*

$$A_f = A + BF_\Lambda \qquad \text{and} \qquad A_b = -A + BF_{b,\Lambda}$$

generate operator semigroups $\mathbb{T}^f$ and $\mathbb{T}^b$ on X that satisfy (1.3) for some $M \geq 1$ and $\mu > 0$. (That A_f and A_b generate operator semigroups follows from Proposition 3.1.) Furthermore, we assume that the triples $(A, B_\mathcal{N}, F)$ and $(-A, B_\mathcal{N}, F_b)$ are regular.

Remark 4.2: We discuss some immediate consequences of Assumption 4.1.

First, from the statement after Proposition 3.1 it follows that $B_\mathcal{N}$ is an admissible control operator for $\mathbb{T}^f$ and $\mathbb{T}^b$, the operator semigroups generated by A_f and A_b.

Let $\omega_{\mathbb{T}^f}$ and $\omega_{\mathbb{T}^b}$ be the growth bounds for $\mathbb{T}^f$ and $\mathbb{T}^b$, respectively. According to (1.3) we have $\|\mathbb{T}^f_t\| \cdot \|\mathbb{T}^b_t\| \leq Me^{-\mu t}$. Clearly this implies that

$$\frac{1}{t}\log\|\mathbb{T}^f_t\| + \frac{1}{t}\log\|\mathbb{T}^b_t\| \leq \frac{1}{t}\log M - \mu \qquad \forall t > 0.$$

Taking limits as $t \to \infty$, and using the expression for the growth bound from Proposition 2.1.2 of Tucsnak and Weiss (2009), we get

$$\omega_{\mathbb{T}^f} + \omega_{\mathbb{T}^b} \leq -\mu < 0. \tag{4.6}$$

Lemma 4.2: *Suppose that Assumption 4.1 holds for the linear operators of the plant (1.1). We use the notation from Remark 4.2. Consider the nonlinear systems*

$$\dot{x}_f(t) = A_f x_f(t) + B_\mathcal{N} h(x_f(t), t) \qquad \forall t \geq 0, \tag{4.7}$$

$$\dot{x}_b(t) = A_b x_b(t) + B_\mathcal{N} h_b(x_b(t), t) \qquad \forall t \geq 0, \tag{4.8}$$

where $h, h_b : X \times [0, \infty) \to U_\mathcal{N}$ are continuous in the second argument t and Lipschitz in the first argument with Lipschitz constant $L_\mathcal{N}$, i.e. for all $x_1, x_2 \in X$ and $t \geq 0$

$$\|h(x_1, t) - h(x_2, t)\| \leq L_\mathcal{N}\|x_1 - x_2\|,$$
$$\|h_b(x_1, t) - h_b(x_2, t)\| \leq L_\mathcal{N}\|x_1 - x_2\|.$$

Moreover, assume that $h(0, t) = h_b(0, t) = 0$ for all $t \geq 0$.

For each $t \geq 0$ and every $\gamma \in \mathbb{R}$, let $\mathbb{S}^{f,\gamma}_t = e^{-\gamma t}\mathbb{T}^f_t$ and $\mathbb{S}^{b,\gamma}_t = e^{-\gamma t}\mathbb{T}^b_t$ and define the maps $\Phi^{f,\gamma}_t, \Phi^{b,\gamma}_t : L^2([0, \infty); U_\mathcal{N}) \to X$ as follows: for each $z \in L^2([0, \infty); U_\mathcal{N})$,

$$\Phi^{f,\gamma}_t z = \int_0^t \mathbb{S}^{f,\gamma}_{t-\sigma} B_\mathcal{N} z(\sigma)\, d\sigma,$$
$$\Phi^{b,\gamma}_t z = \int_0^t \mathbb{S}^{b,\gamma}_{t-\sigma} B_\mathcal{N} z(\sigma)\, d\sigma.$$

For every $\gamma \in \mathbb{R}$ denote

$$\|\Phi^{f,\gamma}_\infty\| = \sup_{t\geq 0}\|\Phi^{f,\gamma}_t\|, \qquad \|\Phi^{b,\gamma}_\infty\| = \sup_{t\geq 0}\|\Phi^{b,\gamma}_t\|.$$

Then for each $\gamma_f > \omega_{\mathbb{T}^f}$ and $\gamma_b > \omega_{\mathbb{T}^b}$, $\|\Phi^{f,\gamma_f}_\infty\| < \infty$ and $\|\Phi^{b,\gamma_b}_\infty\| < \infty$.

If $L_\mathcal{N}$ is sufficiently small, then there exist constants $\gamma_f > \omega_{\mathbb{T}^f}$ and $\gamma_b > \omega_{\mathbb{T}^b}$ such that $\omega_f + \omega_b > 0$, where

$$-\omega_f = \gamma_f + \|\Phi^{f,\gamma_f}_\infty\|^2 L^2_\mathcal{N},$$
$$-\omega_b = \gamma_b + \|\Phi^{b,\gamma_b}_\infty\|^2 L^2_\mathcal{N}, \tag{4.9}$$

and there exist constants $M_f, M_b \geq 1$, which are independent of the specific form of h and h_b, such that the following holds: For each $x_0 \in X$, there exist unique state trajectories x_f for (4.7) and x_b for (4.8) such that $x_f(0) = x_0$, $x_b(0) = x_0$ and for all $t \geq 0$,

$$\|x_f(t)\| \leq M_f\|x_0\|e^{-\omega_f t},$$
$$\|x_b(t)\| \leq M_b\|x_0\|e^{-\omega_b t}. \tag{4.10}$$

Proof: The facts that for any $\gamma_f > \omega_{\mathbb{T}^f}$ and $\gamma_b > \omega_{\mathbb{T}^b}$ we have $\|\Phi^{f,\gamma_f}_\infty\| < \infty$ and $\|\Phi^{b,\gamma_b}_\infty\| < \infty$ follow from the discussion before Lemma 4.1.

To see that it is possible to satisfy $\omega_f + \omega_b > 0$ if $L_\mathcal{N} > 0$ is sufficiently small, notice that if we choose γ_f sufficiently close to $\omega_{\mathbb{T}^f}$ and γ_b sufficiently close to $\omega_{\mathbb{T}^b}$, then from (4.6) we have $\gamma_f + \gamma_b < 0$. Now the claim follows easily from (4.9).

The existence of unique state trajectories x_f for (4.7) and x_b for (4.8) such that $x_f(0) = x_0$ and $x_b(0) = x_0$ follows from Proposition 4.1.

Finally, the claims (4.10) follow by applying Lemma 4.1 to (4.7) with $\gamma = \gamma_f$ and (4.8) with $\gamma = \gamma_b$ and from Remark 4.1. ■

Definition 4.2: The nonlinear plant (1.1) is *exactly controllable in time* $\tau > 0$ if for every $x_0, x_\tau \in X$, there exists a $u \in L^2([0, \tau]; U)$ such that the state trajectory x of (1.1) corresponding to this input u and initial state $x(0) = x_0$ satisfies $x(\tau) = x_\tau$.

The following is the main result of this paper.

