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Brief paper

New stability and exact observability conditions for semilinear wave equations[☆]

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ABSTRACT

The problem of estimating the initial state of 1-D wave equations with globally Lipschitz nonlinearities from boundary measurements on a finite interval was solved recently by using the sequence of forward and backward observers, and deriving the upper bound for exact observability time in terms of Linear Matrix Inequalities (LMIs) (Fridman, 2013). In the present paper, we generalize this result to n-D wave equations on a hypercube. This extension includes new LMI-based exponential stability conditions for n-D wave equations, as well as an upper bound on the minimum exact observability time in terms of LMIs. For 1-D wave equations with locally Lipschitz nonlinearities, we find an estimate on the region of initial conditions that are guaranteed to be uniquely recovered from the measurements. The efficiency of the results is illustrated by numerical examples.

1. Introduction

Lyapunov-based solutions of various control problems for finite-dimensional systems can be formulated in the form of Linear Matrix Inequalities (LMIs) (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). The LMI approach to distributed parameter systems is capable of utilizing nonlinearities and of providing the desired system performance (see e.g. Castillo, Witrant, Prieur, & Dugard, 2012, Fridman & Orlov, 2009b, Lamare, Girard, & Prieur, 2013). For 1-D wave equations, several control problems were solved by using the direct Lyapunov method in terms of LMIs (Fridman, 2013; Fridman & Orlov, 2009a). However, there have not been yet LMI-based results for n-D wave equations, though the exponential stability of the n-D wave equations in bounded spatial domains has been studied in the literature via the direct Lyapunov method (see e.g. Ammari, Nicaise, & Pignotti, 2010, Fridman, Nicaise, & Valein, 2010, Guo, Zhou, & Yao, 2014, Zuazua, 1990).

The problem of estimating the initial state of 1-D wave equations with globally Lipschitz nonlinearities from boundary measurements on a finite interval was solved recently by using the sequence of forward and backward observers, and deriving the upper bound for exact observability time in terms of LMIs (Fridman, 2013). In the present paper, we generalize this result to n-D wave equations on a hypercube. This extension includes new LMI-based exponential stability conditions for n-D wave equations. Their derivation is based on n-D extensions of the Wirtinger (Poincaré) inequality (Hardy, Littlewood, & Pólya, 1988) and of the Sobolev inequality with tight constants, which is crucial for the efficiency of the results. As in 1-D case, the continuous dependence of the reconstructed initial state on the measurements follows from the integral input-to-state stability of the corresponding error system, which is guaranteed by the LMIs for the exponential stability. Some preliminary results on global exact observability of multidimensional wave PDEs will be presented in Fridman and Terushkin (2015).

Another objective of the present paper is to study regional exact observability for systems with locally Lipschitz in the state nonlinearities. Here we restrict our consideration to 1-D case, and find an estimate on the region of initial conditions that are guaranteed to be uniquely recovered from the measurements. Note that our result on the regional observability cannot be extended to multi-dimensional case (see Remark 4 for explanation and for discussion on possible n-D extensions for different classes of nonlinearities). The efficiency of the results is illustrated by numerical examples.

The presented simple finite-dimensional LMI conditions complete the theoretical qualitative results of e.g. Ramdani, Tucsnak, and Weiss (2010) (where exact observability of linear systems in a Hilbert space was studied via a sequence of forward and backward observers) and Baroun, Jacob, Maniar, and Schnaubelt (2013)

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(where local exact observability of abstract semilinear systems was considered).

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the space of $n\times m$ real matrices. The notation $P>0$ with $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. For the symmetric matrix M, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimum and the maximum eigenvalues of M respectively. The symmetric elements of the symmetric matrix will be denoted by $*$. Continuous functions (continuously differentiable) in all arguments, are referred to as of class C (of class C^1). $L^2(\Omega)$ is the Hilbert space of square integrable $f:\Omega\to\mathbb{R}$, where $\Omega\subset\mathbb{R}^n$, with the norm $\|f\|_{L^2}=\sqrt{\int_\Omega |f(x)|^2dx}$. For the scalar smooth function $z=z(t,x_1,\dots,x_n)$ denote by $z_t, z_{x_k}, z_{tt}, z_{x_kx_j}$ $(k,j=1,\dots,n)$ the corresponding partial derivatives. For $z:\Omega\to\mathbb{R}$ define $\nabla z=z_x^T=[z_{x_1}\dots z_{x_n}]^T$, $\Delta z=\sum_{p=1}^n z_{x_px_p}$. $\mathcal{H}^1(\Omega)$ is the Sobolev space of absolutely continuous functions $z:\Omega\to\mathbb{R}$ with the square integrable ∇z. $\mathcal{H}^2(\Omega)$ is the Sobolev space of scalar functions $z:\Omega\to\mathbb{R}$ with absolutely continuous ∇z and with $\Delta z\in L^2(\Omega)$.

2. Observers and exponential stability of n-D wave equations

2.1. System under study and Luenberger type observer

Throughout the paper we denote by Ω the n-D unit hypercube $[0,1]^n$ with the boundary Γ. We use the partition of the boundary:

$$\Gamma_D=\{x=(x_1,\dots,x_n)^T\in\Gamma:\exists p\in 1,\dots,n \text{ s.t. } x_p=0\}$$

$$\Gamma_{N,p}=\{x\in\Gamma: x_p=1\},\qquad \Gamma_N=\bigcup_{p=1,\dots,n}\Gamma_{N,p}.$$

Here subscripts D and N stand for Dirichlet and for Neumann boundary conditions respectively.

We consider the following boundary value problem for the scalar n-D wave equation:

$$\begin{aligned} z_{tt}(x,t)&=\Delta z(x,t)+f(z,x,t) && \text{in } \Omega\times(t_0,\infty),\\ z(x,t)&=0 && \text{on } \Gamma_D\times(t_0,+\infty),\\ \frac{\partial}{\partial\nu}z(x,t)&=0 && \text{on } \Gamma_N\times(t_0,\infty),\end{aligned}\tag{2.1}$$

where f is a C^1 function, ν denotes the outer unit normal vector to the point $x\in\Gamma$ and $\frac{\partial}{\partial\nu}z$ is the normal derivative. Let $g_1>0$ be the known bound on the derivative of $f(z,x,t)$ with respect to z:

$$|f_z(z,x,t)|\le g_1\quad\forall (z,x,t)\in \mathrm{R}^{n+2}.\tag{2.2}$$

Since Ω is a unit hypercube, the boundary conditions on Γ_N can be rewritten as

$$z_{x_p}(x,t)\Big|_{x_p=1}=0\quad \forall x_i\in[0,1],\ i\neq p,\ p=1,\dots,n.$$

Consider the following initial conditions:

$$z(x,t_0)=z_0(x),\qquad z_t(x,t_0)=z_1(x),\quad x\in\Omega.\tag{2.3}$$

The boundary measurements are given by

$$y(x,t)=z_t(x,t)\quad \text{on } \Gamma_N\times(t_0,\infty).\tag{2.4}$$

Similar to Fridman (2013), the boundary-value problem (2.1) can be represented as an abstract differential equation by defining the state $\zeta(t)=[\zeta_0(t)\ \zeta_1(t)]^T=[z(t)\ z_t(t)]^T$ and the operators

$$\mathcal{A}=\begin{bmatrix}0 & I\\ \Delta z & 0\end{bmatrix},\qquad F(\zeta,t)=\begin{bmatrix}0\\ F_1(\zeta_0,t)\end{bmatrix},$$

where $F_1:\mathcal{H}^1(\Omega)\times R\to L^2(\Omega)$ is defined as $F_1(\zeta_0,t)=f(\zeta_0(x),x,t)$ so that it is continuous in t for each $\zeta_0\in\mathcal{H}^1(\Omega)$. The differential equation is

$$\dot\zeta(t)=\mathcal{A}\zeta(t)+F(\zeta(t),t),\quad t\ge t_0\tag{2.5}$$

in the Hilbert space $\mathcal{H}=\mathcal{H}^1_{\Gamma_D}(\Omega)\times L^2(\Omega)$, where

$$\mathcal{H}^1_{\Gamma_D}(\Omega)=\left\{\zeta_0\in\mathcal{H}^1(\Omega)\,\Big|\,\zeta_{0|\Gamma_D}=0\right\}$$

and $\|\zeta\|^2_{\mathcal{H}}=\|\nabla\zeta_0\|^2_{L^2}+\|\zeta_1\|^2_{L^2}$. The operator $\mathcal{A}$ has the dense domain

$$\mathcal{D}(\mathcal{A})=\Big\{(\zeta_0,\zeta_1)^T\in\mathcal{H}^1_{\Gamma_D}(\Omega)\times\mathcal{H}^1_{\Gamma_D}(\Omega)\,\Big|\,\Delta\zeta_0\in L^2(\Omega)\ \text{and}\ \frac{\partial}{\partial\nu}\zeta_{0|\Gamma_N}=-b\zeta_{1|\Gamma_N}\Big\},$$

where $b=0$. Here the boundary condition holds in a weak sense (as defined in Sect. 3.9 of Tucsnak & Weiss, 2009), i.e. the following relation holds:

$$\langle\Delta\zeta_0,\phi\rangle_{L^2(\Omega)}+\langle\nabla\zeta_0,\nabla\phi\rangle_{[L^2(\Omega)]^n}=-b\langle\zeta_0,\phi\rangle_{L^2(\Gamma_N)}\quad \forall\phi\in\mathcal{H}^1_{\Gamma_D}(\Omega).$$

