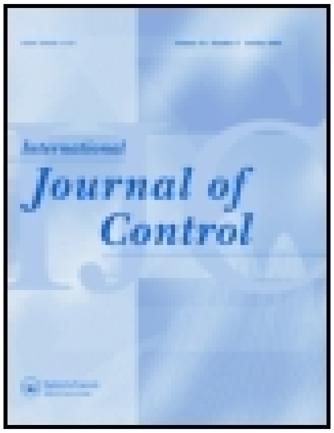

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Stability and passivity analysis of semilinear diffusion PDEs with time-delays

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Stability and passivity analysis of semilinear diffusion PDEs with time-delays

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In the present paper, sufficient conditions for the exponential stability and passivity analysis of nonlinear diffusion partial differential equations (PDEs) with infinite distributed and discrete time-varying delays are derived. Such systems arise in many applications, e.g. in population dynamics and in heat flows. The existing Lyapunov-based results on the stability of diffusion nonlinear PDEs treat either systems with infinite delays or the ones with discrete slowly varying delays (with the delay-derivatives upper bounded by $d < 1$), where the conditions are delay-independent in the discrete delays. In this paper, we introduce the Lyapunov-based analysis of semilinear diffusion PDEs with *fast-varying* (without any constraints on the delay-derivative) discrete and infinite distributed delays. Two novel methods are suggested leading to conditions in terms of linear matrix inequalities. The first one provides delay-independent with respect to discrete delays stability criterion via combination of Lyapunov–Krasovskii functionals and of the Halanay inequality. Note that the Halanay inequality is not applicable to the passivity analysis. Therefore, the second method develops the direct Lyapunov–Krasovskii method via the descriptor approach that leads to *delay-dependent* (in discrete delays) conditions for the exponential stability and passivity. Numerical examples illustrate the efficiency of the methods.

Keywords: diffusion PDEs; time-delays; Lyapunov–Krasovskii method

1. Introduction

Diffusion partial differential equations (PDEs) with time-delays are present in population dynamics (Capasso & Liddo, 1994; Kolmanovskii & Myshkis, 1999), heat flows (Fridman & Blighovsky, 2012) and other engineering applications. The existing results for the delayed diffusion PDEs are mostly restricted to delay-independent conditions with respect to the discrete delays and avoid the performance analysis.

The objective of the present paper is to derive simple and efficient conditions for the exponential stability and passivity analysis of diffusion PDEs with infinite distributed and discrete time-varying delays. Two approaches are presented leading to conditions in terms of linear matrix inequalities (LMIs). The first one provides delay-independent (with respect to discrete fast-varying delays) stability criterion, which is derived via a novel combination of a Lyapunov–Krasovskii functional (LKF) and of the Halanay inequality (Halanay, 1966). Note that the Halanay inequality is not applicable to the passivity and L_2-gain analysis. Moreover, delay-dependent with respect to discrete delays criteria via Krasovskii method are usually less restrictive. Therefore, the second method develops the direct Lyapunov–Krasovskii method via the descriptor approach, which leads to delay-dependent conditions for the exponential stability and passivity analysis. In our results the stabilising effect of the diffusion terms under the Dirichlet boundary conditions is taken into account with the help of Wirtinger's inequality. As a by-product, we provide also novel LMI conditions for the stability and passivity analysis of nonlinear ordinary differential equations (ODEs) with distributed and discrete delays. Numerical examples from the literature illustrate the efficiency of the presented methods and their advantages over the existing results.

The paper is organised as follows. Section 2 presents problem formulation and useful lemmas. Section 3 presents the first method for the exponential stability via a combination of an LKF (that corresponds to the distributed delay) with the Halanay inequality. Delay-dependent with respect to discrete delays stability conditions and the passivity analysis are presented in Sections 4.1 and 4.2, respectively. Finally, Section 5 presents three numerical examples that illustrate the efficiency of the presented methods.

Notation: The superscript 'T' stands for matrix transposition, the subscript 't' stands for differentiation with respect to time, subscript 'x_k' stands for differentiation with respect to spatial variable 'x_k', $k = 1, \dots, m$ and $\nabla_x = [\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}]^T$. $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with the norm $|\cdot|$, $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The notation $P > 0$, for $P \in \mathbf{R}^{n\times n}$ means that P is symmetric and positive definite, whereas $\lambda_{\min}(P)$ ($\lambda_{\max}(P)$) denotes its minimum (maximum)

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eigenvalue. The notation I_n stands for the identity $n \times n$ matrix and $\mathrm{diag}\{l_1, \dots, l_n\}$ stands for a diagonal $n \times n$ matrix with the scalars $l_1, \dots, l_n$ on its main diagonal. In symmetric block matrices we use $*$ for terms that are induced by the symmetry.

Let $\Omega = \{x = (x_1, \dots, x_m)^T\}$ be an open bounded domain in $\mathbf{R}^m$ with a smooth boundary $\partial\Omega$. $L_2(\Omega; \mathbf{R}^n)$ ($n =$ 1,2,...) is the Hilbert space of functions $z : \Omega \to \mathbf{R}^n$ with $\|z\|^2_{L^2} = \int_\Omega |z(x)|^2 dx < \infty$. $\mathbf{H}^1(\Omega)$ is the Sobolev space of absolutely continuous functions $z = [z_1, \dots, z_n]^T : \Omega \to \mathbf{R}^\mathbf{n}$ with $||z||^2_{H^1} = \sum_{i=1}^n ||\nabla_x z_i||^2_{L_2} < \infty$. The space of continuous functions $\phi : (-\infty, 0] \to \mathbf{H}^1(\Omega)$ with the norm $\|\phi\|_C = \sup_{s\in(-\infty,0]} ||\phi(\cdot, s)||_{H^1} < \infty$ is denoted by $C(-\infty, 0; \mathbf{H}^1(\Omega))$. The space of the continuously differentiable functions $\phi : (-\infty, 0] \to \mathbf{H}^1(\Omega)$ with the norm $\|\phi\|_{C^1} = \|\phi\|_C + \|\dot\phi\|_C < \infty$ is denoted by $C^1(-\infty, 0; \mathbf{H}^1(\Omega))$.

2. Problem formulation and useful inequalities

Consider the following system governed by the diffusion PDE

$$\begin{aligned} y_t(x,t) &= \Delta_D y(x,t) - Ay(x,t) + A_1 f(y(x,t)) \\ &\quad + A_2 g(y(x,t-\tau(t))) \\ &\quad + A_d \int_0^\infty K(s)\psi(y(x,t-s))\,ds + I, \\ &\quad t > 0 \end{aligned} \tag{1}$$

under the Dirichlet

$$y(x,t) = y^*, \quad (x,t) \in \partial\Omega \times (0,\infty), \tag{2}$$

or under the Neumann

$$\frac{\partial y(x,t)}{\partial x} = 0\,, \quad (x,t) \in \partial\Omega \times (0,\infty) \tag{3}$$

boundary conditions. Here $y(x,t) = [y_1(x,t) \dots y_n(x,t)]^T \in \mathbf{R}^n$ is the space state vector that depends on the spatial vector $x \in \Omega \subset \mathbf{R}^m$ and the time t, $y_t(x,t)$ stands for the vector derivative with respect to the time t. Vector $y^* \in \mathbf{R}^\mathbf{n}$ is a steady state solution of (1) (see assumption A1 below). Under the Dirichlet boundary conditions we assume that $\Omega = [0, a_1] \times \dots \times [0, a_m]$.

The diffusion term

$$\Delta_D y(x,t)^T = \left[\sum_{k=1}^m D_{1k}\frac{\partial^2 y_1(x,t)}{\partial x_k^2}, \dots, \sum_{k=1}^m D_{nk}\frac{\partial^2 y_n(x,t)}{\partial x_k^2}\right]$$

is a vector Laplacian with constant diffusion coefficients $D_{ik} \ge 0$, $i = 1, \dots, n$, $k = 1, \dots, m$. $A \in \mathbf{R}^{n\times n}$ is a self decay rate matrix, $A_1, A_2, A_d \in \mathbf{R}^{n\times n}$ are connection weight matrices, $K(s) \in L_1(0,\infty; [0,\infty))$ is a kernel function for the distributed delay and I is an external constant input. The continuous, nonlinear functions

$$\begin{aligned} f(y(x,t)) &= [f_1(y_1(x,t)), \dots, f_n(y_n(x,t))]^T, \\ g(y(x,t-\tau(t))) &= [g_1(y_1(x,t-\tau(t))), \dots, \\ &\qquad g_n(y_n(x,t-\tau(t)))]^T, \\ \psi(y(x,s)) &= [\psi_1(y_1(x,s)), \dots, \psi_n(y_n(x,s))]^T \end{aligned}$$

stand for the activation functions, where $\tau(t)$ is a time-varying discrete delay.

Note that in many cases a system with a matrix integrable kernel $K(\theta) \in \mathbf{R}^{n\times n}$ can be presented as a system with multiple delays and scalar kernels $K_i(s) \in [0,\infty)$

$$\begin{aligned} y_t(x,t) &= \Delta_D y(x,t) - Ay(x,t) + A_1 f(y(x,t)) \\ &\quad + A_2 g(y(x,t-\tau(t))) + \sum_{i=1}^q A_{di} \\ &\quad \times \int_0^\infty K_i(s)\psi(y(x,t-s))\,ds + I, \end{aligned} \tag{4}$$

where $A_{di} \in \mathbf{R}^{n\times n}$. The results will also be applicable to a finite distributed delay $h_d < \infty$, where $K(s) = 0$, $s > h_d$.

