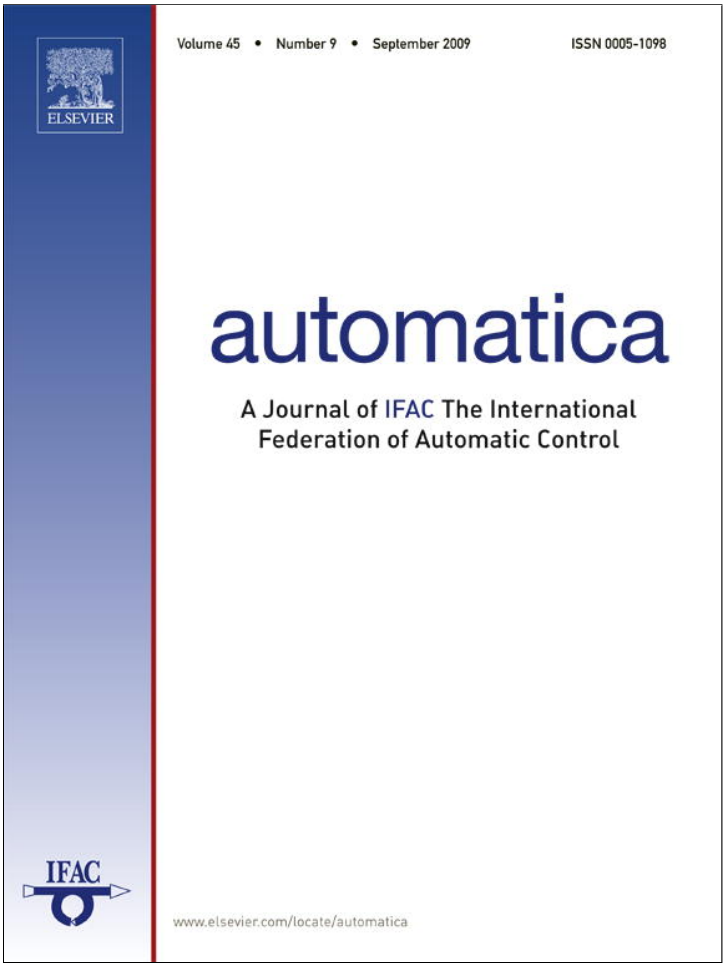
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Brief paper

An LMI approach to $\mathcal{H}_\infty$ boundary control of semilinear parabolic and hyperbolic systems[☆]

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ABSTRACT

Exponential stability analysis and L_2-gain analysis are developed for scalar uncertain distributed parameter systems, governed by semilinear partial differential equations of parabolic and hyperbolic types. Sufficient exponential stability conditions with a given decay rate are derived in the form of Linear Matrix Inequalities (LMIs) for both systems. These conditions are then utilized to synthesize $\mathcal{H}_\infty$ static output feedback boundary controllers of the systems in question.

1. Introduction

Many important plants, such as flexible manipulators and heat transfer processes are governed by partial differential equations and are often described by models with a significant degree of uncertainties. The existing results (Bensoussan, Da Prato, Delfour, & Mitter, 1993; Curtain & Zwart, 1995; Foias, Ozbay, & Tannenbaum, 1996; Ikeda, Azuma, & Uchida, 2001; van Keulen, 1993) on robust control of distributed parameter systems, operating under uncertainty conditions, extend the state space or the frequency domain $\mathcal{H}_\infty$ approach and are confined to the linear case. Thus it is of interest to develop consistent methods that are capable of utilizing nonlinear distributed parameter models and of providing the desired system performance in spite of significant model uncertainties. The LMI approach (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994) is definitely among such methods and its extension to uncertain distributed parameter systems is the primary concern of the present paper.

In our recent paper (Fridman & Orlov, 2009) we have introduced LMI approach to the stability analysis of linear heat and wave equations with the Dirichlet boundary conditions. In the present paper we extend the LMI approach to the Neumann boundary control stabilization and $\mathcal{H}_\infty$ control of uncertain distributed parameter systems. A conference version of the paper has been presented in Fridman and Orlov (2008).

The paper is organized as follows. Exponential stability analysis and L_2-gain analysis are developed side by side in Sections 2 and 3 for scalar parabolic and, respectively, hyperbolic systems. Sufficient exponential stability conditions with a given decay rate are derived in the form of LMIs for these systems. Capabilities of the LMI approach are then tested for designing $\mathcal{H}_\infty$ static output feedback boundary controllers of the systems in question.

1.1. Notation and preliminaries

The notation used throughout is fairly standard. The superscript 'T' stands for matrix transposition, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with the norm $|\cdot|$, $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$ with $P \in \mathbf{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$.

Functions, continuous in all arguments and, respectively, continuously differentiable in all arguments, are referred to as of class C and of class C^1.

$L_2(a, b)$ is the Hilbert space of square integrable functions $z(\xi)$, $\xi \in [a, b]$ with the corresponding norm $\|z\|_{L_2} = \sqrt{\int_a^b z^2(\xi)\mathrm{d}\xi}$. $L_2(0, \infty; L_2(a, b))$ is the Hilbert space of square

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integrable functions $w(\cdot, t) \in L_2(0, \infty)$ with values $w(\xi, \cdot) \in L_2(a, b)$ and with the corresponding norm. $W^{l,2}(a, b)$ is the Sobolev space of absolutely continuous scalar functions $z : [a, b] \to R$ with square integrable derivatives $z^{(l)}$ of the order $l \geq 1$ and with the norm $\|z\|^2_{W^{l,2}} = \int_a^b (z^{(l)})^2(\xi)\mathrm{d}\xi$.

For later use, we recall the following.

