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OSCILLATIONS IN A SECOND-ORDER DISCONTINUOUS SYSTEM WITH DELAY

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Abstract. We consider the equation

$$\alpha x''(t) = -x'(t) + F(x(t), t) - \operatorname{sign} x(t-h), \quad \alpha = \text{const} > 0,\ h = \text{const} > 0,$$

which is a model for a scalar system with a discontinuous negative delayed feedback, and study the dynamics of oscillations with emphasis on the existence, frequency and stability of periodic oscillations. Our main conclusion is that, in the autonomous case $F(x,t) \equiv F(x)$, for $|F(x)| < 1$, there are periodic solutions with different frequencies of oscillations, though only slowly-oscillating solutions are (orbitally) stable. Under extra conditions we show the uniqueness of a periodic slowly-oscillating solution. We also give a criterion for the existence of bounded oscillations in the case of unbounded function $F(x,t)$. Our approach consists basically in reducing the problem to the study of dynamics of some discrete scalar system.

Introduction. Equations with discontinuity often appear in various control theory models [2, 4, 5, 10, 11, 15, 18]. Time delay, always existing in real systems, usually results in oscillations around the discontinuity surface. In the present paper we study various aspects of such oscillations for the system

$$\alpha x''(t) = -x'(t) + F(x(t), t) - \operatorname{sign} x(t-h)\ , \tag{0.1}$$

where F is a smooth function and α, h are positive constants (the positivity of α means that $-\operatorname{sign} x(t-h)$ provides a negative feedback). An important point is a

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choice of $\operatorname{sign} 0$. From the control theory point of view signum should be a binary sensor or actuator, i.e., taking only values ± 1. Following [1], we choose

$$\operatorname{sign} x(t) = \begin{cases} 1 \ , & \text{as } x(t) > 0, \\ -1 \ , & \text{as } x(t) < 0, \\ z(t) \ , & \text{as } x(t) = 0, \end{cases}$$

where $z(t)$ is any measurable function with $|z(t)| = 1$. The convenience of this choice is explained by Lemma 1.1 below, which claims that, for any solution of (0.1), $\operatorname{mes}\{t \geq 0 \ : \ x(t) = 0\} = 0$.

For small α, equation (0.1) can be viewed as a singular perturbation of the equation

$$x'(t) = F(x(t), t) - \operatorname{sign} x(t-h) \ , \tag{0.2}$$

which has thoroughly been studied in [7, 16] (see also [20], where signum is replaced by a close continuous odd function), and we show that equation (0.1) inherits the basic properties of (0.2), especially, in the case

$$|F(x,t)| \leq p = \text{const} < 1 \ . \tag{0.3}$$

First of all, for oscillating solutions of (0.1) we define a frequency function which is non-increasing as a similar function for equation (0.2). Moreover, stability is observed for the only oscillating solutions with a minimal oscillation frequency, which, in fact, is the case for similar systems of the first order [6, 14, 16, 19]. The *autonomous* case $F(x,t) \equiv F(x)$ is of special interest. In contrast to the first-order autonomous system (0.2), whose slowly oscillating solutions coincide up to shifts in t, the variety of slowly oscillating solutions of (0.1) reveals a more complicated structure. We prove the existence of *periodic* slowly oscillating solutions and show, under some conditions, the uniqueness of such a solution, and in the last case, the orbital asymptotic stability of the periodic solution. We show that there always exist periodic solutions with oscillation frequency greater than the minimal one. These fast oscillations seem to be unstable (as occurs in (0.2)), but it is still a question.

We point out the difference between the following two situations. Under assumption (0.3), all solutions oscillate around zero, and the main problem is to describe the periodic and stable solutions. If $|F(x)|$ can be greater than 1, one observes both bounded oscillatory solutions and unbounded solutions, in which case we give a criterion for the existence of bounded solutions. The latter problem reflects the following phenomenon: the associated equation $\alpha x''(t) = -x'(t) + F(x(t))$ does not have nontrivial bounded solutions, whereas equation (0.1) does. This has been studied in detail for the first-order system (0.2) in [6, 16] and for system (0.1) with $F(x) = kx$, $k > 0$, in [17].

We should mention that a similar equation $x''(t) = -\alpha x(t) + f(x(t-h))$ with a two-level piecewise constant function f was considered in [3, 8, 9] with emphasis on the existence and stability of periodic solutions.