We assume the following:

A1: There exists a steady state solution $y^* \in \mathbf{R}^n$ of (1).

A2: The scalar functions $f_i(y_i(x,t))$, $g_i(y_i(x,t))$, $\psi_i(y_i(x,t))$, $i = 1, \dots, n$ are continuous and satisfy

$$\begin{aligned} f_i^- &\le \frac{f_i(u_1) - f_i(u_2)}{u_1 - u_2} \le f_i^+, \\ g_i^- &\le \frac{g_i(u_1) - g_i(u_2)}{u_1 - u_2} \le g_i^+, \\ \psi_i^- &\le \frac{\psi_i(u_1) - \psi_i(u_2)}{u_1 - u_2} \le \psi_i^+, \\ &\forall u_1 \ne u_2,\ u_1, u_2 \in \mathbf{R}^1,\ i = 1, \dots, n, \end{aligned}$$

with some scalars $f_i^-, f_i^+, g_i^-, g_i^+, \psi_i^-, \psi_i^+$.

A3: The kernel $K \in L_1(0,\infty; [0,\infty))$ satisfies the inequality

$$\int_0^\infty K(s)\,e^{2\delta_{\max}s}ds < \infty \tag{5}$$

with some $\delta_{\max} > 0$.

Under A1, we obtain the following differential equation with respect to the deviation $\hat y(x,t) = y(x,t) - y^*$ from the steady state solution y^* of Equation (1):

$$\begin{aligned} \hat y_t(x,t) &= \Delta_D \hat y(x,t) - A\hat y(x,t) + A_1 \hat f(y(x,t)) \\ &\quad + A_2 \hat g(y(x,t-\tau(t))) \\ &\quad + A_d \int_0^\infty K(s)\hat\psi(y(x,t-s))\,ds \end{aligned} \tag{6}$$

where

$$\hat{f}(y(x,t)) = f(y(x,t)) - f(y^*),$$
$$\hat{g}(y(x,t)) = g(y(x,t)) - g(y^*),$$
$$\hat{\psi}(y(x,t)) = \psi(y(x,t)) - \psi(y^*),$$

under the Dirichlet boundary conditions

$$\hat{y}(x,t) = 0\ ,\quad (x,t) \in \partial\Omega \times (0,\infty), \tag{7}$$

or the Neumann boundary conditions

$$\frac{\partial \hat{y}(x,t)}{\partial x} = 0\ ,\quad (x,t) \in \partial\Omega \times (0,\infty). \tag{8}$$

Denote

$$\begin{aligned} F_i &= \max\{|f_i^-|, |f_i^+|\},\ G_i = \max\{|g_i^-|, |g_i^+|\},\\ \Psi_i &= \max\{|\psi_i^-|, |\psi_i^+|\},\ i = 1,\dots,n,\\ F &= \mathrm{diag}\{F_1,\dots,F_n\},\quad G = \mathrm{diag}\{G_1,\dots,G_n\},\\ \Psi &= \mathrm{diag}\{\Psi_1,\dots,\Psi_n\}. \end{aligned}$$

Under A2 for all diagonal $n \times n$ matrices $S_i \geq 0, i = 1, 2, 3$, the following holds (Sabri, 2003; Zhou, Xu, Zhang, & Shen, 2012):

$$\begin{aligned} &\int_\Omega [\hat{y}^T(x,t)F^2 S_1 \hat{y}(x,t) - \hat{f}^T(y(x,t))S_1 \hat{f}(y(x,t))]dx \geq 0,\\ &\int_\Omega [\hat{y}^T(x,t)G^2 S_2 \hat{y}(x,t) - \hat{g}^T(y(x,t))S_2 \hat{g}(y(x,t))]dx \geq 0,\\ &\int_\Omega [\hat{y}^T(x,t)\Psi^2 S_3 \hat{y}(x,t) - \hat{\psi}^T(y(x,t))S_3 \hat{\psi}(y(x,t))]dx \geq 0. \end{aligned} \tag{9}$$

If additionally $f_i^-, g_i^-, \psi_i^- \geq 0,\ \forall i = 1,\dots,n$, then for all diagonal $n \times n$ matrices $S_4, S_5, S_6 \geq 0$, the following holds:

$$\begin{aligned} &2\int_\Omega [\hat{f}^T(y(x,t))F S_4 \hat{y}(x,t) - \hat{f}^T(y(x,t))S_4 \hat{f}(y(x,t))]dx \geq 0,\\ &2\int_\Omega [\hat{g}^T(y(x,t))G S_5 \hat{y}(x,t) - \hat{g}^T(y(x,t))S_5 \hat{g}(y(x,t))]dx \geq 0,\\ &2\int_\Omega [\hat{\psi}^T(y(x,t))\Psi S_6 \hat{y}(x,t) - \hat{\psi}^T(y(x,t))S_6 \hat{\psi}(y(x,t))]dx \geq 0. \end{aligned} \tag{10}$$

System (6) is said to be *exponentially stable with a decay rate* $\delta > 0$ if there exists a constant $b \geq 1$ such that the following exponential estimate holds for the solution of Equation (6) initialised with $\phi - y^* \in C(-\infty, 0; \mathbf{H}^1(\Omega))$:

$$\|\hat{y}(\cdot,t)\|^2_{L_2} \leq b e^{-2\delta t}\|\phi - y^*\|^2 \quad \forall t \geq 0, \tag{11}$$

where $\|\phi - y^*\| \triangleq \|\phi - y^*\|_C$. Here $\phi(x,\theta) = y(x,\theta)$, $x \in \Omega, \theta \in (-\infty, 0]$ is the initial function for Equation (1). The objective of the present paper is to derive sufficient conditions for the exponential stability of Equation (6). For $\phi - y^* \in C^1(-\infty, 0; \mathbf{H}^1(\Omega))$ we will find less restrictive exponential stability conditions, where in Equation (11) $\|\phi - y^*\| = \|\phi - y^*\|_{C^1}$. The results will further be extended to the passivity analysis.

Remark 1: A3 with $\delta_{\max} > 0$ is assumed for the exponential stability with a given decay rate. For the exponential stability with a small enough decay rate, $\delta_{\max}$ can be chosen to be 0 in Equation (5).

We present several useful lemmas:

Lemma 2.1 (**Halanay's inequality**; Halanay, 1966)**:** *Let* $0 < \delta_1 < \delta_0$ *and let* $V : [t_0 - h, \infty) \to [0, \infty)$ *be an absolutely continuous function that satisfies*

$$\dot{V}(t) \leq -2\delta_0 V(t) + 2\delta_1 \sup_{-h\leq\theta\leq 0} V(t+\theta),\ t \geq t_0.$$

Then

$$V(t) \leq e^{-2\delta(t-t_0)} \sup_{-h\leq\theta\leq 0} V(t_0+\theta), \quad t \geq t_0,$$

where $\delta > 0$ *is a unique positive solution of* $\delta = \delta_0 - \delta_1 e^{2\delta h}$.

Lemma 2.2 (**Wirtinger's inequality**; Tucsnak & Weiss, 2009)**:** *Suppose that* $\Omega = [0, a_1] \times \cdots \times [0, a_m]$, $f : \Omega \to \mathbf{R}$ *and* $f \in \mathbf{H}^1_0(\Omega)$*, where*

$$\mathbf{H}^1_0(\Omega) = \{f \in \mathbf{H}^1(\Omega) \mid f|_{\partial\Omega} = 0\},$$

then

$$C_p||f||_{L_2} \leq ||\nabla f||_{L_2}$$

Here $C_p = \sum_{i=1}^m \frac{\pi^2}{a_i^2}$.

Lemma 2.3 (**Extended Jensen's inequality**; Solomon & Fridman, 2013)**:** *Given an* $n \times n$ *matrix* $R > 0$*, a scalar function* $\alpha : [0,\infty) \to (0,\infty)$*, a scalar* $\tau \geq 0$ *and a vector function* $\phi : [0,\infty) \to \mathbf{R}^n$ *such that the integrations concerned are well defined. Then the following inequalities hold:*

$$\begin{aligned} &\int_0^\infty \alpha(\theta)|K(\theta)|\phi^T(\theta)R\phi(\theta)\,d\theta\\ &\quad\geq K_0^{-1}\int_0^\infty K(\theta)\phi^T(\theta)\,d\theta R \int_0^\infty K(\theta)\phi(\theta)\,d\theta,\\ &K_0 = \int_0^\infty \alpha^{-1}(\theta)|K(\theta)|d\theta \end{aligned} \tag{12}$$

and

$$\int_0^\infty \int_{t-\theta}^t \alpha(\theta)|K(\theta)|\phi^T(s)R\phi(s)ds\,d\theta \geq K_1{}^{-1}\int_0^\infty \int_{t-\theta}^t K(\theta)\phi^T(s)\,ds\,d\theta \times R\int_0^\infty \int_{t-\theta}^t K(\theta)\phi(s)\,ds\,d\theta,$$

$$K_1 = \int_0^\infty \alpha^{-1}(\theta)|K(\theta)|\theta d\theta. \quad (13)$$

3. Exponential stability via a combination of LKF and Halanay's inequality

In the absence of discrete delay, the following LKF

$$\begin{aligned} V(t) &= V_{P_1}(t) + V_{G_d}(t),\\ V_{P_1}(t) &= \int_\Omega \hat{y}^T(x,t)P_1\hat{y}(x,t)\,dx,\\ V_{G_d}(t) &= \int_\Omega \int_0^\infty \int_{t-\theta}^t K(\theta)e^{2\delta_0(s-t)}\hat{\psi}^T(y(x,s))\\ &\quad \times G_d\hat{\psi}(y(x,s))\,ds\,d\theta\,dx \end{aligned} \quad (14)$$

with positive $n \times n$ matrices P_1 and G_d can be used for the exponential stability analysis of Equation (6), where $g = \hat{g} = 0$ (i.e. of the system with the distributed delay only). The G_d-term of V generalises the similar construction of Solomon and Fridman (2013) and Kolmanovskii and Richard (1999) to the diffusion nonlinear PDEs. We will use the above $V(t)$ and Halanay's inequality to treat the discrete delay in Equation (6). Note that the combination of an LKF with the Halanay inequality was introduced in Fridman and Blighovsky (2012), where sampled-data stabilisation of a one-dimensional (1D) heat equation was studied.