The operator $\mathcal{A}$ is m-dissipative (see Proposition 3.9.2 of Tucsnak & Weiss, 2009) and hence it generates a strongly continuous semigroup. Due to (2.2), the following Lipschitz condition holds:

$$\|F_1(\zeta_0,t)-F_1(\bar\zeta_0,t)\|_{L^2}\le g_1\|\zeta_0-\bar\zeta_0\|_{L^2}\tag{2.6}$$

where $\zeta_0,\bar\zeta_0\in\mathcal{H}^1_{\Gamma_D}(\Omega)$, $t\in R$. Then by Theorem 6.1.2 of Pazy (1983), a unique continuous mild solution $\zeta(\cdot)$ of (2.5) in $\mathcal{H}$ initialized by

$$\zeta_0(t_0)=z_0\in\mathcal{H}^1_{\Gamma_D}(\Omega),\qquad \zeta_1(t_0)=z_1\in L^2(\Omega)$$

exists in $C([t_0,\infty),\mathcal{H})$. If $\zeta(t_0)\in\mathcal{D}(\mathcal{A})$, then this mild solution is in $C^1([t_0,\infty),\mathcal{H})$ and it is a classical solution of (2.1) with $\zeta(t)\in\mathcal{D}(\mathcal{A})$ (see Theorem 6.1.5 of Pazy, 1983).

We suggest a Luenberger type observer of the form:

$$\hat z_{tt}(x,t)=\Delta\hat z(x,t)+f\left(\hat z,x,t\right),\quad t\ge t_0,\ x\in\Omega\tag{2.7}$$

under the initial conditions $[\hat z(\cdot,t_0),\hat z_t(\cdot,t_0)]^T\in\mathcal{H}$ and the boundary conditions

$$\begin{aligned}\hat z(x,t)&=0 && \text{on } \Gamma_D\times(t_0,\infty)\\ \frac{\partial}{\partial\nu}\hat z(x,t)&=k\Big[y(x,t)-\hat z_t(x,t)\Big] && \text{on } \Gamma_N\times(t_0,\infty)\end{aligned}\tag{2.8}$$

where k is the injection gain.

The well-posedness of (2.7), (2.8) will be established by showing the well-posedness of the estimation error $e=z-\hat z$. Taking into account (2.1), (2.3) we obtain the following PDE for the estimation error $e=z-\hat z$:

$$e_{tt}(x,t)=\Delta e(x,t)+ge(x,t)\quad t\ge t_0,\ x\in\Omega\tag{2.9}$$

under the boundary conditions

$$\begin{aligned}e(x,t)&=0 && \text{on } \Gamma_D\times(t_0,\infty)\\ \frac{\partial}{\partial\nu}e(x,t)&=-ke_t(x,t) && \text{on } \Gamma_N\times(t_0,\infty).\end{aligned}\tag{2.10}$$

Here $ge=f(z,x,t)-f(z-e,x,t)$ and

$$g=g(z,e,x,t)=\int_0^1 f_z(z+(\theta-1)e,x,t)d\theta.$$

The initial conditions for the error are given by

$$e(x,t_0)=z_1(x)-z(\cdot,t_0),$$
$$e_t(x,t_0)=z_2(x)-z_t(\cdot,t_0).$$

The boundary conditions on Γ_N can be presented as

$$e_{x_p}(x,t)\Big|_{x_p=1}=-ke_t(x,t)\quad\forall x_i\in[0,1],$$
$$i\neq p,\quad p=1,\dots,n.$$

Let z be a mild solution of (2.1). Then $z : [t_0, \infty) \to \mathscr{H}^1$ is continuous and, thus, the function $F_2 : \mathscr{H}^1 \times [t_0, \infty) \to L_2(0, 1)$ defined as

$$F_2(\zeta_0, t) = f(z, x, t) - f(z - \zeta_0, x, t)$$

satisfies the Lipschitz condition (2.6), where F_1 is replaced by F_2. By the above arguments, where in the definition of $\mathcal{D}(\mathcal{A})$ we have $b = k$, the error system (2.9), (2.10) has a unique mild solution $\{e, e_t\} \in C([t_0, \infty), \mathscr{H})$ initialized by $[e(\cdot, t_0), e_t(\cdot, t_0)]^T \in \mathscr{H}$. Therefore, there exists a unique mild solution $\{\hat{z}, \hat{z}_t\} \in C([t_0, \infty), \mathscr{H})$ to the observer system (2.7), (2.8) with the initial conditions $[\hat{z}(\cdot, t_0), \hat{z}_t(\cdot, t_0)]^T \in \mathscr{H}$. If $[e(\cdot, t_0), e_t(\cdot, t_0)]^T \in \mathcal{D}(\mathcal{A})$, then $\{e, e_t\} \in C^1([t_0, \infty), \mathscr{H})$ is a classical solution of (2.9), (2.10) with $[e(\cdot, t), e_t(\cdot, t)] \in \mathcal{D}(\mathcal{A})$ for $t \geq t_0$. Hence, if $[\hat{z}(\cdot, t_0), \hat{z}_t(\cdot, t_0)]^T \in \mathcal{D}(\mathcal{A})$ and $[z_0, z_1]^T \in \mathcal{D}(\mathcal{A})$, there exists a unique classical solution $\{\hat{z}, \hat{z}_t\} \in C^1([t_0, \infty), \mathscr{H})$ to the observer system (2.7), (2.8) with $[\hat{z}(\cdot, t), \hat{z}_t(\cdot, t)]^T \in \mathcal{D}(\mathcal{A})$ for $t \geq t_0$.

2.2. Lyapunov function and useful inequalities

We will derive further sufficient conditions for the exponential stability of the error wave equation (2.9) under the boundary conditions (2.10). Let

$$E(t) = \frac{1}{2}\int_\Omega \left[|\nabla e|^2 + e_t^2\right] dx, \tag{2.11}$$

be the energy of the system. Consider the following Lyapunov function for (2.9), (2.10):

$$\begin{aligned} V(t) &= E(t) + \chi \int_\Omega \left[2(x^T \cdot \nabla e) + (n-1)e\right] e_t dx \\ &\quad + \chi \frac{k(n-1)}{2} \int_{\Gamma_N} e^2 d\Gamma \end{aligned}$$

with some constant $\chi > 0$. Note that the above Lyapunov function without the last term was considered in Ammari et al. (2010), Fridman et al. (2010) and Zuazua (1990). The time derivative of this new term of V cancels the same term with the opposite sign in the time derivative of $\chi \int_\Omega [(n-1)e] e_t dx$ (cf. (2.23)) leading to LMI conditions for the exponential convergence of the error wave equation.

We will employ the following n-D extensions of the classical inequalities:

Lemma 1. *Consider $e \in \mathscr{H}^1(\Omega)$ such that $e\big|_{x \in \Gamma_D} = 0$. Then the following n-D Wirtinger's inequality holds:*

$$\int_\Omega \left[\frac{4}{\pi^2 n}|\nabla e|^2 - e^2\right] dx \geq 0. \tag{2.12}$$

Moreover,

$$\int_{\Gamma_N} e^2 d\Gamma \leq \int_\Omega |\nabla e|^2 dx. \tag{2.13}$$

Proof. Since $e\big|_{x_1=0} = 0$, by the classical 1-D Wirtinger's inequality (Hardy et al., 1988)

$$\int_0^1 e^2 dx_1 \leq \frac{4}{\pi^2}\int_0^1 e_{x_1}^2 dx_1.$$

Integrating the latter inequality in $x_2, \ldots, x_n$ we obtain

$$\int_\Omega e^2 dx \leq \frac{4}{\pi^2}\int_\Omega e_{x_p}^2 dx$$

with $p = 1$. Clearly the latter inequality holds for all $p = 1, \ldots, n$, which after summation in p yields (2.12).