The material is organized as follows. Section 1 to 3 contain statements of results and discussion, section 4 contains the proofs.

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1. **Oscillatory solutions.** For equation (0.1) we state the initial value problem by defining the initial data space to be the space $C_0[-h,0]$ of continuous functions $\varphi : [-h,0] \to \mathbb{R}$ differentiable at the origin. We intend to characterize oscillations in (0.1) via the distribution of zeroes of solutions, and the following Lemma provides basics for the further study.

Lemma 1.1. *Equation (0.1), satisfying (0.3), with initial condition*

$$x(t) = \varphi(t),\ t \in [-h,0],\ \varphi \in C_0[-h,0], \quad x'(0) = \varphi'(0), \tag{1.4}$$

has a unique continuous solution $x_\varphi(t)$, $t \in [-h,\infty)$, and this solution satisfies

$$\operatorname{mes}\{t \geq 0\ :\ x_\varphi(t) = 0\} = 0\ . \tag{1.5}$$

Moreover, in the interval $(0,\infty)$, x_φ is differentiable, its derivative is absolutely continuous and differentiable almost everywhere.

1.1. **Frequency of oscillations.** Denote by Z_φ the set of zeroes of $x_\varphi(t)$ in $[-h,\infty)$. We say that x_φ changes sign at an isolated zero $t \in Z_\varphi$ if $x_\varphi(t-\varepsilon)x_\varphi(t+\varepsilon) < 0$ for all sufficiently small $\varepsilon > 0$. Denote by Z_φ^0 the set of zeroes $t \in Z_\varphi$ where x_φ changes sign. A solution $x_\varphi(t)$ is said to be *oscillatory* if Z_φ is unbounded.

Lemma 1.2. *For any $\varphi \in C_0[-h,0]$, a solution of the initial value problem (0.1), (0.3), (1.4) is oscillatory.*

For any (oscillatory) solution x_φ put

$$\nu_\varphi(t) = \begin{cases} \infty, & \text{if } \operatorname{card}(Z_\varphi \cap (t^* - h, t^*)) = \infty, \\ \operatorname{card}(Z_\varphi^0 \cap (t^* - h, t^*)), & \text{if } \operatorname{card}(Z_\varphi \cap (t^* - h, t^*)) < \infty, \end{cases}$$

where $t \geq 0$, $t^* = \max\{\tau \leq t :\ x_\varphi(\tau) = 0\}$. Define the *frequency function* by

$$\widetilde{\nu}_\varphi(t) = \infty \text{ if } \nu_\varphi(t) = \infty, \quad \widetilde{\nu}_\varphi(t) = \left[\frac{\nu_\varphi(t)+1}{2}\right] \text{ if } \nu_\varphi(t) < \infty,$$

where the brackets mean the entire part.

The following property of the frequency function, analogous to one in the case of the first-order equation [7, 12, 13, 16, 19] determines the leading role of this parameter in the theory.

Theorem 1.3. *If x_φ is a solution of the initial value problem (0.1), ((0.3), 1.4), then the function $\widetilde{\nu}_\varphi(t)$ is non-increasing.*

Notice that in the first-order case the function ν_φ itself is non-increasing.

Theorem 1.3 immediately yields

Corollary 1.4. *Under the conditions of Theorem 1.3, for any solution x_φ, there exists the limit frequency (shortly, l-frequency)*

$$N_\varphi = \lim_{t\to\infty} \widetilde{\nu}_\varphi(t) \in \mathbb{N} \cup \{0\} \cup \{\infty\}.$$

In particular, the set of initial functions $C_0[-h,0]$ splits into the disjoint union

$$C_0[-h,0] = \bigcup_{n=0}^{\infty} U_n \cup U_\infty, \quad U_n = \{\varphi \in C_0[-h,0]\ :\ N_\varphi = n\}.$$

We shall see later that the sets U_n, $n > 0$, can be nonempty. However, Theorem 1.3 indicates that the set U_0 constitutes the main part of $C_0[-h,0]$. Namely, the next claim shows that the property of having zero l-frequency is stable in the following sense.

