Finite Horizon H_∞ State-Feedback Control of Continuous-Time Systems with State Delays

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***Abstract*—The finite horizon H_∞ control of time-invariant linear systems with a finite number of point and distributed time delays is considered. The controller is obtained by solving coupled Riccati-type partial differential equations. The solutions to these equations and the resulting controllers are approximated by series expansions in powers of the largest delay. Unlike the infinite horizon case, these approximations possess both regular and boundary layer terms. The performance of the closed-loop system under the memoryless zero-approximation controller is analyzed.**

***Index Terms*—Asymptotic approximations, H_∞–state-feedback control, Riccati type partial differential equations, singular perturbations, time-delay systems.**

I. Problem Formulation

Throughout this paper we denote by $|\cdot|$ the Euclidean norm of a vector or the appropriate norm of a matrix. Given $t_f > 0$, let $L_2[0, t_f]$ be the space of the square integrable functions with the norm $\|\cdot\|_{L_2}$ and let $C[a, b]$ be the space of the continuous functions on $[a, b]$ with the norm $\|\cdot\|_\infty$. We denote $x_t = x(t+\theta), y^t = y(t-\theta), \theta \in [-h, 0]$. Prime denotes the transpose of a matrix and col$\{x, y\}$ denotes a column vector with components x and y.

Consider the system

$$\begin{aligned}\dot{x}(t) &= L(x_t(\cdot)) + Bu(t) + Dw(t)\\ z(t) &= \mathrm{col}\{Cx(t), u(t)\}, \qquad t \geq 0\\ x(\theta) &= x_0(\theta), \qquad \theta \in [-h, 0]\end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^l$ is the control signal, $w(t) \in \mathbf{R}^q$ is the exogenous disturbance, $z(t) \in \mathbf{R}^p$ is the observation vector, and B, C, and D are constant matrices of appropriate dimensions. The R^n-valued function $L(\cdot)$ which carries R^n-valued functions on $[-h, 0]$ into R^n is defined as follows:

$$L(x_t(\cdot)) = \sum_{i=0}^{r} A_i x_t(-h_i) + \int_{-h}^{0} A_{01}(s) x_t(s)\, ds \tag{2}$$

where $-h = -h_r < -h_{r-1} < \cdots < -h_1 < -h_0 = 0$, $A_0, A_1 \cdots, A_r$ are constant matrices and $A_{01}(s)$ is a square integrable matrix function.

Denote

$$F(x_t)(\xi) = \sum_{i=1}^{r} A_i x_t(-h_i - \xi)\chi_i(\xi) + \int_{-h}^{\xi} A_{01}(p) x_t(p - \xi)\, dp \tag{3}$$

and where χ_i is the indicator function for the set $[-h_i, 0]$, i.e., $\chi_i(\xi) = 1$ if $\xi \in [-h_i, 0]$ and $\chi_i(\xi) = 0$ otherwise.

Given $\gamma > 0$, and assuming that $w \in L_2[0, t_f]$ and $x_0 \in L_2[-h, 0]$, we consider the following performance index:

$$J = \|z\|_{L_2}^2 - \gamma^2 \|w\|_{L_2}^2 - W(x_0) \tag{4}$$

where

$$W(x_0) = x'(0) M_1 x(0) + \int_{-h}^{0} F'(x_0)(s) M_2 F(x_0)(s)\, ds \tag{5}$$

and where $M_1 = M_1', M_2 = M_2'$ are matrices denoting the initial weightning condition. The problem is to find a state-feedback controller which ensures that $J \leq 0$ for all $w \in L_2[0, t_f]$ and for all initial conditions $x_0 \in L_2[-h, 0]$.

In the *infinite horizon* case such a controller (for zero initial condition $x_0 = 0$) has been designed in [1]–[6] (see also references therein). In [1] and [2] the controller has been obtained by solving Riccati operator equations. In [3] and [4] delay-independent and in [6] delay-dependent memoryless controllers have been designed. In [5] the controller (with memory) has been derived from Riccati-type partial differential equations (RPDEs) or inequalities, and the solution of the RPDEs has been approximated by expansions in the powers of the delay. In [7] the gradient of J with respect to h at $h = 0$ has been computed. In [8] and [9] bounded real criteria have been obtained. Asymptotic series solutions of systems with small delay have been constructed in [10]–[12].

In many engineering cases (target maneuver, missile guidance, etc.) a control session of limited time length is needed. In such cases the effect of the initial conditions is most important and the results of [1]–[6] cannot provide a satisfactory control strategy. In the present paper, we generalize the results of [5] to the *finite horizon* case. Unlike [5], the required controllers are time-varying, they are obtained by solving coupled finite horizon RPDEs. For small delays, similarly to the case of singularly perturbed systems (see [10]–[13]), the controllers are affected by the boundary-layer phenomenon. The main contribution of the paper is the construction, for the first time, of an asymptotic solution to the important class of finite horizon RPDEs that are encountered with the finite-horizon LQ control (see [14]), and with the H_∞ control. Proofs of Theorem 1 and Lemma 2 are given in the Appendix.