Lemma 1 (Wang, 1994). *Let $z \in W^{1,2}(a, b)$ be a scalar function with $z(a) = 0$. Then*

$$\int_a^b z^2(\xi)\mathrm{d}\xi \leq \frac{(b-a)^2}{2}\int_a^b (z'(\xi))^2\mathrm{d}\xi, \tag{1}$$

$$\max_{\xi\in[a,b]} z^2(\xi) \leq (b-a)\int_a^b (z'(\xi))^2\mathrm{d}\xi. \tag{2}$$

2. Boundary stabilization of a semilinear parabolic system

2.1. Exponential stability

Consider the parabolic equation

$$z_t(\xi, t) = \frac{\partial}{\partial\xi}[a(\xi)z_\xi(\xi, t)] + r_0(\xi, t, z(\xi, t))z(\xi, t) + r_1(\xi, t, z(\xi, t))z(1, t), \quad t \geq t_0, 0 \leq \xi \leq 1 \tag{3}$$

coupled to the mixed boundary condition

$$z(0, t) = 0, \qquad z_\xi(1, t) = -kz(1, t), \quad t \geq t_0, \tag{4}$$

where $t_0 \in R$ is an initial time instant, $k \geq 0$ is a parameter. Functions $a(\xi)$ and $r_i(\xi, t, z)$, $i = 0, 1$ are of class C^1 and may be unknown. These functions satisfy the inequalities

$$|r_i| \leq \beta_i, \qquad a \geq a_1 > 0 \tag{5}$$

for all $(\xi, t, z) \in [0, 1]\times R^2$ and for some constants $\beta_0 \geq 0$, $\beta_1 \geq 0$, $a_1 > 0$, known *a priori*. Hereinafter, the dependence on time t and spatial variable ξ is suppressed whenever possible and the functions a and r_i are written without arguments.

Subject to $r_1 \equiv 0$, Eq. (3) describes the nonlinear propagation of heat in a one-dimensional rod. Heat equation with the term $z(1, t)$ may describe the deviation from the steady state if the steady state depends on the boundary value (similar to that of Boskovic and Krstic (2003)). Due to the presence of the boundary-value term $z(1, t)$ in the state Eq. (3) with $r_1 \neq 0$, the above model particularly captures significant features of thermal instability in solid propellant rockets (Boskovic & Krstic, 2003).

Clearly, the boundary-value problem (3) and (4) can be rewritten as the differential equation

$$\dot{x}(t) = Ax(t) + F(t, x(t)), \quad t \geq t_0 \tag{6}$$

in the Hilbert space $\mathcal{H} = L_2(0, 1)$ where the infinitesimal operator $A = \frac{\partial\left[a(\xi)\frac{\partial}{\partial\xi}\right]}{\partial\xi}$ possesses the dense domain

$$\mathcal{D}(A) = \{x \in W^{2,2}(0, 1) : x(0) = 0, x_\xi(1) = -kx(1)\}, \tag{7}$$

and the nonlinear term $F : R \times W^{1,2}(0, 1) \to L_2(0, 1)$ is defined on potential solutions $x(\cdot, t)$ of (36) according to

$$F(t, x(\cdot)) = r_0(\xi, t, x(\xi, t))x(\xi, t) + r_1(\xi, t, x(\xi, t))\int_0^1 x_\zeta(\zeta, t)\mathrm{d}\zeta.$$

It is well known that the infinitesimal operator A generates an analytical exponentially stable semigroup $T(t)$, the induced norm $\|T(t)\|$ of which satisfies the inequality $\|T(t)\| \leq \kappa e^{-\delta t}$ everywhere with some constant $\kappa > 0$ and decay rate $\delta > 0$ (see, e.g., Curtain and Zwart (1995) for details). The domain $\mathcal{D}(A) = A^{-1}H$ of such an operator A forms another Hilbert space with the graph inner product $(x, y)_{\mathcal{D}(A)} = \langle Ax, Ay\rangle$, $x, y \in \mathcal{D}(A)$. The domain $\mathcal{D}(A)$ of A is thus continuously embedded into H, i.e., $\mathcal{D}(A) \subset H$, $\mathcal{D}(A)$ is dense in H and the inequality $|x| \leq \omega|Ax|$ holds for all $x \in \mathcal{D}(A)$ and some constant $\omega > 0$.

Apart from this, the square root $\sqrt{A}$ of the operator A is rigorously introduced on $\mathcal{D}(A)$ as a positive definite solution X of the algebraic operator equation $X^2 = A$. Being extended by continuity, this operator is well posed on the domain

$$\mathcal{D}(\sqrt{A}) = \{x \in W^{1,2}(0, 1) : x(0) = 0, x_\xi(1) = -kx(1)\}, \tag{8}$$

and continuously embedded into H whereas $\mathcal{D}(A)$ turns out to be continuously embedded into $\mathcal{D}(\sqrt{A})$. Thus, $\mathcal{D}(A) \subset \mathcal{D}(\sqrt{A}) \subset H$ and the following inequalities

$$|x| \leq \omega|\sqrt{A}x| \quad \text{for all } x \in \mathcal{D}(\sqrt{A}) \tag{9}$$

$$|\sqrt{A}x| \leq \omega|Ax| \quad \text{for all } x \in \mathcal{D}(A) \tag{10}$$

hold with a generic constant $\omega > 0$. All relevant background material on fractional operator degrees can be found, e.g., in Krasnoselskii, Zabreyko, Pustylnik, and Sobolevski (1976).

Since the functions r_0 and r_1 are smooth in their arguments the following Lipschitz condition

$$\|F(t_1, x_1) - F(t_2, x_2)\|_{L_2} \leq L[|t_1 - t_2| + \|\sqrt{A}(x_1 - x_2)\|_{L_2}] \tag{11}$$

on the nonlinear term F with some positive constant L is derived locally in $(t_i, x_i) \in R \times \mathcal{D}(\sqrt{A})$, $i = 1, 2$ by employing (1) and (2). Thus, Theorem 3.3.3 of Henry (1993) proves to be applicable to (6), and by applying this theorem, a unique strong solution of (6), initialized with $x(t_0) \in \mathcal{D}(\sqrt{A})$, is established to locally exist. The latter implies the local existence of the strong solution to the boundary-value problem (3) and (4) for an arbitrary initial condition

$$z(\xi, t_0) = \phi(\xi) \in \mathcal{D}(\sqrt{A}). \tag{12}$$

It is well known (Boskovic & Krstic, 2003) that the linear system (3) and (4) with $k = 0$ and with constant coefficients $r_0 = 0$, $a = 1$ and $r_1 > 2$ is unstable. We are looking for exponential stability conditions for uncertain nonlinear system (3) and (4) with $k \geq 0$.