Since $e\big|_{x_1=0} = 0$ we have by Sobolev's inequality

$$e^2(x)\Big|_{x_1=1} \leq \int_0^1 e_{x_1}^2 dx_1 \quad \forall x_i \in [0, 1],\ i \neq 1,$$

that after integration in $x_2, \ldots, x_n$ leads to

$$\int_{\Gamma_{N,p}} e^2 d\Gamma \leq \int_\Omega e_{x_p}^2 dx$$

with $p = 1$. The latter inequality holds $\forall p = 1, \ldots, n$ leading after summation in p to (2.13). □

2.3. Exponential stability of n-D wave equation

In this section we derive LMI conditions for the exponential stability of the estimation error equation. We start with the conditions for the positivity of the Lyapunov function:

Lemma 2. *Let there exist positive scalars χ and λ_0 such that*

$$\Phi_0 \triangleq \begin{bmatrix} \frac{1}{2} - \lambda_0 \frac{4}{\pi^2 n} & \sqrt{n}\chi & 0 \\ * & \frac{1}{2} & \frac{n-1}{2}\chi \\ * & * & \lambda_0 \end{bmatrix} > 0. \tag{2.14}$$

Then the Lyapunov function $V(t)$ is bounded as follows:

$$\begin{aligned} &\alpha E(t) \leq V(t) \leq \beta E(t), \quad \alpha = 2\lambda_{\min}(\Phi_0), \\ &\beta = 2\left(1 + \frac{2}{\pi^2 n}\right)\lambda_{\max}(\Phi_0) + \chi k(n-1). \end{aligned} \tag{2.15}$$

Proof. By Cauchy–Schwarz inequality we have

$$|x^T \cdot \nabla e| \leq |x|\,|\nabla e| \leq \sqrt{n}|\nabla e|. \tag{2.16}$$

Then

$$\begin{aligned} &\left|\chi \int_\Omega \left[2(x^T \cdot \nabla e) + (n-1)e\right] e_t dxr\right| \\ &\quad \leq \chi \int_\Omega [2\sqrt{n}|\nabla e|\,|e_t| + (n-1)|e|\,|e_t|]dx, \end{aligned}$$

leading to

$$\begin{aligned} V(t) &\geq \frac{1}{2}\int_\Omega \left[e_t^2 + |\nabla e|^2\right] dx \\ &\quad - \chi \int_\Omega [2\sqrt{n}|\nabla e|\,|e_t| + (n-1)|e|\,|e_t|]dx. \end{aligned} \tag{2.17}$$

Taking into account the n-D Wirtinger inequality (2.12), we further apply S-procedure (Yakubovich, 1971),[1] where we subtract from the right-hand side of (2.17) the nonnegative term

$$\lambda_0 \int_\Omega \left[\frac{4}{\pi^2 n}|\nabla e|^2 - e^2\right] dx \tag{2.18}$$

with $\lambda_0 > 0$:

$$\begin{aligned} V(t) &\geq \frac{1}{2}\int_\Omega \{e_t^2 + |\nabla e|^2\} dx \\ &\quad - \chi \int_\Omega [2\sqrt{n}|\nabla e|\,|e_t| + (n-1)|e|\,|e_t|]dx \\ &\quad - \lambda_0 \int_\Omega \left[\frac{4}{\pi^2 n}|\nabla e|^2 - e^2\right] dx = \int_\Omega \eta^T \Phi_0 \eta, \end{aligned}$$

where $\eta = col\{|\nabla e|, -|e_t|, |e|\}$.

[1] Let $A_i \in \mathbb{R}^{p \times p}$, $i = 0, 1$. Then the inequality $\xi^T A_0 \xi \geq 0$ holds for any $\xi \in \mathbb{R}^p$ satisfying $\xi^T A_1 \xi \geq 0$ iff there exists a real scalar $\lambda \geq 0$, such that $A_0 - \lambda A_1 \geq 0$.

Similarly

$$
\begin{aligned}
V(t) &\le \frac{1}{2}\int_{\Omega}\left[e_t^2+|\nabla e|^2\right]dx \\
&\quad+\chi\int_{\Omega}\left[2\sqrt{n}|\nabla e|\,|e_t|+(n-1)|e|\,|e_t|\right]dx \\
&\quad+\chi\frac{k(n-1)}{2}\int_{\Gamma_N}e^2d\Gamma \\
&\le \eta_1^T\Phi_0\eta_1+\chi\frac{k(n-1)}{2}\int_{\Gamma_N}e^2d\Gamma
\end{aligned}
\tag{2.19}
$$

with $\eta_1=col\{|\nabla e|,|e_t|,|e|\}$.

Then (2.15) follows from

$$
\begin{aligned}
&\lambda_{\min}(\Phi_0)\left[2E(t)+\int_{\Omega}e^2dx\right]\le V(t) \\
&\le \lambda_{\max}(\Phi_0)\left[2E(t)+\int_{\Omega}e^2dx\right]+\chi\frac{k(n-1)}{2}\int_{\Gamma_N}e^2d\Gamma
\end{aligned}
$$

and from the inequalities (2.12) and (2.13). □

We are looking next for conditions that guarantee $\dot V(t)+2\delta V(t)\le 0$ along the classical solutions of the wave equation initiated from $[z_0,z_1]^T,[\hat z(\cdot,t_0),\hat z_t(\cdot,t_0)]^T\in\mathcal{D}(\mathcal{A})$. Then $V(t)\le e^{-2\delta(t-t_0)}V(t_0)$ and, thus, (2.15) yields

$$
\begin{aligned}
&\int_{\Omega}[|\nabla e|^2(x,t)+e_t^2(x,t)]dx \\
&\le\frac{\beta}{\alpha}e^{-2\delta(t-t_0)}\int_{\Omega}[|\nabla(z_0(x)-\hat z(x,t_0))|^2 \\
&\quad+(z_1(x)-\hat z_t(x,t_0))^2]dx.
\end{aligned}
\tag{2.20}
$$

Since $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ the same estimate (2.20) remains true (by continuous extension) for any initial conditions $[z_0,z_1]^T$, $[\hat z(\cdot,t_0),\hat z_t(\cdot,t_0)]^T\in\mathcal{H}$. For such initial conditions we have mild solutions of (2.1), (2.3).

Theorem 1. *Given $k>0$ and $\delta>0$, assume that there exist positive constants χ, λ_0 and λ_1 that satisfy the LMI (2.14) and the following LMIs:*

$$
\begin{aligned}
&\Psi_1\triangleq -k+(1+k^2n)\chi\le 0, \\
&\Psi_2\triangleq\begin{bmatrix}\psi_2 & 2\delta\sqrt{n}\chi & \sqrt{n}g_1\chi \\ * & -\chi+\delta & \frac{1}{2}g_1+\delta(n-1)\chi \\ * & * & -\lambda_1+g_1(n-1)\chi\end{bmatrix}\le 0, \\
&\psi_2=-\chi+\delta(1+\chi k(n-1))+\lambda_1\frac{4}{\pi^2 n}.
\end{aligned}
\tag{2.21}
$$

Then, under the condition (2.2), solutions of the boundary-value problem (2.9), (2.10) satisfy (2.20), where α and β are given by (2.15), i.e. the system governed by (2.9), (2.10) is exponentially stable with a decay rate $\delta>0$.

Proof. Differentiating V in time we obtain

$$
\begin{aligned}
\dot V(t)&=\dot E(t)+\chi\frac{d}{dt}\Big[\int_{\Omega}\left[2(x^T\cdot\nabla e)+(n-1)e\right]e_tdx\Big] \\
&\quad+\chi k(n-1)\int_{\Gamma_N}ee_td\Gamma.
\end{aligned}
$$

We have

$$
\dot E(t)=\int_{\Omega}\left((\nabla e)^T\nabla(e_t)+e_te_{tt}\right)dx.
$$

Applying Green's formula to the first integral term, substituting $e_{tt}=\Delta e+ge$ and taking into account (2.2), we find

$$
\begin{aligned}
\dot E(t)&=\int_{\Gamma}e_t\frac{\partial e}{\partial\nu}d\Gamma-\int_{\Omega}e_t\Delta edx+\int_{\Omega}e_t[\Delta e+ge]dx \\
&\le -k\int_{\Gamma_N}e_t^2d\Gamma+g_1\int_{\Omega}|e|\,|e_t|dx.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_{\Omega}[2x^T\nabla e+(n-1)e]e_tdx\right\} \\
&=\int_{\Omega}\frac{d}{dt}[2x^T\nabla e+(n-1)e]e_tdx \\
&\quad+\int_{\Omega}[2x^T\nabla e+(n-1)e][\Delta e+ge]dx.
\end{aligned}
$$

Then Green's formula leads to (see (11.35) of Lions, 1988)

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_{\Omega}[x^T\nabla e+(n-1)e]e_tdx\right\} \\
&=2\int_{\Gamma_N}x^T\nabla e\frac{\partial e}{\partial\nu}d\Gamma-\int_{\Gamma_N}(x^T\nu)|\nabla e|^2d\Gamma \\
&\quad+(n-2)\int_{\Omega}|\nabla e|^2dx+\int_{\Gamma_N}(x^T\nu)e_t^2d\Gamma-n\int_{\Omega}e_t^2dx \\
&\quad+(n-1)\int_{\Omega}e_t^2dx+(n-1)\int_{\Gamma_N}e\frac{\partial e}{\partial\nu}d\Gamma \\
&\quad-(n-1)\int_{\Omega}|\nabla e|^2dx+\int_{\Omega}[2x^T\nabla e+(n-1)e]gedx.
\end{aligned}
\tag{2.22}
$$