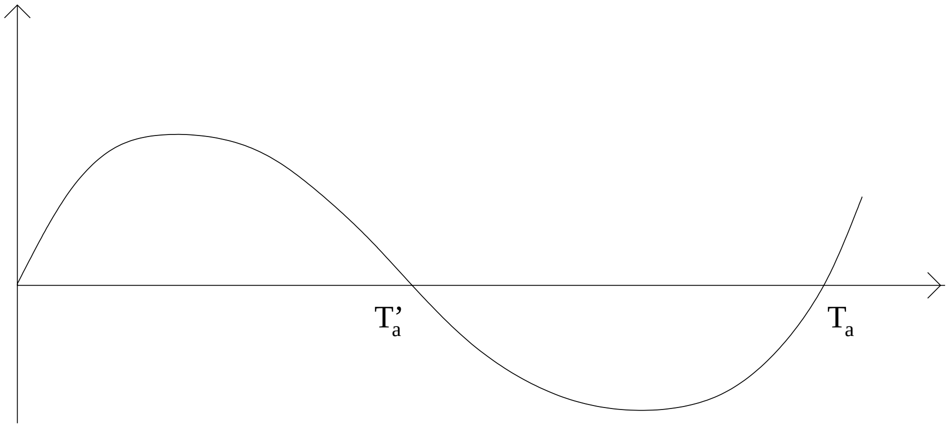

FIGURE 1. Solution x_a

Theorem 1.5. *Under the condition of Theorem 1.3, the (nonempty) set*

$$U_0^0 = \{\varphi \in C_0[-h, 0] \ : \ N_\varphi = 0, \ \text{mes}(\varphi^{-1}(0)) = 0\}$$

is contained in the interior of U_0 *in* $C_0[-h, 0]$ *with respect to the norm*

$$||\varphi||_\infty^0 = \max_{[-h,0]} |\varphi| + |\varphi'(0)| \ .$$

For the first-order system (0.2), satisfying (0.3), the set $\bigcup_{n>0} U_n$ (under some mild conditions [7, 16]) is nowhere dense in $C[-h, 0]$. We *conjecture* that the similar result holds for the second order system (0.1), satisfying (0.3).

2. **Periodic and stable solutions of the autonomous equation.** We intend to study in more detail solutions of the autonomous equation

$$\alpha x''(t) = -x'(t) + F(x(t)) - \text{sign}\, x(t - h) \ . \tag{2.6}$$

Any solution of the first-order autonomous system (0.2) satisfying (0.3) becomes periodic after a period of time [7, 16]. This is not the case in our situation, however, periodic solutions play the central role in the study.

Theorem 2.1. *For any integer* $n \geq 0$*, equation (2.6), satisfying (0.3), has a periodic solution with a constant frequency function* $\widetilde{\nu}(t) \equiv n$*.*

Theorem 1.5 states, in particular, that the frequency function $\widetilde{\nu}_\varphi(t)$ with $N_\varphi = 0$ is stable if φ has finitely many zeroes in $[-h, 0]$, but this does not imply that the corresponding solution $x_\varphi(t)$ is stable in any sense. The zero l-frequency periodic solutions are natural candidates to be stable, and we show that this is, in fact, the case. Moreover, the stability problem is related to the question on the uniqueness of the zero l-frequency periodic solution.

Since the solutions of (2.6) are invariant with respect to shifts in t, we consider only solutions with zero l-frequency such that

$$x(0) = 0, \quad x(t) < 0, \ t \in (-h, 0) \ , \tag{2.7}$$

and ask the question:

How many periodic solutions of (2.6) with zero l-frequency, satisfying (2.7), are there ?

We reformulate the question as follows. As a consequence of Lemma 1.2 and Theorem 1.3, we obtain that for any $a \geq 0$, there is a zero l-frequency solution $x_a(t)$ to (2.6), (0.3), satisfying (2.7) and the condition $x'(0) = a$. Moreover, this solution is uniquely defined in the interval $[0, \infty)$. Define the map $\Phi : [0, \infty) \to [0, \infty)$ by $\Phi(a) = x_a'(T_a)$, where T_a is the second positive zero of $x_a(t)$ (see Figure 1). Clearly, if $\Phi(a) = a$ then x_a is periodic. In the proof of Theorem 2.4, section 4.7 below, we shall show that all periodic zero l-frequency solutions correspond to the roots of the equation $\Phi(a) = a$. So, the above question reduces to counting roots of the latter equation. Since the function $F(x)$ in (2.6) is smooth, so is Φ. A periodic zero l-frequency solution $x_a(t)$ is said to be simple if the root a of the function $\Phi(a) - a$ has multiplicity one.