II. Main Results

A. H_∞-Controller Design

Consider the following RPDEs with respect to the $n \times n$-matrices $P(t), Q(t, \xi)$, and $R(t, \xi, s)$:

$$\begin{aligned}&\dot{P}(t) + A_0' P(t) + P(t) A_0 + \sum_{i=1}^{r} A_i' Q'(t, -h_i)\\ &\quad + \sum_{i=1}^{r} Q(t, -h_i) A_i + P(t) S P(t) + C'C\\ &\quad + \int_{-h}^{0} Q(t, \theta) A_{01}(\theta)\, d\theta\\ &\quad + \int_{-h}^{0} A_{01}'(\theta) Q'(t, \theta)\, d\theta = 0\end{aligned} \tag{6}$$

$$\begin{aligned}&\frac{\partial}{\partial t} Q(t, \xi) + \frac{\partial}{\partial \xi} Q(t, \xi)\\ &\quad = -[A_0' + P(t) S] Q(t, \xi) - \sum_{i=1}^{r} A_i' R(t, -h_i, \xi)\\ &\quad\quad - \int_{-h}^{0} A_{01}'(s) R(t, s, \xi)\, ds\end{aligned} \tag{7}$$

$$\begin{aligned}&\frac{\partial}{\partial t} R(t, \xi, s) + \frac{\partial}{\partial \xi} R(t, \xi, s) + \frac{\partial}{\partial s} R(t, \xi, s)\\ &\quad = -Q'(t, \xi) S Q(t, s)\end{aligned} \tag{8}$$

$$\begin{aligned}&P(t) = Q(t, 0), \quad Q(t, \xi) = R(t, 0, \xi),\\ &R(t, \xi, s) = R'(t, s, \xi), \qquad \xi \in [0, h],\ s \in [0, h]\end{aligned} \tag{9}$$

$$P(t_f) = 0, \quad Q(t_f, \xi) = 0, \quad R(t_f, \xi, s) = 0 \tag{10}$$

where $S = \gamma^{-2} DD' - BB'$.

A solution of (6)–(10) is a triple of $n \times n$-matrices $\{P(t), Q(t, \xi), R(t, \xi, s)\}$ $t \in [0, t_f], \xi \in [-h, 0], s \in [-h, 0]$,

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where $P(t), Q(t,\xi)$, and $R(t,\xi,s)$ are continuous and piecewise continuously differentiable functions of their arguments that satisfy (6)–(10) for almost every t, ξ, and s.

Lemma 1: Given $\gamma > 0$. Let (6)–(10) have a solution on $[0, t_f]$ that for some $n \times n$ matrices $\Delta_1 > 0$ and $\Delta_2 > 0$ satisfies the following conditions:

$$\begin{aligned} &M_1 - 0.5P(0) - 0.5P'(0) - h\Delta_1 \geq 0 \\ &M_2 - Q'(0,s)\Delta_1^{-1}Q(0,s) \\ &\quad - 0.5h\Delta_2 - 0.5\int_{-h}^{0} R(0,s,\xi)\Delta_2^{-1}R(0,\xi,s)\,d\xi \geq 0, \\ &\qquad \forall\, s \in [-h, 0]. \end{aligned} \tag{11}$$

Then, the controller

$$u^*(t) = -B'\left[P(t)x(t) + \int_{-h}^{0} Q(t,\xi)F(x_t)(\xi)\,d\xi\right] \tag{12}$$

solves the H_∞-control problem with the performance level of γ.

Proof: Let $x(t)$ be a solution of (1). Consider the following Lyapunov–Krasovskii functional [14]:

$$\begin{aligned} V(t,x_t) = x(t)'P(t)x(t) + 2x'(t)\int_{-h}^{0} Q(t,\xi)F(x_t)(\xi)\,d\xi \\ + \int_{-h}^{0}\int_{-h}^{0} F'(x_t)(s)R(t,s,\xi)F(x_t)(\xi)\,ds\,d\xi. \end{aligned} \tag{13}$$

Differentiating $V(t,x_t)$ with respect to t and integrating by parts, we obtain, similarly to [5], that

$$\begin{aligned} \frac{d}{dt}V(t,x_t) = -x'(t)C'Cx(t) - \gamma^2|w(t) - w^*(t)|^2 \\ + \gamma^2|w(t)|^2 + |u(t) - u^*(t)|^2 - |u(t)|^2 \end{aligned} \tag{14}$$

where

$$w^*(t) = \gamma^{-2}D'\left[P(t)x(t) + \int_{-h}^{0} Q(t,\xi)F(x_t)(\xi)\,d\xi\right].$$

It follows from (14) that

$$\begin{aligned} V(t_f, x_{t_f}) - V(0,x_0) + \int_0^{t_f} [|z|^2 - \gamma^2|w|^2]\,dt \\ = -\gamma^2\|w - w^*\|_{L_2} + \|u - u^*\|_{L_2}. \end{aligned} \tag{15}$$

We show next that (11) implies

$$d = W(x_0) - V(0,x_0) \geq 0. \tag{16}$$

Denoting $v = x(0)$ and $y(s) = F(x_0)(s)$ we have

$$\begin{aligned} d = v'(M_1 - P(0))v + \int_{-h}^{0} y'(s)M_2y(s)\,ds \\ - 2v'\int_{-h}^{0} Q(0,s)y(s)\,ds \\ - \int_{-h}^{0}\int_{-h}^{0} y'(s)R(0,s,\xi)y(\xi)\,d\xi\,ds. \end{aligned} \tag{17}$$

Then, (16) follows from (17), the inequalities

$$\begin{aligned} &2\int_{-h}^{0} v'Q(0,s)y(s)\,ds \\ &\quad \leq hv'\Delta_1 v + \int_{-h}^{0} y'(s)Q'(0,s)\Delta_1^{-1}Q(0,s)y(s)\,ds \\ &2\int_{-h}^{0} y'(s)R(0,s,\xi)y(\xi)\,d\xi \\ &\quad \leq hy'(s)\Delta_2 y(s) + y'(s)\int_{-h}^{0} R(0,s,\xi)\Delta_2^{-1}R(0,\xi,s)\,d\xi y(s) \end{aligned} \tag{18}$$

and (11). Finally, (16), (15), and (10) imply that $J \leq 0$ for $u = u^*$. □

Remark 1: In the case of LQ problem from (15) it follows that $\min_u \|z\|_{L_2}^2 = V(0,x_0)$ and similarly

$$\begin{aligned} &\min_u \int_{t_0}^{t_f} |z|^2\,dt \\ &\quad = V(t_0,x_0), \qquad \forall\, t_0 \leq t_f,\ \forall\, x_0 \in L_2[-h,0]. \end{aligned}$$

Hence, $V(t,x) \geq 0$ in the LQ case.

