# New Bounded Real Lemma Representations for Time-Delay Systems and Their Applications

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***Abstract*—New delay-dependent/delay-independent bounded real criteria are derived for linear continuous-time systems with delay in the dynamics and in the objective function. A sufficient condition for the system to possess a $H_\infty$-norm that is less than a prescribed level, is given in terms of a linear matrix inequality (LMI). The proposed criteria are less conservative than other existing criteria since they are based on an augmented model that is equivalent to the original system and since they require bounds for fewer terms. We apply the new bounded real criteria to state-feedback $H_\infty$ control. The advantage of the new criteria is demonstrated by four examples. The first two compare our results with those obtained in the literature for the bounded real lemma and the other two examples treat the state-feedback control problem and compare our result with recently published designs.**

***Index Terms*—Bounded real lemma, delay-dependent criteria, $H_\infty$-state-feedback control, linear matrix inequalities (LMIs), time-delay systems.**

## I. INTRODUCTION

It is well-known (see, e.g., [1]–[3]) that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving stability criteria. The same is true concerning bounded real criteria. Thus the general form of this functional leads to a complicated system of Riccati type partial differential equations [4], [5] or inequalities [6]. Special forms of Lyapunov–Krasovskii functionals lead to simpler (but more conservative) delay-independent [7]–[12] and delay-dependent sufficient conditions [10], [11], [13]. In this note, we derive a delay-dependent/delay-independent bounded real lemma (BRL).

The conservatism of the delay-dependent criteria of [11] and [13] is twofold: the transformed system is not equivalent to the original one (see [14]) and the bounds put on certain terms, when developing the required criteria, are quite wasteful. In the present note we apply a new type of Lyapunov–Krasovskii functional based on an equivalent augmented model—a "descriptor form" representation of the system. Such a representation has been introduced in [3] for stability analysis. Our approach significantly reduces the overdesign entailed in the existing methods since it is based on a model that is equivalent to the original system and since fewer bounds are applied. We demonstrate the applicability of our approach with three examples, where we compare our method to the most efficient method published to date.

*Notation:* Throughout this note, the superscript "$T$" stands for matrix transposition, $\mathcal{R}^n$ denotes the $n$ dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that $P$ is symmetric and positive–definite. The space of functions in $\mathcal{R}^q$ that are square integrable over $[0\ \ \infty)$ is denoted by $\mathcal{L}_2^q[0,\ \infty)$.

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## II. THE BOUNDED REAL LEMMA

### A. Delay-Dependent BRL

Given the following system:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{m} A_i x(t-h_i) + B_1 w(t)\\ z(t) &= col\{C_0 x(t),\ C_1 x(t-h_1),\ \ldots,\ C_m x(t-h_m)\}\\ x(t) &= 0 \qquad \forall t\in[-h\ 0]\end{aligned}\tag{1}$$

where $x(t)\in\mathcal{R}^n$ is the system state vector, $w(t)\in\mathcal{L}_2^q[0,\ \infty]$ is the exogenous disturbance signal and $z(t)\in\mathcal{R}^p$ is the state combination (objective function signal) to be attenuated. The time delays $0=h_0<h_i\le h$, $i=1,\ldots,m$ are assumed to be known and the matrices $A_i,\ i=0,\ldots,m$, $B_1$ and $C_i,\ i=0,\ldots,m$ are constant matrices of appropriate dimensions. For a prescribed scalar $\gamma>0$, we define the performance index

$$J(w)=\int_0^\infty (z^T z-\gamma^2 w^T w)\,d\tau. \tag{2}$$

Following [3], we represent (1) in the equivalent descriptor form

$$\dot{x}(t)=y(t),\qquad y(t)=\sum_{i=0}^{m}A_i x(t-h_i)+B_1 w(t). \tag{3}$$

The latter is equivalent to the following descriptor system with distributed delay in the variable $y$:

$$\begin{aligned}\dot{x}(t)=y(t),\quad 0=-y(t)+\left(\sum_{i=0}^{m}A_i\right)x(t)\\ -\sum_{i=1}^{m}A_i\int_{t-h_i}^{t}y(s)\,ds+B_1 w(t).\end{aligned}\tag{4}$$

A Lyapunov–Krasovskii functional for the system (4) has the form

$$\begin{aligned}V(t)=[x^T(t)\ y^T(t)]EP\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}\\ +\sum_{i=1}^{m}\int_{-h_i}^{0}\int_{t+\theta}^{t}y^T(s)R_i y(s)\,ds\,d\theta\end{aligned}\tag{5}$$

where

$$E=\begin{bmatrix}I_n & 0\\ 0 & 0\end{bmatrix}\quad P=\begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix},\qquad P_1>0,\qquad R_i>0. \tag{6}$$

The second term of (5) corresponds to the delay-dependent condition that we shall derive below.