Consider the following Lyapunov–Krasovskii functional

$$V(z(\cdot, t)) = \int_0^1 z^2(\xi, t)\mathrm{d}\xi. \tag{13}$$

We aim to find conditions guaranteeing that along the solutions $z(\xi, t)$ of (3) and (4) the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}V(z(\cdot, t)) + 2\delta V(z(\cdot, t)) \leq 0 \tag{14}$$

holds. By the comparison principle argument (Khalil, 1992), it would follow

$$\int_0^1 z^2(\xi, t)\mathrm{d}\xi = V(z(\cdot, t)) \leq V(z(\cdot, t_0))e^{-2\delta(t-t_0)} = e^{-2\delta(t-t_0)}\int_0^1 \phi^2(\xi)\mathrm{d}\xi.$$

By virtue of this, the solution of (3), (4) and (12) would be uniformly bounded in $L_2(0, 1)$ on its domain of existence and it would satisfy, due to (11), the following inequality

$$\|F(t, x)\|_{L_2} \leq L\|\sqrt{A}x\|_{L_2} \tag{15}$$

for all $t \geq t_0$. Then according to Exercise 1 on p. 58 of Henry (1993), such a solution of the boundary-value problem (3) and (4) would be globally continuable to the right. Hence, this solution would satisfy the inequality

$$\|z(\cdot,t)\|_{L_2} \leq e^{-\delta(t-t_0)}\|\phi(\cdot)\|_{L_2}, \quad \forall t \geq t_0, \tag{16}$$

thereby ensuring that the parabolic process (3) and (4) is exponentially stable in $L_2(0,1)$ with the decay rate δ.

Differentiating V along (3), integrating by parts and taking into account (4), we find that

$$\begin{aligned}
\frac{d}{dt}V + 2\delta V &= 2\int_0^1 z(\xi,t)z_t(\xi,t)d\xi + 2\delta\int_0^1 z^2(\xi,t)d\xi \\
&= 2\int_0^1 z(\xi,t)\left[\frac{\partial}{\partial\xi}[az_\xi(\xi,t)] + r_0 z(\xi,t) + r_1 z(1,t)\right]d\xi \\
&\quad + 2\delta\int_0^1 z^2(\xi,t)d\xi = -2kaz^2(1,t) - 2\int_0^1 az_\xi^2(\xi,t)d\xi \\
&\quad + 2\int_0^1(\delta + r_0)z^2(\xi,t)d\xi + 2r_1\int_0^1 z(\xi,t)z(1,t)d\xi \\
&\leq -2ka_1 z^2(1,t) - 2\int_0^1 a_1 z_\xi^2(\xi,t)d\xi \\
&\quad + 2(\delta+\beta_0)\int_0^1 z^2(\xi,t)d\xi + 2r_1\int_0^1 z(\xi,t)z(1,t)d\xi.
\end{aligned} \tag{17}$$

Applying inequality (1) yields

$$-2a_1\int_0^1 z_\xi^2 d\xi \leq -4a_1\int_0^1 z^2(\xi,t)d\xi.$$

We thus derive that

$$\begin{aligned}
\frac{d}{dt}V + 2\delta V &\leq \int_0^1 [z(\xi,t)\ z(1,t)]\,\Psi\,[z(\xi,t)\ z(1,t)]^T d\xi \\
&\leq 0
\end{aligned} \tag{18}$$

provided that the following LMI

$$\Psi \triangleq \begin{bmatrix} -4a_1 + 2(\delta+\beta_0) & r_1 \\ r_1 & -2ka_1 \end{bmatrix} \leq 0 \tag{19}$$

is feasible. Since LMI (19) is affine in r_1 and $r_1 \in [-\beta_1, \beta_1]$, the latter LMI is feasible if the following LMI

$$\begin{bmatrix} -4a_1 + 2(\delta+\beta_0) & \beta_1 \\ \beta_1 & -2ka_1 \end{bmatrix} \leq 0 \tag{20}$$

is feasible.

We note that the condition $\beta_0 < 2a_1$ is necessary for the feasibility of (20). For $\beta_1 = 0$ the system (3) and (4) is exponentially stable for all $k \geq 0$ with $\delta = 2a_1 - \beta_0$. For $\beta_1 > 0$ (3) and (4) is exponentially stable with the decay rate $0 < \delta < 2a_1 - \beta_0$ for large enough $k > 0$ that can be found from the inequality

$$-4a_1 + 2(\delta+\beta_0) + \frac{\beta_1^2}{2ka_1} \leq 0. \tag{21}$$

Summarizing, the following result is concluded.

Theorem 1. *Consider the boundary-value problem (3), (4) and (12) with the assumptions above and with $\beta_0 < 2a_1$. Given $\delta \in (0, 2a_1 - \beta_0]$, let there exist k such that LMI (20) is feasible. Then a unique strong solution of (3), (4) and (12) is globally continuable to the right and it satisfies (16).*

2.2. $\mathcal{H}_\infty$ boundary control

Let us, along with the homogeneous parabolic process (3), consider its perturbed version

$$\begin{aligned}
z_t(\xi,t) &= \frac{\partial}{\partial\xi}[az_\xi(\xi,t)] + r_0 z(\xi,t) + r_1 z(1,t) \\
&\quad + bw(\xi,t),\ t \geq t_0, 0 \leq \xi \leq 1
\end{aligned} \tag{22}$$

where $w(\xi,t) \in L_2(0,\infty; L_2(0,1))$ is an external disturbance; $b = b(\xi,t,z)$ is a function of class C^1, which is assumed to be uniformly bounded, i.e., $|b(\xi,t,z)| \leq b_1$ for all $(\xi,t,z) \in [0,1] \times R^2$ and some $b_1 > 0$.