Remark 2: Note that a certain amount of overdesign is introduced by the conditions of (11). This overdesign stems from the bounding in (18). In the case of the zero initial conditions $x(\theta) = 0, \theta \in [-h, 0]$ the conditions of (11) are not relevant and the controller of (12) solves the H_∞ control problem under the sole assumption that (6)–(10) have a solution on $[0, t_f]$.

B. Asymptotic Solutions to the RPDEs

For simplicity we assume that $A_{01} = 0$ further on. The H_∞ controller has been found above by solving a set of coupled PRDEs. Finding a solution to the latter is not an easy task and we are, therefore, looking for a solution to the RPDEs in a form of asymptotic expansion in the powers of the delay h

$$\begin{aligned} P(t) &= P_0(t) + h[P_1(t) + \Pi_{1P}(\tau)] \\ &\quad + h^2[P_2(t) + \Pi_{2P}(\tau)] + \cdots, \\ Q(t,h\zeta) &= Q_0(t,\zeta) + h[Q_1(t,\zeta) + \Pi_{1Q}(\tau,\zeta)] \\ &\quad + h^2[Q_2(t,\zeta) + \Pi_{2Q}(\tau,\zeta)] + \cdots, \\ R(t,h\zeta,h\theta) &= R_0(t,\zeta,\theta) + h[R_1(t,\zeta,\theta) + \Pi_{1R}(\tau,\zeta,\theta)] \\ &\quad + h^2[R_2(t,\zeta,\theta) + \Pi_{2R}(\tau,\zeta,\theta)] + \cdots, \\ \tau &= \frac{t_f - t}{h}, \qquad \zeta \in [-1,0],\ \theta \in [-1,0]. \end{aligned} \tag{19}$$

Expansion (19) has a *typical for singular perturbations* form: it includes the "outer expansion" (regular) terms $\{P_i, Q_i, R_i\}, i = 0, 1\cdots$ and the boundary-layer correction terms Π_{iP}, Π_{iQ}, and $\Pi_{iR}, i = 1, 2\cdots$ The "outer expansion" terms constitute the major part of the solution that satisfies (6)–(9) for $t \in [0, t_f], \theta \in [-1, 0], \zeta \in [-1, 0]$. The *boundary-layer correction* terms will be chosen such that (19) satisfies the *terminal conditions* of (10) and that

$$\begin{aligned} &|\Pi_{iP}(\tau)| + \sup_{\zeta\in[-1,0]} |\Pi_{iQ}(\tau,\zeta)| \\ &\quad + \sup_{\zeta,\theta\in[-1,0]} |\Pi_{iR}(\tau,\zeta,\theta)| \to 0 \qquad \text{as } \tau \to \infty. \end{aligned} \tag{20}$$

The boundary-layer correction terms depend on τ as on the *independent variable* and do not depend on h. Since τ is a stretched-time variable around $t = t_f$, (20) asserts that Π_{iP}, Π_{iQ}, and Π_{iR} are essential only around $t = t_f$ and they thus provide a correction to the outer expansion at the terminal point $t = t_f$.

We substitute (19) in (6)–(9) and *equate*, *separately*, outer expansion and boundary-layer correction terms with the same powers of h. We notice that for $t = t_f - h\tau, \xi = h\zeta$, and $s = h\theta$ we have $\partial/\partial t = -h^{-1}\partial/\partial\tau, \partial/\partial\xi = h^{-1}\partial/\partial\zeta$, and $\partial/\partial s = h^{-1}\partial/\partial\theta$. Thus, for the zero-order terms we obtain from (7)-(9)

$$\begin{aligned} &\frac{\partial}{\partial\zeta}Q_0(t,\zeta) = 0 \\ &\frac{\partial}{\partial\zeta}R_0(t,\zeta,\theta) + \frac{\partial}{\partial\theta}R_0(t,\zeta,\theta) = 0 \\ &Q_0(t,0) = P_0(t), \quad R_0(t,0,\zeta) = Q_0(t,\zeta) \end{aligned}$$

and hence

$$Q_0(t,\zeta) = P_0(t), \quad R_0(t,\zeta,\theta) = P_0(t). \tag{21}$$

Then, from (6), we have

$$\begin{aligned} \dot{P}_0(t) + \sum_{i=0}^{r} A_i'P_0(t) + \sum_{i=0}^{r} P_0(t)A_i + P_0(t)SP_0(t) \\ + C'C = 0, \qquad P_0(t_f) = 0. \end{aligned} \tag{22}$$

The latter is the well-known Riccati differential equation (RDE) that corresponds to (1) for $h = 0$. Our main assumption is as follows.

A1. For a specified value of $\gamma > 0$, the RDE of (22) has a bounded solution on $[0, t_f]$.

Assumption A1 means that the H_∞ state-feedback control problem for (1) without delay has a solution. If this were not the case, even P_0, the zero-order term in (19), would not exist. Note that $P_0 = P_0'$.