Denote

$$K_0 = \int_0^\infty K(\theta)d\theta\ ,\ K_{\delta_0} = \int_0^\infty e^{2\delta_0\theta}K(\theta)d\theta.$$

Differentiation of Equation (14) along the trajectories of Equation (6) yields

$$\begin{aligned} \dot{V}(t) + 2\delta_0 V(t) &\leq 2\int_\Omega \hat{y}^T(x,t)P_1\Big[\Delta_D\hat{y}(x,t)\\ &\quad - A\hat{y}(x,t) + A_1\hat{f}(y(x,t)) + A_2\hat{g}(y(x,t-\tau(t)))\\ &\quad + A_d\int_0^\infty K(\theta)\hat{\psi}(y(x,t-\theta))\,d\theta\Big]\,dx\\ &\quad + \int_\Omega \hat{\psi}^T(y(x,t))K_0G_d\hat{\psi}(y(x,t))\,dx\\ &\quad - \int_\Omega\int_0^\infty K(\theta)e^{-2\delta_0\theta}\hat{\psi}^T(y(x,t-\theta))\\ &\quad \times G_d\hat{\psi}(y(x,t-\theta))\,d\theta\,dx\\ &\quad + 2\delta_0\int_\Omega \hat{y}^T(x,t)P_1\hat{y}(x,t)\,dx. \end{aligned} \quad (15)$$

By using the extended Jensen's inequality we have

$$\begin{aligned} &-\int_\Omega\int_0^\infty K(\theta)e^{-2\delta_0\theta}\hat{\psi}^T(y(x,t-\theta))G_d\hat{\psi}\\ &\times(y(x,t-\theta))\,d\theta\,dx\\ &\leq -K_{\delta_0}^{-1}\int_\Omega\int_0^\infty K(\theta)\hat{\psi}^T(y(x,t-\theta))\,d\theta G_d\\ &\times\int_0^\infty K(\theta)\hat{\psi}(y(x,t-\theta))\,d\theta\,dx. \end{aligned} \quad (16)$$

Denote $D_k = \text{diag}\{D_{1k}, \ldots, D_{nk}\}$. Taking into account the boundary conditions (7) or (8) and applying Green's formula, we obtain

$$\begin{aligned} &2\int_\Omega \hat{y}^T(x,t)P_1\Delta_D\hat{y}(x,t)\,dx\\ &= -2\sum_{k=1}^m\int_\Omega \hat{y}_{x_k}^T(x,t)D_kP_1\hat{y}_{x_k}(x,t)\,dx\\ &= -\sum_{k=1}^m\int_\Omega \hat{y}_{x_k}^T(x,t)(D_kP_1 + P_1D_k)\hat{y}_{x_k}(x,t)\,dx. \end{aligned} \quad (17)$$

Assume that for some $\lambda \geq 0$

$$D_kP_1 + P_1D_k \geq \lambda I, \quad k = 1, \ldots, m. \quad (18)$$

Then

$$\begin{aligned} &-\sum_{k=1}^m\int_\Omega \hat{y}_{x_k}^T(x,t)(D_kP_1 + P_1D_k)\hat{y}_{x_k}(x,t)\,dx\\ &\leq -\lambda\sum_{k=1}^m\int_\Omega \hat{y}_{x_k}^T(x,t)\hat{y}_{x_k}(x,t)dx\\ &\leq -\lambda\sum_{i=1}^n\int_\Omega \nabla_x^T\hat{y}_i(x,t)\nabla_x\hat{y}_i(x,t)dx\\ &\leq -\lambda C_p\int_\Omega \hat{y}^T(x,t)\hat{y}(x,t)\,dx, \end{aligned} \quad (19)$$

where $C_p = \sum_{i=1}^m \frac{\pi^2}{a_i^2}$ under the Dirichlet and $C_p = 0$ under the Neumann boundary conditions. Note that the last inequality of Equation (19) follows from the Wirtinger inequality (Lemma 2.2).

We further apply the Halanay inequality, where

$$\begin{aligned} &-2\delta_1\sup_{\theta\in[-h,0]} V(t+\theta) \leq -2\delta_1 V(t-\tau(t))\\ &\leq -2\delta_1\int_\Omega \hat{y}^T(x,t-\tau(t))P_1\hat{y}(x,t-\tau(t))\,dx. \end{aligned} \quad (20)$$

Denote

$$\eta(t) = \text{col}\Big\{\hat{y}(x,t),\ \hat{f}(y(x,t)),\ \hat{g}(y(x,t-\tau(t))),\ \int_0^\infty K(\theta)\hat{\psi}(y(x,t-\theta))\,d\theta),\ \hat{\psi}(y(x,t)\Big\}.$$

Taking into account Equations (15), (16), (19) and (20), and adding to $\dot V(t) + 2\delta V(t)$ the nonnegative left-hand sides of Equation (9) and, for the case of $f_i^-, g_i^- \geq 0$ and $\psi_i^- \geq 0$ $(i = 1, \ldots, n)$, the nonnegative left-hand sides of Equation (10), we arrive at

$$\dot V(t) + 2\delta_0 V(t) - 2\delta_1 \int_\Omega \hat y^T(x, t - \tau(t)) P_1 \hat y(x, t - \tau(t))\, dx$$

$$\leq \int_\Omega [\eta^T(t) \Upsilon \eta(t) + \hat y^T(x, t - \tau(t)) \Upsilon_0 \hat y(x, t - \tau(t))] dx \leq 0$$

if the following LMIs hold:

$$\Upsilon_0 = -2\delta_1 P_1 + G^2 S_2 < 0 \tag{21}$$

and

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & P_1A_1 + FS_4 & P_1A_2 + GS_5 & P_1A_d & \Psi S_6 \\ * & -S_1 - 2S_4 & 0 & 0 & 0 \\ * & * & -S_2 - 2S_5 & 0 & 0 \\ * & * & * & -K_{\delta_0}^{-1} G_d & 0 \\ * & * & * & * & K_0 G_d - S_3 - 2S_6 \end{bmatrix} < 0, \tag{22}$$

where

$$\Upsilon_{11} = 2\delta_0 P_1 - \lambda C_p I - (P_1 A + A^T P_1) + F^2 S_1 + \Psi^2 S_3. \tag{23}$$

Let $y(x, \theta) = \phi(x, \theta) \in C(-\infty, 0; \mathbf{H}^1(\Omega))$ be the initial condition for Equation (1). If Equations (21)–(23) are satisfied, then by Halanay's inequality

$$\lambda_{\min}(P) \int_\Omega |y(x, t) - y^*|^2 dx \leq V(t) \leq e^{-2\delta t} \sup_{-h \leq \theta \leq 0} V(\theta)$$

where

$$\sup_{-h\leq\theta\leq 0} V(\theta) \leq \lambda_{\max}(P) \sup_{-h\leq\theta\leq 0} \int_\Omega |\phi(x, \theta) - y^*|^2 dx$$

$$+ \lambda_{\max}(G_d) \sup_{-\infty\leq\theta\leq 0} \int_\Omega |\psi(\phi(x, \theta)) - \psi(y^*)|^2 dx$$

$$\times \int_0^\infty sK(s) ds$$

and $\delta = \delta_0 - \delta_1 e^{2\delta h}$. Therefore, the following bound is achieved:

$$\int_\Omega |y(x, t) - y^*|^2 dx \leq \beta e^{-2\delta t} \sup_{-\infty\leq\theta\leq 0} \int_\Omega |\phi(x, \theta) - y^*|^2 dx,$$

$$\beta = \left[\lambda_{\max}(P) + \lambda_{\max}(\Psi^2)\lambda_{\max}(G_d) \int_0^\infty sK(s) ds\right] \Big/ \lambda_{\min}(P),$$

i.e. the system (6) is exponentially stable with a decay rate δ. We have proved the following:

Theorem 3.1 *Assume A1–A3. Given $\delta_0 > \delta_1 > 0$, let there exist $n \times n$ matrices $P_1 > 0$ and $G_d > 0$, diagonal matrices $S_i > 0$ $(i = 1, \ldots, 6)$ and a scalar $\lambda \geq 0$ such that the LMIs* (18), (21) *and* (22) *with notation* (23) *are feasible. Then for all $\tau(t) \leq h$ and for f, g and ψ with nonnegative f_i^-, g_i^-, ψ_i^- $(i = 1, \ldots, n)$, the system* (6) *is exponentially stable with a decay rate δ, where δ is a unique solution of $\delta = \delta_0 - \delta_1 e^{2\delta h}$. If the above inequalities hold with $\delta_1 = \delta_0$ then the system* (6) *is exponentially stable with a small enough decay rate. Moreover, if the above LMIs hold with $S_4 = S_5 = S_6 = 0$, then the system* (6) *is exponentially stable for f, g and ψ with any sign of f_i^-, g_i^-, ψ_i^- $(i = 1, \ldots, n)$.*