Theorem 4.1: *Suppose that Assumption 4.1 holds for the linear operators of the plant (1.1). Then for each $L_{\mathcal{N}}$ sufficiently small, if $\mathcal{N}$ is continuous in the second argument and satisfies (1.2), then there exists $\tau^* > 0$ (independent of $\mathcal{N}$) such that for every $\tau > \tau^*$, the system (1.1) is exactly controllable in time τ.*

Proof: We follow the same four steps used to prove Theorem 2.1 to establish the null controllability of the nonlinear plant (1.1) in some time $\tau > 0$ and then we mimic Remark 2.1 to conclude the exact controllability of the plant in the same time. We use the notation from Assumption 4.1 and Remark 4.2.

In the first step, we consider the nonlinear differential equations (4.7) and (4.8), where h and h_b are as described in Lemma 4.2. Each equation (except the first) in the sequence of differential equations that we will associate with the nonlinear plant (1.1) in the next step resembles either (4.7) or (4.8). We know from Lemma 4.2 that if $L_{\mathcal{N}}$ is sufficiently small, then there exist constants $M_f, M_b \geq 1$ and $\omega_f, \omega_b \in \mathbb{R}$ (independent of the initial state x_0 for (4.7) and (4.8) and the specific forms of h and h_b) such that (4.10) holds and $\omega_f + \omega_b > 0$. For the remainder of this proof, we suppose that $L_{\mathcal{N}}$ is sufficiently small so that (4.10) holds with $\omega_f + \omega_b > 0$.

Let $\tau^* \geq 0$ be such that $M_f M_b e^{-(\omega_f+\omega_b)\tau^*} = 1$. Fix $\tau > \tau^*$, then clearly (2.4) holds. We will show that (1.1) is null controllable in time τ.

Next we derive some useful properties for the state trajectories x_f of (4.7) and x_b of (4.8) that satisfy $x_f(0) = x_b(0) = x_0$. The discussion below Proposition 3.1 implies that $(A_f, B_{\mathcal{N}}, F_\Lambda)$ and $(A_b, B_{\mathcal{N}}, F_{b,\Lambda})$ are regular triples and the Λ-extensions of F_Λ and $F_{b,\Lambda}$ with respect to A_f and A_b, respectively, are F_Λ with domain $\mathcal{D}(F_\Lambda)$ and $F_{b,\Lambda}$ with domain $\mathcal{D}(F_{b,\Lambda})$. It now follows from the last part of Lemma 4.1 that $x_f(t) \in \mathcal{D}(F_\Lambda)$ and $x_b(t) \in \mathcal{D}(F_{b,\Lambda})$ for almost all $t \geq 0$ and there exist constants $c_f, d_f, c_b, d_b > 0$ (independent of x_0 and the specific forms of h and h_b, see Remark 4.1) such that

$$\|\mathbf{P}_\tau F_\Lambda x_f\|_{L^2} \leq c_f \|x_0\| + d_f \|\mathbf{P}_\tau x_f\|_{L^2}, \qquad (4.11)$$

$$\|\mathbf{P}_\tau F_{b,\Lambda} x_b\|_{L^2} \leq c_b \|x_0\| + d_b \|\mathbf{P}_\tau x_b\|_{L^2}. \qquad (4.12)$$

Furthermore, using Theorem 6.1 in Weiss (1994b) (in particular (6.1) there), we can show that x_f is the unique strong solution in X_{-1} to the equation

$$\dot{x}(t) = Ax(t) + BF_\Lambda x_f(t) + B_{\mathcal{N}} h(x(t), t) \quad \forall t \geq 0, \qquad x(0) = x_0,$$

and x_b is the unique strong solution in X_{-1} to the equation

$$\dot{x}(t) = -Ax(t) + BF_{b,\Lambda} x_b(t) + B_{\mathcal{N}} h_b(x(t), t) \quad \forall t \geq 0, \qquad x(0) = x_0.$$

We remark that all the claims in this paragraph hold trivially when X is finite-dimensional and hence were not discussed in the proof of Theorem 2.1.

The second step is to introduce a sequence of nonlinear differential equations defined recursively, meaning that each equation in the sequence (except the first) is defined using the solutions of the previous equations in the sequence. Fix $x_0 \in X$. The first equation is (2.5) (with the current meaning of the symbols). For each $n \in \mathbb{N}$ and all $t \in [0, \tau]$ we consider the 'backward' equation

$$\dot{x}_b^n(t) = -A_b x_b^n(t) + B_{\mathcal{N}} h_b^n(x_b^n(t), t), \quad x_b^n(\tau) = x^{n-1}(\tau), \qquad (4.13)$$

where h_b^n is defined using the solutions of the previous equations and is given by (2.7). For each $n \in \mathbb{N}$ and all $t \in [0, \tau]$ we consider the 'forward' equation

$$\dot{x}^n(t) = A_f x^n(t) + B_{\mathcal{N}} h^n(x^n(t), t), \quad x^n(0) = x_b^n(0), \qquad (4.14)$$

where h^n is defined using the solutions of the previous equations and is given by (2.9). The structure of h_b^n and h^n implies that these equations must be solved sequentially. Hence we solve the above equations for x^0, x_b^1, x^1, x_b^2, x^2, x_b^3, x^3 and so on, in this order, on the time interval $[0, \tau]$. Let X_{-1}^f and X_{-1}^b be the extensions of X defined similarly to X_{-1} but using A_f and A_b, respectively, in place of A. By *solving* the above equations, we mean finding the functions in $C([0, \tau]; X)$ which satisfy the differential equations in (2.5), (4.13) and (4.14) pointwise in X_{-1}, X_{-1}^b and X_{-1}^f, respectively, for almost all $t \in [0, \tau]$ and which also satisfy the boundary conditions. Using these solutions we will construct the required control signal.

The third step is to obtain estimates for the solutions of this sequence of equations. From Proposition 4.1 we get that there exists a unique strong solution x^0 in X_{-1} for (2.5) such that $x^0 \in C([0, \tau]; X)$ and $x^0(0) = x_0$. Next notice that for $t \in [0, \tau]$, (4.14) is an equation of the form (4.7), with $h = h^n$. Also (4.13) can be formally rewritten, using the notation $z(t) = x_b^n(\tau - t)$, as

$$\dot{z}(t) = A_b z(t) - B_{\mathcal{N}} h_b^n(z(t), \tau - t), \quad z(0) = x^{n-1}(\tau), \tag{4.15}$$

which for $t \in [0, \tau]$ is an equation of the form (4.8), with

$$h_b(x, t) = -h_b^n(x, \tau - t) \qquad \forall x \in X.$$

It is easy to see that the nonlinear functions h and h_b just defined possess all the properties assumed after the equations (4.7) and (4.8). It now follows from Lemma 4.2 that there exist unique state trajectories x^n for (4.14) and z for (4.15) for which the estimates in (4.10) hold. This, and the relation $z(t) = x_b^n(\tau - t)$ for all $t \in [0, \tau]$, means that for every $n \in \mathbb{N}$ and $t \in [0, \tau]$,

$$\|x_b^n(\tau - t)\| \le M_b e^{-\omega_b t}\|x^{n-1}(\tau)\|, \\ \|x^n(t)\| \le M_f e^{-\omega_f t}\|x_b^n(0)\|. \tag{4.16}$$

It is easy to see using Proposition 4.1 that the functions x^0, x_b^n and x^n mentioned above are solutions of (2.5), (4.13) and (4.14) in the sense discussed earlier. From (4.16) we can deduce (like in the proof of Theorem 2.1) that x^0, x_b^n and x^n satisfy the estimates in (2.11), (2.12) and (2.13).