While internally stabilizing the parabolic process, the influence of the admissible external disturbance $w(\xi,t) \in L_2(0,\infty; L_2(0,1))$ on the controlled output

$$\bar{z}(\xi,t) = [\alpha(\xi,t,z(\xi,t))z(\xi,t) \quad d(t,z(1,t))u(t)]^T, \tag{23}$$

is to be attenuated through the boundary actuation at $\xi = 1$:

$$z(0,t) = 0, \qquad z_\xi(1,t) = u(t), \quad t \geq t_0. \tag{24}$$

Hereinafter, $u(t)$ is the control input, d and α are continuous functions, which are uniformly bounded

$$|\alpha(\xi,t,z)| \leq \alpha_1, \qquad |d(t,z)| \leq d_1, \tag{25}$$

for all $(\xi,t,z) \in [0,1] \times R^2$, where $\alpha_1 \geq 0$ and $d_1 \geq 0$ are some constants. Collocated sensing $y(t) = z(1,t)$ at the boundary $\xi = 1$ is the only available information on the process.

The following $\mathcal{H}_\infty$ control problem is thus under study. Given $\gamma > 0$, it is required to find a linear static output feedback

$$u(t) = -kz(1,t), \tag{26}$$

that exponentially stabilizes the unperturbed process (4) and (22) and leads to a negative performance index

$$J = \int_{t_0}^{\infty}\int_0^1 [\bar{z}^T(\xi,t)\bar{z}(\xi,t) - \gamma^2 w^2(\xi,t)]d\xi dt < 0 \tag{27}$$

for all admissible external disturbances $0 \neq w(\xi,t) \in L_2(t_0,\infty; L_2(0,1))$, under which the solutions of (22) and (24), being initialized with the zero data $z(\xi,t_0) = 0$, are globally continuable to the right. We note that if $u(t)$ is stabilizing and w is C^1 in ξ, t, then by arguments of the previous section, the strong solutions of (22) and (24) exist and they are continuable for $t \geq t_0$.

In order to solve the problem we carry out conditions that guarantee the following:

$$\begin{aligned}
W(t) &\triangleq p\frac{d}{dt}V + \int_0^1 [\bar{z}^T(\xi,t)\bar{z}(\xi,t) - \gamma^2 w^2(\xi,t)]d\xi \\
&< 0,
\end{aligned} \tag{28}$$

where $p > 0$, V is given by (13) and the temporal derivative is computed along the trajectories of the closed-loop system (4) and (22). Then integrating (17) in t from t_0 to ∞ and taking into account that $V \geq 0$ and $V(0) = 0$ would yield (27).

It is worth noticing that

$$\int_0^1 \bar{z}^T(\xi,t)\bar{z}(\xi,t)d\xi \leq \int_0^1 \alpha_1^2 z^2(\xi,t)d\xi + d_1^2 k^2 z^2(1,t).$$

Then using (18) and (19) and setting $\zeta = [z(\xi,t)\ z(1,t)\ w(\xi,t)]^T$, we find that

$$W \leq \int_0^1 \zeta^T \Psi_\gamma \zeta d\xi < 0$$

if

$$\Psi_\gamma \triangleq \begin{bmatrix} -4a_1 p + 2\beta_0 p + \alpha_1^2 & r_1 p & bp \\ * & -2kap + d_1^2 k^2 & 0 \\ * & * & -\gamma^2 \end{bmatrix} < 0 \tag{29}$$

is feasible. By Schur complements, the latter inequality holds if

$$\begin{bmatrix} -4a_1p+2\beta_0 p+\alpha_1^2 & r_1 p & bp & 0 \\ * & -2ka_1 p & 0 & d_1 k \\ * & * & -\gamma^2 & 0 \\ * & * & * & -1 \end{bmatrix} < 0. \tag{30}$$

Multiplying (30) by $diag\{p^{-1}, p^{-1}, 1, 1\}$ from the right and from the left, we denote $q = p^{-1}$ and $g = p^{-1}k$. By Schur complements formula we arrive at

$$\begin{bmatrix} -4a_1q+2\beta_0 q & r_1 & b & 0 & q\alpha_1^2 \\ * & -2ga_1 & 0 & d_1 g & 0 \\ * & * & -\gamma^2 & 0 & 0 \\ * & * & * & -1 & 0 \\ * & * & * & * & -\alpha_1^2 \end{bmatrix} < 0. \tag{31}$$

LMI (31) is affine in r_1 and b and it is therefore feasible for all $r_1 \in [-\beta_1, \beta_1]$, $b \in [-b_1, b_1]$ if it is feasible for $r_1 = \pm\beta_1$ and $b = \pm b_1$, thereby yielding 4 LMIs. It is easy to see that these 4 LMIs are equivalent to the following LMI

$$\begin{bmatrix} -4a_1q+2\beta_0 q & \beta_1 & b_1 & 0 & q\alpha_1^2 \\ * & -2ga_1 & 0 & d_1 g & 0 \\ * & * & -\gamma^2 & 0 & 0 \\ * & * & * & -1 & 0 \\ * & * & * & * & -\alpha_1^2 \end{bmatrix} < 0. \tag{32}$$

Thus, we proved the following.

Theorem 2. *Consider the perturbed input–output system (22)–(24) with the assumptions above and with $\beta_0 < 2a_1$. Given $\gamma > 0$, let there exist $q > 0$ and g such that the LMI (32) is satisfied. Then the static output feedback (26) with $k = q^{-1}g$ internally exponentially stabilizes the boundary-value problem (22) and (24) and attenuates the admissible perturbations $w(\xi, t) \in L_2(0, \infty; L_2(0, 1))$ in the sense of (27).*

2.3. Example

Consider (22)–(25) with

$$a_1 = 1, \quad b_1 = 1, \quad \beta_0 = 1, \quad \beta_1 = 3,$$
$$d_1 = 0.1, \quad \alpha_1 = 1.$$

In this example $\beta_0 < 2a_1$ and $\beta_1 > 0$. Therefore, by Theorem 1 the static output feedback (26) with large enough $k > 0$ internally exponentially stabilizes the system which appears to be unstable for $k = 0$ (since $\beta_1 > 2a_1$; cf. Boskovic and Krstic (2003)). By using LMI toolbox of Matlab to verify the feasibility of LMI (32), we find that the static output feedback (26) with $k = 10.1744$ internally exponentially stabilizes the system and leads to the disturbance attenuation level $\gamma = 3$. Substituting the resulting k into (21), we find that this gain exponentially stabilizes the system with $\delta = 0.7789$.

A lower L_2-gain $\gamma = 1.1$ is achieved by a higher gain $k = 106.01$. The decay rate by the latter gain is found to be $\delta = 0.9788$.