Remark 2: The results can be easily extended to the case of multiple distributed delays as in Equation (4), under the assumption that there exists $\delta_{\max} > 0$ such that $\int_0^\infty K_i(\theta) e^{2\delta_{\max}\theta} d\theta < \infty$, $i = 1, \ldots, q$. Then the corresponding LMIs have the form

$$\Upsilon_0 = -2\delta_1 P_1 + G^2 S_2 < 0 \tag{24}$$

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & P_1A_1 + FS_4 & P_1A_2 + GS_5 & \ldots & P_1A_{dq} & \Psi S_6 \\ * & -S_1 - 2S_4 & 0 & \ldots & 0 & 0 \\ * & * & -S_2 - 2S_5 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & & -(K_{\delta_0}^q)^{-1} G_{dq} & 0 \\ * & * & * & & 0 & \sum_{i=1}^q K_{0i} G_{di} - S_3 - 2S_6 \end{bmatrix} < 0, \tag{25}$$

where Υ_{11} is given by Equation (23) and

$$K_{0i} = \int_0^\infty K_i(\theta) d\theta, \quad K_{\delta_0}^i = \int_0^\infty e^{2\delta_0\theta} K_i(\theta) d\theta.$$

Remark 3: The exponential stability criterion of Theorem 3.1 is novel also for finite-dimensional delayed models governed by ODEs with delays

$$\dot{y}(t) = -Ay(t) + A_1 f(y(t)) + A_2 g(y(t-\tau(t))) + A_d \int_0^\infty K(s)\psi(y(t-s))\,ds. \quad (26)$$

In this case the conditions of Theorem 3.1 with $\lambda = 0$ guarantee the exponential stability of Equation (26).

4. Delay-dependent stability and passivity: the descriptor approach

In this section we assume that the discrete delay is bounded from above: $\tau(t) \leq h$ and that it is either slowly varying (differentiable with $\dot{\tau}(t) \leq d < 1$) or fast-varying (without any constrains on the delay derivative). The function ψ is supposed to be differentiable.

Remark 4: For ODE systems with time-delay there are two main delay-dependent LKF-based methods (Fridman & Orlov, 2009) (for computation of $\dot{V}$): the first one is based on the direct substitution of the state derivative by the right-hand side of the ODE, the second one is the descriptor method (Fridman, 2001) that avoids this substitution and uses some additional free matrices P_2 and P_3. The first method for diffusion PDEs leads to the quadratic positive diffusion terms that may complicate the analysis. In 1D scalar case it was shown in Fridman and Orlov (2009) that the descriptor method, where (differently from ODE) P_3 appears in V, leads to efficient conditions for diffusion PDEs.

We will derive below the delay-dependent conditions by extending the descriptor method to m-D vector diffusion PDEs.

4.1 *Delay-dependent stability*

Consider the following LKF:

$$\begin{aligned}
V(t) &= V_{P_1}(t) + V_{P_3}(t) + V_R(t) + V_Q(t) + V_S(t) + V_{G_d}(t) + V_H(t),\\
V_{P_1}(t) &= \int_\Omega \hat{y}^T(x,t)P_1\hat{y}(x,t)\,dx,\\
V_{P_3}(t) &= \frac{1}{2}\sum_{k=1}^m \int_\Omega \hat{y}_{x_k}^T(x,t)M_k\hat{y}_{x_k}(x,t)\,dx,\\
M_k &= D_k P_3 + P_3^T D_k,\\
V_R(t) &= h\int_\Omega \int_{-h}^0 \int_{t+\theta}^t e^{2\delta(s-t)}\hat{y}_s^T(x,s)R\hat{y}_s(x,s)\,ds\,d\theta\,dx,\\
V_Q(t) &= \int_\Omega \int_{t-\tau(t)}^t e^{2\delta(s-t)}\hat{y}^T(x,s)Q\hat{y}(x,s)\,ds\,dx,\\
V_S(t) &= \int_\Omega \int_{t-h}^t e^{2\delta(s-t)}\hat{y}^T(x,s)S\hat{y}(x,s)\,ds\,dx,\\
V_{G_d}(t) &= \int_\Omega \int_0^\infty \int_{t-\theta}^t K(\theta)e^{2\delta(s-t)}\hat{\psi}^T(y(x,s))G_d\hat{\psi} \times (y(x,s))\,ds\,d\theta\,dx,\\
V_H(t) &= \int_\Omega \int_0^\infty \int_0^\theta \int_{t-\zeta}^t K(\theta)e^{2\delta(s-t)}\frac{\partial}{\partial s}\left[\hat{\psi}^T(y(x,s))\right] \times H\frac{\partial}{\partial s}\left[\hat{\psi}(y(x,s))\right]ds\,d\zeta\,d\theta\,dx
\end{aligned} \quad (27)$$

with $n \times n$ constant matrices $P_1 > 0$, $S > 0$, $R > 0$, $G_d > 0$, P_3 and a diagonal matrix $H > 0$. To guarantee that $V(t) \geq \alpha||\hat{y}||_{L_2}$ for some $\alpha > 0$ we assume the following:

$$M_k = D_k P_3 + P_3^T D_k \geq 0, \; k = 1, \ldots, m. \quad (28)$$

Here $V_{P_1}, V_{P_3}, V_R, V_Q$ and V_S extend the Lyapunov construction of Fridman and Orlov (2009) to n-D vector state. V_{G_d} and V_H are added to $V(t)$ to treat the distributed delay. The latter terms extend the constructions of Solomon and Fridman (2013), Chen and Zheng (2007) and Sun, Liu, and Chen (2009) to diffusion nonlinear PDEs.

Denote

$$\begin{aligned}
K_0 &= \int_0^\infty K(\theta)d\theta, \quad K_{0\delta} = \int_0^\infty e^{2\delta\theta}K(\theta)d\theta,\\
K_1 &= \int_0^\infty K(\theta)\theta d\theta, \quad K_{1\delta} = \int_0^\infty e^{2\delta\theta}K(\theta)\theta d\theta.
\end{aligned}$$

Differentiating Equation (27) along the trajectories of Equation (6) we have

$$\begin{aligned}
\dot{V}(t) + 2\delta V(t) \leq{}& 2\int_\Omega \hat{y}^T(x,t)P_1\hat{y}_t(x,t)\,dx\\
&+ \sum_{k=1}^m \int_\Omega \hat{y}_{x_k}^T(x,t)M_k\hat{y}_{x_k t}(x,t)\,dx\\
&+ h^2\int_\Omega \hat{y}_t^T(x,t)R\hat{y}_t(x,t)\,dx\\
&- h\int_\Omega \int_{t-h}^t e^{2\delta(s-t)}\hat{y}_s^T(x,s)R\hat{y}_s(x,s)\,ds\,dx\\
&+ \int_\Omega \hat{y}^T(x,t)S\hat{y}(x,t)\,dx - e^{-2\delta h} \times \int_\Omega \hat{y}^T(x,t-h)S\hat{y}(x,t-h)\,dx\\
&+ \int_\Omega \hat{y}^T(x,t)Q\hat{y}(x,t)\,dx - (1-d)e^{-2\delta h} \times \int_\Omega \hat{y}^T(x,t-\tau(t))Q\hat{y}(x,t-\tau(t))\,dx\\
&+ K_0\int_\Omega \hat{\psi}^T(y(x,t))G_d\hat{\psi}(y(x,t))\,dx\\
&- \int_\Omega \int_0^\infty K(\theta)e^{-2\delta\theta}\hat{\psi}^T(y(x,t-\theta))G_d\hat{\psi} \times (y(x,t-\theta))\,d\theta\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ 2\delta \int_\Omega \hat{y}^T(x,t)P_1\hat{y}(x,t)\,dx \\
&+ \delta \sum_{k=1}^m \int_\Omega \hat{y}^T_{x_k}(x,t)M_k\hat{y}_{x_k}(x,t)\,dx \\
&+ K_1 \int_\Omega \frac{\partial}{\partial t}\big[\hat{\psi}^T(y(x,t))\big]H\frac{\partial}{\partial t}\big[\hat{\psi}(y(x,t))\big]dx \\
&- \int_\Omega\int_0^\infty\int_{t-\theta}^t e^{-2\delta\theta}K(\theta)\frac{\partial}{\partial s}\big[\hat{\psi}^T(y(x,s))\big] \\
&\times H\frac{\partial}{\partial s}\big[\hat{\psi}(y(x,s))\big]dsd\theta dx. \qquad (29)
\end{aligned}
$$

Denote $\xi(x,t) = \mathrm{col}\{(\hat{y}(x,t) - \hat{y}(x,t-\tau(t)), (\hat{y}(x,t-\tau(t)) - \hat{y}(x,t-h))\}$. Following Park, Ko, and Jeong (2011) and by using the extended Jensen's inequality we obtain

$$
\begin{aligned}
&-h\int_\Omega\int_{t-h}^t e^{2\delta(s-t)}\hat{y}_s^T(x,s)R\hat{y}_s(x,s)\,ds\,dx \le \\
&-he^{-2\delta h}\int_\Omega\int_{t-h}^{t-\tau(t)}\hat{y}_s^T(x,s)R\hat{y}_s(x,s)\,ds\,dx \\
&-he^{-2\delta h}\int_\Omega\int_{t-\tau(t)}^{t}\hat{y}_s^T(x,s)R\hat{y}_s(x,s)\,ds\,dx \le \\
&-e^{-2\delta h}\int_\Omega\left[\frac{h}{\tau(t)}\xi^T(x,t)\begin{bmatrix} R & 0\\ 0 & 0\end{bmatrix}\xi(x,t)\right. \\
&\left.+\frac{h}{h-\tau(t)}\xi^T(x,t)\begin{bmatrix} 0 & 0\\ 0 & R\end{bmatrix}\xi(x,t)dx\right] \le \\
&-\int_\Omega e^{-2\delta h}\xi^T(x,t)\Gamma\xi(x,t)dx,
\end{aligned} \qquad (30)
$$

if

$$
\Gamma = \begin{bmatrix} R & S_{12}\\ * & R\end{bmatrix} \ge 0. \qquad (31)
$$