The fourth step is to find the desired control signal u which will ensure that the state trajectory x of the plant (1.1) corresponding to the initial state $x(0) = x_0$ satisfies $x(\tau) = 0$. We will regard x^n, x_b^n, $F_\Lambda x^n$ and $F_{b,\Lambda} x_b^n$ as functions on the interval $[0, \tau]$. Applying (4.11) to (4.14) and (4.12) to (4.15) (and recalling the notation $z(t) = x_b^n(\tau - t)$ for all $t \in [0, \tau]$), we get that for each $n \ge 1$

$$\|F_\Lambda x^n\|_{L^2[0,\tau]} \le c_f \|x_b^n(0)\| + d_f \|x^n\|_{L^2[0,\tau]}, \tag{4.17}$$

$$\|F_{b,\Lambda} x_b^n\|_{L^2[0,\tau]} \le c_b \|x^{n-1}(\tau)\| + d_b \|x_b^n\|_{L^2[0,\tau]}. \tag{4.18}$$

For each $n \in \mathbb{N}$, we define $x_n \in C([0, \tau]; X)$ and $u_n \in L^2([0, \tau]; U)$ as follows:

$$x_n = \sum_{k=0}^{n} x^k - \sum_{k=1}^{n} x_b^k, \qquad u_n = \sum_{k=1}^{n} F_\Lambda x^k + \sum_{k=1}^{n} F_{b,\Lambda} x_b^k.$$

It can be verified using the structure of the nonlinear terms in (2.5), (4.13) and (4.14) and the discussion below (4.11)–(4.12) that x_n is a strong solution in X_{-1} for (1.1) on the time interval $[0, \tau]$ with $x_n(0) = x_0$ and $u = u_n$, i.e. for almost all $t \in [0, \tau]$,

$$\dot{x}_n(t) = Ax_n(t) + Bu_n(t) + B_{\mathcal{N}} \mathcal{N}(x_n(t), t) \\ + f(t) \quad \text{in } X_{-1}$$

and moreover $x_n(\tau) = x^n(\tau)$. From Proposition 4.1 we have

$$x_n(t) = \mathbb{T}_t x_0 + \Phi_t u_n + \Phi_t^{\mathcal{N}}(\mathcal{N} x_n)(t) + \mathcal{F}(t) \\ \forall t \in [0, \tau]. \tag{4.19}$$

Here $(\mathcal{N} x_n)(t) = \mathcal{N}(x_n(t), t)$ for all $t \in [0, \tau]$. It follows from (2.12), (2.13), (4.17) and (4.18) that $(u_n)_{n=1}^\infty$ is a Cauchy sequence in $L^2([0, \tau]; U)$. Define $u \in L^2([0, \tau]; U)$ to be its limit, i.e.

$$u = \sum_{k=1}^{\infty} F_\Lambda x^k + \sum_{k=1}^{\infty} F_{b,\Lambda} x_b^k. \tag{4.20}$$

Let x be the strong solution of (1.1) on $[0, \tau]$ for this u when $x(0) = x_0$. Then, according to Proposition 4.1, (4.2) holds. From (4.19) and (4.2) we get that for all $t \in [0, \tau]$,

$$\|x(t) - x_n(t)\| \le \alpha \|u - u_n\|_{L^2[0,t]} \\ + \beta L_{\mathcal{N}} \|x - x_n\|_{L^2[0,t]},$$

where $\alpha = \sup_{t \in [0,\tau]} \|\Phi_t\| = \|\Phi_\tau\|$ and $\beta = \sup_{t \in [0,\tau]} \|\Phi_t^{\mathcal{N}}\| = \|\Phi_\tau^{\mathcal{N}}\|$. Hence for all $t \in [0, \tau]$,

$$\|x(t) - x_n(t)\|^2 \le 2\alpha^2 \|u - u_n\|_{L^2[0,\tau]}^2 \\ + 2\beta^2 L_{\mathcal{N}}^2 \int_0^t \|x(\sigma) - x_n(\sigma)\|^2 \, d\sigma.$$

Applying Gronwall's inequality to the above estimate we get that

$$\|x(\tau) - x_n(\tau)\|^2 \le 2\alpha^2 \|u - u_n\|_{L^2[0,\tau]}^2 e^{2\beta^2 L_{\mathcal{N}}^2 \tau}.$$

Since $\|u - u_n\|_{L^2[0,\tau]} \to 0$ as $n \to \infty$, $x_n(\tau) = x^n(\tau)$ and $x^n(\tau) \to 0$ as $n \to \infty$ (see (2.11)), it follows from the above equation that $x(\tau) = 0$. Hence the control signal u takes the nonlinear plant (1.1) from the given initial state x_0 to 0 in time τ, i.e. we have established the null controllability of the plant (1.1) in every time $\tau > \tau^*$.

Next we show that (1.1) is exactly controllable in any time $\tau > \tau^*$. Suppose that for (1.1) the given initial state is $x(0) = x_0$ and the desired final state is $x(\tau) = x_\tau$. Define $\mathcal{M}(z, t) = \mathcal{N}(z + x_\tau, t)$ for each $z \in X$ and $t \ge 0$. Then $\mathcal{M}$ is continuous in its second argument and Lipschitz in the first argument with the same Lipschitz constant as $\mathcal{N}$, i.e. $\|\mathcal{M}(x_1, t) - \mathcal{M}(x_2, t)\| \le L_{\mathcal{N}} \|x_1 - x_2\|$ for all x_1, $x_2 \in X$ and each $t \ge 0$. Consider the modified nonlinear plant

$$\dot{z}(t) = Az(t) + Bu(t) + B_{\mathcal{N}} \mathcal{M}(z(t), t) + f(t) \\ + Ax_\tau \qquad \forall t \ge 0, \tag{4.21}$$

where $f(t) + Ax_\tau$ is the drift term. The proof described above for (1.1) applies to (4.21) and hence we can conclude that (4.21) is null controllable in each time $\tau > \tau^*$. Following the algorithm in the proof above, construct $u \in L^2([0, \tau]; U)$ such that the state trajectory z of (4.21) corresponding to $z(0) = x_0 - x_\tau$ satisfies $z(\tau) = 0$. It is easy to verify that x defined by $x(t) = z(t) + x_\tau$ for each $t \in [0, \tau]$ solves (1.1) for the same u and satisfies $x(0) = x_0$ and $x(\tau) = x_\tau$. Hence this control signal u takes (1.1) from x_0 to x_τ in time τ (which is our control objective). ■

5. Examples

In this section we present two numerical examples to illustrate our main result. In the first example the plant is described by a sine-Gordon equation on the interval [0, 1] with Neumann boundary control at $x = 1$. In this example B is unbounded and $B_{\mathcal{N}}$ is bounded. We obtain estimates relating the Lipschitz constant $L_{\mathcal{N}}$ of the nonlinearity to the controllability time τ (recall that we are interested in the exact controllability of the plant in time τ). For $L_{\mathcal{N}} = 0.2$ we verify analytically that the condition (2.4) for exact controllability holds when $\tau = 14$. Then we numerically demonstrate the proof of Theorem 4.1 by constructing a control signal u that takes the plant from an initial state to a final state in 14 seconds. The second example is a wave equation on the interval [0, 1] with Neumann boundary control at $x = 1$ and the nonlinearity entering at $x = 0$ as part of a Robin boundary condition. Hence in this case both B and $B_{\mathcal{N}}$ are unbounded. We verify Assumption 4.1 to show that the conclusions of Theorem 4.1 apply to this plant. We then demonstrate the exact controllability of the plant numerically for $L_{\mathcal{N}} = 0.3$.