We obtain the following.

*Theorem 2.1:* Consider the system of (1). For a prescribed $\gamma>0$, the cost function (2) achieves $J(w)<0$ for all nonzero $w\in\mathcal{L}_2^q[0,\ \infty)$, if there exist $n\times n$-matrices $0<P_1,\ P_2,\ P_3$, and $R_i=R_i^T$, $i=1,\ldots,m$ that satisfy the linear matrix inequality (LMI) shown in (7) at the bottom of the next page.

*Proof:* Note that if (7) holds, then the following LMI is feasible as shown in (8) at the bottom of the next page and, thus, (1) is asymptotically stable [3].

To prove that $J<0$, we note that

$$[x^T\ y^T]EP\begin{bmatrix}x\\ y\end{bmatrix}=x^T P_1 x$$

and, hence, differentiation of the first term of (5) with respect to $t$ gives us, due to (4)

$$\begin{aligned} \frac{d}{dt}&\left\{[x^T(t)\ y^T(t)]EP\begin{bmatrix} x(t) \\ y(t)\end{bmatrix}\right\} \\ &= 2x^T(t)P_1\dot{x}(t) \\ &= 2[x^T(t)\ y^T(t)]P^T\begin{bmatrix}\dot{x}(t) \\ 0\end{bmatrix} \\ &= 2[x^T(t)\ y^T(t)]P^T \\ &\cdot\begin{bmatrix} y(t) \\ -y(t)+\left(\sum_{i=0}^m A_i\right)x(t) - \sum_{i=1}^m A_i\int_{t-h_i}^t y(s)\,ds + B_1w(t)\end{bmatrix}. \end{aligned} \tag{9}$$

From (9), we obtain (10) shown at the bottom of the page, where $\psi \triangleq col\{x(t),\, y(t),\, w(t)\}$ and

$$\eta_i(t) \triangleq -2\int_{t-h_i}^t [x^T(t)\,y^T(t)]P^T\begin{bmatrix}0 \\ A_i\end{bmatrix}y(s)\,ds.$$

For any matrix $0 < R_i \in \mathcal{R}^{n\times n}$

$$\eta_i \le h_i[x^T\ y^T]P^T\begin{bmatrix}0 \\ A_i\end{bmatrix}R_i^{-1}[0\ A_i^T]P\begin{bmatrix}x \\ y\end{bmatrix} + \int_{t-h_i}^t y^T(s)R_iy(s)\,ds. \tag{11}$$

From (3) and the fact, [due to the asymptotic stability of $x(t)$ from (8)], that $x(t)$ and $w(t)$ are square integrable on $[0,\,\infty)$, it follows that $y \in \mathcal{L}_2^n[0,\,\infty)$. We substitute (11) into (10) and integrate the resulting inequality in $t$ from 0 to $\infty$. Because

$$\begin{aligned}\int_0^\infty z^Tz\,dt &= \sum_{i=0}^m\int_0^\infty x^T(t-h_i)C_i^TC_ix(t-h_i)\,dt \\ &= \sum_{i=0}^m\int_0^\infty x^T(t)C_i^TC_ix(t)\,dt\end{aligned}$$

we obtain (by Schur complements) that $J < 0$ if the following LMI holds, as shown in (12) at the bottom of the next page, where

$$\Psi = P^T\begin{bmatrix}0 & I \\ \left(\sum_{i=0}^m A_i\right) & -I\end{bmatrix} + \begin{bmatrix}0 & \left(\sum_{i=0}^m A_i^T\right) \\ I & -I\end{bmatrix}P + \begin{bmatrix}\sum_{i=0}^m C_i^TC_i & 0 \\ 0 & \sum_{i=1}^m h_iR_i\end{bmatrix}. \tag{13}$$

LMI (7) results from the latter LMI by expansion of the block matrices. □

$$\begin{bmatrix}
\left(\sum_{i=0}^m A_i^T\right)P_2 + P_2^T\left(\sum_{i=0}^m A_i\right) + \sum_{i=0}^m C_i^TC_i & P_1 - P_2^T + \left(\sum_{i=0}^m A_i^T\right)P_3 & P_2^TB_1 & h_1P_2^TA_1 & \cdots & h_mP_2^TA_m \\
P_1 - P_2 + P_3^T\left(\sum_{i=0}^m A_i\right) & -P_3 - P_3^T + \sum_{i=1}^m h_iR_i & P_3^TB_1 & h_1P_3^TA_1 & \cdots & h_mP_3^TA_m \\
B_1^TP_2 & B_1^TP_3 & -\gamma^2I_q & 0 & \cdots & 0 \\
h_1A_1^TP_2 & h_1A_1^TP_3 & 0 & -h_1R_1 & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & 0 \\
h_mA_m^TP_2 & h_mA_m^TP_3 & \cdot & \cdot & \cdots & -h_mR_m
\end{bmatrix} < 0. \tag{7}$$