Conjecture 2.2. *Equation (2.6), satisfying (0.3), has only one zero l-frequency periodic solution with property (2.7), and this solution is simple.*

According to [17], Conjecture 2.2 holds in the case $F(x) = kx$, provided the constant k satisfies

$$0 \geq k \geq -\frac{h^2 + 4\alpha^2\pi^2}{4\alpha h},$$

or

$$k > 0, \quad \exp\left(-h\frac{\sqrt{1+4\alpha k} - 1}{2\alpha}\right) > \frac{\sqrt{1+4\alpha k}}{\sqrt{1+4\alpha k} + 1}.$$

Here we supplement this result with

Theorem 2.3. *(1) Conjecture 2.2 holds if $F(x) \equiv \text{const}$.*

(2) There exists a positive function $A(M, p, h)$, $p \in [0, 1)$, $M \geq 0$, such that Conjecture 2.2 holds for any $\alpha \leq A(M, p, h)$, provided, $|F(x)| \leq p$, $|dF/dx| \leq M$.

A solution $x_\varphi(t)$ of equation (2.6) is called *orbitally asymptotically stable* if, for any $\psi \in C_0[-h, 0]$ in a small neighborhood of φ there is t_0 with

$$\lim_{t\to\infty} |x_\psi(t) - x_\varphi(t + t_0)| = 0 . \tag{2.8}$$

A periodic zero l-frequency solution $x_a(t)$ is called *quasi-stable* if a is an attractor of the dynamical system defined by Φ. Clearly, an orbitally-stable periodic zero l-frequency solution is quasi-stable.

Theorem 2.4. *(1) If the function $F(x)$ is analytic and satisfies (0.3), then equation (2.6) has a quasi-stable zero l-frequency periodic solution.*

(2) If equation (2.6), satisfying (0.3), has only one zero l-frequency periodic solution with property (2.7), and this solution is simple, then it is orbitally asymptotically stable. Moreover, any other zero l-frequency solution approaches the periodic solution in the sense of (2.8).

We should like to comment the difference between quasi-stable and orbitally-stable solutions. Assume, for example, that F is twice differentiable, and so is Φ, and that a non-simple stable root a_0 of the function $\Phi(a) - a$ has multiplicity ≥ 2, in particular, that, for $\varepsilon > 0$ small,

$$a_0 + \varepsilon > \Phi(a_0 + \varepsilon) \geq a_0 + \varepsilon - \alpha\varepsilon^2, \quad \alpha > 0 .$$

Let $a_1 = a_0 + \varepsilon_0$ be close to a_0. Then Φ generates the sequence $a_{n+1} = \Phi(a_n)$, $n \geq 1$, such that $\varepsilon_n = a_{n+1} - a_0$, $n \geq 1$, satisfy

$$\varepsilon_{n+1} \geq \varepsilon_n - \alpha\varepsilon_n^2, \quad n \geq 1 .$$

This in turn implies that

$$\varepsilon_n \geq \frac{C}{n}, \quad n \geq 1, \quad C = \text{const} > 0, \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \varepsilon_n = \infty \ . \tag{2.9}$$

Suppose, in addition, that $dT_a/da > 0$ at the point $a = a_0$, where T_a is the second positive zero of x_a. Then, for any $a_1 > a_0$, the $2n$-th positive zero T of the solution $x_{a_1}(t)$ satisfies

$$T \geq nT_{a_0} + C_1 \cdot \sum_{i=0}^{n-1} \varepsilon_i, \quad C_1 = \text{const} > 0,$$

which in view of (2.9) says that the solution x_{a_0} is not orbitally stable.

Remark 2.5. *One may ask what is common in the three proven cases of Conjecture 2.2, when $F(x) = kx$, or $F(x) = \text{const}$, or α is sufficiently small, and what is the difficulty in handling the general situation. In principle, one could write certain implicit equations for the function Φ introduced above, but such general expressions seem to be hardly treatable. It is easy to show (see section 5) that Φ takes the interval $[0, 2]$ into itself, so one can expect the uniqueness and simplicity of the root of the equation $\Phi(a) = a$ when $-1 < \Phi'(a) < 1$. This, for instance, holds for small values of α (the case (2) of Theorem 2.3). Indeed, then equation (0.1) is close to (0.2), which corresponds to the constant function Φ. We omit the proof of Theorem 2.3(2), since it is just a routine computation.*

For (relatively) large values of α, $\Phi'(a)$ can be greater than 1, the second derivative $\Phi''(a)$ can change sign etc. One can produce such examples playing with formulas of section 4.8 below. If $F(x) = kx$, or $F(x) = \text{const}$, we integrate equation (0.1) and find implicit formulas for Φ via elementary functions, which finally reduce (in different ways) to one equation in one unknown with two parameters. To prove the uniqueness of the root of $\Phi(a) = a$, we then perform rather delicate (and rather different) computations, which do not indicate if this argument can be generalized in any way.