Example 5.1: We consider the sine-Gordon equation on the unit interval with Neumann boundary control $u(t)$ at the right end point ($x = 1$) and homogenous Dirichlet boundary condition at the left end point ($x = 0$):

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) + \sigma \sin(w(x,t)), & x \in (0,1), \\ w(x,0) = w_{10}(x), \quad w_t(x,0) = w_{20}(x), \\ w(0,t) = 0, \quad w_x(1,t) = u(t). \end{cases} \tag{5.1}$$

Here $w_{10} \in H_L^1(0,1)$ and $w_{20} \in L^2(0,1)$ describe the initial state, $\sigma \in \mathbb{R}$ is a constant and $H_L^1(0,1) = \{f \in H^1(0,1) \mid f(0) = 0\}$.

Let $X = H_L^1(0,1) \times L^2(0,1)$ be the state space with the following inner product:

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \langle f_{1,x}, f_{2,x} \rangle_{L^2} + \langle g_1, g_2 \rangle_{L^2} \qquad \forall \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \in X.$$

If we let $z = \begin{bmatrix} w & w_t \end{bmatrix}^\top$, then the plant (5.1) can be written as an abstract differential equation on X as follows:

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) + B_{\mathcal{N}} \sigma \sin(z_1(t)), \\ z(0) &= \begin{bmatrix} w_{10} & w_{20} \end{bmatrix}^\top, \end{aligned} \tag{5.2}$$

where we use the notation $z(t) = \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^\top$ and A, B and $B_{\mathcal{N}}$ are defined by

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H^2(0,1) \times H_L^1(0,1) \,\middle|\, f(0) = 0, \; f'(1) = 0 \right\},$$

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}^2 x} \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A),$$

$$B = \begin{bmatrix} 0 \\ \delta_1 \end{bmatrix}, \qquad B_{\mathcal{N}} = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Here δ_1 is the Dirac pulse at $x = 1$ and I is the identity operator on $L^2(0, 1)$. This follows from the material in Section 10.2.2 of Tucsnak and Weiss (2009), where the linear system corresponding to $\sigma = 0$ is shown to be a well-posed boundary control system having the above abstract description. We have added the nonlinear term into the abstract description to obtain (5.2).

From the same reference we know that the operator A is skew-adjoint and generates a strongly continuous group $\mathbb{T}$ on X and B is an admissible control operator for $\mathbb{T}$. Moreover, if we denote the adjoint of B by B^*, then

$$B^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \psi(1) \qquad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A).$$

The operator $B_{\mathcal{N}}$, being bounded, is also an admissible control operator for $\mathbb{T}$. The nonlinearity $\sigma \sin(\cdot) : H_L^1(0,1) \to L^2(0,1)$ is globally Lipschitz with Lipschitz constant $L_{\mathcal{N}} = |\sigma|$.

The pairs (A, B) and $(-A, B)$ are exponentially stabilisable in the sense of Definition 3.3. In fact, for each $k > 0$, the operator $F = -kB^*$ is a stabilising state feedback operator both for (A, B) and for $(-A, B)$. This follows from Curtain and Weiss (2006). Moreover, since $B_{\mathcal{N}}$ is bounded, it is clear that $(A, B_{\mathcal{N}}, F)$ and $(-A, B_{\mathcal{N}}, F)$ are regular. Thus, the system (5.2) satisfies Assumption 4.1. It follows from Theorem 4.1 that if (2.4) holds for some $\tau > 0$, then (5.2) is exactly controllable in time τ. We can then use the algorithm described in the proof of that theorem to construct the control signal which takes the plant from any given initial state to any final state in time τ.

In this example, it is possible to find explicit bounds for τ. For this we consider differential equations (4.7) and (4.8) (with minor changes) corresponding to the operators defined in this example and find expressions for M_f, M_b, ω_f, ω_b. Fix $k > 0$ such that $A - kBB_\Lambda^*$ and $-A - kBB_\Lambda^*$ are exponentially stable. Then (4.7) can be written as

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t) + g(v(x,t),t), & x \in (0,1), \\ v(x,0) = v_{10}(x), \quad v_t(x,0) = v_{20}(x), \\ v(0,t) = 0, \quad v_x(1,t) = -kv_t(1,t), \end{cases} \tag{5.3}$$

where $\begin{bmatrix} v_{10} & v_{20} \end{bmatrix}^\top \in X$ and $g : H_L^1(0,1) \times [0,\infty) \to L^2(0,1)$ is continuous in its second argument and satisfies $g(0,t) = 0$ and $\|g(v_1,t) - g(v_2,t)\| \le |\sigma| \|v_1 - v_2\|$ for all $v_1, v_2 \in H_L^1(0,1)$ and each $t \ge 0$. Define $V(t) = E(t) + \gamma\rho(t)$, where $\gamma \in (0,1)$ is a constant and

$$E(t) = \frac{1}{2}\int_0^1 \left[v_x^2(x,t) + v_t^2(x,t)\right] \mathrm{d}x,$$

and

$$\rho(t) = \int_0^1 x v_x(x,t) v_t(x,t) \mathrm{d}x.$$

Clearly $(1-\gamma)E(t) \le V(t) \le (1+\gamma)E(t)$ for all $t \ge 0$. Differentiating V along a solution of (5.3), we get after a simple calculation that if $2k > \gamma(k^2+1)$, then

$$\begin{aligned} \dot V(t) &\le -(\gamma(1-2|\sigma|) - |\sigma|)E(t) \\ &\quad -(k - \gamma(k^2+1)/2)v_t^2(1,t) \\ &\le -\left(\frac{\gamma(1-2|\sigma|) - |\sigma|}{1+\gamma}\right) V(t) \end{aligned}$$

which implies that

$$V(t) \le V(0)\exp\left(-\frac{\gamma(1-2|\sigma|) - |\sigma|}{1+\gamma} t\right) \qquad \forall t \ge 0. \tag{5.4}$$