To determine the first-order terms we start with the equations for Q_1

$$\begin{aligned} \frac{\partial}{\partial\zeta} Q_1(t,\zeta) &= -\mathcal{M}'(t)P_0(t) - \dot{P}_0(t) \\ Q_1(t,0) &= P_1(t), \qquad \mathcal{M} = \sum_{i=0}^{r} A_i + SP_0. \end{aligned} \tag{23}$$

Then

$$Q_1(t,\zeta) = P_1(t) - [\mathcal{M}'(t)P_0(t) + \dot{P}_0(t)]\zeta.$$

Substituting this expression into the equation for P_1, we obtain

$$\begin{aligned} &\dot{P}_1 + \mathcal{M}'P_1 + P_1\mathcal{M} + \sum_{i=1}^{r} g_i A_i'(P_0\mathcal{M} + \dot{P}_0) \\ &\quad + \sum_{i=1}^{r} g_i(\mathcal{M}'P_0 + \dot{P}_0)A_i = 0 \\ &P_1(t_f) + \Pi_{1P}(0) = 0, \quad g_i = h_i/h \end{aligned} \tag{24}$$

It follows from (6) that $\dot{\Pi}_{1P}(\tau) = 0$. Since Π_{1P} vanishes for $\tau \to \infty$, we have $\Pi_{1P}(\tau) \equiv 0, \tau \geq 0$. Hence, $P_1(t_f) = 0$, and P_1 is a solution to the linear differential equation (24) with the latter terminal condition. For Π_{1Q}, R_1, and Π_{1R} we obtain from (7), (8), and (21)

$$\begin{aligned} \frac{\partial}{\partial\tau} \Pi_{1Q}(\tau,\zeta) - \frac{\partial}{\partial\zeta} \Pi_{1Q}(\tau,\zeta) &= 0 \\ Q_1(t_f,\zeta) + \Pi_{1Q}(0,\zeta) &= 0 \\ \Pi_{1Q}(\tau,0) = \Pi_{1P}(\tau) &= 0 \\ \frac{\partial}{\partial\zeta} R_1(t,\zeta,\theta) + \frac{\partial}{\partial\theta} R_1(t,\zeta,\theta) &= -P_0(t)SP_0(t) - \dot{P}_0(t) \\ R_1(\tau,0,\theta) &= Q_1(\tau,\theta) \end{aligned} \tag{25}$$

and

$$\begin{aligned} \frac{\partial}{\partial\tau}\Pi_{1R}(\tau,\zeta,\theta) - \frac{\partial}{\partial\zeta}\Pi_{1R}(\tau,\zeta,\theta) - \frac{\partial}{\partial\theta}\Pi_{1R}(\tau,\zeta,\theta) &= 0 \\ R_1(t_f,\zeta,\theta) + \Pi_{1R}(0,\zeta,\theta) &= 0 \\ \Pi_{1R}(\tau,0,\theta) &= \Pi_{1Q}(\tau,\theta). \end{aligned} \tag{26}$$

Note that $Q_1(t_f,\zeta) = -\dot{P}_0(t_f)\zeta$. Then, for $\tau \geq 0$ and $t \in [0, t_f]$, we find successively

$$\begin{aligned} \Pi_{1Q}(\tau,\zeta) &= \begin{cases} (\zeta+\tau)\dot{P}_0(t_f), & \text{if } \tau \leq -\zeta \\ 0, & \text{if } \tau > -\zeta \end{cases} \\ R_1(t,\zeta,\theta) &= R_1'(t,\theta,\zeta) \\ &= -\zeta[P_0(t)SP_0(t) + \dot{P}_0(t)] + Q_1(t,\theta-\zeta), \\ &\qquad \zeta \geq \theta \\ \Pi_{1R}(0,\zeta,\theta) &= \theta\dot{P}_0(t_f) \\ \Pi_{1R}(\tau,\zeta,\theta) &= \Pi_{1R}'(\tau,\theta,\zeta) \\ &= \begin{cases} (\theta+\tau)\dot{P}_0(t_f), & \text{if } \tau \leq -\theta, \theta \leq \zeta \\ 0, & \text{if } \tau > -\theta, \theta \leq \zeta. \end{cases} \end{aligned} \tag{27}$$

Therefore

$$\begin{aligned} \Pi_{1Q}(\tau,\zeta) &= 0, \qquad \tau + \zeta > 0 \\ \Pi_{1R}(\tau,\zeta,\theta) &= \Pi_{1Q}(\tau+\zeta, \theta-\zeta) = 0 \\ &\qquad \tau + \theta > 0, \ \theta \leq \zeta. \end{aligned} \tag{28}$$

The higher order terms of the outer expansions can be similarly found. We obtain next the boundary-layer terms and show by induction that

$$\begin{aligned} \Pi_{iP}(\tau) &= 0, \qquad \tau > i-1 \\ \Pi_{iQ}(\tau,\zeta) &= 0, \qquad \tau+\zeta > i-1 \\ \Pi_{iR}(\tau,\zeta,\theta) &= 0, \qquad \tau+\theta > i-1, \ \theta \leq \zeta. \end{aligned} \tag{29}$$

We assume that (29) is satisfied for all $i \leq m-1$. We derive the following equations for Π_{mP}, Π_{mQ}, and Π_{mR}:

$$\begin{aligned} \dot{\Pi}_{mP}(\tau) &= f_m(\tau) \\ \Pi_{mP}(m-1) &= 0 \\ \frac{\partial}{\partial\tau}\Pi_{mQ}(\tau,\zeta) - \frac{\partial}{\partial\zeta}\Pi_{mQ}(\tau,\zeta) &= \phi_m(\tau,\zeta) \\ \Pi_{mQ}(\tau,0) &= \Pi_{mP}(\tau) \\ Q_m(t_f,\zeta) + \Pi_{mQ}(0,\zeta) &= 0 \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial\tau}\Pi_{mR}(\tau,\zeta,\theta) - \frac{\partial}{\partial\zeta}\Pi_{mR}(\tau,\zeta,\theta) - \frac{\partial}{\partial\theta}\Pi_{mR}(\tau,\zeta,\theta) \\ &\quad = \psi_m(\tau,\zeta,\theta) \\ &\Pi_{mR}(\tau,0,\theta) = \Pi_{mQ}(\tau,\theta), \quad R_m(t_f,\zeta,\theta) + \Pi_{mR}(0,\zeta,\theta) = 0 \end{aligned}$$

where f_m and ϕ_m are known functions that vanish for $\tau > m-1$, and ψ_m is a known function that vanishes for $\tau+\theta > m-2, \theta \leq \zeta$.