The distributed kernel term can be upper bounded by using Jensen's extended inequalities (12) and (13) as follows:

$$
\begin{aligned}
&-\int_\Omega\int_0^\infty K(\theta)e^{-2\delta\theta}\hat{\psi}^T(y(x,t-\theta))G_d\hat{\psi} \\
&\times(y(x,t-\theta))\,d\theta\,dx \le \\
&-K_{0\delta}^{-1}\int_\Omega\int_0^\infty K(\theta)\hat{\psi}^T(y(x,t-\theta))\,d\theta \\
&\times G_d\int_0^\infty K(\theta)\hat{\psi}(y(x,t-\theta))\,d\theta\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_\Omega\int_0^\infty\int_{t-\theta}^t e^{-2\delta\theta}K(\theta)\frac{\partial}{\partial s}\big[\hat{\psi}^T(y(x,s))\big] \\
&\times H\frac{\partial}{\partial s}\big[\hat{\psi}(y(x,s))\big]dsd\theta dx \le \\
&-K_{1\delta}^{-1}\int_\Omega\int_0^\infty\int_{t-\theta}^t K(\theta)\frac{\partial}{\partial s}\big[\hat{\psi}^T(y(x,s))\big]dsd\theta \\
&\times H\int_0^\infty\int_{t-\theta}^t K(\theta)\frac{\partial}{\partial s}\big[\hat{\psi}(y(x,s))\big]dsd\theta dx \\
&= -K_{1\delta}^{-1}\int_\Omega\zeta^T(x,t)\begin{bmatrix} K_0 I\\ -I\end{bmatrix}H[K_0 I \;\; -I]\zeta(x,t)dx,
\end{aligned}
$$

where $\zeta(x,t) = \mathrm{col}\{\hat{\psi}(y(x,t)) \;\; \int_0^\infty K(\theta)\hat{\psi}(y(x,t-\theta))d\theta\}$.

Note that $\hat{\psi}_y(y) = \mathrm{diag}\{\psi_{y_1}(y_1),\ldots,\psi_{y_n}(y_n)\}$ and, thus,

$$
\hat{\psi}_y^T(y(x,t))H\hat{\psi}_y(y(x,t))\hat{y}_t(x,t) \le \hat{y}_t^T(x,t)\Psi^2H\hat{y}_t(x,t).
$$

Then

$$
\begin{aligned}
&K_1\int_\Omega\frac{\partial}{\partial t}\big[\hat{\psi}^T(y(x,t))\big]H\frac{\partial}{\partial t}\big[\hat{\psi}(y(x,t))\big]dx \\
&= K_1\int_\Omega\hat{y}_t^T(x,t)\hat{\psi}_y^T(y(x,t))H\hat{\psi}_y(y(x,t))\hat{y}_t(x,t)dx \\
&\le K_1\int_\Omega\hat{y}_t^T(x,t)\Psi^2H\hat{y}_t(x,t)dx. \qquad (32)
\end{aligned}
$$

We take into account the dynamics of Equation (6) through the use of the descriptor method (Fridman, 2001; Fridman & Orlov, 2009), where we add to $\dot{V}(t) + 2\delta V(t)$ the right-hand side of

$$
\begin{aligned}
0 = 2\big[\hat{y}^T(x,t)P_2^T + \hat{y}_t^T(x,t)P_3^T\big]\Big[&-\hat{y}_t(x,t) + \Delta_D\hat{y}(x,t) \\
&- A\hat{y}(x,t) + A_1\hat{f}(y(x,t)) + A_2\hat{g}(y(x,t-\tau(t))) \\
&+ A_d\int_0^\infty K(s)\hat{\psi}(y(x,t-s))\,ds\Big] \qquad (33)
\end{aligned}
$$

with a free $n \times n$ matrix P_2. Taking into account the boundary conditions (7) or (8) and applying Green's formula, we obtain

$$
\begin{aligned}
&2\int_\Omega\hat{y}_t^T(x,t)P_3^T\Delta_D\hat{y}(x,t)\,dx \\
&= -2\sum_{k=1}^m\int_\Omega\hat{y}^T_{x_kt}(x,t)D_kP_3\hat{y}_{x_k}(x,t)\,dx \\
&= -\sum_{k=1}^m\int_\Omega\hat{y}^T_{x_k}(x,t)M_k\hat{y}_{x_kt}(x,t)\,dx. \qquad (34)
\end{aligned}
$$

When combined with Equation (29), Equation (34) cancels the term $\sum_{k=1}^m\int_\Omega\hat{y}^T_{x_k}(x,t)M_k\hat{y}_{x_{kt}}(x,t)\,dx$.

Similarly

$$
\begin{aligned}
&\int_\Omega\hat{y}^T(x,t)P_2^T\Delta_D\hat{y}(x,t) \\
&= -\sum_{k=1}^m\int_\Omega\hat{y}^T_{x_k}(x,t)[D_kP_2 + P_2^TD_k]\hat{y}_{x_k}(x,t)\,dx.
\end{aligned}
$$

Denote $J_k = D_k P_2 + P_2^T D_k$ and $\chi_k = J_k - \delta M_k = D_k(P_2 - \delta P_3) + (P_2 - \delta P_3)^T D_k$, where M_k and J_k are symmetric $n \times n$ matrices. Assume that for some $\lambda \geq 0$ and every D_k , $k = 1, \ldots, m$,

$$D_k(P_2 - \delta P_3) + (P_2 - \delta P_3)^T D_k \geq \lambda I. \quad (35)$$

Then by combining the terms

$$\delta \sum_{k=1}^{m} \int_\Omega \hat{y}_{x_k}^T(x,t) M_k \hat{y}_{x_k}(x,t)\,dx - \int_\Omega \hat{y}_{x_k}^T(x,t) J_k \hat{y}_{x_k}(x,t)\,dx = -\sum_{k=1}^{m} \int_\Omega \hat{y}_{x_k}^T(x,t)\chi_k \hat{y}_{x_k}(x,t)\,dx \quad (36)$$

and applying Wirtinger's inequality we arrive at

$$\begin{aligned} -\sum_{k=1}^{m} \int_\Omega \hat{y}_{x_k}^T(x,t)\chi_k \hat{y}_{x_k}(x,t)\,dx &\leq -\lambda \sum_{k=1}^{m} \int_\Omega \hat{y}_{x_k}^T(x,t)\hat{y}_{x_k}(x,t)\,dx \\ &\leq -\lambda \sum_{i=1}^{n} \int_\Omega \nabla_x^T \hat{y}_i^T(x,t)\nabla_x \hat{y}_i(x,t)\,dx \\ &\leq -\lambda C_p \int_\Omega \hat{y}^T(x,t)\hat{y}(x,t)\,dx. \end{aligned}$$

Further denoting

$$\eta(x,t) = \mathrm{col}\Big\{\hat{y}(x,t), \hat{y}_t(x,t), \hat{y}(x,t-h), \hat{y}(x,t-\tau(t)), \hat{f}(y(x,t)), \hat{g}(y(x,t-\tau(t))), \hat{\psi}(y(x,t)), \int_0^\infty K(\theta)\hat{\psi}(y(x,t-\theta))\,d\theta\Big\}$$

and adding to $\dot{V}(t) + 2\delta V(t)$ the nonnegative left-hand sides of Equation (9) and, for the case of $f_i^-, g_i^- \geq 0$ and $\psi_i^- \geq 0$ $(i = 1, \ldots, n)$, the nonnegative left-hand sides of Equation (10), we find that $\dot{V}(t) + 2\delta V \leq \int_\Omega \eta^T(t)\Psi\eta(t)dx \leq 0$ if the following LMI is feasible

$$\Psi = \begin{bmatrix} \Psi_{11} & P_1 - P_2^T - A^T P_3 & e^{-2\delta h} S_{12} & e^{-2\delta h}(R - S_{12}) & \Psi_{15} & \Psi_{16} & \Psi S_6 & P_2^T A_d \\ * & \Psi_{22} & 0 & 0 & P_3^T A_1 & P_3^T A_2 & 0 & P_3^T A_d \\ * & * & -e^{-2\delta h}(S+R) & e^{-2\delta h}(R - S_{12}^T) & 0 & 0 & 0 & 0 \\ * & * & * & \Psi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1 - 2S_4 & 0 & 0 & 0 \\ * & * & * & * & * & -S_2 - 2S_5 & 0 & 0 \\ * & * & * & * & * & * & \Psi_{77} & \Psi_{78} \\ * & * & * & * & * & * & * & \Psi_{88} \end{bmatrix} < 0 \quad (37)$$