3. Boundary stabilization of a semilinear hyperbolic equation

3.1. Exponential stability

Consider the hyperbolic equation

$$z_{tt}(\xi, t) = \frac{\partial}{\partial\xi}[az_\xi(\xi, t)] + r_0 z_t(\xi, t) + r_1 z_\xi(\xi, t),$$
$$t \geq t_0, 0 \leq \xi \leq 1 \tag{33}$$

coupled to the mixed boundary condition

$$z(0, t) = 0, \qquad z_\xi(1, t) = -kz_t(1, t), \quad t \geq 0, \tag{34}$$

with a parameter $k > 0$. where $k > 0$ is a parameter, $a = a(\xi)$, $r_0 = r_0(\xi, t, z, z_t)$, and $r_1 = r_1(\xi)$ are functions of class C^1. Subject to $r_1 \equiv 0$, Eq. (33) describes nonlinear oscillations of a string whereas its general form is of academic interest. As in the parabolic equation (3), the functions r_i, $i = 0, 1$ are admitted to be unknown subject to inequalities (5) that hold for all $(\xi, t, z, z_t) \in [0, 1] \times R^3$ with *a priori* known constants $\beta_i \geq 0$, $i = 0, 1$. Function a satisfies the bound

$$0 < a_1 \leq a(\xi) \leq a_2, \quad \forall \xi \in [0, 1] \tag{35}$$

with *a priori* known constants a_1, a_2.

To facilitate exposition, we have ignored restoring stiffness of the string, implicitly assuming that the corresponding term $r(\xi, t, z, z_t)z(\xi, t)$ is negligible. Since the above simplified model captures all the essential features of the general treatment, the extension to a hyperbolic model with a nontrivial stiffness is indeed possible.

The boundary-value problem (33) and (34) can be represented as the differential equation

$$\dot{x}(t) = \mathcal{A}x(t) + F(t, x_1(t), x_2(t)), \quad t \geq t_0 \tag{36}$$

in the Hilbert space $\mathcal{H} = W^{1,2}(0, 1) \times L_2(0, 1)$. In the above equation, the infinitesimal operator

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ \dfrac{\partial\left[a(\xi)\frac{\partial}{\partial\xi}\right]}{\partial\xi} + r_2\dfrac{\partial}{\partial\xi} & 0 \end{bmatrix} \tag{37}$$

possesses the dense domain

$$\mathcal{D}(\mathcal{A}) = \{(x_1, x_2) \in W^{2,2}(0, 1) \times L_2(0, 1) : x_i(0) = 0,$$
$$x_{i_\xi}(1) = -kx_i(1), i = 1, 2\} \tag{38}$$

and generates a strongly continuous semigroup whereas the second component

$$F_2(t, x_1, x_2) : R \times W^{1,2}(0, 1) \times L_2(0, 1) \to L_2(0, 1)$$

of the nonlinear term $F = (0, F_2)$ is defined on potential solutions $(x_1(\xi, t), x_2(\xi, t))^T$ of (6) according to

$$F_2(t, x_1, x_2) = r_0(\xi, t, x_1(\xi, t), x_2(\xi, t))x_2(\xi, t)$$
$$+ r_1(\xi, t, x_1(\xi, t), x_2(\xi, t))x_{1\xi}(\xi, t). \tag{39}$$

Since r_0 and r_1 are smooth, and hence, the following Lipschitz condition

$$\|F_2(t_1, x_{11}, x_{12}) - F_2(t_2, x_{21}, x_{22})\|_{L_2}$$
$$\leq L[|t_1 - t_2| + \|x_{11} - x_{21}\|_{W^{1,2}} + \|x_{21} - x_{22}\|_{L_2}] \tag{40}$$

holds locally in $(t_i, x_{i1}, x_{i2}) \in R \times W^{1,2}(0, 1) \times L_2(0, 1)$, $i = 1, 2$ with some generic constant $L > 0$, a unique strong solution of (36), initialized with $(x_1(t_0), x_2(t_0)) \in W^{1,2}(0, 1) \times L_2(0, 1)$, $x_i(0) = 0$, $x_{i_\xi}(1) = -kx_i(1)$ $(i = 1, 2)$ turns out to locally exist (see, e.g., Theorem 23.4 of Krasnoselskii et al. (1976)). Thus, there exists a unique local strong solution to the boundary-value problem (33) and (34) for an arbitrary initial condition

$$z(\xi, t_0) = \phi(\xi) \in W^{1,2}(0, 1) : \phi(0) = 0,$$
$$\phi_\xi(1) = -k\phi(1),$$
$$z_t(\xi, t_0) = \phi_1(\xi) \in L_2(0, 1) : \phi_1(0) = 0,$$
$$\phi_{1_\xi}(1) = -k\phi_1(1). \tag{41}$$

As in the heat equation case, only strong solutions of (33), (34) and (41) are under study.

We note that the linear system (33) and (34), specified with $k = 0, a = 1$, and $r_i = 0, i = 0, 1$, generates oscillating solutions and it is therefore asymptotically unstable. Our aim is to carry out exponential stability conditions for uncertain nonlinear system (33) and (34) with $k > 0$.

On solutions of (33) and (34), consider the Lyapunov–Krasovskii functional

$$V(z_\xi(\cdot,t), z_t(\cdot,t)) = \int_0^1 [z_\xi\ z_t]P[z_\xi\ z_t]^{\mathrm{T}} \mathrm{d}\xi, \tag{42}$$

proposed in Nicaise and Pignotti (2006) with some constants $p > 0$ and $\chi > 0$, and

$$P = \begin{bmatrix} ap & \chi\xi \\ \chi\xi & p \end{bmatrix} \geq \begin{bmatrix} a_1 p & \chi\xi \\ \chi\xi & p \end{bmatrix} > 0, \quad \forall \xi \in [0,1].$$

The latter inequality holds for all $\xi \in [0, 1]$ iff

$$\begin{bmatrix} a_1 p & \chi \\ \chi & p \end{bmatrix} > 0. \tag{43}$$

Our aim is to find conditions that would guarantee that along (33) the inequality $\frac{\mathrm{d}}{\mathrm{d}t}V + 2\delta V \leq 0$ holds. Then by the comparison principle argument (Khalil, 1992), it would follow that

$$V(z_\xi(\cdot,t), z_t(\cdot,t)) \leq V(z_\xi(\cdot,t_0), z_t(\cdot,t_0))\mathrm{e}^{-2\delta(t-t_0)}. \tag{44}$$