From these equations we find

$$\Pi_{mP}(\tau) = \int_{m-1}^{\tau} f_m(s)\,ds$$

and thus (29) holds for Π_{mP} since $f_m(s) = 0$ for $\tau > m-1$. Further

$$\Pi_{mQ}(\tau,\zeta) = \begin{cases} \Pi_{mQ}(0,\zeta+\tau) \\ \quad + \int_0^{\tau} \phi_m(s, -s+\tau+\zeta)\,ds, & \text{if } \tau \leq -\zeta \\ \Pi_{mP}(\zeta+\tau) \\ \quad - \int_0^{\zeta} \phi_m(-s+\tau+\zeta, s)\,ds, & \text{if } \tau > -\zeta \end{cases}$$

and Π_{mQ} satisfies (29) since $\Pi_{mP}(\zeta+\tau) = 0$ for $\zeta+\tau > m-1$ and $\phi_m(\tau,\zeta) = 0$ for $\tau > m-1$. Finally

$$\begin{aligned} &\Pi_{mR}(\tau,\zeta,\theta) \\ &\quad = \Pi_{mR}'(\tau,\theta,\zeta) \\ &\quad = \begin{cases} \int_0^{\tau} \psi_m(s, -s+\zeta+\tau, -s+\theta+\tau)\,ds \\ \quad + \Pi_{mR}(0, \zeta+\tau, \theta+\tau), & \text{if } \tau \leq -\zeta, \theta \leq \zeta \\ -\int_0^{\zeta} \psi_m(-s+\zeta+\tau, s, s+\theta-\zeta)\,ds \\ \quad + \Pi_{mQ}(\tau+\zeta, \theta-\zeta), & \text{if } \tau > -\zeta, \theta \leq \zeta. \end{cases} \end{aligned} \tag{30}$$

Conditions (29) for Π_{mR} readily follow from (30) and the properties of Π_{mQ} and ψ_m.

C. Near-Optimal H_∞ Control

Theorem 1: Under A1 the following holds for all small enough time-delay h

i) The system of (6)–(10) has a solution. This solution is approximated, for any integer m, by

$$\begin{aligned} P(t) &= P_0(t) + \sum_{i=1}^{m} h^i [P_i(t) + \Pi_{iP}(\tau)] + O(h^{m+1}) \\ Q(t, h\zeta) &= P_0(t) + \sum_{i=1}^{m} h^i [Q_i(t,\zeta) + \Pi_{iQ}(\tau,\zeta)] + O(h^{m+1}) \\ R(t, h\zeta, h\theta) &= P_0(t) + \sum_{i=1}^{m} h^i [R_i(t,\zeta,\theta) + \Pi_{iR}(\tau,\zeta,\theta)] \\ &\quad + O(h^{m+1}) \\ \tau &= \frac{t_f - t}{h}, \qquad \zeta \in [-1, 0],\ \theta \in [-1, 0] \end{aligned} \tag{31}$$

where the boundary-layer terms satisfy (29), and $|O(h^{m+1})| \le ch^{m+1}$, where c is a positive scalar which is independent of h, t, ζ, and θ.

ii) If additionally

$$P_0(0) < M_1, \qquad M_2 > 0 \tag{32}$$

then the controller of (12) is approximated by

$$\begin{aligned} u(x_t) &= u_m(x_t) + O(h^{m+1}) \\ u_m(x_t) &= -\sum_{i=0}^{m} h^i B' \Big\{ [P_i(t) + \Pi_{iP}(\tau)] x(t) \\ &\quad + \int_{-1}^{0} [Q_{i-1}(t,\zeta) + \Pi_{i-1,Q}(\tau,\zeta)] x(t + h\zeta)\, d\zeta \Big\} \end{aligned} \tag{33}$$

where $\Pi_{0P} = 0, Q_{-1} = \Pi_{-1,Q} = \Pi_{0,Q} = 0, Q_0 = P_0$. The approximate controller u_m guarantees an attenuation level of $\gamma + O(h^{m+1})$.

It follows from Theorem 1 that a high-order approximate controller improves the performance polynomially in the size of the small time-delay h.

D. The Zero-Order Controller Performance

We study the performance of the system under the zero-order controller $u_0(t) = -B'P_0(t)x(t)$ which solves the H_∞-control problem for (1) without delay. For simplicity we consider the case of $x_0 = 0$. Applying u_0 to (1), we obtain

$$\begin{aligned} \dot{x}(t) &= \overline{A}(t)x(t) + \sum_{i=1}^{r} A_i x(t - h_i) + Dw(t) \\ \overline{A}(t) &= A_0 - BB'P_0(t) \\ z &= \tilde{C}x(t), \quad \tilde{C}(t) = \mathrm{col}\{C, -B'P_0(t)\}. \end{aligned} \tag{34}$$

Let $X(t, t_0)$ be the transition matrix of the system of (34), i.e., $X(t, t_0) = 0$ for $t < t_0$, $X(t_0, t_0) = I_n$ and $X(t, t_0)$ satisfies (34) for $t \ge t_0$. Let $X_0(t, t_0)$ be the transition matrix of (34) without delay, i.e., where $h_i = 0$. Then there exist scalars $\beta_0 > 0, \beta > 0, \alpha$, and δ such that for small enough h the following inequalities are valid:

$$|X_0(t, t_0)| \le \beta_0 e^{\alpha(t - t_0)} \tag{35a}$$

$$|X(t, t_0)| \le \beta e^{\delta(t - t_0)}, \qquad t, t_0 \in [0, t_f]. \tag{35b}$$