Here

$$\begin{aligned} \Psi_{11} &= S + Q + 2\delta P_1 - e^{-2\delta h} R - (P_2^T A + A^T P_2) \\ &\quad - \lambda C_p I + F^2 S_1 + \Psi^2 S_3, \\ \Psi_{15} &= P_2^T A_1 + F S_4, \quad \Psi_{16} = P_2^T A_2 + G S_5, \\ \Psi_{22} &= h^2 R - P_3 - P_3^T + K_1 \Psi_d H, \\ \Psi_{44} &= G^2 S_2 - e^{-2\delta h}[2R - S_{12} - S_{12}^T + (1-d)Q], \\ \Psi_{77} &= K_0 G_d - S_3 - 2S_6 - K_{1\delta}^{-1} K_0^2 H, \\ \Psi_{78} &= K_{1\delta}^{-1} K_0 H, \quad \Psi_{88} = -K_{0\delta}^{-1} G_d - K_{1\delta}^{-1} H. \end{aligned} \quad (38)$$

Let $y(x,\theta) = \phi(x,\theta) \in C^1(-\infty, 0; \mathbf{H}^1(\Omega))$ be the initial condition for Equation (1). If Equations (37) and (38) are satisfied, then

$$\lambda_{\min}(P_1) \int_\Omega |y(x,t) - y^*|^2 dx \leq V(t) \leq e^{-2\delta t} V(0),$$

$$\begin{aligned} V(0) &\leq \lambda_{\max}(P_1) \int_\Omega |\phi(x,0) - y^*|^2 dx \\ &+ \frac{1}{2}\sum_{k=1}^{m} \lambda_{\max}(M_k) \int_\Omega |\phi_{x_k}(x,0)|^2 dx \\ &+ \Big[h(\lambda_{\max}(Q) + \lambda_{\max}(S)) + \lambda_{\max}(G_d)\lambda_{\max}(\Psi^2) \\ &\times \int_0^\infty sK(s)ds\Big] \sup_{-\infty \leq \theta \leq 0} \int_\Omega |\phi(x,\theta) - y^*|^2 dx \\ &+ \Big[\frac{h^3}{2}\lambda_{\max}(R) + \frac{1}{2}\lambda_{\max}(H\Psi^2) \\ &\times \int_0^\infty s^2 K(s)ds\Big] \sup_{-\infty \leq \theta \leq 0} \int_\Omega |\phi_\theta(x,\theta)|^2 dx, \end{aligned}$$

i.e. the system is exponentially stable with a decay rate δ. We have proved the following:

Theorem 4.1: *Assume A1–A3. Given $\delta > 0$, $h > 0$, $0 \leq d < 1$, let there exist $n \times n$ matrices $P_1 > 0$, $S > 0$, $R > 0$, $Q > 0$, $G_d > 0$, P_2, P_3 and S_{12}, diagonal $n \times n$ matrices $S_i > 0$ $(i = 1, \ldots, 6)$, $H > 0$ and a scalar $\lambda \geq 0$, such that the LMIs* (28), (31), (35) *and* (37) *with notation* (38) *are feasible. Then for all discrete delays $\tau(t) \in [0, h]$, the system* (6) *with nonnegative f_i^-, g_i^-, ψ_i^- $(i = 1, \ldots, n)$ is exponentially stable with a decay rate δ. If the LMI* (37) *is feasible for $Q = 0$, then the system* (6) *is exponentially stable for all fast-varying delays $\tau(t) \in [0, h]$. Moreover,* *if the above LMIs hold with* $S_4 = S_5 = S_6 = 0$, *then the system* (6) *is exponentially stable for any sign of* f_i^-, g_i^-, ψ_i^- $(i = 1, \dots, n)$.

Remark 5: The results can be easily extended to the case of multiple distributed delays as in Equation (4). Here it is assumed that there exists $\delta_{\max} > 0$ such that $\int_0^\infty K_i(\theta) e^{2\delta_{\max}\theta} d\theta < \infty$, $i = 1, \dots, q$. Then the corresponding LMI has the form

$$\begin{bmatrix} \Psi_{11} & P_1 - P_2^T - A^T P_3 & e^{-2\delta h} S_{12} & e^{-2\delta h}(R - S_{12}) & \Psi_{15} & \Psi_{16} & \Psi S_6 & \dots & P_2^T A_{dq} \\ * & \Psi_{22} & 0 & 0 & P_3^T A_1 & P_3^T A_2 & 0 & \dots & P_3^T A_{dq} \\ * & * & -e^{-2\delta h}(S + R) & e^{-2\delta h}(R - S_{12}^T) & 0 & 0 & 0 & \dots & 0 \\ * & * & * & \Psi_{44} & 0 & 0 & 0 & \dots & 0 \\ * & * & * & * & -S_1 - 2S_4 & 0 & 0 & \dots & 0 \\ * & * & * & * & * & -S_2 - 2S_5 & 0 & \dots & 0 \\ * & * & * & * & * & * & \Psi_{77} & \dots & \Psi_{7q} \\ * & * & * & * & * & * & * & \ddots & \Psi_{8q} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & * & * & * & \dots & \Psi_{qq} \end{bmatrix} < 0 \quad (39)$$

where

$$K_{0i} = \int_0^\infty K_i(\theta) d\theta\ ,\ K_{0\delta}^i = \int_0^\infty e^{2\delta\theta} K_i(\theta) d\theta,$$
$$K_{1i} = \int_0^\infty K_i(\theta)\theta d\theta\ ,\ K_{1\delta}^i = \int_0^\infty e^{2\delta\theta} K_i(\theta)\theta d\theta,$$

and

$$\begin{aligned} \Psi_{11} &= S + Q + 2\delta P_1 - e^{-2\delta h} R - (P_2^T A + A^T P_2) \\ &\quad -\lambda C_p I + F^2 S_1 + \Psi^2 S_3, \\ \Psi_{22} &= h^2 R - P_3 - P_3^T + \sum_{i=1}^q K_{1i} \Psi_d H_i, \\ \Psi_{44} &= G^2 S_2 - e^{-2\delta h}[2R - S_{12} - S_{12}^T + (1 - d)Q], \\ \Psi_{15} &= P_2^T A_1 + F S_4, \quad \Psi_{16} = P_2^T A_2 + G S_5, \\ \Psi_{77} &= \sum_{i=1}^q K_{0i} G_{di} - S_3 - 2S_6 - \sum_{i=1}^q (K_{1\delta}^i)^{-1} K_{0i}^2 H_i, \\ \Psi_{7q} &= (K_{1\delta}^q)^{-1} K_{0q} H_q, \quad \Psi_{8q} = 0, \\ \Psi_{qq} &= -(K_{0\delta}^q)^{-1} G_{dq} - (K_{1\delta}^q)^{-1} H_q. \end{aligned} \quad (40)$$

Remark 6: Consider the case of $A_d \in \mathbf{R}^{n\times n}$ that can be presented in the form of $A_d = D^T A_d^l C$, with $D \in \mathbf{R}^{l\times n}$, $C \in \mathbf{R}^{p\times n}$, $A_d^l \in \mathbf{R}^{l\times p}$ and $l, p < n$. Here the reduced-order LMIs can be derived by choosing LKFs as above, where H and G_d are changed by $C^T \bar{H} C$ (provided this matrix is diagonal) and $C^T \bar{G}_d C$, respectively, with the positive $\bar{H}, \bar{G}_d \in \mathbf{R}^{p\times p}$. Changing $\int_0^\infty K(\theta)\hat{\psi}(y(x, t - \theta))\, d\theta$ by $\int_0^\infty K(\theta) C \hat{\psi}(y(x, t - \theta))\, d\theta$ in $\eta(x, t)$, one can arrive at the modified LMIs. Thus, in the delay-independent LMIs (21) and (22) we obtain

$$\Upsilon_{14} = P_1 D^T A_d^l, \quad \Upsilon_{44} = -K_{\delta_0}^{-1} \bar{G}_d,$$
$$\Upsilon_{55} = K_0 C^T \bar{G}_d C - S_3 - 2S_6,$$

and in the delay-dependent condition (37) we have

$$\begin{aligned} \Psi_{18} &= P_2^T D^T A_d^l, \\ \Psi_{22} &= h^2 R - P_3 - P_3^T + K_1 \Psi^2 C^T \bar{H} C, \\ \Psi_{28} &= P_3^T D^T A_d^l, \\ \Psi_{77} &= K_0 C^T \bar{G}_d C - S_3 - 2S_6 - K_{1\delta}^{-1} K_0^2 C^T \bar{H} C, \\ \Psi_{78} &= K_{1\delta}^{-1} K_0 \bar{H} C^T, \quad \Psi_{88} = -K_{0\delta}^{-1} \bar{G}_d - K_{1\delta}^{-1} \bar{H}, \end{aligned}$$

4.2 *Passivity analysis*

Consider the following system:

$$\begin{aligned} \hat{y}_t(x, t) &= \Delta_D \hat{y}(x, t) - A\hat{y}(x, t) + A_1 \hat{f}(y(x, t)) \\ &\quad + A_2 \hat{g}(y(x, t - \tau(t))) \\ &\quad + A_d \int_0^\infty K(s) \hat{\psi}(y(x, t - s))\, ds + B_1 u(x, t), \end{aligned} \quad (41)$$

where the notations defined previously hold with the additional input $u(x, t) \in \mathbf{R}^q$ and $B_1 \in \mathbf{R}^{n\times q}$. Consider also the following output of the system:

$$z(x, t) = [C_1 \ \ C_2 \ \ C_3] \begin{bmatrix} \hat{y}(x, t) \\ \hat{f}(y(x, t)) \\ \hat{\psi}(y(x, t)) \end{bmatrix} + Zu(x, t), \quad (42)$$

where $C_1, C_2, C_3 \in \mathbf{R}^{q\times n}$ and $Z \in \mathbf{R}^{q\times q}$ are constant matrices.