Strictly speaking, we can differentiate V only along classical solutions of (5.3). But we can show using regularity results (such as Theorem 1.5 in Chapter 6 of Pazy (1983)) that the estimate in (5.4) is valid along all solutions of (5.3). Hence from (5.4) we get that

$$E(t) \le E(0)\left(\frac{1+\gamma}{1-\gamma}\right)\exp\left(-\frac{\gamma(1-2|\sigma|) - |\sigma|}{1+\gamma} t\right) \quad \forall t \ge 0.$$

Therefore

$$M_f = \frac{1+\gamma}{1-\gamma}, \qquad \omega_f = \frac{\gamma(1-2|\sigma|) - |\sigma|}{1+\gamma}.$$

We can similarly show that M_b and ω_b are also given by the above expressions, i.e. $M_b = M_f$ and $\omega_b = \omega_f$. So for any $\tau > 0$ the formula for Γ in (2.4) takes the form

$$\Gamma = \left(\frac{1+\gamma}{1-\gamma}\right)^2 \exp\left(-\frac{\gamma(1-2|\sigma|) - |\sigma|}{1+\gamma} 2\tau\right).$$

Clearly for $\Gamma < 1$ to hold, we require that

$$\tau > \frac{1+\gamma}{\gamma(1-2|\sigma|) - |\sigma|} \ln \frac{1+\gamma}{1-\gamma} > 0. \tag{5.5}$$

The restrictions $\gamma > 0$ and $2k > \gamma(k^2+1)$ imply that if $|\sigma| < 1/3$, then there exist $k, \gamma, \tau > 0$ (which can be computed) such that $\Gamma < 1$, i.e. the plant (5.1) is exactly controllable in time τ. Suppose $\sigma = 0$ and let $k = 1$, then it is easy to see that for any $\tau > 2$ there exists $\gamma > 0$ sufficiently small such that $\Gamma < 1$. In other words, we recover the well-known result that the 1D wave equation on [0, 1] with Neumann boundary control is exactly controllable in any time $\tau > 2$. For $|\sigma| > 0$, our estimate for τ based on the above discussion is typically conservative.

To illustrate our results numerically we choose $\sigma = 0.2$ in (5.1). Then for $k = 1$, $\gamma = 0.7$ and $\tau = 14$ we have $\Gamma < 1$. Let the initial and final states for the plant be

$$\begin{bmatrix} w_{10}(x) \\ w_{20}(x) \end{bmatrix} = \begin{bmatrix} 2x - x^2 \\ 1 - 2x + x^2 \end{bmatrix} \quad \text{and}$$
$$\begin{bmatrix} w_{1\tau}(x) \\ w_{2\tau}(x) \end{bmatrix} = \begin{bmatrix} -2x + x^2 \\ 2x \end{bmatrix} \quad \forall x \in (0,1), \tag{5.6}$$

respectively. Using the algorithm in Theorem 4.1 we numerically construct the control signal u that takes the plant from this initial state to the final state in $\tau = 14$ seconds (see Figure 2). To discretise the plant and the associated sequence of forward and backward systems, we use the backward Euler scheme for the time variable with step size 0.002 and the Chebyshev spectral method for the spatial variable with 20 grid points. Simulations are performed in MATLAB and the results are shown in Figures 2–6. As seen in Figures 5 and 6, the constructed u ensures that the plant state at $\tau = 14$ seconds is close to the desired final state. To construct u we have taken $n = 1$ (i.e. only one set of forward and backward systems), which is enough partly due to the fact that our estimate for τ is conservative. For smaller τ, larger n will be required to get a good approximation for u.

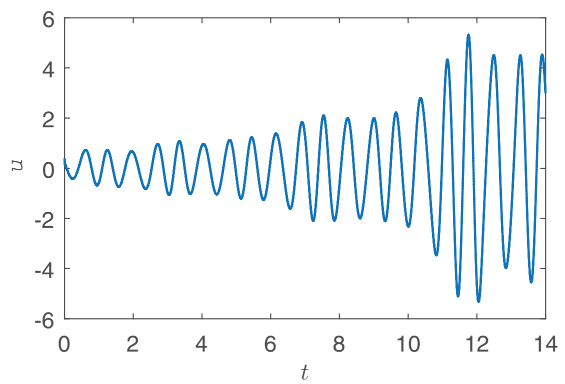

Figure 2. The control signal u that takes the plant (5.1) from the initial state $\left[\, w_{10} \;\; w_{20} \,\right]^{\top}$ to the final state $\left[\, w_{1\tau} \;\; w_{2\tau} \,\right]^{\top}$ defined in (5.6).

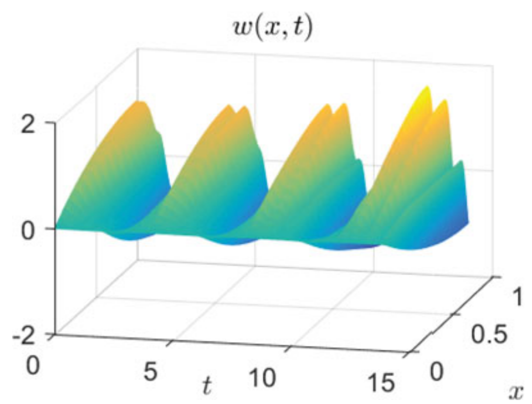

Figure 3. The displacement w of the plant (5.1) under the control signal shown in Figure 2.

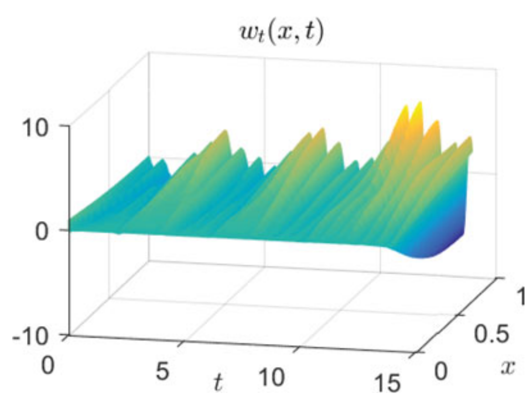

Figure 4. The velocity w_t of the plant (5.1) under the control signal shown in Figure 2.

Remark 5.1: A less restrictive bound on τ than the one given by (5.5) can be derived via Lyapunov functions, as was suggested in Fridman (2013), Fridman and Terushkin (2016). By using arguments from Lemma 3 of Fridman and Terushkin (2016), the following can be proved. Let $V_b(t)$ and $V_f(t)$ be Lyapunov functions corresponding to (4.7) and (4.8), respectively. Assume that there exist $\omega_f \geq 0$ and $\omega_b \geq 0$ such that along solutions of

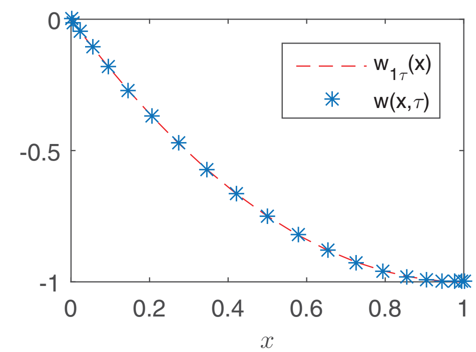

Figure 5. The actual and desired displacement w at $\tau = 14$ seconds.