$$\begin{bmatrix}
\left(\sum_{i=0}^m A_i^T\right)P_2 + P_2^T\left(\sum_{i=0}^m A_i\right) & P_1 - P_2^T + \left(\sum_{i=0}^m A_i^T\right)P_3 & h_1P_2^TA_1 & \cdots & h_mP_2^TA_m \\
P_1 - P_2 + P_3^T\left(\sum_{i=0}^m A_i\right) & -P_3 - P_3^T + \sum_{i=1}^m h_iR_i & h_1P_3^TA_1 & \cdots & h_mP_3^TA_m \\
h_1A_1^TP_2 & h_1A_1^TP_3 & -h_1R_1 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & 0 \\
h_mA_m^TP_2 & h_mA_m^TP_3 & \cdot & \cdots & -h_mR_m
\end{bmatrix} < 0. \tag{8}$$

$$\begin{aligned}\frac{dV(t)}{dt} + z^T(t)z(t) - \gamma^2w^T(t)w(t) &= \psi^T\begin{bmatrix} P^T\begin{bmatrix}0 & I \\ \left(\sum_{i=0}^m A_i\right) & -I\end{bmatrix} + \begin{bmatrix}0 & \left(\sum_{i=0}^m A_i^T\right) \\ I & -I\end{bmatrix}P & P^T\begin{bmatrix}0 \\ B_1\end{bmatrix} \\ [0\ B_1^T]\,P & -\gamma^2I_q\end{bmatrix}\psi \\ &\quad + \sum_{i=1}^m\left[h_iy^T(t)R_iy(t) - \int_{t-h_i}^t y^T(s)R_iy(s)\,ds + \eta_i\right] + z^Tz\end{aligned} \tag{10}$$

*Remark 1:* Our method entails an overdesign due to the overbounding in (11). It is, however, less conservative than the method of [11], where more terms have to be bounded. In the case of a single delay (which is considered in [11]) by using the relation

$$x(t-h_1) = x(t) - \int_{t-h_1}^{t} [A_0 x(\tau) + A_1 x(\tau - h_1)]\, d\tau - \int_{t-h_1}^{t} B_1 w\, d\tau$$

the following system transformation is obtained:

$$\dot{x}(t) = [A_0 + A_1]x(t) - A_1 \int_{t-h_1}^{t} [A_0 x(\tau) + A_1 x(\tau - h_1)]\, d\tau - A_1 \int_{\tau-h_1}^{t} B_1 w\, d\tau + B_1 w(t).$$

By choosing (as was done in [11]) a Lyapunov–Krasovskii functional of the form $V(t) \triangleq x^T(t)P_1 x(t)$, where $P_1 = P_1^T \geq 0$, it is found that

$$\frac{dV(t)}{dt} = x^T \left[ (A_0 + A_1)^T P_1 + P_1(A_0 + A_1) \right] x + (\eta_1 + \eta_2 + \eta_3)$$

where

$$\eta_1(t) \triangleq -2 \int_{t-h_1}^{t} x^T(t) P A_1 A_0 x(\tau)\, d\tau,$$
$$\eta_2(t) \triangleq -2 \int_{t-h_1}^{t} x^T(t) P A_1 A_1 x(\tau - h_1)\, d\tau$$
$$\eta_3(t) \triangleq -2 \int_{t-h_1}^{t} x^T(t) P A_1 B_1 w(\tau)\, d\tau + 2x^T(t) P B_1 w(t).$$

Three cross terms: $\eta_1$, $\eta_2$ and $\eta_3$ should then be bounded, while by the method of Theorem 2.1 only one such term, $\eta_1$ of (11), should be coped with. In the case of $1 < m$ delays there are $m^2 + m$ more cross terms to be overbounded by the method of [11] than by our method.

The result of Theorem 2.1 can be used to verify whether or not system (1) is stable and, in case it is, to find its $H_\infty$-norm. In the following example we apply the theorem to a simple system, taken from the literature, and compare our results to those obtained there.

*Example 1 [11]:* We consider the following system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_1 w, \quad z(t) = C_0 x(t)$$

where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$
$$B_1 = [-0.5 \quad 1]^T \quad C_0 = [1 \quad 0].$$

In [11], for $h = 0.846$ a minimum value of $\gamma = 2$ is found. The actual $H_\infty$ norm of the system turns out to be $0.2364$. Applying Theorem 2.1 for $h = 0.846$ we obtain a minimum value of $\gamma = 0.32$, a result that is quite close to the actual $H_\infty$-norm. For $h = 0.8$ we obtain $\gamma = 0.28$ and for $h \leq 0.7$ our minimum value of $\gamma = 0.24$ is almost equal to the actual $H_\infty$-norm of the system.