Lemma 2: Under A1 the controller $u_0(t) = -B'P_0(t)x(t)$ for the zero initial condition $x_0 = 0$ guarantees

i) for all small enough h a performance level of γ;

ii) for all h, a performance level of $\overline{\gamma}$, where

$$\begin{aligned} \overline{\gamma}^2 &= \frac{\beta^2}{2\delta} t_f \left(e^{2\delta t_f} - 1\right) |D|^2 \left[|C|^2 + \|B'P_0\|_{L_2}^2\right] \left[\sum_{i=1}^{r} |A_i|^2 h_i^2\right] \\ &\quad \cdot \left[1 + \frac{\beta_0^2}{2\alpha} t_f \left(e^{2\alpha t_f} - 1\right) \left\| \overline{A} + \sum_{i=1}^{r} A_i \right\|_\infty^2 \right] + \gamma^2 \end{aligned} \tag{36}$$

and where for $\alpha = 0$ ($\delta = 0$) one has to take limit $\alpha \to 0$ ($\delta \to 0$).

It follows from Lemma 2 that the controller u_0 guarantees a γ performance level for all small time delays and it guarantees a performance level $\overline{\gamma}$ for all delays. Note that $\overline{\gamma} \to \gamma$ for $h \to 0$. Given $\gamma > 0$ and h, in order to make certain that u_0 leads to a performance level of γ one can verify conditions in terms of differential linear matrix inequalities or Riccati differential inequalities (RDI) that were formulated for the case of one delay in [8] and can be easily generalized to the case of r delays.

E. Example

Consider the following system:

$$\dot{x}(t) = x(t) - x(t - h) + 2u - w, \qquad z = \mathrm{col}\{x, u\} \tag{37}$$

and J of (4) with $M_1 > \tan t_f$ and $M_2 > 0$. From (22) we obtain

$$\dot{P}_0(t) + (\gamma^{-2} - 4)P_0^2 + 1 = 0, \qquad P_0(t_f) = 0. \tag{38}$$

Note that for $\gamma^2 \ge 1/4$ the latter RDE has a bounded solution on $[0, t_f]$ for all $t_f > 0$. Choosing $\gamma^2 = 1/5 < 1/4$ we find that $P_0 = \tan(t_f - t)$ and thus for $t_f < \pi/2$ (38) has a bounded solution on $[0, t_f]$. It is readily seen that conditions (32) hold. Equation (24) has the form

$$\dot{P}_1 + 2\tan(t_f - t)P_1 + 2 = 0, \qquad P_1(t_f) = 0.$$

From the latter equation, (27), and (28) we find

$$\begin{aligned} P_1 &= \tan(t_f - t) + \frac{(t_f - t)}{\cos^2(t_f - t)}, \quad Q_1 = P_1 + \zeta \\ \Pi_{1Q} &= -(\tau + \zeta)\chi(-\tau - \zeta) \\ R_1(t, \zeta, \theta) &= R_1'(t, \theta, \zeta) = \theta + P_1(t) \\ \Pi_{1R}(\tau, \zeta, \theta) &= \Pi_{1R}'(\tau, \zeta, \theta) = -(\tau + \theta)\chi(-\tau - \theta), \qquad \theta \le \zeta \end{aligned}$$

where $\chi(s) = 1$ for $s \ge 0$ and $\chi(s) = 0$ for $s < 0$. Since $Q_0 = P_0$ we obtain for $0 < t_f < \pi/2$ that

$$u_0(t) = -2\tan(t_f - t)x(t)$$

and

$$\begin{aligned} u_1(t) &= u_0(t) - 2h \\ &\quad \cdot \left[P_1(t)x(t) + \tan(t_f - t) \int_{-1}^{0} x(t + h\zeta)\, d\zeta\right]. \end{aligned}$$

Consider now the performance of (37) under $u = u_0$ and $x_0 = 0$ for $t_f = 1.1$. Applying the delay-dependent criterion of [8] on the closed-loop system we find that u_0 achieves $\gamma = (1/\sqrt{5})$ for all delays $h \in (0, 0.027]$, since the corresponding RDIs have bounded solutions on [0, 1.1]. For $h = 0.028$ the solutions to the RDIs encounter escape points and the criterion of [8], which provide a sufficient condition only, cannot therefore be used to verify the level $\gamma = (1/\sqrt{5})$.

III. Conclusions

A solution to the state-feedback H_∞ control of linear time-invariant systems with state time delays in the finite horizon case is presented. The controller is obtained by solving RPDEs. An approximate solution to the RPDEs has been constructed by expansion in powers of the largest delay. The theory that has been developed in this paper shows that similarly to the case of singularly perturbed systems [13], for small delays our controllers are affected by the boundary-layer phenomenon. The high order approximate controller improves the performance of the closed-loop system polinomially in the size of the delay. The memoryless zero-approximation may, in many cases, be sufficient for robustly achieving the required performance. It is shown that the performance of the system under such a controller is robust for small time delays. Explicit formula for the guaranteed performance level is obtained for this case in terms of the coefficients of the system.