It is said that system (41), (42) is passive if there exists a scalar $\gamma \geq 0$ such that,

$$2\int_{\Omega}\int_{0}^{t_p} z^T(x,t)u(x,t)\,dt\,dx \geq -\gamma \int_{\Omega}\int_{0}^{t_p} u^T(x,t)u(x,t)\,dt\,dx \quad \forall t_p \geq 0, \quad (43)$$

for all the solutions of system (41) with the zero initial conditions and under the boundary conditions (7) or (8).

Consider the LKF (27) with $\delta = 0$. Then the following inequality

$$\dot{V}(t) - 2\int_{\Omega} z^T(x,t)u(x,t)dx - \gamma \int_{\Omega} u^T(x,t)u(x,t)dx \leq 0 \quad (44)$$

yields Equation (43). Indeed, integration of Equation (44) implies

$$V(t_p) - V(0) - \gamma \int_{\Omega}\int_{0}^{t_p} u^T(x,t)u(x,t)dt\,dx \leq 2\int_{\Omega}\int_{0}^{t_p} z^T(x,t)u(x,t)dt\,dx, \quad \forall t_p > 0$$

and thus Equation (43), because $V(t_p) \geq 0$ and $V(0) = 0$.

By applying the descriptor method, we arrive at the LMIs (28), (31), (35) and

$$\Upsilon = \left[\begin{array}{c|c} & P_2^T B_1 - C_1^T \\ & P_3^T B_1 \\ \Psi & 0 \\ & 0 \\ & -C_2^T \\ & 0 \\ & -C_3^T \\ & 0 \\ \hline * & -\gamma I - Z - Z^T \end{array}\right] < 0. \quad (45)$$

We have proved the following:

Corollary 4.2: *Assume A1–A3. Given $\gamma > 0, \delta = 0$, $h > 0$ and $0 \leq d < 1$, let there exist $n \times n$ matrices $P_1 > 0, S > 0, R > 0, Q > 0, G_d > 0 \in R^{n\times n}$, full matrices $P_2, P_3, S_{12} \in R^{n\times n}$, diagonal matrices $n \times n$ matrices $S_i > 0$ $(i = 1, \ldots, 6)$, $H > 0$ and a scalar $\lambda \geq 0$, such that the LMIs* (28)*,* (31)*,* (35) *and* (45) *with notations* (37) *and* (40) *are feasible. Then the system* (6) *is internally exponentially stable and is passive for all discrete delays $\tau(t) \in [0, h]$ with $\dot{\tau}(t) \leq d$ and for f, g and ψ with nonnegative $f_i^-, g_i^-, \psi_i^- (i = 1, \ldots, n)$. If the LMI* (45) *is feasible for $Q = 0$, then Equation* (6) *is exponentially stable and passive for all fast-varying delays $\tau(t) \in [0, h]$. If the above LMIs hold with $S_4 = S_5 = S_6 = 0$, then Equation* (6) *is exponentially stable and passive for any sign of $f_i^-, g_i^-, \psi_i^- (i = 1, \ldots, n)$.*

Remark 7: The conditions of Theorem 4.1 and of Corollary 4.2, where the LMIs (28) and (35) are omitted and $\lambda = 0$, can be applied to the exponential stability and passivity analysis of the ODE delayed model (26). Moreover, for the ODE model the LMIs (37) and (45) can be modified by using additional free-weighting matrices. Thus, using P_4

$$0 = 2u^T(t)P_4^T\Big[-\dot{y}(t) - Ay(t) + A_1\hat{f}(y(t)) + A_2\hat{g}(y(t-\tau(t))) + A_d\int_0^{\infty} K(s)\hat{\psi}(y(t-s))ds + B_1u(t)\Big]$$

leads to the following modified terms in the last column and row of the matrix in LMI (45):

$$\Psi_{19} = P_2^T B_1 - C_1^T - A^T P_4,$$
$$\Psi_{59} = A_1^T P_4 - C_2^T, \Psi_{69} = A_2^T P_4,$$
$$\Psi_{89} = A_d^T P_4, \ \Psi_{99} = -\gamma I - Z - Z^T + P_4^T B_1 + B_1^T P_4.$$

Note that for the ODE model, the corresponding LMIs can be derived without the descriptor slack variables P_2 and P_3 – by the direct substitution of $\dot{y}(t)$ by the right-hand side of Equation (26). The resulting LMI for the passivity and stability analysis has the following form:

$$\begin{bmatrix} \Psi_{11} & e^{-2\delta h}S_{12} & e^{-2\delta h}(R - S_{12}) & \Psi_{14} & \Psi_{15} & \Psi S_6 & P_1A_d & \Psi_{18} & \Psi_{19} \\ * & -e^{-2\delta h}(S+R) & e^{-2\delta h}(R - S_{12}^T) & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Psi_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -S_1 - 2S_4 & 0 & 0 & 0 & -C_2^T & \Psi_{49} \\ * & * & * & * & -S_2 - 2S_5 & 0 & 0 & 0 & \Psi_{59} \\ * & * & * & * & * & \Psi_{66} & \Psi_{67} & -C_3^T & 0 \\ * & * & * & * & * & * & \Psi_{77} & 0 & \Psi_{79} \\ * & * & * & * & * & * & * & \Psi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Psi_{99} \end{bmatrix} < 0, \quad (46)$$

where

$$\begin{aligned}
\Psi_{11} &= S + Q + 2\delta P_1 - e^{-2\delta h} R - (P_1 A + A^T P_1) \\
&\quad + F^2 S_1 + \Psi^2 S_3, \\
\Psi_{14} &= P_1^T A_1 + F S_4, \quad \Psi_{15} = P_1^T A_2 + G S_5, \\
\Psi_{18} &= P_1 B_1 - C_1^T, \\
\Psi_{19} &= -A^T (Rh^2 + K_1 \Psi^2 H), \\
\Psi_{33} &= G^2 S_2 - e^{-2\delta h}[2R - S_{12} - S_{12}^T + (1-d)Q], \\
\Psi_{49} &= A_1^T (Rh^2 + K_1 \Psi^2 H), \quad \Psi_{59} = A_2^T (Rh^2 + K_1 \Psi^2 H), \\
\Psi_{66} &= K_0 G_d - S_3 - 2S_6 - K_{1\delta}^{-1} K_0^2 H, \quad \Psi_{67} = K_{1\delta}^{-1} K_0 H, \\
\Psi_{77} &= -K_{0\delta}^{-1} G_d - K_{1\delta}^{-1} H, \qquad \Psi_{88} = -\gamma I - Z - Z^T, \\
\Psi_{79} &= A_d^T (Rh^2 + K_1 \Psi^2 H), \quad \Psi_{99} = -(Rh^2 + K_1 \Psi^2 H).
\end{aligned} \tag{47}$$

Example 5.3 below illustrates that P_4 may improve the passivity analysis (comparatively to the direct substitution-based analysis).

For the passivity of discrete-time nonlinear systems with discrete and distributed delays see Wu, Shi, Su, and Chu (2011).

5. Examples

In this section, examples from the literature illustrate the efficiency of the presented methods.

Example 5.1 (Gue & Liu, 2011)**:** Consider the two cells finite-dimensional delayed neural-network

$$\begin{aligned}
\dot{y}(t) = &-\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} y(t) + \begin{pmatrix} \frac{1}{5} & \frac{1}{4} \\ 1 & \frac{3}{2} \end{pmatrix} f(y(t)) \\
&+ \begin{pmatrix} \frac{2}{5} & \frac{1}{2} \\ \frac{4}{5} & \frac{1}{2} \end{pmatrix} g(y(t-\tau(t))) \\
&+ \begin{pmatrix} \frac{1}{5} & \frac{2}{3} \\ \frac{2}{5} & 1 \end{pmatrix} \int_0^{h_d} K(\theta)\psi(y(t-\theta)) + \begin{pmatrix} 3 \\ 2 \end{pmatrix},
\end{aligned}$$

where $y(t) = [y_1(t), y_2(t)]^T$ denotes the potential (or voltage) of each cell at time t, $K(s) = 2se^{-s^2}$, $f_1(y_1(t)) = g_1(y_1(t)) = \psi_1(y_1(t)) = \cos(y_1(t)/3) + y_1(t)/3$ and $f_2(y_2(t)) = g_2(y_2(t)) = \psi_2(y_2(t)) = \cos(y_2(t)/2) + y_2(t)/4$. The coefficients of the matrices that multiply the nonlinear terms denote the strengths of connectivity between the cells. By using the Brouwer's fixed point theorem, it was shown in Gue and Liu (2011) that for $\tau(t) = 0.06|\sin(t)|$ and $h_d = 0.06$ the system has an exponentially stable steady state y^*.

By verifying the LMIs of the Halanay-based Theorem 3.1 we will show that y^* is exponentially stable for all $\tau(t)$ and essentially larger h_d and will find the resulting decay rate. Note that $f_2(y_2(t)) = g_2(y_2(t)) = \psi_2(y_2(t))$ has a sign changing time-derivative, so the LMIs are considered

Table 1. Example 1 – maximum decay rate for different delay upper bounds.

h	h_d	δ_0	δ_1	$\delta_{\max}$
0.4	0.1	1	0.307	0.5306
0.4	0.5	1	0.692	0.1926
0.4	0.8	0.5	0.341	0.1235

with $S_4 = S_5 = S_6 = 0$. Since this is the ODE model, we take the zero Wirtinger's constant: $C_p = 0$ (or $\lambda = 0$). The diagonal matrices F, G and Ψ are chosen with

$$F_1 = G_1 = \Psi_1 = \frac{2}{3}, \quad F_2 = G_2 = \Psi_2 = \frac{3}{4}.$$

It appears that for $\delta_0 = \delta_1 = 0.135$ the LMIs are feasible, i.e. the system preserves the exponential stability for $h_{d_{\max}} = 1.134$. We have also verified Theorem 3.1 for the exponential stability. Table 1 shows the maximum achievable decay rate, for different delay upper bounds h and h_d.