Due to (43) and (35), we have $0 < mI < P < MI$ for some scalars $0 < m < M$. Therefore, the solution of (33), (34) and (41) would satisfy the bound

$$\int_0^1 [z_\xi^2(\xi,t) + z_t^2(\xi,t)]\mathrm{d}\xi \leq \frac{M}{m}\mathrm{e}^{-2\delta(t-t_0)} \int_0^1 [\phi_\xi^2(\xi) + \phi_1^2(\xi)]\mathrm{d}\xi \tag{45}$$

and would be globally continuable to the right (see Theorem 23.5 of Krasnoselskii et al. (1976)).

For later use, we derive that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(2\int_0^1 \xi z_t z_\xi \mathrm{d}\xi\right) &= 2\int_0^1 \xi z_{tt} z_\xi \mathrm{d}\xi + 2\int_0^1 \xi z_t z_{\xi t}\mathrm{d}\xi \\
&= 2\int_0^1 \xi \frac{\partial}{\partial\xi}[az_\xi(\xi,t)]z_\xi \mathrm{d}\xi + 2\int_0^1 \xi z_t z_{\xi t}\mathrm{d}\xi \\
&\quad + 2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]z_\xi \mathrm{d}\xi \\
&= \int_0^1 \frac{1}{a}\xi\frac{\partial}{\partial_\xi}(az_\xi)^2 \mathrm{d}\xi + 2\int_0^1 \xi z_t z_{\xi t}\mathrm{d}\xi \\
&\quad + 2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]z_\xi \mathrm{d}\xi \\
&= -\int_0^1 az_\xi^2 \mathrm{d}\xi + a_{|\xi=1} z_\xi^2(1,t) + 2\int_0^1 \xi z_t(\xi,t) z_{\xi t}\mathrm{d}\xi \\
&\quad + 2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]z_\xi \mathrm{d}\xi.
\end{aligned}$$

After integrating by parts, we obtain

$$2\int_0^1 \xi z_t z_{\xi t}\mathrm{d}\xi = -2\int_0^1 \xi z_{\xi t} z_t \mathrm{d}\xi - 2\int_0^1 z_t^2 \mathrm{d}\xi + 2z_t^2(1,t).$$

Therefore, $2\int_0^1 \xi z_t z_{\xi t}\mathrm{d}\xi = -\int_0^1 z_t^2 \mathrm{d}\xi + z_t^2(1,t)$, that results in

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(2\int_0^1 \xi z_t z_\xi \mathrm{d}\xi\right) &= -\int_0^1 (z_t^2 + az_\xi^2)\mathrm{d}\xi + z_t^2(1,t) \\
&\quad + a_{|\xi=1} z_\xi^2(1,t) + 2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]z_\xi \mathrm{d}\xi.
\end{aligned}$$

Thus, differentiating V along (33), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V + 2\delta V &\leq 2p\int_0^1 az_\xi(\xi,t) z_{t\xi}(\xi,t)\mathrm{d}\xi \\
&\quad + 2p\int_0^1 z_t(\xi,t) z_{tt}(\xi,t)\mathrm{d}\xi + \frac{\mathrm{d}}{\mathrm{d}t}\left(2\chi\int_0^1 \xi z_t z_\xi \mathrm{d}\xi\right) \\
&\quad + \int_0^1 2\delta[apz_\xi^2(\xi,t) + 2\chi\xi z_\xi(\xi,t) z_t^2(\xi,t) + pz_t^2(\xi,t)]\mathrm{d}\xi \\
&= 2p\int_0^1 [az_\xi(\xi,t) z_{t\xi}(\xi,t) + z_t(\xi,t)\frac{\partial}{\partial\xi}[az_\xi(\xi,t)]]\mathrm{d}\xi \\
&\quad + 2p\int_0^1 z_t(\xi,t)[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]\mathrm{d}\xi \\
&\quad + \chi\left[-\int_0^1 (z_t^2 + az_\xi^2)\mathrm{d}\xi + z_t^2(1,t) + a_{|\xi=1}k^2 z_t^2(1,t)\right. \\
&\quad \left. + 2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]z_\xi \mathrm{d}\xi\right] \\
&\quad + \int_0^1 2\delta[az_\xi^2(\xi,t) + 2\chi\xi z_\xi(\xi,t) z_t(\xi,t) + z_t^2(\xi,t)]\mathrm{d}\xi.
\end{aligned}$$

Now integrating by parts and taking into account (33) and (34) yield

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V + 2\delta V &\leq -2a_{|\xi=1} kp z_t^2(1,t) \\
&\quad + 2p\int_0^1 z_t(\xi,t)[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]\mathrm{d}\xi \\
&\quad + \chi\left[-\int_0^1 (z_t^2 + az_\xi^2)\mathrm{d}\xi + (1 + a_{|\xi=1}k^2) z_t^2(1,t)\right. \\
&\quad \left. + 2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]z_\xi \mathrm{d}\xi\right] \\
&\quad + \int_0^1 2\delta[apz_\xi^2(\xi,t) + 2\chi\xi z_\xi(\xi,t) z_t(\xi,t) + pz_t^2(\xi,t)]\mathrm{d}\xi.
\end{aligned} \tag{46}$$