3. Existence of bounded solutions in the presence of an unbounded perturbation.

If the function $F(x,t)$ (which we call *perturbation*) in equation (0.1) does not satisfy $|F| \leq 1$, one can observe unbounded solution. In this situation, we intend to provide a sufficient condition for the existence of bounded oscillatory solutions, which, furthermore, form an open set in the space of all solutions.

Theorem 3.1. *Let a smooth function $F(x,t)$ satisfy for all x, t,*

$$F(0,t) = p_0, \ p_0 = \text{const} \in (-1,1), \quad 0 \leq \frac{F(x,t) - p_0}{x} \leq k \ , \tag{3.10}$$

where the positive constant k satisfies

$$\exp\left(-h\frac{\sqrt{1+4\alpha k}-1}{2\alpha}\right) \geq \frac{(1+|p_0|)(2+(\sqrt{1+4\alpha k}-1)(1-e^{-\tau/\alpha}))}{4} \ , \tag{3.11}$$

and τ is the positive root of the equation

$$\alpha(1 - e^{-\tau/\alpha}) = \tau - \frac{1+|p_0|}{k} \ . \tag{3.12}$$

Then any solution $x(t)$ of equation (0.1) such that

$$x(t) < 0, \ t \in [-h, 0), \quad x(0) = 0, \quad 0 \leq x'(0) < \xi(h, \alpha, k, p_0) \tag{3.13}$$

or

$$x(t) > 0,\ t \in [-h, 0), \quad x(0) = 0, \quad 0 \geq x'(0) > -\xi(h, \alpha, k, -p_0)\ , \tag{3.14}$$

is bounded, oscillatory, and has zero l-frequency, where

$$\xi(h, \alpha, k, p) = \left(2 \exp\left(-h\frac{\sqrt{1+4\alpha k}-1}{2\alpha}\right) - 1 - p\right) \cdot \frac{2}{\sqrt{1+4\alpha k}-1}\ .$$

In the following special case we can weaken condition (3.11)

Theorem 3.2. *The statement of Theorem 3.1 holds if* $F(x,t) = kx + p_0$, *where* $p_0 \in (-1, 1)$ *and* k *is positive, satisfying*

$$\exp\left(-h\frac{\sqrt{1+4\alpha k}-1}{2\alpha}\right) > \frac{\sqrt{1+4\alpha k}+|p_0|}{\sqrt{1+4\alpha k}+1}\ . \tag{3.15}$$

Remark 3.3. *(1) The left-hand side of (3.15) is less than that of (3.11), i.e., Theorem 3.2 strengthens Theorem 3.1 for a particular case.*

(2) Restriction (3.11) may, in principle, be weakened, but cannot be replaced by (3.15) even in the case $F(x,t) \equiv F(x)$, $F(0) = 0$. *Indeed, in an example with* $F(x) = k_2 x$, $x \geq 0$, *and* $F(x) = k_1 x$, $x < 0$, *where* $k_0 > k_1 > k_2 < 0$, *substitution of* $k = k_0 > 0$ *in (3.15) turns it into an equality,* k_1 *is close to* k_0, $k_1 - k_2 \ll k_0 - k_1$, *a direct integration shows that there are no oscillating solutions with zero l-frequency.*

(3) In fact, under condition (3.13) or (3.14), we have $|F(x(t), t)| \leq p_1 < 1$ *with some constant* p_1, *i.e., such solutions* $x(t)$ *possess the properties asserted in sections 1, 2. In particular, by Theorem 1.5 all solutions of (0.1) with an initial function* $\varphi \in C_0[-h, 0]$, *close to* $x(t)\big|_{[-h,0]}$ *satisfying (3.13) or (3.14), are bounded, oscillatory, and have zero l-frequency.*

(4) If α *tends to zero, both conditions (3.10) and (3.15) converge to*

$$kh < \log \frac{2}{1+|p_0|}\ ,$$

known to be sufficient for the existence of bounded zero l-frequency oscillations in the first-order system (0.2) [7, 16, 6].