Noting that $x^T\nu=1$ on Γ_N and taking into account the boundary conditions we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_{\Omega}[2x^T\nabla e+(n-1)e]e_tdx\right\} \\
&=-\int_{\Omega}\{e_t^2+|\nabla e|^2+[2x^T\nabla e+(n-1)e]ge\}dx \\
&\quad-\int_{\Gamma_N}\left[|\nabla e|^2+2kx^T\nabla ee_t\right]d\Gamma \\
&\quad+\int_{\Gamma_N}\left[e_t^2-k(n-1)ee_t\right]d\Gamma.
\end{aligned}
\tag{2.23}
$$

By inequalities (2.16) and (2.2) we have

$$
\begin{aligned}
&\int_{\Omega}[2x^T\nabla e+(n-1)e]gedx \\
&\le\int_{\Omega}\left[2|x^T\nabla e|\,|g|\,|e|dx+(n-1)g_1e^2\right]dx \\
&\le\int_{\Omega}[2\sqrt{n}g_1|\nabla e|\,|e|+(n-1)g_1e^2]dx.
\end{aligned}
$$

Further due to (2.16)

$$
\begin{aligned}
&-\int_{\Gamma_N}2kx^T\nabla ee_td\Gamma \\
&\le 2k\int_{\Gamma_N}|x^T\nabla e|\,|e_t|d\Gamma\le 2k\sqrt{n}\int_{\Gamma_N}|\nabla e|\,|e_t|d\Gamma.
\end{aligned}
$$

Then by completion of squares we find

$$
\begin{aligned}
&-\int_{\Gamma_N}\left[|\nabla e|^2+2kx^T\nabla ee_t\right]d\Gamma \\
&\le\int_{\Gamma_N}\left[k^2ne_t^2-\left[|\nabla e|-k\sqrt{n}|e_t|\right]^2\right]d\Gamma\le k^2n\int_{\Gamma_N}e_t^2d\Gamma.
\end{aligned}
$$

Summarizing we obtain

$$\dot V(t) \le [\chi(1+k^2 n)-k]\int_{\Gamma_N} e_t^2 d\Gamma - \int_\Omega \Big[\chi[e_t^2+|\nabla e|^2] - \left[2\sqrt{n}\chi g_1|\nabla e|\,|e| + (n-1)\chi g_1 e^2 + g_1|e|\,|e_t|\right]\Big]dx. \quad (2.24)$$

Therefore, employing (2.19) we arrive at

$$\begin{aligned}\dot V(t)+2\delta V(t) &\le \int_{\Gamma_N}[\Psi_1 e_t^2 + \delta\chi k(n-1)e^2]d\Gamma \\ &\quad -(\chi-\delta)\int_\Omega \left[e_t^2+|\nabla e|^2\right]dx \\ &\quad + \int_\Omega \Big[2\sqrt{n}\chi g_1|\nabla e|\,|e| + (n-1)\chi g_1 e^2 \\ &\quad + [g_1+2\delta\chi(n-1)]|e|\,|e_t| + 4\delta\chi\sqrt{n}|\nabla e|\,|e_t|\Big]dx. \end{aligned} \quad (2.25)$$

By taking into account Wirtinger's inequality (2.12), we add to (2.25) the nonnegative term (2.18), where λ_0 is replaced by $\lambda_1 > 0$. Denote $\eta_2 = col\{|\nabla e|, |e_t|, |e|\}$. Then after employing the bound (2.13) we arrive at

$$\frac{d}{dt}V(t)+2\delta V(t) \le \Psi_1\int_{\Gamma_N} e_t^2 d\Gamma + \int_\Omega \eta_2^T\Psi_2\eta_2 dx \le 0$$

if the LMIs (2.21) are feasible. □

Remark 1. For $n > 1$ the term $\chi\int_\Omega (n-1)ee_t dx$ of V leads to $-\chi\int_\Omega |\nabla e|^2 dx$ in $\dot V$ (cf. (2.22)).

3. Exact observability of n-D wave equation

Our next objective is to recover (if possible) the unique initial state (2.3) of the solution to (2.1)–(2.3) from the measurements on the finite time interval

$$y(x,t) = z_t(x,t) \quad \text{on } \Gamma_N \times [t_0, t_0+T],\ T>0. \quad (3.1)$$

Definition 1 (*Fridman, 2013*)**.** The system (2.1), (2.3) with the measurements (3.1) is called exactly observable in time T, if

(i) for any initial condition $[z_0,z_1]^T \in \mathcal{H} = \mathcal{H}^1_{\Gamma_D}(\Omega)\times L^2(\Omega)$ it is possible to find a sequence $[z_0^m, z_1^m]^T \in \mathcal{H}(m=1,2,\dots)$ from the measurements (3.1) such that $\lim_{m\to\infty}\|[z_0^m,z_1^m]^T - [z_0,z_1]^T\|_{\mathcal{H}} = 0$ (i.e. it is possible to recover the unique initial state as $[z_0,z_1]^T = \lim_{m\to\infty}[z_0^m,z_1^m]^T$);

(ii) there exists a constant $C>0$ such that for any initial conditions $[z_0,z_1]^T \in \mathcal{H}$ and $[\bar z_0,\bar z_1]^T \in \mathcal{H}$ leading to the measurements $y(x,t)$ and $\bar y(x,t)$ and to the corresponding sequences $[z_0^m,z_1^m]^T$ and $[\bar z_0^m,\bar z_1^m]^T$, the following holds:

$$\|\lim_{m\to\infty}[z_0^m,z_1^m]^T - \lim_{m\to\infty}[\bar z_0^m,\bar z_1^m]^T\|^2_{\mathcal{H}} \le C\int_{t_0}^{t_0+T}\int_{\Gamma_N}|y(x,t)-\bar y(x,t)|^2 d\Gamma dt. \quad (3.2)$$

The time T is called the observability time.

The system is called *regionally exactly observable* if the above conditions hold for all $[z_0,z_1]^T \in \mathcal{H}$ with $\|[z_0,z_1]^T\|_{\mathcal{H}} \le d_0$ for some $d_0>0$.

Note that (3.2) means the continuous dependence of the reconstructed initial state on the measurements. In this section we will derive LMI sufficient conditions for n-D wave equations with globally Lipschitz in the first argument f, where (2.2) holds globally in z. In Section 4, we will present LMI-based conditions for the regional observability for 1-D wave equation, where (2.2) holds locally in z.

3.1. Iterative forward and backward observer design

In order to recover the initial state of the solution to (2.1) from the measurements (3.1) we use the iterative procedure as in Ramdani et al. (2010). Define the sequences of forward $z^{(m)}$ and backward observers $z^{b(m)}$, $m=1,2,\dots$ with the injection gain $k>0$:

$$\begin{aligned} &z_{tt}^{(m)}(x,t) = \Delta z^{(m)}(x,t) + f(z^{(m)}(x,t),x,t),\\ &z^{(m)}(x,t)=0, \quad x\in\Gamma_D,\\ &\frac{\partial}{\partial\nu}z^{(m)}(x,t) = k[y(x,t)-z_t^{(m)}(x,t)], \quad x\in\Gamma_N,\\ &\quad t\in[t_0,t_0+T],\\ &z^{(m)}(x,t_0) = z^{b(m-1)}(x,t_0),\\ &z_t^{(m)}(x,t_0) = z_t^{b(m-1)}(x,t_0), \end{aligned} \quad (3.3)$$

where $z^{b(0)}(x,t_0) = z_t^{b(0)}(x,t_0) \equiv 0$, and

$$\begin{aligned} &z_{tt}^{b(m)}(x,t) = \Delta z^{b(m)}(x,t) + f(z^{b(m)}(x,t),x,t),\\ &z^{b(m)}(x,t)=0, \quad x\in\Gamma_D,\\ &\frac{\partial}{\partial\nu}z^{b(m)}(x,t) = -k[y(x,t)-z_t^{b(m)}(x,t)], \quad x\in\Gamma_N,\\ &\quad t\in[t_0,t_0+T],\\ &z^{b(m)}(x,t_0+T) = z^{(m)}(x,t_0+T),\\ &z_t^{b(m)}(x,t_0+T) = z_t^{(m)}(x,t_0+T). \end{aligned} \quad (3.4)$$