Taking into account

$$\begin{aligned}
&2\int_0^1 \chi\xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t)]z_\xi(\xi,t)\mathrm{d}\xi \\
&\leq \int_0^1 \xi\left[\frac{\chi^2\beta_0^2}{s_0} z_t^2(\xi,t) + s_0 + 2\chi\beta_1 z_\xi^2(\xi,t)\right]\mathrm{d}\xi
\end{aligned}$$

for some $s_0 > 0$, setting $\zeta^{\mathrm{T}}(\xi,t) = [z_t(1,t)\ z_\xi(\xi,t)\ z_t(\xi,t)]$ and using $a \geq a_1$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}V + 2\delta V \leq \int_0^1 \zeta^{\mathrm{T}}(\xi,t)\Psi\zeta(\xi,t)\mathrm{d}\xi \leq 0,$$

if

$$\Psi = \begin{bmatrix} \psi_1 & 0 & 0 \\ * & \psi_2 & 2\chi\delta\xi + pr_1 \\ * & * & \psi_3 + \dfrac{\chi^2\beta_0^2}{s_0}\xi \end{bmatrix} \leq 0 \tag{47}$$

where

$$\begin{aligned}
\psi_1 &= -2a_1 kp + (1 + a_1 k^2)\chi, \\
\psi_2 &= -a_1\chi + 2\delta a_1 p + s_0\xi + 2\chi\xi\beta_1, \\
\psi_3 &= -\chi + 2p\beta_0 + 2\delta p.
\end{aligned} \tag{48}$$

By Schur complements (47) holds if

$$\begin{bmatrix} \psi_1 & 0 & 0 & 0 \\ * & \psi_2 & 2\chi\delta\xi + pr_1 & 0 \\ * & * & \psi_3 & \beta_0\chi\xi \\ * & * & * & -s_0\xi \end{bmatrix} \leq 0. \tag{49}$$

It is worth noticing that given k, (49) is LMI which is affine in $\xi \in [0,1]$, $r_1 \in [-\beta_1, \beta_1]$. Therefore, LMI (49) is feasible if the following LMIs in the four vertices are feasible:

$$\begin{bmatrix} \psi_1 & 0 & 0 & 0 \\ * & \psi_2^{(j)} & 2\chi\delta\xi^{(j)} + pr_1^{(l)} & 0 \\ * & * & \psi_3 & \beta_0\chi\xi^{(j)} \\ * & * & * & -s_0 \end{bmatrix} \le 0,$$

$$j = 1,2;\ l = 1,2; \tag{50}$$

$$\psi_2^{(j)} = -a_1\chi + 2\delta a_1 p + s_0 + 2\chi\xi^{(j)}\beta_1,$$

$$r_1^{(1)} = \beta_1, \quad r_1^{(2)} = -\beta_1, \quad \xi^{(1)} = 0, \quad \xi^{(2)} = 1.$$

We note that for the stability analysis, $p = 1$ can be chosen. Moreover, feasibility of (50) implies $\psi_3 \le 0$ and, thus, $\chi > 0$. Summarizing, we obtain the following:

Theorem 3. *Given $k > 0$ and $\delta > 0$, let the LMIs (43) and (50) with notations (48) and $p = 1$ hold for some χ and s_0. Then a unique strong solution of the boundary-value problem (33), (34) and (41) is globally continuable to the right and it satisfies (45) for all $t \ge t_0$.*

3.2. $\mathcal{H}_\infty$ boundary control

In addition to the hyperbolic equation (33), let us now consider its perturbed version

$$z_{tt}(\xi,t) = \frac{\partial}{\partial\xi}[az_\xi] + r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t) + bw(\xi,t), \quad t \ge 0, 0 \le \xi \le 1 \tag{51}$$

where $w(\xi,t) \in L_2(0,\infty; L_2(0,1))$ is an external disturbance; $b = b(\xi,t,z)$ is a function of class C^1, which is assumed to be uniformly bounded, i.e., $|b(\xi,t,z)| \le b_1$ for all $(\xi,t,z) \in [0,1] \times R^2$ and some $b_1 > 0$. While internally stabilizing the hyperbolic process, the influence of the admissible external disturbance on the controlled output

$$\bar{z}(\xi,t) = [\alpha z(\xi,t) \quad \bar{\alpha} z_t(\xi,t) \quad du(t)]^{\mathrm{T}}, \tag{52}$$

is to be attenuated through the boundary actuation at $\xi = 1$:

$$z(0,t) = 0, \qquad z_\xi(1,t) = u(t), \quad t \ge t_0. \tag{53}$$

Hereinafter, $u(t)$ is the control input, $d = d(t, z_t(1,t))$ and $\alpha = \alpha(\xi,t,z,z_t)$, $\bar{\alpha} = \bar{\alpha}(\xi,t,z,z_t)$ are continuous functions, which are uniformly bounded

$$|\alpha(\xi,t,z,z_t)| \le \alpha_0, \qquad |\bar{\alpha}(\xi,t,z,z_t)| \le \alpha_1,$$
$$|d(t,z_t(1,t))| \le d_1,$$

for all $(\xi,t,z,z_t) \in [0,1] \times R^3$, where $\alpha_i \ge 0, i = 0,1$ and $d_1 \ge 0$ are some constants. Collocated sensing $y(t) = z_t(1,t)$ at the boundary $\xi = 1$ is the only available information on the process.

The $\mathcal{H}_\infty$ control problem of interest is stated as follows. Given $\gamma > 0$, find a linear static output feedback

$$u(t) = -kz_t(1,t), \tag{54}$$

that exponentially stabilizes the unperturbed system (33) and (34) and leads to a negative performance index

$$J = \int_{t_0}^{\infty}\int_0^1 [\bar{z}^{\mathrm{T}}(\xi,t)\bar{z}(\xi,t) - \gamma^2 w^2(\xi,t)]\mathrm{d}\xi\mathrm{d}t < 0 \tag{55}$$

for all admissible external disturbances $0 \ne w(\xi,t) \in L_2(0,\infty; L_2(0,1))$, under which the solutions of (51) and (53), being initialized with the zero data $z(\xi,t_0) = z_t(\xi,t_0) = 0$, are globally continuable to the right.