By Theorem 2.1 we find that the system is stable for all $h < 1$. The same result is obtained using the method of [11]. We then look for the $H_\infty$-norm of the system for, say, $\mathrm{h} = 0.99$ sec. The BRL of [11] possesses a solution for $100 \leq \gamma$, whereas our BRL has a solution for $1.2 \leq \gamma$.

### B. Delay-Dependent/Delay-Independent BRL

The above results can be generalized for the following system:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{m} A_i x(t-h_i) + \sum_{i=1}^{k} F_i x(t-g_i) + B_1 w \\ z &= col\{C_0 x(t), C_1 x(t-h_1), \ldots, C_m x(t-h_m) \\ &\qquad C_{m+1} x(t-g_1), \ldots, C_{m+k} x(t-g_k)\} \end{aligned} \tag{14}$$

where $g_i \geq 0$, $i = 1, \ldots, k$. We are looking for a bounded real criterion which is delay-dependent with respect to $h_i$, $i = 1, \ldots m$ and delay-independent with respect to $g_i$, $i = 1, \ldots k$. We represent this system in the following descriptor form:

$$\dot{x}(t) = y(t), \quad y(t) = \left( \sum_{i=0}^{m} A_i \right) x(t) - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} y(s)\, ds + \sum_{i=1}^{k} F_i x(t-g_i) + B_1 w(t). \tag{15}$$

The corresponding Lyapunov–Krasovskii functional has the following form:

$$\begin{aligned} V(t) &= [x^T(t)\ y^T(t)] E P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &\quad + \sum_{i=1}^{m} \int_{-h_i}^{0} \int_{t+\theta}^{t} y^T(s) R_i y(s)\, ds\, d\theta \\ &\quad + \sum_{i=1}^{k} \int_{t-g_i}^{t} x^T(s) U_i x(s)\, ds \end{aligned} \tag{16}$$

where $R_i > 0$, $U_i > 0$, and $E$ and $P$ are given by (6). Similar to Theorem 2.1 and [3], we obtain the following.

*Theorem 2.2:* Consider the system of (14). For a prescribed $\gamma > 0$, the cost function (2) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$, if there exist $n \times n$-matrices $P_1 > 0$, $P_2$, $P_3$, $R_i = R_i^T$, $i = 1, \ldots, m$ and $U_i = U_i^T, i = 1, \ldots, k$ that satisfy the LMI shown in (17) at the bottom of the next page.

The LMI (17) has the block-matrix form shown in (18) at the bottom of the next page, where

$$\Psi_1 = \Psi + \begin{bmatrix} \left( \sum_{i=1}^{k} U_i \right) & 0 \\ 0 & 0 \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} \Psi & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_1 P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & \cdots & h_m P^T \begin{bmatrix} 0 \\ A_m \end{bmatrix} \\ [0\ B_1^T]\, P & -\gamma^2 I & 0 & \cdots & 0 \\ h_1 [0\ A_1^T] P & 0 & -h_1 R_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ h_m [0\ A_m^T] P & 0 & 0 & \cdots & -h_m R_m \end{bmatrix} < 0 \tag{12}$$

and $\Psi$ is given by (13). This form will be utilized in the next section to obtain a representation which is amenable to state-feedback control solutions.

*Example 2 [11]:* Consider the system

$$\dot{x}(t) = A_0 x(t) + F_1 x(t - g_1) + B_1 w, \qquad z(t) = C_0 x(t)$$

with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad F_1 = \begin{bmatrix} 0 & 0.9 \\ -1.3 & -1.9 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_0 = [1 \;\; 0].$$

In [11], the conditions of the delay-independent BRL are not satisfied in this case for all $\gamma > 0$. Applying theorem 2.2, we obtain the minimum achievable value of $\gamma$ is $\gamma = 4.37$.

## III. State-Feedback Control

We apply the results of the previous section to the infinite-horizon state-feedback control problem. Given the system $\mathcal{S}(\overline{A}_0, \overline{A}_1, B_1, B_2, \overline{C}_1, D_{12})$

$$\begin{aligned} \dot{x}(t) &= \overline{A}_0 x(t) + \overline{A}_1 x(t-h) + B_1 w(t) + B_2 u(t), \\ z &= col\{\overline{C}_1 x, D_{12} u\}, \quad x(t) = 0 \; \forall t \in [-h \;\; 0] \end{aligned} \tag{20}$$

where $x$ and $w$ are defined in Section II, $u \in \mathcal{R}^\ell$ is the control input, $\overline{A}_0, \overline{A}_1, B_1, B_2$ are constant matrices of appropriate dimension, $z$ is