This results in the sequence of the forward $e^{(m)} = z - z^{(m)}$ and the backward $e^{b(m)} = z - z^{b(m)}$, $m=1,2,\dots$ errors satisfying

$$\begin{aligned} &e_{tt}^{(m)}(x,t) = \Delta e_x^{(m)}(x,t) + g^{(m)}e^{(m)}(x,t),\\ &e^{(m)}(x,t)\Big|_{x\in\Gamma_D} = 0, \qquad \frac{\partial}{\partial\nu}e^{(m)}(x,t) = -ke_t^{(m)}(x,t)\Big|_{x\in\Gamma_N},\\ &\quad t\in[t_0,t_0+T],\\ &e^{(m)}(x,t_0) = e^{b(m-1)}(x,t_0),\\ &e_t^{(m)}(x,t_0) = e_t^{b(m-1)}(x,t_0), \end{aligned} \quad (3.5)$$

where $e^{b(0)}(x,t_0) = z_0(x)$, $e_t^{b(0)}(x,t_0) = z_1(x)$ and

$$\begin{aligned} &e_{tt}^{b(m)}(x,t) = \Delta e^{b(m)}(x,t) + g^{b(m)}e^{b(m)}(x,t),\\ &e^{b(m)}(x,t)\Big|_{x\in\Gamma_D} = 0, \qquad \frac{\partial}{\partial\nu}e^{(m)}(x,t) = ke_t^{(m)}(x,t)\Big|_{x\in\Gamma_N},\\ &\quad t\in[t_0,t_0+T],\\ &e^{b(m)}(x,t_0+T) = e^{(m)}(x,t_0+T),\\ &e_t^{b(m)}(x,t_0+T) = e_t^{(m)}(x,t_0+T). \end{aligned} \quad (3.6)$$

Here

$$\begin{aligned} g^{(m)} &= g(z,e^{(m)},x,t) = \int_0^1 f_z(z+(\theta-1)e^{(m)},x,t)d\theta,\\ g^{b(m)} &= g(z,e^{b(m)},x,t) = \int_0^1 f_z(z+(\theta-1)e^{b(m)},x,t)d\theta. \end{aligned} \quad (3.7)$$

3.2. LMIs for the exact observability time

For (3.5) and (3.6) we consider for $t\in[t_0,t_0+T]$ the Lyapunov functions

$$\begin{aligned} V^{(m)}(t) &= E^{(m)}(t) + \chi\frac{k(n-1)}{2}\int_{\Gamma_N}(e^{(m)})^2 d\Gamma \\ &\quad + \chi\int_\Omega \left[2(x^T\cdot\nabla e^{(m)}) + (n-1)e^{(m)}\right]e_t^{(m)}dx, \end{aligned} \quad (3.8)$$

$$E^{(m)}(t) = \frac{1}{2}\int_\Omega\left[|\nabla e^{(m)}|^2 + (e_t^{(m)})^2\right]dx$$

and

$$V^{b(m)}(t) = E^{b(m)}(t) - \chi \frac{k(n-1)}{2} \int_{\Gamma_N} (e^{b(m)})^2 d\Gamma - \chi \int_{\Omega} \left[2(x^T \cdot \nabla e^{b(m)}) + (n-1)e^{b(m)}\right] e_t^{b(m)} dx, \quad (3.9)$$

$$E^{b(m)}(t) = \frac{1}{2} \int_{\Omega} \left[|\nabla e^{b(m)}|^2 + (e_t^{b(m)})^2\right] dx$$

with some constant $\chi > 0$. Then for χ and $\lambda_0 > 0$ subject to (2.14) we have (cf. (2.15))

$$\begin{aligned} \alpha E^{(m)}(t) &\leq V^{(m)}(t) \leq \beta E^{(m)}(t), \quad t \geq t_0, \\ \alpha E^{b(m)}(t) &\leq V^{b(m)}(t) \leq \beta E^{b(m)}(t), \end{aligned} \quad (3.10)$$

where α and β are given by (2.15).

Lemma 3. *Consider $V^{(m)}$ and $V^{b(m)}$ given by (3.8) and (3.9) respectively with $\chi > 0$ satisfying (2.14). Assume there exist $\delta > 0$ and $T > 0$ such that for all $m = 1, 2, \ldots$ and for all $t \in [t_0, t_0 + T]$ the inequalities*

$$\dot{V}^{(m)}(t) + 2\delta V^{(m)}(t) \leq 0 \quad (3.11)$$

and

$$\dot{V}^{b(m)}(t) - 2\delta V^{b(m)}(t) \geq 0 \quad (3.12)$$

hold along (3.5) and (3.6) respectively. Assume additionally that for some $T^* \in (0, T)$

$$\begin{aligned} V^{(m)}(t_0)e^{-2\delta T^*} &\leq V^{b(m-1)}(t_0), \\ V^{b(m)}(t_0 + T)e^{-2\delta T^*} &\leq V^{(m)}(t_0 + T). \end{aligned} \quad (3.13)$$

Then the iterative algorithm converges on $[t_0, t_0 + T]$:

$$V^{b(m)}(t_0) \leq qV^{b(m-1)}(t_0) \leq q^m V^{b(0)}(t_0), \quad (3.14)$$

$q = e^{-4\delta(T-T^*)}$ is the convergence rate.

Moreover, for all $t \in [t_0, t_0 + T]$ and $m = 1, 2, \ldots$

$$\max\{V^{(m)}(t),\ V^{b(m)}(t)\} \leq e^{2\delta T^*} V^{b(0)}(t_0). \quad (3.15)$$

Proof. The inequalities (3.11), (3.12) yield

$$V^{b(m)}(t_0) \leq V^{b(m)}(t_0 + T)e^{-2\delta T},$$
$$V^{(m)}(t_0 + T) \leq V^{(m)}(t_0)e^{-2\delta T}.$$

Hence, (3.13) implies (3.14):

$$\begin{aligned} V^{b(m)}(t_0) &\leq V^{b(m)}(t_0 + T)e^{-2\delta T} \leq V^{(m)}(t_0 + T)\sqrt{q} \\ &\leq V^{(m)}(t_0)\sqrt{q}e^{-2\delta T} \leq V^{b(m-1)}(t_0)q. \end{aligned}$$

The bound (3.15) follows from the following inequalities:

$$\begin{aligned} V^{(m+1)}(t) &\leq V^{(m+1)}(t_0) \leq e^{2\delta T^*} V^{b(m)}(t_0) \\ &\leq V^{b(m)}(t_0 + T) \leq V^{(m)}(t_0 + T)e^{2\delta T^*} \\ &\leq V^{(m)}(t_0) \leq \cdots \leq V^{(1)}(t_0) \leq e^{2\delta T^*} V^{b(0)}(t_0), \end{aligned}$$
$$V^{b(m)}(t) \leq V^{b(m)}(t_0 + T) \leq e^{2\delta T^*} V^{b(0)}(t_0). \quad \square$$

We are in a position to formulate sufficient conditions for the exact observability:

Theorem 2. *Given positive tuning parameters T^* and δ, let there exist positive constants χ, λ_1 and λ_2 that satisfy the LMIs (2.21) and*

$$\Phi \triangleq \begin{bmatrix} \Phi_{11} & \sqrt{n}[1 + e^{-2\delta T^*}]\chi & 0 \\ * & -\frac{1}{2}[1 - e^{-2\delta T^*}] & \frac{n-1}{2}[1 + e^{-2\delta T^*}]\chi \\ * & * & -\lambda_2 \end{bmatrix} < 0. \quad (3.16)$$

$$\Phi_{11} = -\frac{1}{2}[1 - e^{-2\delta T^*}] + \lambda_2 \frac{4}{\pi^2 n}.$$

Then

(i) the system (2.1)–(2.3) with the measurements (2.4) is exactly observable in time T^*;

(ii) for all $\Delta T > 0$ the iterative algorithm with $T = T^* + \Delta T$ converges

$$\int_{\Omega} \left[|\nabla e^{b(m)}(x, t_0)|^2 + [e_t^{b(m)}(x, t_0)]^2\right] dx \leq \frac{\beta}{\alpha} q^m \int_{\Omega} \left[|\nabla z_0|^2(x) + z_1^2(x)\right] dx, \quad (3.17)$$

where $q = e^{-4\delta \Delta T}$, and the following bound holds:

$$\begin{aligned} \max\Big\{ &\int_{\Omega} \left[|\nabla e^{b(m)}(x, t)|^2 + [e_t^{b(m)}(x, t)]^2\right] dx, \\ &\int_{\Omega} \left[|\nabla e^{(m)}(x, t)|^2 + [e_t^{(m)}(x, t)]^2\right] dx \Big\} \\ &\leq \frac{\beta}{\alpha} e^{2\delta T^*} \int_{\Omega} \left[|\nabla z_0|^2(x) + z_1^2(x)\right] dx, \end{aligned} \quad (3.18)$$

$t \in [t_0, t_0 + T]$.

Here α and β are given by (2.15).