Appendix

Proof of Theorem 1:

i) To prove the validity of (31) we consider the equations for the remainders

$$\begin{aligned} &h^{m+1}P_{m+1} \\ &\quad = P - \sum_{i=0}^{m} h^i P_i \\ &h^{m+1}Q_{m+1}(t,\xi) \\ &\quad = Q(t,\xi) - \sum_{i=0}^{m} h^i \left[Q_i\left(t, h^{-1}\xi\right) + \Pi_{iQ}\left[h^{-1}(t_f - t), h^{-1}\xi\right]\right] \\ &h^{m+1}R_{m+1}(t,\xi,s) \\ &\quad = R(t,\xi,s) - \sum_{i=0}^{m} h^i \left[R_i\left(t, h^{-1}\xi, h^{-1}s\right)\right. \\ &\qquad \left. + \Pi_{iR}\left[h^{-1}(t_f - t), h^{-1}\xi, h^{-1}s\right)\right]\right] \end{aligned}$$

in the following expansions:

$$\begin{aligned} &\dot{P}_{m+1} + P_{m+1}\mathcal{M} + \mathcal{M}'P_{m+1} \\ &\quad + \sum_{i=1}^{r} A_i'[Q_{m+1}'(t,-h_i) - Q_{m+1}'(t,0)] \\ &\quad + \sum_{i=1}^{r} [Q_{m+1}(t,-h_i) - Q_{m+1}(t,0)]A_i \\ &\quad + E_m(t,h,hP_{m+1}(t)) = 0 \end{aligned} \tag{39}$$

$$\begin{aligned} &\frac{\partial}{\partial t} Q_{m+1}(t,\xi) + \frac{\partial}{\partial \xi} Q_{m+1}(t,\xi) \\ &\quad = -\mathcal{M}'Q_{m+1}(t,\xi) - \sum_{i=1}^{r} A_i'[R_{m+1}(t,-h_i,\xi) \\ &\qquad - R_{m+1}(t,0,\xi)] \\ &\qquad + h^{-1}g_m(t,\xi) + G_m(t,h,hP_{m+1}(t),hQ_{m+1}(t,\xi)) \end{aligned} \tag{40}$$

$$\begin{aligned} &\frac{\partial}{\partial t} R_{m+1}(t,\xi,s) \\ &\quad + \frac{\partial}{\partial \xi} R_{m+1}(t,\xi,s) \\ &\quad + \frac{\partial}{\partial s} R_{m+1}(t,\xi,s) + h^{-1}k_m(t,\xi,s) \\ &\quad + K_m(t,h,\xi,s,hQ_{m+1}(t,\xi),hQ_{m+1}(t,s)) = 0 \end{aligned} \tag{41}$$

$$\begin{aligned} &P_{m+1}(t) = Q_{m+1}(t,0 \\ &Q_{m+1}(t,\xi) = R_{m+1}(t,0,\xi) \\ &R_{m+1}(t,\xi,s) = R_{m+1}'(t,s,\xi) \\ &P_{m+1}(t_f) = 0, \quad Q_{m+1}(t_f,\xi) = 0, \quad R(t_f,\xi,s) = 0. \end{aligned} \tag{42}$$

Note that P_{m+1}, Q_{m+1}, and R_{m+1} depend on h. The known matrix functions E_m, G_m, and K_m are continuous on t,h,ξ,s and contain linear and quadratic terms in hP_{m+1} and hQ_{m+1}. The known matrix functions g_m and k_m are continuous on t,ξ,s.

Let $\Phi(t,s)$ be the transition matrix of the system $\dot{x}(t) = -\mathcal{M}'(t)x(t)$. Denote

$$\begin{aligned} \overline{E}_m(t) &= E_m(t,h,hP_{m+1}(t)) \\ \overline{G}_m(t,\xi) &= h^{-1}g_m(t,\xi) \\ &\quad + G_m(t,h,hP_{m+1}(t),hQ_{m+1}(t,\xi)) \\ \overline{K}_m(t,\xi,s) &= h^{-1}k_m(t,\xi,s) \\ &\quad + K_m(t,h,\xi,s,hQ_{m+1}(t,\xi),hQ_{m+1}(t,s)). \end{aligned}$$

Then, the system of (39)–(42) implies the following integral system for the determination of P_{m+1}, R_{m+1}, and Q_{m+1}:

$$\begin{aligned} &P_{m+1}(t) \\ &\quad = -\int_{t_f}^{t} \Phi(t,p)\left\{\sum_{i=1}^{r} A_i'[Q_{m+1}'(p,-h_i) - Q_{m+1}'(p,0)]\right. \\ &\qquad + \sum_{i=1}^{r} [Q_{m+1}(p,-h_i) - Q_{m+1}(p,0)]A_i \\ &\qquad \left. + \overline{E}_m(p,h,hP_{m+1}(p))\right\}\Phi'(t,p)\,dp \\ &Q_{m+1}(t,\xi) \\ &\quad = \begin{cases} \displaystyle\int_{t_f}^{t} \Phi(t,p)\overline{G}_m(p,p-t+\xi)\,dp, \\ \qquad \text{if } t-\xi \geq t_f \\ \Phi(t,t-\xi)P_{m+1}(t-\xi) \\ \quad \displaystyle + \int_0^{\xi} \Phi(\xi,p)\overline{G}_m(p+t-\xi,p)\,dp, \\ \qquad \text{if } t-\xi < t_f \end{cases} \\ &R_{m+1}(t,\xi,s) \\ &\quad = R_{m+1}'(t,s,\xi) \\ &\quad = \int_{t_f}^{t} \overline{K}_m(p,p+\xi-t,p+s-t)\,dp, \\ &\quad \text{if } t_f - t \leq -\xi, s \leq \xi \\ &R_{m+1}(t,\xi,s) \\ &\quad = R_{m+1}'(t,s,\xi) \\ &\quad = -\int_0^{\xi} \overline{K}_m(p-\xi+t,p,p+s-\xi)\,dp \\ &\qquad + Q_{m+1}(t-\xi,s-\xi), \quad \text{if } t_f - t > -\xi, s \leq \xi. \end{aligned}$$

Applying the contraction principle argument on the latter system, one can show that for all small enough $h > 0$ this system has a unique solution P_{m+1}, Q_{m+1}, and R_{m+1}, uniformly bounded and continuously depending on $h > 0, t, s$, and ξ. Hence, the approximation of (31) is uniform on h, t, ζ, and θ.