$$\left[\begin{array}{cc|cccccccc}
\left(\sum_{i=0}^{m} A_i^T\right) P_2 + P_2^T \left(\sum_{i=0}^{m} A_i\right) + \sum_{i=0}^{m+k} C_i^T C_i + \sum_{i=1}^{k} U_i & P_1 - P_2^T + \left(\sum_{i=0}^{m} A_i^T\right) P_3 & P_2^T B_1 & h_1 P_2^T A_1 & \cdots & h_m P_2^T A_m & P_2^T F_1 & \cdots & P_2^T F_k \\
P_1 - P_2 + P_3^T \left(\sum_{i=0}^{m} A_i\right) & -P_3 - P_3^T + \sum_{i=1}^{m} h_i R_i & P_3^T B_1 & h_1 P_3^T A_1 & \cdots & h_m P_3^T A_m & P_3^T F_1 & \cdots & P_3^T F_k \\
B_1^T P_2 & B_1^T P_3 & -\gamma^2 I & 0 & \cdots & 0 & 0 & \cdots & 0 \\
h_1 A_1^T P_2 & h_1 A_1^T P_3 & 0 & -h_1 R_1 & \cdots & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & 0 & 0 & \cdots & 0 \\
h_m A_m^T P_2 & h_m A_m^T P_3 & \cdot & \cdot & \cdots & -h_m R_m & 0 & \cdots & 0 \\
F_1^T P_2 & F_1^T P_3 & \cdot & \cdot & \cdots & 0 & -U_1 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
F_k^T P_2 & F_k^T P_3 & \cdot & \cdot & \cdots & 0 & 0 & \cdots & -U_k
\end{array}\right] < 0. \tag{17}$$

$$\begin{bmatrix}
\Psi_1 & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_1 P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & \cdots & h_m P^T \begin{bmatrix} 0 \\ A_m \end{bmatrix} & P^T \begin{bmatrix} 0 \\ F_1 \end{bmatrix} & \cdots & P^T \begin{bmatrix} 0 \\ F_k \end{bmatrix} \\
\begin{bmatrix} 0 & B_1^T \end{bmatrix} P & -\gamma^2 I & 0 & \cdots & 0 & 0 & \cdots & 0 \\
h_1 \begin{bmatrix} 0 & A_1^T \end{bmatrix} P & 0 & -h_1 R_1 & \cdots & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
h_m \begin{bmatrix} 0 & A_m^T \end{bmatrix} P & 0 & 0 & \cdots & -h_m R_m & 0 & \cdots & 0 \\
\begin{bmatrix} 0 & F_1^T \end{bmatrix} P & 0 & 0 & \cdots & 0 & -U_1 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\begin{bmatrix} 0 & F_k^T \end{bmatrix} P & 0 & 0 & \cdots & 0 & 0 & \cdots & -U_k
\end{bmatrix} < 0 \tag{18}$$

the objective vector, $\overline{C}_1 \in \mathcal{R}^{p\times n}$ and $D_{12} \in \mathcal{R}^{r\times \ell}$. For a prescribed scalar $\gamma > 0$, we consider the performance index of (2). We treat two different cases. The first one allows for instantaneous state-feedback while the second case is based on a delayed measurement of the state.

### A. Instantaneous State Feedback

We look for the state-feedback gain matrix $K$ which, via the control law

$$u(t) = Kx(t) \tag{21}$$

achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$.

Substituting (21) into (20), we obtain the structure of (1) with

$$\begin{aligned} A_0 &= \overline{A}_0 + B_2 K, \quad A_1 = \overline{A}_1, \\ C_0^T C_0 &= \overline{C}_1^T \overline{C}_1 + K^T D_{12}^T D_{12} K. \end{aligned} \tag{22}$$

Applying the BRL of Section II to the above matrices, results in a nonlinear matrix inequality because of the terms $P_2^T B_2 K$ and $P_3^T B_2 K$. We, therefore, consider another version of the BRL which is derived from (12).

It is obvious from the requirement of $0 < P_1$, and the fact that in (7) $-(P_3 + P_3^T)$ must be negative definite, that $P$ is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \Delta = diag\{Q,\ I_{q+m\times n}\}$$

and

$$\tilde{C} = [\, C_0^T \quad \cdots \quad C_m^T \,]^T \tag{23a-c}$$

we multiply (12) by $\Delta^T$ and $\Delta$, on the left and on the right, respectively. Applying the Schur formula to the quadratic term in $Q$ and to $\sum_{i=1}^m h_i R_i$, we obtain the following inequality:

$$\Gamma_h < 0$$

where it is shown in (24) at the bottom of the page.

Noticing that in (20) $m = 1$ [and in (23) $C_1 = 0$] we substitute (22) into (24), denote $KQ_1$ by $Y$, and obtain the following.