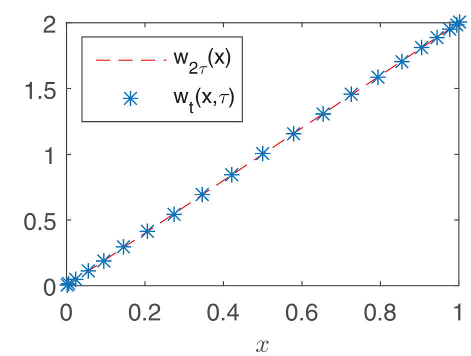

Figure 6. The actual and desired velocity w_t at $\tau = 14$ seconds.

(4.7) and (4.8) the following conditions hold:

$$\dot{V}_f + 2\omega_f V_f \leq 0, \qquad \dot{V}_b + 2\omega_b V_b \leq 0 \qquad \forall\, t \geq 0. \tag{5.7}$$

Assume additionally that $\omega_f + \omega_b > 0$, and for some $\tau^* > 0$ and all $t \geq 0$

$$V_f(t) e^{-2\omega_f \tau^*} \leq V_b(t), \qquad V_b(t) e^{-2\omega_b \tau^*} \leq V_f(t). \tag{5.8}$$

Then (1.1) is exactly controllable in any time $\tau > \tau^*$.

Thus, in Example 5.1 with $\sigma = 0.1$, by verifying linear matrix inequalities that guarantee (5.7) and (5.8) (see Theorem 2 of Fridman & Terushkin, 2016) we arrive at $\tau > \tau^* = 3.78$, which is less (i.e. gives a stronger result) than the estimate $\tau > 6$ that follows from (5.5) and guarantees (2.4).

Example 5.2: Consider the following wave equation on $[0, 1]$ with Neumann boundary control at $x = 1$ and Robin boundary condition, with a nonlinear boundary term, at $x = 0$:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1), \\ w(x,0) = w_{10}(x), \quad w_t(x,0) = w_{20}(x), \\ w_x(0,t) = w(0,t) + \sigma \sin(w(0,t)), \quad w_x(1,t) = u(t). \end{cases} \tag{5.9}$$

The functions $w_{10} \in H^1(0, 1)$ and $w_{20} \in L^2(0, 1)$ describe the initial state, $\sigma \in \mathbb{R}$ is a constant and u is the control input.

Let $X = H^1(0, 1) \times L^2(0, 1)$ be the state space with the following inner product:

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \langle f_{1,x}, f_{2,x}\rangle_{L^2} + \langle g_1, g_2\rangle_{L^2} + f_1(0)\overline{f_2(0)} \quad \forall \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \in X.$$

Here a *bar* denotes a complex conjugate. Defining $z = [w \ w_t]^\top$, (5.9) can be written as an abstract differential equation on X as follows:

$$\dot{z}(t) = Az(t) + Bu(t) + B_{\mathcal{N}}\sigma \sin(Q_0 z(t)), \quad (5.10)$$

with initial state $z(0) = [w_{10} \ w_{20}]^\top$. Here A and Q_0 are operators defined by

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H^2(0, 1) \times H^1(0, 1) \middle| \begin{matrix} f'(0) = f(0), \\ f'(1) = 0 \end{matrix} \right\},$$

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{d^2 f}{d^2 x} \end{bmatrix} \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A),$$

$$Q_0 \begin{bmatrix} f \\ g \end{bmatrix} = f(0) \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X.$$

Denoting the adjoint of an operator using *, a simple computation shows that $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A^* = -A$. From Tucsnak and Weiss (2009, Theorem 3.8.6), A generates a unitary group $\mathbb{T}$ on X. The operators B and $B_{\mathcal{N}}$ can be computed using the theory of boundary control systems in Tucsnak and Weiss (2009, Sect. 10.1). It is more straightforward to compute $[B^* \ B^*_{\mathcal{N}}]^\top$ first, using formula (10.1.7) in Tucsnak and Weiss (2009), where $G[f \ g]^\top = [f'(1) \ f'(0) - f(0)]^\top$. After some simple integrations by parts, we obtain that

$$B^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \psi(1), \qquad B^*_{\mathcal{N}} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = -\psi(0) \quad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A^*).$$

This implies that we have

$$B = \begin{bmatrix} 0 \\ \delta_1 \end{bmatrix}, \qquad B_{\mathcal{N}} = \begin{bmatrix} 0 \\ -\delta_0 \end{bmatrix},$$

where δ_a is the Dirac pulse at $x = a$, with a suitable interpretation.

We now prove that B and $B_{\mathcal{N}}$ are admissible control operators for $\mathbb{T}$. For this, it suffices to show that B^* and $B^*_{\mathcal{N}}$ are admissible observation operators for the adjoint semigroup $\mathbb{T}^*$ (Tucsnak and Weiss (2009, Theorem 4.4.3)) or equivalently that (i) $B^*, B^*_{\mathcal{N}} \in \mathcal{L}(X_1^d, \mathbb{C})$, where $X_1^d = \mathcal{D}(A^*)$ with the graph norm, and (ii) for each $T > 0$ there exists $M_T > 0$ such that for every initial state $[v_{10} \ v_{20}]^\top \in \mathcal{D}(A^*)$, the outputs $v_t(0, t)$ and $v_t(1, t)$ of the system

$$\begin{cases} v_{tt}(x, t) = v_{xx}(x, t), & x \in (0, 1), \\ v(x, 0) = v_{10}(x), & v_t(x, 0) = v_{20}(x), \\ v_x(0, t) = v(0, t), & v_x(1, t) = 0, \end{cases} \quad (5.11)$$

satisfy

$$\int_0^T |v_t(0, t)|^2 dt + \int_0^T |v_t(1, t)|^2 dt \le M_T \left\| \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} \right\|_X^2. \quad (5.12)$$

From the definitions of B^*, $B^*_{\mathcal{N}}$ and X_1^d, it is easy to see that (i) holds. To establish (ii), we define the following functions along the solution $[v \ v_t]^\top$ of (5.11):

$$E(t) = \frac{1}{2}\int_0^1 \left[v_x^2(x, t) + v_t^2(x, t)\right] dx + \frac{v^2(0, t)}{2},$$

$$\rho(t) = \int_0^1 (x - 0.5)v_x(x, t)v_t(x, t)dx.$$

Clearly $|\rho(t)| \le E(t)$ for all $t \ge 0$. It is easy to check that $\dot{E}(t) = 0$ and therefore $E(T) = E(0)$ for all $T > 0$. All this, and the expression

$$\dot{\rho}(t) = \frac{1}{4}\left[v_t^2(0, t) + v_t^2(1, t) + v^2(0, t)\right] - \frac{1}{2}\int_0^1 \left[v_x^2(x, t) + v_t^2(x, t)\right] dx$$

implies that (5.12) holds with $M_T = 4(T + 2)$. This completes the proof of the claim that B and $B_{\mathcal{N}}$ are admissible for $\mathbb{T}$. Now the equivalence of (5.10) and (5.9) can be shown using Tucsnak and Weiss (2009, Proposition 4.2.5 and Remark 4.2.6). The map $\mathcal{N} = \sigma \sin(Q_0(\cdot)) : X \to \mathbb{C}$ is Lipschitz with Lipschitz constant $|\sigma|$.