(5) It is easy to verify that (3.15) holds for all $\alpha > 0$, *provided* $2hk \leq 1 - |p_0|$.

4. **Proofs.**

4.1. **Proof of Lemma 1.1.** Assume that a continuous solution x_φ of (0.1), (1.4) is uniquely defined on $[-h, T]$ and is differentiable on the interval $[0, T]$. Show that x_φ can be extended to $[T, T+h]$ as a continuous solution of (0.1), and that such an extension is unique and differentiable.

The latter requirement is reduced to solving the initial value problem

$$\alpha x'' = -x' + F(x,t) + \psi(t), \quad x(0) = a_0,\ x'(0) = a_1, \quad t \in [0, h], \tag{4.16}$$

where $\psi(t)$ is a measurable bounded function. Integrating (4.16) twice, one successively obtains

$$x'(t) = a_1 e^{-t/\alpha} + \frac{1}{\alpha}\int_0^t (F(x(\tau), \tau) + \psi(\tau)) e^{(\tau - t)/\alpha} d\tau \quad t \geq 0\ , \tag{4.17}$$

$$x(t) = a_0 + \alpha a_1 (1 - e^{-t/\alpha}) + \int_0^t (1 - e^{(\tau - t)/\alpha})(F(x(\tau), \tau) + \psi(\tau)) d\tau, \quad t \geq 0\ . \tag{4.18}$$

Solutions to (4.16) are, clearly, bounded on the segment $[0, h]$ by some constant. Hence we can assume $|F_x(x,t)| < M = \text{const} > 0$ in our situation. Denote by

$C[0,1/M]$ the space of continuous functions on $[0,1/(2M)]$, equipped with the sup-norm $||x||_\infty = \sup|x(\tau)|$. Then the right hand side of (4.18) defines an operator $Q : C[0,1/(2M)] \to C[0,1/(2M)]$, which is contraction due to

$$||Q(x_1) - Q(x_2)||_\infty = \sup\left|\int_0^t (1 - e^{(\tau-t)/\alpha})(F(x_1(\tau)) - F(x_2(\tau)))\right| d\tau$$
$$< Mt \cdot ||x_1 - x_2||_\infty \le \frac{1}{2}||x_1 - x_2||_\infty .$$

Hence there exists a unique continuous solution of (4.16) on the interval $[0,1/(2M)]$. Repeating the procedure $[2Mh]+1$ times, one covers the interval $[0,h]$. The differentiability properties of x_φ can be extracted from (4.17), (4.18) and (1.5), which in turn can be proven in the same way as Lemma 5 in [1].

4.2. **Proof of Lemma 1.2.** Assume on the contrary that Z_φ is bounded, and that $T = \max Z_\varphi$. Without loss of generality suppose that $x_\varphi(t) > 0$ as $t > T$. Then one derives from (0.1), (0.3) that, for $t \ge T + h$,

$$\alpha x'' + x' = -1 + F(x,t) \le -1 + p \quad \Longrightarrow \quad \left(x' e^{t/\alpha}\right)' \le -\frac{1-p}{\alpha} e^{t/\alpha}$$
$$\Longrightarrow \quad x(t) \le C_0 + C_1 e^{-t/\alpha} - (1-p)t \ ,$$

with some constants C_0, C_1, which implies $x_\varphi(t) < 0$ as $t > (|C_0| + |C_1|)/(1-p)$, contradicting the above assumption and proving Lemma.

4.3. **Proof of Theorem 1.3.** (1) Assume that $x_\varphi(T) = 0$, $\widetilde{\nu}_\varphi(T) = 0$, and $x_\varphi(t) \le 0$, $t \in [T-h,T]$, whereas $x_\varphi^{-1}(0) \cap [T-h,T]$ is finite. We shall show that $x_\varphi(t) > 0$ as $t \in (T, T+h)$, which will imply that $\widetilde{\nu}_\varphi(t) = 0$ for $t > T$.