For solving the stated problem, let us find conditions that guarantee the following:

$$W(t) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}V + \int_0^1 [\bar{z}^{\mathrm{T}}(\xi,t)\bar{z}(\xi,t) - \gamma^2 w^2(\xi,t)]\mathrm{d}\xi < 0, \tag{56}$$

where V is given by (42), and the temporal derivative is computed along the closed-loop system (51) and (53). First, employing (1), we obtain

$$\int_0^1 \bar{z}^{\mathrm{T}}(\xi,t)\bar{z}(\xi,t)\mathrm{d}\xi$$
$$\le \int_0^1 \left[\alpha_0^2 z^2(\xi,t) + \alpha_1^2 z_t^2(\xi,t) + d_1^2k^2z_t^2(1,t)\right]\mathrm{d}\xi$$
$$\le \int_0^1 \left[\frac{1}{2}\alpha_0^2 z_\xi^2(\xi,t) + \alpha_1^2 z_t^2(\xi,t) + d_1^2k^2z_t^2(1,t)\right]\mathrm{d}\xi.$$

Then, in analogy to (46), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V \le -2a_{|\xi=1}kpz_t^2(1,t)$$
$$+ 2p\int_0^1 z_t(\xi,t)[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t) + bw]\mathrm{d}\xi$$
$$+ \chi\left[-\int_0^1 (z_t^2 + az_\xi^2)\mathrm{d}\xi + (1 + a_{|\xi=1}k^2)z_t^2(1,t)\right.$$
$$\left. + \ 2\int_0^1 \xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t) + bw]z_\xi \mathrm{d}\xi\right].$$

Along with this, for some $s_0 > 0$ and $s_1 > 0$, the following holds:

$$2\int_0^1 \chi\xi[r_0 z_t(\xi,t) + r_1 z_\xi(\xi,t) + bw]z_\xi(\xi,t)\mathrm{d}\xi$$
$$\le \int_0^1 \left[\frac{\chi^2\beta_0^2}{s_0} z_t^2(\xi,t) + \frac{\chi^2 b_1^2}{s_1}w^2 \right.$$
$$\left. + (s_0 + s_1 + 2\chi\beta_1)z_\xi^2(\xi,t)\right]\mathrm{d}\xi.$$

Finally, by taking into account $a \ge a_1$, we conclude that

$$W = \frac{\mathrm{d}}{\mathrm{d}t}V + \int_0^1 [\bar{z}^{\mathrm{T}}\bar{z} - \gamma^2 w^2]\mathrm{d}\xi \le \bar{\zeta}^{\mathrm{T}}\Psi_\gamma\bar{\zeta},$$

where

$$\bar{\zeta}^{\mathrm{T}} = [z_t(1,t)\ z_\xi(\xi,t)\ z_t(\xi,t)\ w(\xi,t)],$$

$$\Psi_\gamma = \begin{bmatrix} \psi_1 + d_1^2k^2 & 0 & 0 & 0 \\ * & \psi_{2\gamma} & pr_1 & 0 \\ * & * & \psi_{3\gamma} + \dfrac{\beta_0^2\chi^2}{s_0} & pb \\ * & * & * & -\gamma^2 + \dfrac{b_1^2\chi^2}{s_1} \end{bmatrix},$$

and

$$\psi_1 = -2a_1kp + (1 + a_1k^2)\chi,$$
$$\psi_{2\gamma} = -a_1\chi + s_0 + s_1 + \frac{1}{2}\alpha_0^2 + 2\chi\beta_1, \tag{57}$$
$$\psi_{3\gamma} = -\chi + 2p\beta_0 + \alpha_1^2.$$

Hence, $W < 0$ if $\Psi_\gamma < 0$, i.e. by Schur complements, if

$$\begin{bmatrix} \psi_1 + d_1^2k^2 & 0 & 0 & 0 & 0 & 0 \\ * & \psi_{2\gamma} & pr_1 & 0 & 0 & 0 \\ * & * & \psi_{3\gamma} & pb & \beta_0\chi & 0 \\ * & * & * & -\gamma^2 & 0 & b_1\chi \\ * & * & * & * & -s_0 & 0 \\ * & * & * & * & * & -s_1 \end{bmatrix} < 0. \tag{58}$$

LMI (58) is affine in $r_1 \in [-\beta_1, \beta_1]$, and $b \in [-b_1, b_1]$. Therefore, it is feasible if it holds in the vertices $r_1 = \pm\beta_1$ and $b = \pm b_1$. It is easy to see that the four LMIs in the vertices are equivalent to the following one

$$\begin{bmatrix} \psi_1 + d_1^2 k^2 & 0 & 0 & 0 & 0 & 0 \\ * & \psi_{2\gamma} & p\beta_1 & 0 & 0 & 0 \\ * & * & \psi_{3\gamma} & pb_1 & \beta_0\chi & 0 \\ * & * & * & -\gamma^2 & 0 & b_1\chi \\ * & * & * & * & -s_0 & 0 \\ * & * & * & * & * & -s_1 \end{bmatrix} < 0. \tag{59}$$

We note that if (59) is feasible, then the LMIs (50) for exponential stability hold with small enough $\delta > 0$. We thus proved the following.

Theorem 4. *Consider the perturbed input–output system (51)–(53) with the assumptions above. Given $\gamma > 0$ and $k > 0$, let there exist $p > 0$, χ, s_0 and s_1 such that the LMIs (43) and (59) are satisfied with the notations given by (57). Then the static output feedback (54) internally exponentially stabilizes the boundary-value problem (51) and (53) and attenuates the external disturbances $w(\xi, t) \in L_2(0, \infty; L_2(0, 1))$ in the sense of (55).*

3.3. Example

Consider (51)–(53) with

$$a_1 = 2, \quad \beta_0 = 0.2, \quad \beta_1 = 0.3, \quad \alpha_0 = \alpha_1 = 1, \quad d_1 = 0.1.$$

As mentioned above, the open-loop system is unstable. By using LMI toolbox of Matlab to verify LMIs (43) and (59), we find that the static output feedback (54) with $k = 1$ internally exponentially stabilizes the system and attenuates the external disturbances with $\gamma = 4.3$. By verifying (50) in the four vertices, we find that the resulting closed-loop system is internally exponentially stable with the decay rate $\delta = 0.13$.

4. Conclusions

In the present paper an LMI approach is extended to $\mathcal{H}_\infty$ boundary control of uncertain semilinear parabolic and hyperbolic systems. The uncertainties are admitted to be time-, space- and state-dependent with *a priori* known upper/lower bounds. Sufficient conditions for static output feedback stabilization are given in terms of LMIs. Numerical examples illustrate the efficiency of the method.

The proposed method seems to be extendible to dynamic output feedback $\mathcal{H}_\infty$ control and to other classes of distributed parameter systems. LMIs are thus expected to provide effective tools for robust control of distributed parameter systems.

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