Proof. (i) From Theorem 1 it follows that LMIs (2.21) yield (3.11). By the similar derivations, LMIs (2.21) imply (3.12) for the backward system. Taking into account that $e^{(m)}(x, t_0 + T) = e^{b(m)}(x, t_0 + T)$ and $e_t^{(m)}(x, t_0 + T) = e_t^{b(m)}(x, t_0 + T)$, the bound (2.19) and the n-D Wirtinger inequality we obtain for some $\lambda_2 > 0$

$$\begin{aligned} &V^{b(m)}(t_0 + T)e^{-2\delta T^*} - V^{(m)}(t_0 + T) \\ &= \frac{1}{2}[-1 + e^{-2\delta T^*}]\Big\{\int_{\Omega} [(e_t^{(m)})^2 + |\nabla e^{(m)}|^2] dx + k(n-1) \int_{\Gamma_N} (e^{(m)})^2 d\Gamma\Big\} \\ &\quad - \chi[1 + e^{-2\delta T^*}] \int_{\Omega} \left[2(x^T \cdot \nabla e^{(m)}) + (n-1)e\right] e_t^{(m)} dx \\ &\leq \frac{1}{2}[-1 + e^{-2\delta T^*}] \int_{\Omega} [(e_t^{(m)})^2 + |\nabla e^{(m)}|^2] dx \\ &\quad + \chi[1 + e^{-2\delta T^*}] \int_{\Omega} \left[2\sqrt{n}|\nabla e^{(m)}| + (n-1)|e^{(m)}|\right] |e_t^{(m)}| dx \\ &\quad + \lambda_2 \int_{\Omega} \left[\frac{4}{\pi^2 n}|\nabla e^{(m)}|^2 - (e^{(m)})^2\right] dx \leq \int_{\Omega} \eta_2^T \Phi \eta_2 dx \leq 0, \end{aligned}$$

where

$$\eta_2 = col\{|\nabla e^{(m)}(x, t)|, |e_t^{(m)}(x, t)|, |e^{(m)}(x, t)|\} \quad (3.19)$$

and where $t = t_0 + T$, if (3.16) is feasible. Similarly (3.16) guarantees $V^{(m)}(t_0)e^{-2\delta T^*} \leq V^{b(m-1)}(t_0)$. The feasibility of the LMI (3.16) yields the feasibility of (2.14), i.e. the positivity of $V^{(m)}$ and $V^{b(m)}$. Moreover, the strict LMI (3.16) guarantees (3.13) with T^* changed by $T^* - \Delta T$, where $\Delta T > 0$ is small enough, implying due to Lemma 3 the convergence of the iterative algorithm with $T = T^*$.

To prove the exact observability in time T^*, consider initial states $\zeta(t_0) \in \mathcal{H}$ and $\bar{\zeta}(t_0) \in \mathcal{H}$ of (2.1)–(2.3) that lead to the measurements $y(x, t)$ and $\bar{y}(x, t)$ and to the corresponding forward and backward observers $z^{(m)}, z^{b(m)}$ and $\bar{z}^{(m)}, \bar{z}^{b(m)}$. Note that $\bar{z}^{(m)}, \bar{z}^{b(m)}$ satisfy (3.3) and (3.4), where $z^{(m)}, z^{b(m)}$ and y are replaced by $\bar{z}^{(m)}, \bar{z}^{b(m)}$ and $\bar{y}$. The resulting $e^{(m)} = z^{(m)} - \bar{z}^{(m)}$, $e^{b(m)} = z^{b(m)} - \bar{z}^{b(m)}$ satisfy (3.5), (3.6) with the perturbed boundary conditions at $x \in \Gamma_N$:

$$\begin{aligned} &\frac{\partial}{\partial \nu} e^{(m)} = -k e_t^{(m)} + w, \qquad w \triangleq k[y(x,t) - \bar{y}(x,t)], \\ &\frac{\partial}{\partial \nu} e^{b(m)} = k e_t^{b(m)} - w, \qquad x \in \Gamma_N,\ t \geq t_0. \end{aligned} \tag{3.20}$$

Let $V^{(m)}$ and $V^{b(m)}$ be defined by (3.8) and (3.9). LMI (3.16) implies inequalities (3.13).

We will show next that the feasibility of (2.21) implies

$$\begin{aligned} &\dot{V}^{(m)}(t) + 2\delta V^{(m)}(t) - \gamma \int_{\Gamma_N} w^2 d\Gamma \leq 0, \\ &\dot{V}^{b(m)}(t) - 2\delta V^{b(m)}(t) + \gamma \int_{\Gamma_N} w^2 d\Gamma \geq 0 \end{aligned} \tag{3.21}$$

for $t \geq t_0$ and some $\gamma > 0$. Taking into account w-term in (3.20), by the arguments of Theorem 1 we have

$$\dot{E}^{(m)}(t) = \int_{\Gamma_N} \left[-k \left(e_t^{(m)} \right)^2 + e_t^{(m)} w \right] d\Gamma + g_1 \int_{\Omega} |e^{(m)}|\,|e_t^{(m)}| dx,$$

and

$$\begin{aligned} &\frac{d}{dt} \left\{ \int_{\Omega} [2x^T \nabla e^{(m)} + (n-1) e^{(m)}] e_t^{(m)} dx \right\} \\ &= - \int_{\Omega} \{ (e_t^{(m)})^2 + |\nabla e^{(m)}|^2 + [2x^T \nabla e^{(m)} + (n-1) e^{(m)}] g e^{(m)} \} dx \\ &\quad - \int_{\Gamma_N} \left[|\nabla e^{(m)}|^2 + 2x^T \nabla e^{(m)} [k e_t^{(m)} - w] \right] d\Gamma \\ &\quad + \int_{\Gamma_N} \left[(e_t^{(m)})^2 - (n-1) e [k e_t^{(m)} - w] \right] d\Gamma. \end{aligned}$$

Then after bounding and completion of squares we find

$$\begin{aligned} &\frac{d}{dt} V^{(m)}(t) + 2\delta V^{(m)}(t) \\ &\leq \Psi_1 \int_{\Gamma_N} (e_t^{(m)})^2 d\Gamma + \int_{\Omega} \eta_2^T \Psi_2 \eta_2 dx \\ &\quad + \int_{\Gamma_N} \Big\{ |e_t^{(m)}|\,|w| + \chi (n-1) |e^{(m)}|\,|w| \\ &\quad + \chi k^2 n \Big[2 |e_t^{(m)}|\,|w| + w^2 \Big] \Big\} d\Gamma \leq 0, \end{aligned}$$

where η_2 is given by (3.19). By Young's inequality with some $r > 0$ and by (2.13)

$$\begin{aligned} &\chi (n-1) \int_{\Gamma_N} |e^{(m)}|\,|w| d\Gamma \leq \frac{\chi (n-1)}{r} \int_{\Gamma_N} (e^{(m)})^2 d\Gamma \\ &\quad + \chi (n-1)\, r \int_{\Gamma_N} w^2 d\Gamma \\ &\leq \frac{\chi (n-1)}{r} \int_{\Omega} |\nabla e^{(m)}|^2 dx + \chi (n-1)\, r \int_{\Gamma_N} w^2 d\Gamma. \end{aligned}$$

Then the first inequality (3.21) holds if

$$\begin{aligned} &\begin{bmatrix} \Psi_1 & \chi \left(\frac{1}{2} + k^2 n \right) \\ * & -\gamma + \chi k^2 n + \chi (n-1)\, r \end{bmatrix} < 0, \\ &\Psi_2 + \frac{\chi (n-1)}{r} [1 \quad 0 \quad 0]^T [1 \quad 0 \quad 0] < 0. \end{aligned} \tag{3.22}$$

It is easy to see that the latter inequalities are feasible for large enough r and γ if $\Psi_1 < 0$ and $\Psi_2 < 0$, i.e. if LMIs (2.21) are satisfied.

Table 1
Nonlinearity vs. minimal observability time.

g_1	δ	T^*
0	0.0001	3.28
0.01	0.01	4.3
0.1	0.01	12.2
0.3	0.01	38

Then, by the comparison principle (see e.g. Khalil, 2002),

$$V^{(m)}(t) \leq e^{-2\delta(t-t_0)} V^{(m)}(t_0) + \gamma \int_{t_0}^{t} \int_{\Gamma_N} |w(x,s)|^2 d\Gamma ds.$$

Similarly, LMIs (2.21) guarantee the second inequality (3.21) for large enough $\gamma > 0$, and, thus,

$$\begin{aligned} V^{b(m)}(t) \geq\ & e^{2\delta(t-t_0)} V^{b(m)}(t_0) \\ & - \gamma \int_{t_0}^{t} \int_{\Gamma_N} e^{2\delta(t-s)} |w(x,s)|^2 d\Gamma ds. \end{aligned}$$

Note that the strict inequalities (3.16) guarantee (3.13) with δ changed by $\delta + \delta_0$ for small enough $\delta_0 > 0$. Therefore,

$$\begin{aligned} V^{b(m)}(t_0) &\leq e^{-2(\delta+\delta_0)T^*} V^{b(m)}(t_0 + T^*) \\ &\quad + \gamma \int_{t_0}^{t_0+T^*} \int_{\Gamma_N} |w(x,s)|^2 d\Gamma ds \\ &\leq e^{-2\delta_0 T^*} V^{(m)}(t_0 + T^*) + \gamma \int_{t_0}^{t_0+T^*} \int_{\Gamma_N} |w(x,s)|^2 d\Gamma ds \\ &\leq e^{-4\delta_0 T^*} V^{b(m-1)}(t_0) \\ &\quad + (e^{-2\delta_0 T^*} + 1)\gamma \int_{t_0}^{t_0+T^*} | \int_{\Gamma_N} |w(x,s)|^2 d\Gamma ds. \end{aligned}$$

We arrive at

$$\begin{aligned} &\alpha \int_0^1 \Big[[e_x^{b(m)}(x,t_0)]^2 + [e_t^{b(m)}(x,t_0)]^2 \Big] dx \\ &\leq V^{b(m)}(t_0) \leq (e^{-4\delta_0 T^*})^2 V^{b(m-2)}(t_0) \\ &\quad + (e^{-6\delta_0 T^*} + e^{-4\delta_0 T^*} + e^{-2\delta_0 T^*} + 1)\gamma \int_{t_0}^{t} \int_{\Gamma_N} |w(s)|^2 d\Gamma ds \\ &\leq e^{-4m\delta_0 T^*} V^{b(0)}(t_0) + \alpha C \int_{t_0}^{t_0+T^*} \int_{\Gamma_N} |w(s)|^2 d\Gamma ds \end{aligned}$$

which implies (3.2), where $C = \frac{\gamma}{\alpha[1-e^{-2\delta_0 T^*}]}$.