Example 5.2 (Wang & Zhang, 2010)**:** Consider the two cells neural-network in the 1D domain $\Omega = [0, \pi]$ governed by the diffusion PDE

$$\begin{aligned}
y_t(x,t) = \Delta_D y(x,t) &- \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} y(x,t) \\
&+ \begin{pmatrix} 0.2787 & 0.5743 \\ -0.7458 & -3.3207 \end{pmatrix} f(y(x,t)) \\
&+ \begin{pmatrix} 0.6465 & 0.4423 \\ -0.1892 & 0.2912 \end{pmatrix} g(y(x,t-\tau(t))) \\
&+ \int_0^\infty \begin{pmatrix} -1.5e^{-\theta} & 0.55e^{-1.1\theta} \\ 3.33e^{-0.9\theta} & -3.8e^{-1.9\theta} \end{pmatrix} \\
&\times \psi(y(x,t-\theta))d\theta.
\end{aligned}$$

Here $D_1 = \mathrm{diag}\{1, 1\}$ and

$$\begin{aligned}
f_i(y_i(x,t)) &= g_i(y_i(x,t)) = \psi_i(y_i(x,t)) \\
&= \tanh(y_i(x,t)), \ i = 1, 2.
\end{aligned}$$

In Wang and Zhang (2010) the delay-independent with respect to slowly varying delays LMIs were derived (see Theorem 3.1). It was found that the system under the Neumann boundary conditions remains globally asymptotically stable for all $\tau(t)$ with $\dot{\tau}(t) \le 0.5$.

We show below that the Halanay-based method guarantees the exponential stability of the system for all fast-varying delays. The system can be presented in the form of Equation (4) with four distributed delays, where

$$\begin{aligned}
&K_1(\theta) = e^{-\theta}, \ K_2(\theta) = 1.1e^{-1.1\theta}, \\
&K_3(\theta) = 0.9e^{-0.9\theta}, \ K_4(\theta) = 1.9e^{-1.9\theta}
\end{aligned}$$

Table 2. Example 2 – decay rate for $h = 1$.

	Halanay			Descriptor ($S_6 = 0$)	Descriptor ($S_6 \neq 0$)
	δ_0	δ_1	$\delta_{\max}$	$\delta_{\max}$	$\delta_{\max}$
Neumann	0.192	10^{-5}	0.192	0.196	0.231
Dirichlet	0.261	10^{-5}	0.261	0.265	0.284

and

$$
\begin{aligned}
A_{d1} &= \begin{pmatrix} -1.5 & 0 \\ 0 & 0 \end{pmatrix} = -1.5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1\ 0], \\
A_{d2} &= \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix} = 0.5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} [0\ 1], \\
A_{d3} &= \begin{pmatrix} 0 & 0 \\ 3.7 & 0 \end{pmatrix} = 3.7 \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1\ 0], \\
A_{d4} &= \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} = -2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0\ 1].
\end{aligned}
\tag{48}
$$

Here $\frac{d}{dt}\tanh(y_i) = \cosh^{-2}(y_i) \in (0, 1)$ and, thus, $F = G = \Psi = I_2$.

We verify the reduced-order LMIs with $l = 1$ (as explained in Remark 6), where the decision variables $G^1_{d1}, \ldots, G^1_{d4}$ and $H^1_1, \ldots, H^1_4$ are scalars. Under the Neumann ($C_p = 0, \lambda = 0$) and the Dirichlet ($C_p = 1$) boundary conditions, the Halanay-based conditions (Remark 2) with $\delta_0 = \delta_1 = 0.1$ and $S_1 = S_2 = S_6 = 0$ guarantee the exponential stability for all fast-varying discrete delays τ, whereas the descriptor-based conditions (Remark 5 with $\delta = 0$ and $S_1 = S_2 = S_6 = 0$) guarantee the exponential stability for all $\tau(t) \leq h \approx 10^{17}$. Note that nonzero S_1, S_2 and S_6 in this example do not improve the results. However, the LMIs with $S_4 = S_5 = 0$ and nonzero S_1, S_2, S_3 appeared to be unfeasible.

Table 2 compares the maximum decay rate achieved using the Halanay approach and the descriptor approach for fast-varying delays and $h = 1$ under the both boundary conditions. It is seen that for the both approaches, the decay rate under the Dirichlet boundary conditions is larger than the one under the Neumann boundary conditions. Moreover, the descriptor-based approach may yield larger values for the decay rate δ over the Halanay-based one. However, the latter approach has less decision variables. Thus, under the Neumann boundary conditions the number of the scalar decision variables for the asymptotic stability is 13 for Wang and Zhang (2010) and 14 for the Halanay-based exponential stability conditions (including the tuning parameter $\delta_0 = \delta_1$), whereas it is 40 for the descriptor-based LMIs (including S_6).

Example 5.3: Consider the following model of a two cells neural-network:

$$
\begin{aligned}
y_t(x,t) &= \Delta_D y(x,t) - \begin{pmatrix} 2.2 & 0 \\ 0 & 1.8 \end{pmatrix} y(x,t) \\
&+ \begin{pmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{pmatrix} f(y(x,t)) \\
&+ \begin{pmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{pmatrix} g(y(x,t-\tau(t))) \\
&+ A_d \int_0^{h_d} K(\theta)\psi(y(x,t-\theta))d\theta + u(x,t), \\
z(x,t) &= f(y(x,t))
\end{aligned}
\tag{49}
$$

with $y(x,t) = [y_1(x,t),\ y_2(x,t)]^T$, $f_i(y) = g_i(y) = 0.5(|y+1| - |y-1|)$ $(i = 1, 2)$, $x \in \Omega \subset \mathbf{R}^m$ under the Neumann boundary conditions. The activation functions have nonnegative f_i^-, g_i^- and $F = G = I_2$. The finite-dimensional counterpart of Equation (49) with $y = y(t)$ and $A_d = 0$ (i.e. ODE with discrete delays) was analysed in Xu, Zheng, and Zou (2009). It was found that for the slowly varying delay with $\dot{\tau} \leq 0.2$ the system is exponentially stable and, thus, passive for $\tau(t) \leq 0.4683$. Here Theorem 4.1 guarantees the internal exponential stability for all $\tau(t) \leq h \simeq 10^{19}$. For $h = 10$, Theorem 4.1 leads to a larger decay rate $\delta = 0.1999$ than the one $\delta = 0.023$ guaranteed by Theorem 3.1.

Consider next a nonzero distributed delay matrix $A_d = \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$, $\psi_i(y_i(x,t)) = \tanh(y_i(x,t))$ $(i = 1, 2)$, i.e. $\Psi = I_2$, under Neumann boundary conditions and $K(\theta) = 1$. For $h = 0.4683$, $d = 0.2$ and $h_d = 0.5$ the minimum values of γ guaranteed by Corollary 4.2 are given in Table 3. It is seen that in the case of Hurwitz A_d, the additional H-term

Table 3. Example 3 – passivity factor γ.

PDE	$G_d, H \neq 0$	$G_d = 0, H \neq 0$	$G_d \neq 0, H = 0$	ODE	LMI (45), $P_4 \neq 0$	LMI (46)
γ	0.517	0.626	Not feasible	γ	0.147	0.403

in V may lead to a better performance (to smaller values of γ) than the use of the G_d-term only.

Finally for the case of the ODE model with $h = 0.4683$, $d = 0.2$, $h_d = 0.5$ and $H \neq 0$, $G_d \neq 0$, the maximum decay rate (of the input-free model) achieved by the LMI (46) of Remark 7 and the descriptor method was the same: $\delta_{\max} = 0.155$. For the passivity analysis, the LMI (46) guarantees a smaller value of $\gamma = 0.403$ comparatively to $\gamma = 0.517$ that was achieved via the descriptor method with P_2 and P_3 only (see Table 3). The LMI of Remark 7 with the additional matrix P_4 essentially improves the passivity analysis leading to a smaller value of $\gamma = 0.147$ comparatively to $\gamma = 0.403$ that follows from Equation (46).

6. Conclusions

In the present paper, simple LMI conditions have been derived for the exponential stability and passivity of nonlinear, infinite-dimensional diffusion PDEs with infinite distributed and discrete time-varying delays. Such systems are motivated by various applications in biology and engineering, such as population dynamics and heat transfer processes. For the first time, the exponential stability and the passivity analysis have been provided for diffusion PDEs with fast-varying discrete and infinite distributed delays. Two novel Lyapunov-based methods have been developed: the delay-independent with respect to the discrete-delay that combines the LKFs with the Halanay inequality, and the direct descriptor-based Lyapunov–Krasovskii method that has been applied both, to the stability and to the passivity analysis. The stabilising effect of diffusion terms under the Dirichlet boundary conditions has been taken into account by using Wirtinger's inequality. As a by-product, new LMI conditions have been derived for the stability and passivity analysis of nonlinear ODEs with distributed and discrete delays. Three numerical examples have illustrated the efficiency of the new methods.

Topics for future research may include application of the presented stability and passivity analysis to different control problems.

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