*Lemma 3.1:* Consider the system of (20) and the cost function of (2). For a prescribed $0 < \gamma$, the state-feedback law of (21) achieves, $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ if there exist $Q_1 > 0$, $\overline{R} = \overline{R}^T = R_1^{-1}$, $Q_2, Q_3, \in \mathcal{R}^{n\times n}$ and $Y \in \mathcal{R}^{\ell \times n}$ that satisfy the LMI shown in (25) at the bottom of the page.

The state-feedback gain is then given by

$$K = YQ_1^{-1}. \tag{26}$$

*Example 3:* We consider a state-feedback example, taken from [13], where

$$\overline{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \overline{A}_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \overline{C}_1 = [\,0 \quad 1\,] \quad \text{and} \quad D_{12} = 0.1.$$

Applying the method of [13] (Corollary 3.2 there) we find that the system is stabilizable for all $h < 1$. For, say, $h = 0.999$ a minimum value of $\gamma = 1.8822$ results for $K = -[0.104\,520\,342 \quad 749\,057.7]$.

$$\Gamma_h \triangleq \begin{bmatrix} Q_2 + Q_2^T & Q_3 - Q_2^T + Q_1\left(\sum_{i=0}^m A_i^T\right) & 0 & 0 & \cdots & 0 & Q_1\tilde{C}^T & Q_2^T \\ Q_3^T - Q_2 + \left(\sum_{i=0}^m A_i\right)Q_1 & -Q_3 - Q_3^T & B_1 & h_1 A_1 & \cdots & h_m A_m & 0 & Q_3^T \\ 0 & B_1^T & -\gamma^2 I_q & 0 & \cdots & \cdot & \cdot & 0 \\ 0 & h_1 A_1^T & 0 & -h_1 R_1 & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & h_m A_m^T & \cdot & \cdot & \cdots & -h_m R_m & \cdot & \cdot \\ \tilde{C}Q_1 & 0 & \cdot & \cdot & \cdots & \cdot & -I_p & 0 \\ Q_2 & Q_3 & 0 & \cdot & \cdots & 0 & 0 & -\left(\sum_{i=1}^m h_i R_i\right)^{-1} \end{bmatrix}. \tag{24}$$

$$\begin{bmatrix} Q_2 + Q_2^T & Q_3 - Q_2^T + Q_1(\overline{A}_0^T + \overline{A}_1^T) + Y^T B_2^T & 0 & 0 & Q_1\overline{C}_1^T & Y^T D_{12}^T & hQ_2^T \\ Q_3^T - Q_2 + (\overline{A}_0 + \overline{A}_1)Q_1 + B_2 Y & -Q_3 - Q_3^T & B_1 & hA_1\overline{R} & 0 & 0 & hQ_3^T \\ 0 & B_1^T & -\gamma^2 I_q & 0 & \cdot & \cdot & 0 \\ 0 & h\overline{R}A_1^T & 0 & -h\overline{R} & \cdot & \cdot & \cdot \\ \overline{C}_1 Q_1 & 0 & \cdot & \cdot & -I_p & \cdot & 0 \\ D_{12}Y & 0 & \cdot & \cdot & \cdot & -I_r & 0 \\ hQ_2 & hQ_3 & 0 & \cdot & 0 & 0 & -h\overline{R} \end{bmatrix} < 0. \tag{25}$$

By Lemma 3.1, we obtain, for the same value of $h$, a minimum value of $\gamma = 0.228\,44$ with the state-feedback gain of $K = [0 - 182\,193.8]$.

*B. Delayed State-Feedback*

The situation wherein the time delay h appears in the state measurement equation (or in the actuators), results in the state-feedback law of the form

$$u(t) = Kx(t-h) \tag{27}$$

which can also be solved via the LMI of (25). This is accomplished by considering the following asymptotically stable subsystem:

$$\dot{\overline{u}}(t) = -\rho\overline{u}(t) + \rho u(t)$$

for $1 \ll \rho$. The state of this subsystem is almost identical to $u(t)$ when $\rho \to \infty$ and the open-loop system of (20) can, therefore, be approximated by the following augmented system:

$$\dot{\xi}(t) = \tilde{A}_0\xi + \tilde{A}_1\xi(t-h) + \tilde{B}_2u(t) + \tilde{B}_1w(t) \tag{28}$$

where $\xi \triangleq col\{x, \overline{u}\}$

$$\tilde{A}_0 = \begin{bmatrix} \overline{A}_0 & 0 \\ 0 & -\rho I_\ell \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} \overline{A}_1 & B_2 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 \\ \rho I_\ell \end{bmatrix},$$
$$\tilde{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad \text{and} \quad 1 \ll \rho.$$

The objective vector that corresponds to the one in (20) is then given by

$$z(t) = col\{\tilde{C}_0\xi(t),\ \tilde{C}_1\xi(t-h)\}$$

where $\tilde{C}_0 = [\overline{C}_1 \quad 0]$ and $\tilde{C}_1 = [0 \quad D_{12}]$. The state-feedback control problem then becomes one of finding the gain matrix $\tilde{K} = [K_1 \quad \overline{K}]$ which, via the control law of

$$u(t) = \tilde{K}\xi(t) \tag{29}$$

achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$, where $J$ is defined in (2).