(ii) follows from (3.14), (3.15) and (3.10). □

Remark 2. As a by-product, we have derived new LMI conditions (3.22) for input-to-state stability of the n-D wave equation (3.5) with the perturbed boundary condition on Γ_N as in (3.20).

Remark 3. Note that for $n = 1$ and $g = 0$ the LMIs of Theorem 2 are equivalent to the corresponding conditions of Fridman (2013) that are not conservative (in the sense that they lead to the analytical value of the minimal observability time T^*_{an}). However, for $n = 2$ and $g = 0$ the conditions of Theorem 2 lead to an upper bound on T^*_{an} only (see Example 1). This mirrors the conservatism of the conditions for $n > 1$.

Example 1. Consider (2.1)–(2.3), where $n = 2$ with the values of g_1 as given in Table 1. We use the sequence of forward and backward observers (3.3) and (3.4) with $k = 1$. By verifying the conditions of Theorem 2, we find the minimal values of T^* and the corresponding δ for the convergence of the iterative algorithm and, thus, for the exact observability. Note that for $g_1 = 0$ the observability time is $T^* = 3.28$, which is not too far from the analytical value $2\sqrt{2} \approx 2.82$. For simulation results in the linear case see Example 2 of Ramdani et al. (2010).

4. Regional observability of 1-D wave equation with locally Lipschitz nonlinearity

In this section we consider 1-D wave equation (2.1), where $\Omega = [0, 1]$:

$$z_{tt}(x,t) = z_{xx}(x,t) + f(z,x,t), \quad x \in [0,1],\ t > t_0,$$
$$z(0,t) = 0, \qquad z_x(1,t) = 0, \tag{4.1}$$

whereas the measurements are given by

$$y(t) = z_t(1,t), \quad t \in [t_0, t_0 + T]. \tag{4.2}$$

Assume that $f(0,x,t) \equiv 0$ and that f is *locally Lipschitz* in the first argument uniformly on the others. The latter means that we can find a $d > 0$ such that

$$|f_z| \le g_1 \quad \forall |z| \le d,\ x \in [0,1],\ t \ge t_0. \tag{4.3}$$

We present

$$f(z,x,t) = f_1 z, \qquad f_1 = \int_0^1 f_z(\theta z, x, t)d\theta. \tag{4.4}$$

Recall that in 1-D case $\mathcal{H} = \mathcal{H}^1_{\Gamma_D}(0,1) \times L^2(0,1)$, where

$$\mathcal{H}^1_{\Gamma_D}(0,1) = \left\{ \zeta_0 \in \mathcal{H}^1(0,1) \middle| \zeta_0(0) = 0 \right\}$$

and

$$\mathcal{D}(\mathcal{A}) = \Big\{ (\zeta_0, \zeta_1)^T \in \mathcal{H}^2(0,1) \bigcap \mathcal{H}^1_{\Gamma_D}(0,1) \times \mathcal{H}^1_{\Gamma_D}(0,1)$$
$$\Big| \zeta_{0x}(1) = 0 \Big\}.$$

Consider a region of initial conditions defined by

$$\mathcal{X}_{d_0} = \left\{ \left[z_0, z_1\right]^T \in \mathcal{H} \middle| \int_0^1 \left[z_{0x}^2 + z_1^2\right] dx \le d_0^2 \right\}, \tag{4.5}$$

where $d_0 > 0$ is some constant. We are looking for an estimate $\mathcal{X}_{d_0}$ (with d_0 as large as possible) on the region of initial conditions, for which the iterative algorithm defined in Section 3 converges. This gives an estimate on the region of exact observability, where the initial conditions of the system can be recovered uniquely from the measurements on the interval $[t_0, t_0 + T]$.

The convergence of the iterative algorithm in Theorem 2 has been proved for the forward and the backward error systems (3.5) and (3.6) with globally Lipschitz nonlinearities given by (3.7) subject to

$$|f_z(z + (\theta - 1)e^{(m)}, x, t)| \le g_1,$$
$$|f_z(z + (\theta - 1)e^{b(m)}, x, t)| \le g_1, \tag{4.6}$$
$$\forall t \in [t_0, t_0 + T],\ x, \theta \in [0,1],\ z, e^{(m)}, e^{b(m)} \in \mathbb{R}.$$

For the locally bounded nonlinearity as in (4.3) we have to find a region $\mathcal{X}_{d_0}$ of initial conditions starting from which solutions of (4.1), (3.5) and (3.6) satisfy the bound

$$[z_0, z_1]^T \in \mathcal{X}_{d_0} \Rightarrow |z(x,t) + (\theta - 1)e^{(m)}(x,t)| \le d,$$
$$|z(x,t) + (\theta - 1)e^{b(m)}(x,t)| \le d,$$
$$\forall t \in [t_0, t_0 + T],\ x, \theta \in [0,1]. \tag{4.7}$$

The latter implication yields

$$[z_0, z_1]^T \in \mathcal{X}_{d_0} \Rightarrow \max\{|f_1|, |g^{(m)}|, |g^{b(m)}|\} \le g_1,$$
$$\forall t \in [t_0, t_0 + T], \quad x, \theta \in [0,1]. \tag{4.8}$$

We will employ Sobolev's inequality

$$\max_{x\in[0,1]} z^2(x,t) \le \int_0^1 z_x^2(x,t)dx, \quad t \ge t_0 \tag{4.9}$$

that holds since $z\big|_{x=0} = 0$, and similar bounds on $e^{(m)}$ and $e^{b(m)}$. In order to guarantee (4.7) we start with a bound on the solutions of (4.1). Since this system is not stable we give a simple energy-based bound on the exponential growth of z. Define the energy

$$E_{z_{eq}}(t) = \frac{1}{2}\int_0^1 \left(z_x^2 + z_t^2\right)dx.$$

Proposition 1. *Consider (4.1) with $f(0,x,t) \equiv 0$ subject to $|f_z| \le g_1$ for all $(z,x,t) \in R^3$. Then solutions of this system satisfy the following inequality:*

$$E_{z_{eq}}(t) \le e^{\frac{2g_1}{\pi}(t-t_0)} E_{z_{eq}}(t_0), \quad t \ge t_0.$$

Proof. It is sufficient to show that

$$W \triangleq \dot{E}_{z_{eq}} - \frac{2g_1}{\pi}E_{z_{eq}} \le 0$$

along (4.1). Differentiating, integrating by parts, taking into account the boundary conditions (that imply $z_x(1,t) = z_t(0,t) = 0$) and further applying Wirtinger's inequality we have

$$\begin{aligned} W &= \int_0^1 [z_x z_{xt} + z_t(z_{xx} + f)]dx - \frac{2g_1}{\pi}E_{z_{eq}} \\ &= \int_0^1 z_t f_1 z dx - \frac{2g_1}{\pi}E_{z_{eq}} \\ &\le g_1 \int_0^1 |z_t|\,|z| dx - \frac{g_1}{\pi}\int_0^1 \left(\frac{\pi^2}{4}z^2 + z_t^2\right)dx \\ &= -\frac{g_1}{\pi}\int_0^1 \left(\frac{\pi}{2}|z| - |z_t|\right)^2 dx \le 0. \quad \square \end{aligned}$$

Due to (4.9), given $d > 0$ the solution z of (4.1) satisfies the bound

$$z^2(x,t) \le 0.25d^2 \quad \forall x \in [0,1],\ t \in [t_0, t_0 + T] \tag{4.10}$$

if

$$\begin{aligned} \max_{x\in[0,1]} z^2(x,t) &\le \int_0^1 \left[z_x(x,t)^2 + z_t(x,t)^2\right]dx \\ &\le e^{\frac{2g_1}{\pi}(t-t_0)} \int_\Omega \left[|z_0(x)|^2 + z_1(x)^2\right]dx \le \frac{d^2}{4}. \end{aligned} \tag{4.11}$$