Observe that the replacement of $x_\varphi\big|_{[T-h,T]}$ by any smooth function, which is negative in $[T-h,T)$ and zero at T, does not influence on $x_\varphi\big|_{[T,\infty)}$ in view of (4.18). Thus, $x = x_\varphi$ satisfies

$$\alpha x'' = -x' + F(x,t) + 1, \ t \in [T, T+h], \quad x(T) = 0, \ x'(T) \ge 0.$$

Hence

$$x'(t) = x'(T)e^{(T-t)/\alpha} + \frac{1}{\alpha}\int_T^t (1 + F(x(\tau),\tau))e^{(\tau-t)/\alpha} d\tau$$
$$\ge \frac{1-p}{\alpha}\int_T^t e^{(\tau-t)/\alpha} d\tau = (1-p)(1 - e^{(T-t)/\alpha}), \quad t \in [T, T+h]. \tag{4.19}$$

In particular, $x(t)$ strictly increases in $(T, T+h]$, and so is positive there.

(2) Assume that $\nu_\varphi(T) = n > 0$, n is even, $x_\varphi(T) = 0$, $x_\varphi(t) < 0$ as $T - \varepsilon < t < T$. Denote

$$T^* = \min\{t \in (T-h, T) \ : \ x_\varphi \text{ changes sign at } t\} \ . \tag{4.20}$$

Since n is even, $x_\varphi(t) \le 0$ in the interval $[T-h, T^*]$. Hence x_φ satisfies

$$\alpha x'' = -x' + F(x,t) + 1, \ t \in [T, T^* + h], \quad x(T) = 0, \ x'(T) \ge 0,$$

which as above implies that x_φ is positive in $(T, T^*+h]$. That means $T^{**} > T^* + h$, where

$$T^{**} = \min\{t > T \ : \ x_\varphi(t) = 0\}, \tag{4.21}$$

implying $\nu_\varphi(T^{**}) \le \nu_\varphi(T)$, and we are done in this case.

(3) Assume that $\nu_\varphi(T) = n > 0$, n is odd, $x_\varphi(T) = 0$, $x_\varphi(t) < 0$ as $T - \varepsilon < t < T$. Introduce T^* by (4.20), and T^{**} by (4.21). If $T^{**} \ge T^* + h$, then $\nu_\varphi(T^{**}) \le \nu_\varphi(T)$. If $T^{**} < T^* + h$ then $\nu_\varphi(T^{**}) = n + 1$. In both the cases one has $\widetilde{\nu}_\varphi(T^{**}) \le \widetilde{\nu}_\varphi(T)$.

4.4. **Proof of Theorem 1.5.** Let $\varphi \in U_0^0$. Then there exists $T > h$ such that $x_\varphi(T) = 0$, $\nu_\varphi(T) = 0$ and, for instance, $x_\varphi(t) \leq 0$ as $t \in [T-h, T]$. Then (4.19) tells us that $x_\varphi(t) > 0$ as $t \in (T, T+h+2\varepsilon)$ for some $\varepsilon > 0$. If $\psi \in C_0[-h; 0]$ is close to φ, then $\psi^{-1}(0)$ is contained in a sufficiently small neighborhood of $\varphi^{-1}(0)$, and

$$\mathrm{mes}(\{\varphi > 0\} \circ \{\psi > 0\}), \quad \mathrm{mes}(\{\varphi < 0\} \circ \{\psi < 0\})$$

are small enough, where $A \circ B$ denotes $(A \backslash B) \cup (B \backslash A)$. Hence $Z_\psi \cap [0; T+2h]$ is contained in a sufficiently small neighborhood of $Z_\varphi \cap [0; T+2h]$. Therefore $x_\psi(t) > 0$ as $t \in (T+\varepsilon/2, T+h+3\varepsilon/2)$, which, as in the proof of Theorem 1.3, implies $N_\psi = 0$.

4.5. **Proof of Theorem 2.1 in the zero l-frequency case.** Since by Lemma 1.2 any solution x_φ of (2.6), satisfying (0.3), oscillates around zero, so does x'_φ. Hence

$$\sup_{t \geq T} x'_\varphi(t) = \sup_{t^*} x'_\varphi(t^*),$$

where t^* runs through the local maxima $t^* \geq T$, and T is the first local extremum of x'_φ. In particular, in a neighborhood of a local maximum t^* of x'_φ we have $x''_\varphi(t_k) \geq 0$, $t_k \to t^*$, $t_k < t^*$; hence $-x'_\varphi(t^*) + F(x_\varphi(t^*)) \pm 1 \geq 0$, and thus,

$$x'_\varphi(t^*) \leq 1 + p \,. \tag{4.22}$$

Let $\varphi \in C_0[-h, 0]$ be such that $\varphi(t) < 0$, $t < 0$, $\varphi(0) = 0$, $\varphi'(0) = a \geq 0$. By Theorem 1.3, $\nu_\varphi(t) \equiv 0$. Note also that in this case the solution $x_\varphi\big|_{[0,\infty)}$ is completely determined by the value of a. Due to (4.22), the map Φ, introduced in section 2, takes the segment $[0, 2]$ into itself. Hence there exists $a \in [0, 1+p]$ such that $\Phi(a) = a$, thereby defining a periodic zero l-frequency solution.