Based on the result of Lemma 3.1 we obtain the following.

*Corollary 3.2:* Consider the system of (28) for $0 < \rho$. For a prescribed $0 < \gamma$, the state-feedback law of (29) achieves, $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ if there exist $Q_1 > 0$, $\overline{R} = \overline{R}^T$, $Q_2, Q_3, \in \mathcal{R}^{(n+\ell)\times(n+\ell)}$ and $Y \in \mathcal{R}^{\ell\times(n+\ell)}$ that satisfy the LMI shown in (30) at the bottom of the page. The state-feedback gain of (29) is then given by $\tilde{K} = YQ_1^{-1}$.

Denoting further

$$M \triangleq Q_1 - \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} Y \tag{31}$$

there exists $1 \ll \rho$ for which the solution of (30) (if exists) provides a state-feedback gain of (27), given by

$$K = YM^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix} \tag{32}$$

which achieves $J(w) < 0$ for the system of (20) and (27).

*Proof:* The result of (32) stems from $[K_1 \quad \overline{K}] = YQ_1^{-1}$ and from the fact that $u(t) = K_1x(t) + \overline{K}\,\overline{u}(t)$. For $s = j\omega, \omega \in [0\ \omega_B]$, $\omega_B$ being the maximum of the open and the closed-loop bandwidths of the system, the transfer function matrix from $x(t-h)$ to $\overline{u}$ is thus given by $T_{\overline{u},x} = (I - \overline{K}(s+\rho)^{-1}\rho)^{-1}\rho(s+\rho)^{-1}K_1 = (I-\overline{K})^{-1}K_1 + O(\rho^{-1})$, where $s$ is the Laplace transform variable. For $\rho \gg \omega_B, T_{\overline{u},x}$ is almost identical to $(I-\overline{K})^{-1}K_1$ in the significant frequency range of $[0\ \omega_B]$ where the $H_\infty$-norm of the closed-loop system is determined by the maximum singular value of $T_{\overline{u},x}$ over this range. The transfer function matrix $(I-\overline{K})^{-1}K_1$ is expressed by $K$ of (32) using the definition of $M$ in (31). □

The nonsingularity of $M$ is not always guaranteed. However, since a nearly singular $M$ implies large state-feedback gains and since the latter is encountered either when $D_{12}$ is nearly singular or when we compute the gains for the minimum $\gamma$, a possible singularity of $M$ can be avoided in cases where $D_{12}$ is not singular and $\gamma$ is above the minimum possible level of attenuation.

*Example 4:* We consider the system of Example 3 but, instead of applying the state-feedback law of (21), we use (27). We take $\rho = 10^{10}$ and obtain a solution for all $h \le 0.99$ s. For $h = 0.9$ s, a near minimum value of $\gamma = 52.74$ results with

$$Q_1 = \begin{bmatrix} 3.185\times10^{8} & -7.648\times10^{-2} & 1.7635\times10^{-2} \\ -7.648\times10^{-2} & 7.447\times10^{-2} & -1.299\times10^{-2} \\ 1.763\times10^{-2} & -1.299\times10^{-2} & 2.267\times10^{-3} \end{bmatrix}$$
$$Y = [\,1.763\times10^{-2} \quad -1.299\times10^{-2} \quad 2.267\times10^{-3}\,].$$

The corresponding values of $\tilde{K}$ and $M$ are then given by

$$\tilde{K} = [0;\ -3.24\times10^{-6} \quad 0.999\,981\,4]$$

and

$$M = \begin{bmatrix} 3.185\times10^{8} & -7.648\times10^{-2} & 1.763\times10^{-2} \\ -7.648\times10^{-2} & 7.447\times10^{-2} & -1.299\times10^{-2} \\ 4.416\times10^{-12} & 0 & 1.040\times10^{-13} \end{bmatrix}.$$

The resulting state-feedback gain then becomes: $K = [0 \quad -0.1745]$, and the actual $H_\infty$-norm that is achieved is 45.05.