In order to bound $e^{(m)}$ and $e^{b(m)}$, we use Theorem 2. The LMIs (2.21) for $n = 1$ are reduced to

$$-k + (1 + k^2)\chi < 0,$$
$$\begin{bmatrix} -\chi + \delta + \lambda_1 \frac{4}{\pi^2} & 2\delta\chi & g_1\chi \\ * & -\chi + \delta & \frac{1}{2}g_1 \\ * & * & -\lambda_1 \end{bmatrix} \le 0, \tag{4.12}$$

where χ and λ_1 are positive scalars. The LMI (3.16) for $n = 1$ has a form

$$\begin{bmatrix} -\frac{1}{2}[1 - e^{-2\delta T^*}] & [1 + e^{-2\delta T^*}]\chi \\ * & -\frac{1}{2}[1 - e^{-2\delta T^*}] \end{bmatrix} < 0. \tag{4.13}$$

The LMI (2.14) has a form $\Phi_0 > 0$, where $2\Phi_0 = \begin{bmatrix} 1 & 2\chi \\ * & 1 \end{bmatrix}$, leading to $\alpha = 2\lambda_{\min}(\Phi_0)$ and $\beta = 2\lambda_{\max}(\Phi_0)$ in the bounds (3.10). Hence, $\alpha = (1-2\chi)$ and $\beta = (1+2\chi)$.

Similarly to (4.11), if the LMIs (4.12) are feasible, then

$$\max\left\{[e^{(m)}(x,t)]^2, [e^{b(m)}(x,t)]^2\right\} \leq \frac{d^2}{4}$$

$\forall x \in [0,1],\ t \in [t_0, t_0+T]$ provided (cf. (3.18))

$$\begin{aligned} &\max\left\{\max_{x\in[0,1]}[e^{(m)}(x,t)]^2, \max_{x\in[0,1]}[e^{b(m)}(x,t)]^2\right\} \\ &\leq \frac{1+2\chi}{1-2\chi}e^{2\delta T^*}\int_0^1 \left[z_{0x}^2(x) + z_1^2(x)\right]dx \leq \frac{d^2}{4}. \end{aligned} \tag{4.14}$$

Denote

$$d_0 \triangleq \frac{d}{2}\cdot\min\left\{e^{-\frac{g_1}{\pi}T}, \sqrt{\frac{1-2\chi}{1+2\chi}}e^{-\delta T^*}\right\}. \tag{4.15}$$

Then due to (4.11) for all solutions of (4.1) initiated from (4.5) the bound (4.10) holds. Moreover, due to (4.14) for all the resulting $e^{(m)}(x,t)$ and $e^{b(m)}(x,t)$ that satisfy (3.5) and (3.6) respectively the implication (4.7) holds:

$$\begin{aligned} &|z(x,t) + (\theta-1)e^{(m)}(x,t)|^2 \leq 2z^2(x,t) + 2[e^{(m)}(x,t)]^2 \leq d^2, \\ &|z(x,t) + (\theta-1)e^{b(m)}(x,t)|^2 \leq 2z^2(x,t) + 2[e^{b(m)}(x,t)]^2 \leq d^2, \\ &\forall t \in [t_0, t_0+T], \quad x \in [0,1], \theta \in [0,1]. \end{aligned}$$

The latter bounds guarantee (4.8). Then from Theorem 2 we conclude the following:

Corollary 1. *Given g_1 and positive tuning parameters T^* and δ, let there exist positive constants χ and λ_1 that satisfy the LMIs (4.12) and (4.13). Then for all $T \geq T^*$ the system (4.1) subject to $f(0,x,t) \equiv 0$ and (4.3) with the measurements (4.2) is regionally exactly observable on $[t_0, t_0+T]$ for all initial conditions from $\mathcal{X}_{d_0}$ given by (4.5), where d_0 is defined by (4.15).*

Remark 4. The result on the regional observability cannot be extended to multi-dimensional case since the bound (4.9) does not hold in n-D case. One could extend the regional result to n-D case if f would depend on $\int_\Omega |\nabla z|^2 dx$ or on $\int_\Omega z^2 dx$, $\int_{\Gamma_N} z^2 d\Gamma$ (by employing the inequalities of Lemma 1).

The global results of Sections 2 and 3 can be extended to more general functions $f = f(z, \nabla z, z_t)$ with uniformly bounded f_z, $|f_{\nabla z}|$ and f_{z_t}. Note that in Fridman (2013) such more general functions were considered for 1-D wave and for beam equations. However, the regional result in 1-D case seems to be not extendable to these more general nonlinearities due to difficulties of employing the bound (4.9) with z replaced by z_x or z_t.

Remark 5. The result on the regional observability can be easily extended to 1-D wave equations with variable coefficients as considered in Fridman (2013)

$$\begin{aligned} &z_{tt}(x,t) = \frac{\partial}{\partial x}[a(x)z_x(x,t)] + f(z(x,t),x,t), \\ &t \geq t_0, \quad x \in [0,1], \end{aligned}$$

where a is a C^1 function with $a_x \leq 0$ and $a(1) > 0$. This can be done by modifying Lyapunov and energy functions, where the square of the partial derivative in x should be multiplied by $a(x)$. Note that an extension of forward and backward observers to observability of 1-D wave equations with non-Lipschitz coefficients (as studied e.g. in Castro & Zuazua, 2002; Fanelli & Zuazua, 2013) seems to be problematic.

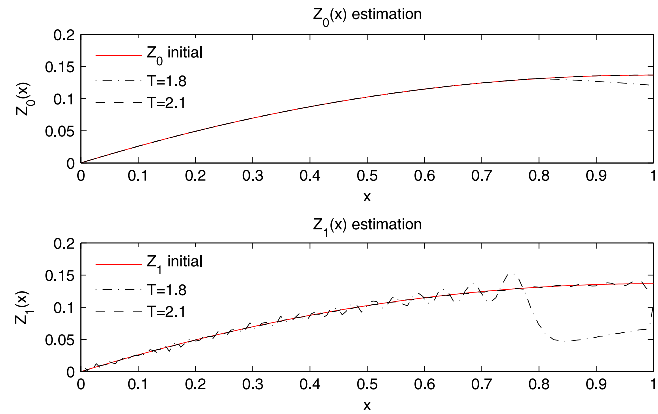

Fig. 1. Initial condition recovery after 10 iterations.

Table 2
Computation time for 10 iterations.

T	Computation time (s)
2.10	3.0469
3.00	3.6875
5.00	4.2813
10.00	5.9219

Example 2. Consider (4.1) with $f = 0.05z^2$. Here $|f_z| = |0.1z| \leq g_1$ if $|z| \leq 10g_1 = d$. Choose $g_1 = 0.1$, meaning that (4.3) holds with $d = 1$. Also here we use the sequence of forward and backward observers (3.3) and (3.4) with $k = 1$. Verifying the feasibility of LMIs (4.12) and (4.13) (subject to minimization of χ that enlarges the resulting d_0), we find that the system is exactly observable in time $T^* = 3.78$, where $\delta = 0.1$ and $\chi = 0.1803$. This leads to the estimate (4.5) with $d_0 = 0.2348$ for the region of exact observability, where the initial conditions of the system can be recovered uniquely from the measurements on the interval $[0, T]$ for all $T \in [3.78, 23.5]$. Note that the convergence of the iterative algorithm is faster for larger T (in the sense that (3.14) holds with a smaller q). Increasing the nonlinearity twice to $f = 0.1z^2$ and choosing $g_1 = 0.2$, we find $d = 1$. The LMIs (4.12) and (4.13) are feasible with $\delta = 0.09$, $T^* = 5.49$ and $\chi = 0.2275$. We arrive at a smaller $d_0 = 0.1867$, whereas $T \in [5.49, 15.4]$.

Simulations of the initial state recovery in the case of $f = 0.1z^2$ and $z_0(x) = z_1(x) = 0.2733 \cdot x(1-\frac{x}{2})$, where $\int_0^1 \left[z_{0x}^2 + z_1^2\right]dx = 0.1867^2$, show the convergence of the iterative algorithm on the predicted observation interval $[0, 5.49]$. Moreover, the algorithm converges on shorter observation intervals with $T \geq 2.1$ that illustrates the conservatism of the LMI conditions. See Fig. 1 for the case of 10 forward and backward iterations with $T = 1.8$ (no convergence) and $T = 2.1$ (convergence). The computation times for 10 iterations for several values of T are given in Table 2.

5. Conclusions

The LMI approach to observers and initial state recovering of semilinear N-D wave equations on a hypercube has been presented. In the linear 2-D case our results lead to an upper bound on the exact observability time, which is close to the analytical value, but does not recover it as it happened in 1-D case. For 1-D systems with locally Lipschitz nonlinearities we have found a (lower) bound on the region of initial values that are uniquely recovered from the measurements on the finite interval.

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