Define $F = -kB^*$ for some $k > 0$. From the discussion above, F is an admissible observation operator for $\mathbb{T}$. Let $F_b = -kB^*$. We claim that (A, B, F), $(-A, B, F_b)$, $(A, B_{\mathcal{N}}, F)$ and $(-A, B_{\mathcal{N}}, F_b)$ are all regular triples. We will now discuss the proof of the regularity of $(A, B_{\mathcal{N}}, F)$ briefly, the regularity of the other three triples can be established similarly. We will complete the proof of regularity of $(A, B_{\mathcal{N}}, F)$ in two steps. In the first step we will compute $(sI - A)^{-1}B_{\mathcal{N}}$. In the second step we will show that $(sI - A)^{-1}B_{\mathcal{N}} \in \mathcal{D}(F_\Lambda)$ and compute $F_\Lambda(sI - A)^{-1}B_{\mathcal{N}}$. Consider the following system with input u,

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t), & x \in (0,1), \\ v(x,0) = 0, \quad v_t(x,0) = 0, \\ v_x(0,t) = v(0,t) + u(t), \quad v_x(1,t) = 0, \end{cases} \tag{5.13}$$

which can be written as an abstract differential equation on X as follows:

$$\dot{p}(t) = Ap(t) + B_{\mathcal{N}}u(t), \qquad p(0) = 0, \tag{5.14}$$

where $p = [v\ v_t]^\top$. For each $\alpha \in \mathbb{R}$ and $u \in L^2_\alpha([0,\infty);\mathbb{R})$, we have

$$\hat{p}(s) = (sI - A)^{-1}B_{\mathcal{N}}\hat{u}(s) \tag{5.15}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. Here a hat denotes the Laplace transform. It follows from Tucsnak and Weiss (2009, Remark 4.2.6) that the mild solution to (5.14) is the unique solution to the weak formulation of (5.13) written as

$$\langle v_t, \varphi\rangle_{L^2} = -\int_0^t \langle v_x, \varphi_x\rangle \mathrm{d}t - \int_0^t (v(0,t) + u(t))\varphi(0)\mathrm{d}t \quad \forall \varphi \in H^1(0,1).$$

This means that for all s with $\operatorname{Re} s$ sufficiently large and all $\varphi \in H^1(0,1)$,

$$\int_0^1 s^2\hat{v}(x,s)\varphi(x)\mathrm{d}x = -\int_0^1 \hat{v}_x(x,s)\varphi_x(x)\,\mathrm{d}x - (\hat{v}(0,s) + \hat{u}(s))\varphi(0). \tag{5.16}$$

Taking the Laplace transform of the expressions in (5.13) formally, we get that for each $u \in L^2_\alpha([0,\infty);\mathbb{R})$

$$\hat{v}(x,s) = \big[\mathbf{H}(s)\big](x)\hat{u}(s) \qquad \forall x \in [0,1], \quad \forall s \in \{z \in \mathbb{C} \big| \operatorname{Re} z > \alpha\}.$$

Here

$$\big[\mathbf{H}(s)\big](x) = \frac{e^{sx-s} + e^{-sx+s}}{(s-1)e^{-s} - (s+1)e^{s}}.$$

It is now easy to check that $\hat{v}(x,s)$ defined above satisfies (5.16) which, along with (5.15), implies that

$$(sI - A)^{-1}B_{\mathcal{N}} = \begin{bmatrix} \mathbf{H}(s) \\ s\mathbf{H}(s) \end{bmatrix} \qquad \forall s \text{ satisfying } \operatorname{Re} s > 0.$$

This completes the first step in the proof of regularity of $(A, B_{\mathcal{N}}, F)$. We now proceed with the second step. Define $V(x) = x^2 - 2x - 3$ for $x \in [0,1]$. It is easy to check that

$$\begin{bmatrix} \mathbf{H}(s) \\ s\mathbf{H}(s) \end{bmatrix} - \begin{bmatrix} V \\ 0 \end{bmatrix} \in \mathcal{D}(A) \subset \mathcal{D}(F_\Lambda).$$

Hence to show that $(sI - A)^{-1}B_{\mathcal{N}} \in \mathcal{D}(F_\Lambda)$, it is sufficient to check that $[V\ \ 0]^\top \in \mathcal{D}(F_\Lambda)$. For each $\lambda > 0$, it follows from an elementary calculation that

$$(\lambda I - A)^{-1}\begin{bmatrix} V \\ 0 \end{bmatrix} = \begin{bmatrix} q^\lambda \\ \lambda q^\lambda - V \end{bmatrix}, \tag{5.17}$$

where q^λ is defined as follows: For each $x \in [0,1]$,

$$q^\lambda(x) = e^{\lambda x}f_1(\lambda) + e^{-\lambda x}f_2(\lambda) + \frac{V(x)}{\lambda} + \frac{2}{\lambda^3},$$
$$f_1(\lambda) = \frac{q^\lambda(0)}{2} + \frac{q^\lambda(0)}{2\lambda} + \frac{3}{2\lambda} + \frac{1}{\lambda^2} - \frac{1}{\lambda^3},$$
$$f_2(\lambda) = \frac{q^\lambda(0)}{2} - \frac{q^\lambda(0)}{2\lambda} + \frac{3}{2\lambda} - \frac{1}{\lambda^2} - \frac{1}{\lambda^3},$$

where $q^\lambda(0)$ must be found using the condition $q^\lambda_x(1) = 0$ which can be equivalently written as

$$e^\lambda f_1(\lambda) = e^{-\lambda}f_2(\lambda). \tag{5.18}$$

It is easy to check using (5.18) that as $\lambda \to \infty$, $q^\lambda(0) \to 0$ and $f_1(\lambda)e^{2\lambda} \to 0$. This, along with the expression

$$F\lambda(\lambda I - A)^{-1}\begin{bmatrix} V \\ 0 \end{bmatrix} = \lambda^2 e^\lambda f_1(\lambda) + \lambda^2 f_2(\lambda)e^{-\lambda} + \frac{2}{\lambda}$$

which follows from (5.17) and the definition of F, implies that $[V\ \ 0]^\top \in \mathcal{D}(F_\Lambda)$ and $F_\Lambda[V\ 0]^\top = 0$. We now get from the discussion above (5.17) that $(sI - A)^{-1}B_{\mathcal{N}} \in \mathcal{D}(F_\Lambda)$ for all s with $\operatorname{Re} s > 0$ and

$$F_\Lambda(sI - A)^{-1}B_{\mathcal{N}} = -ks\big[\mathbf{H}(s)\big](1) = \frac{2ks}{(s+1)e^s - (s-1)e^{-s}}.$$

It now follows from Definition 3.1 that $(A, B_{\mathcal{N}}, F)$ is a regular triple.