ii) Equation (33) follows from (31) and the rest of ii) is similar to [5]. □

Proof of Lemma 2: i) Applying u_0 to (1), we obtain the system of (34). Note that in (34) only the matrices $\overline{A}(t)$ and $\tilde{C}(t)$ are time-varying and thus the corresponding $F(x_t)$ is given by (3) and is time-invariant. Similarly to Remark 2 it can be shown that for this closed-loop system $\|z\|_{L_2}^2 \leq \gamma^2\|w\|_{L_2}^2$ for all $w \in L_2[0,t_f]$ and $x_0 = 0$ if the corresponding RPDEs of (6)–(10), where $A_0 = \overline{A}(t), C = \tilde{C}(t)$, and

$S = DD'/\gamma^2$, have a solution. Similarly to i) of Theorem 1 it can be proved that the resulting RPDEs have a solution, approximated by

$$\begin{aligned} P(t) &= P_0(t) + O(h) \\ Q(t, h\zeta) &= P_0(t) + O(h) \\ R(t, h\zeta, h\theta) &= P_0(t) + O(h) \end{aligned}$$

where $P_0(t)$ satisfies (22).

ii) Let $x(t)$ be a solution of (1) with $u = u_0$ and with $h > 0$ and let $y(t)$ be a solution of (1) with $u = u_0$ and with $h = 0$. Then, $v(t) = x(t) - y(t)$ satisfies the following equation:

$$\begin{aligned} \dot{v}(t) = \overline{A}(t)v(t) &+ \sum_{i=1}^{r} A_i v(t - h_i) \\ &+ \sum_{i=1}^{r} A_i[y(t-h_i) - y(t)], \qquad v_0 = 0 \end{aligned} \tag{43}$$

where

$$y(t) - y(t-h_i) = \int_{t-h_i}^{t} \left[(\overline{A}(s) + \sum_{i=1}^{r} A_i) y(s) + Dw(s) \right] ds. \tag{44}$$

From (44) and (35b) it follows that

$$\begin{aligned} &|y(t) - y(t-h_i)| \\ &\quad \le \int_{t-h_i}^{t} \left[|D|\,|w(s)| + \left| \overline{A}(s) + \sum_{i=1}^{r} A_i \right| |y(s)| \right] ds. \end{aligned} \tag{45}$$

By the variation of constants formula [15], (43) is equivalent to the integral equation

$$v(t) = \int_0^t X(t,s) \sum_{i=1}^{r} A_i[y(s-h_i) - y(s)]\, ds. \tag{46}$$

Applying (35a) and changing the order of integration we find

$$\begin{aligned} \|y\|_{L_2}^2 &= \int_0^{t_f} \left| \int_0^t X_0(t,s) Dw(s)\, ds \right|^2 dt \\ &\le \beta_0^2 \int_0^{t_f} dt \int_0^t ds_1 \int_0^t |D|^2 e^{\alpha(t-s_2)} |w(s_1)| \\ &\quad \times e^{\alpha(t-s_1)} |w(s_2)|\, ds_2 \\ &\le \beta_0^2 |D|^2 \int_0^{t_f} dt \int_0^t ds_1 \int_0^t e^{2\alpha(t-s_1)} |w(s_2)|^2 ds_2 \\ &= \frac{\beta_0^2 |D|^2}{2\alpha} \int_0^{t_f} \int_0^t \left(e^{2\alpha t} - 1 \right) |w(s_2)|^2\, ds_2\, dt \\ &\le \frac{\beta_0^2 |D|^2}{2\alpha} \left(e^{2\alpha t_f} - 1 \right) t_f \|w\|_{L_2}^2. \end{aligned}$$

From (45), (46), (35b), and the latter inequality, we obtain

$$\begin{aligned} \|v\|_{L_2}^2 &\le \beta^2 \sum_{i=1}^{r} \left[|A_i|^2 \int_0^{t_f} \int_0^t e^{\delta(t-s)} \int_{s-h_i}^{s} \right. \\ &\quad \cdot \left[|D|\,|w(\tau)| + \left| \overline{A}(\tau) + \sum_{i=1}^{r} A_i \right| |y(\tau)| \right] d\tau\, ds \\ &\quad \times \int_0^t e^{\delta(t-p)} \int_{p-h_i}^{p} \left[|D|\,|w(r)| \right. \\ &\quad \left. \left. + \left| \overline{A}(r) + \sum_{i=1}^{r} A_i \right| |y(r)| \right] dr\, dp\, dt \right] \\ &\le \beta^2 \sum_{i=1}^{r} |A_i|^2 |h_i|^2 |D|^2 \\ &\quad \cdot \left[1 + \left\| \overline{A} + \sum_{i=1}^{r} A_i \right\|_\infty^2 \left(e^{2\alpha t_f} - 1 \right) t_f \frac{\beta_0^2}{2\alpha} \right] \\ &\quad \cdot \int_0^{t_f} \int_p^{t_f} \int_0^t e^{\delta(2t-2\tau)}\, d\tau\, dt |w(p)|^2\, dp \\ &\le \frac{\beta^2}{2\delta} t_f \left(e^{2\delta t_f} - 1 \right) \sum_{i=1}^{r} |A_i|^2 h_i^2 |D|^2 \\ &\quad \cdot \left[1 + \frac{\beta_0^2}{2\alpha} t_f \left(e^{2\alpha t_f} - 1 \right) \left\| \overline{A} + \sum_{i=1}^{r} A_i \right\|_\infty^2 \right] \|w\|_{L_2}^2. \end{aligned}$$

Hence,

$$\begin{aligned} \|z\|_{L_2}^2 &\le \|Cv\|_{L_2}^2 + \|B'P_0 v\|_{L_2}^2 + \|Cy\|_{L_2}^2 \\ &\quad + \|B'P_0 y\|_{L_2}^2 \le \overline{\gamma}^2 \|w\|_{L_2}^2. \end{aligned} \qquad \square$$

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