$$\begin{bmatrix} Q_2+Q_2^T & Q_3-Q_2^T+Q_1(\tilde{A}_0^T+\tilde{A}_1^T)+Y^T\tilde{B}_2^T & 0 & 0 & Q_1\tilde{C}_0^T & Q_1\tilde{C}_1^T & hQ_2^T \\ Q_3^T-Q_2+(\tilde{A}_0+\tilde{A}_1)Q_1+\tilde{B}_2Y & -Q_3-Q_3^T & \tilde{B}_1 & h\tilde{A}_1\overline{R} & 0 & 0 & hQ_3^T \\ 0 & \tilde{B}_1^T & -\gamma^2I_q & 0 & \cdot & \cdot & 0 \\ 0 & h\overline{R}\tilde{A}_1^T & 0 & -h\overline{R} & \cdot & \cdot & \cdot \\ \tilde{C}_0Q_1 & 0 & \cdot & \cdot & -I_p & \cdot & 0 \\ \tilde{C}_1Q_1 & 0 & \cdot & \cdot & \cdot & -I_r & 0 \\ hQ_2 & hQ_3 & 0 & \cdot & 0 & 0 & -h\overline{R} \end{bmatrix} < 0. \tag{30}$$

In the delay-dependent/delay-independent case of Section II-B, denoting $Q = P^{-1}$, we obtain the following inequality by multiplying (18) by $\Delta = diag\{Q, I_{q+(m+k)\times n}\}$, on the right, and by $\Delta^T$ on the left, respectively

$$\begin{bmatrix} & 0 & & 0 & Q_1 \\ & F_1 & & F_k & 0 \\ \Gamma_h & \cdot & \cdots & \cdot & \cdot \\ & \cdot & & \cdot & \cdot \\ & 0 & & 0 & 0 \\ 0\ F_1^T\ \cdot\cdot 0 & -U_1 & \cdots & \cdot & 0 \\ \cdot\cdot\cdot\cdot\cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot\cdot\cdot\cdot\cdot & \cdot & \cdots & \cdot & \cdot \\ 0\ F_k^T\ \cdot\cdot 0 & \cdot & \cdots & -U_k & \cdot \\ Q_1\ 0\ \cdot\cdot 0 & \cdot & \cdots & \cdot & -\left(\sum_0^k U_i\right)^{-1} \end{bmatrix} < 0 \qquad (33)$$

where $\Gamma_h$ is defined in (24). The latter inequality can be used, as above, to solve a delayed state-feedback control problem where $u$ is given by (27) and the delay in the dynamics is $g \neq h$. Here, a time-dependent result for $h$ and a time-independent result for $g$ can be easily derived from (33), by noting that $m = k = 1$. In this situation, there is hardly any interest in solving for $u = Kx(t - g)$, since a state-feedback control law that is independent of the delay implies, in fact, $u \equiv 0$.

## IV. Conclusion

A new delay-dependent BRL has been proposed for linear time-invariant systems with multiple time delays in the system dynamics and in the objective function. The ensuing lemma provides a sufficient condition, in the form of a LMI, in order for the system to possess an $H_\infty$-norm that is less than a prescribed value. Although this condition is not necessary, the overdesign entailed is minimal since it is based on an equivalent system transformation and on the bounding of a small number of terms. The new BRL extends the results of [3] and applies Lyapunov–Krasovskii functionals depending on derivatives. It is most efficient in analyzing the stability and finding the $H_\infty$-norm of time-delay systems. It also provides a solution to the state-feedback control problem.

The LMI representation of the new BRL also allows solutions for the uncertain case where the system parameters lie within an uncertainty polytope. The convex nature of the LMI obtained ensures that a simultaneous solution to the LMIs that correspond to the vertices of the polytope, if it exists, will lead to an attenuation level that is smaller than the prescribed level for all of the parameters in the polytope.

The convexity of the LMI of the BRL with respect to the delays implies that a solution, if it exists, will hold for all delays less than or equal to the one solved for.

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# Speed Control of Electrical Machines: Unknown Load Torque Case

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***Abstract*—The problem of specifying a desired torque trajectory to achieve speed tracking in passivity-based control of induction motors is addressed. This note presents a solution to the problem that does not require an acceleration measurement nor knowledge of the load torque. To prove the main result a variant of Sontag's input to state stability is used.**

***Index Terms*—Author, please supply your own keywords or send a blank e-mail to keywords@ieee.org to receive a list of suggested keywords.**

## I. Introduction

Passivity-based control methods, already successful in robotics applications [1], have been proposed by several authors as a very powerful tool to solve the induction motor torque control problem [2]–[6]. The method has also been shown to be adequate for a more general class of smooth air gap electromechanical machines, generally known as Blondel–Park transformable [7]. Passivity-based methods are characterized by their robustness, a property that is usually lacking in feedback linearization methods (see among others [8], [9] and more recently [10]) that heavily depend on cancelling of the system nonlinear-

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