By computing $F_\Lambda(sI - A)^{-1}B$ and $F_{b,\Lambda}(sI + A)^{-1}B$, similarly to how we computed $F_\Lambda(sI - A)^{-1}B_{\mathcal{N}}$ above, we can check that both $[I - F_\Lambda(sI - A)^{-1}B]^{-1}$ and $[I - F_{b,\Lambda}(sI + A)^{-1}B]^{-1}$ exist and are bounded on some right half-plane. The exponential stability of $A + BF_\Lambda$ and $-A + BF_{b,\Lambda}$ can be inferred from Krstic, Guo, Balogh, and Smyshlyaev (2008, Lemma 1). Thus F and F_b are stabilising feedback operators for (A, B) and $(-A, B)$, respectively. From all the above discussions, we get that Assumption 4.1 holds for (5.9). It now follows from Theorem 4.1 that if $|\sigma|$

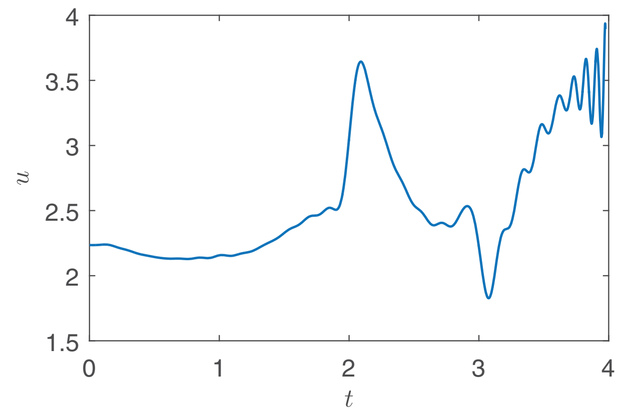

Figure 7. The control signal u that takes the plant (5.9) from the initial state $\begin{bmatrix} w_{10} & w_{20} \end{bmatrix}^\top$ to the final state $\begin{bmatrix} w_{1\tau} & w_{2\tau} \end{bmatrix}^\top$ defined in (5.19).

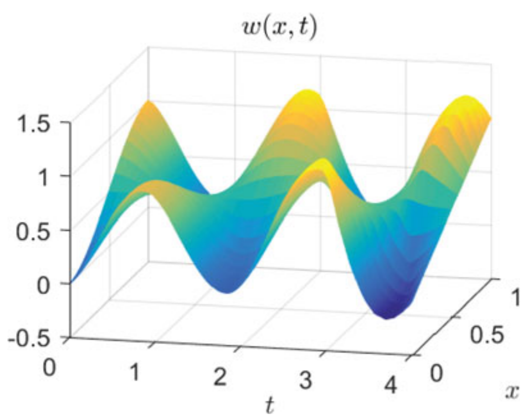

Figure 8. The displacement w of the plant (5.9) under the control signal shown in Figure 7.

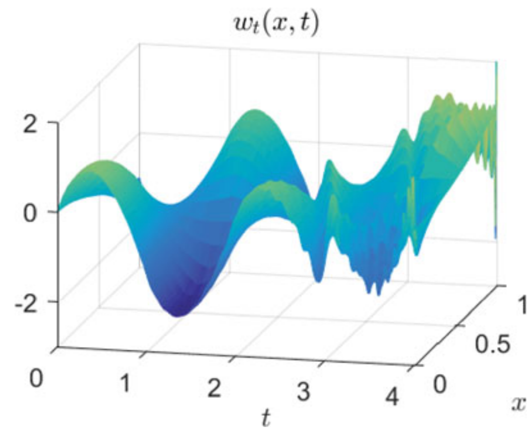

Figure 9. The velocity w_t of the plant (5.9) under the control signal shown in Figure 7.

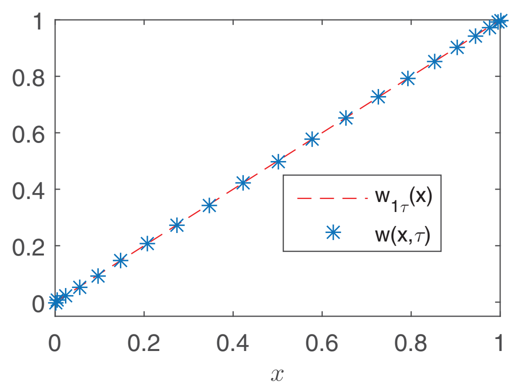

Figure 10. The actual and desired displacement w at $\tau = 4$ seconds.

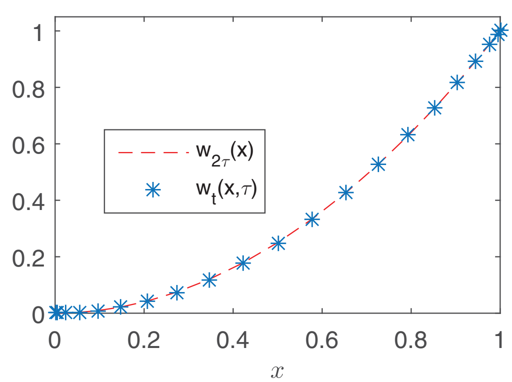

Figure 11. The actual and desired velocity w_t at $\tau = 4$ seconds.

is small and τ is large so that (2.4) holds, then we can construct a control signal u which takes (5.9) from any given initial state to any desired final state in time τ.

To illustrate our theory numerically, we let $\sigma = 0.3$, $k = 1$, $\tau = 4$. The initial and final states are

$$\begin{bmatrix} w_{10}(x) \\ w_{20}(x) \end{bmatrix} = \begin{bmatrix} 3x^2 - 2x^3 \\ -3x^2 + 2x^3 \end{bmatrix} \quad \text{and}$$
$$\begin{bmatrix} w_{1\tau}(x) \\ w_{2\tau}(x) \end{bmatrix} = \begin{bmatrix} x \\ x^2 \end{bmatrix} \qquad \forall\, x \in (0,1), \qquad (5.19)$$

respectively. We implement our algorithm for constructing u in MATLAB. To approximate u we have taken $n = 5$. To discretise the plant and the associated sequence of forward and backward systems, we use the backward Euler scheme for the time variable with step size 0.001 and the Chebyshev spectral method for the spatial variable with 20 grid points. The simulation results are shown in Figures 7–11. As seen in Figures 10 and 11, the constructed u ensures that the plant state at $\tau = 4$ seconds is close to the desired final state.

6. Conclusions

In this work we have derived conditions for the exact controllability of a class of nonlinear distributed parameter systems. In general, the control operator B for the plants in this class are unbounded and the time-varying nonlinear term $B_{\mathcal{N}}\mathcal{N}(x,t)$ is not a Lipschitz in x map from $X \times [0,\infty)$ to the state space X. Our proof of exact controllability is constructive and can be regarded as an extension of Russell's principle to a class of nonlinear systems. Future work could focus on relaxing the regularity requirements in Assumption 4.1. Another interesting direction to explore is the applicability of the techniques in this paper to study the exact controllability of plants in which the nonlinear function $\mathcal{N}$ is only assumed to be locally Lipschitz in the state, or it is a polynomial function of the state.

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ORCID

Hua-Cheng Zhou http://orcid.org/0000-0001-6856-2358

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