4.6. **Proof of Theorem 2.1 in the positive l-frequency case.** *Step 1.* Fix an integer $n > 0$ and consider the family of equations

$$\alpha x''(t) = -x'(t) + F(x(t)) - \operatorname{sign} x(t - h\kappa), \quad \kappa \in (0, \infty) \,, \tag{4.23}$$

with the initial conditions

$$x(t) = \varphi(t), \quad \varphi \in C_0[-\kappa h], \quad \varphi(0) = 0, \quad \varphi(t) < 0 \text{ as } t < 0 \,. \tag{4.24}$$

Let $\Psi : (0, \infty) \times [0, 2] \to [0, 2]$ be a map such, for any $\kappa \in (0, \infty)$, the restriction of Ψ on the segment $\{\kappa\} \times [0, 2]$ is the map Φ for system (4.23) as defined in section 2. Since it is continuous, the set

$$A = \{(\kappa, a) \in (0, \infty) \times [0, 2] \ : \ \Phi_\kappa(a) = a\}$$

is closed in $(0, \infty) \times [0, 2]$. Introduce one more map, $\Pi : (0, \infty) \times [0, 2] \to \mathbb{R}$, which sends a point (κ, a) to the second positive zero of the solution to (4.23), (4.24). It is continuous as well, and sends any point $(\kappa, a) \in A$ to the minimal positive period of the corresponding periodic solution. We intend to show that the image of the map

$$\Theta : A \to \mathbb{R}, \quad \Theta(\kappa, a) = h\kappa + n\Pi(\kappa, a)$$

is the whole interval $(0, \infty)$. In particular, there exists $(\kappa, a) \in A$ such that

$$h\kappa + n\Pi(\kappa, a) = h \,,$$

and thus, for the corresponding $\Pi(\kappa, a)$-periodic solution to (4.23), we have

$$\begin{aligned} \alpha x''(t) &= -x'(t) + F(x(t)) - \operatorname{sign} x(t - h\kappa) \\ &= -x'(t) + F(x(t)) - \operatorname{sign} x(t - h\kappa - n\Pi(\kappa, a)) \end{aligned}$$

$$= -x'(t) + F(x(t)) - \operatorname{sign} x(t-h) \ ,$$

i.e., it is a periodic solution to (2.6) with $\widetilde{\nu} \equiv n$.

Step 2. Show that

$$\lim_{\kappa\to\infty} \Pi(\kappa, a) = \infty, \qquad \lim_{\substack{\kappa\to 0\\ (\kappa,a)\in A}} \Pi(\kappa, a) = 0$$

uniformly on $a \in [0, 2]$. The first relation immediately follows from $\Pi(\kappa, a) > h\kappa$. Assume that the second relation does not hold. Then there exists a sequence $(\kappa_k, a_k) \in A$ such that $\kappa_k \to 0$, $a_k \to a^* \geq 0$, $\Pi(\kappa_k, a_k) \to T^* > 0$ as $k \to \infty$.

Observe that $T^* < \infty$. Indeed, equation (4.23) on the segment $[0, h\kappa]$ turns into

$$\alpha x'' = -x' + F(x) + 1, \quad x(0) = 0, \quad x'(0) = a \in [0, 2] \ .$$

Hence by formulas (4.17), (4.18),

$$x'(h\kappa) \leq a + \frac{(1+p)h\kappa}{\alpha} \leq 3 + \frac{(1+p)h\kappa}{\alpha} \ ,$$

$$x(h\kappa) \leq \alpha a(1-e^{-h\kappa/\alpha}) + (1+p)\int_0^{h\kappa}(1-e^{(\tau-h\kappa)/\alpha})d\tau \leq (a+1+p)h\kappa \leq (3+p)h\kappa \ .$$

On the segment defined by $t \geq h\kappa$, $x(t) \geq 0$, we have

$$\alpha x'' = -x' + F(x) - 1, \quad x(h\kappa) \leq (3+p)h\kappa, \quad x'(0) \leq 3 + \frac{(1+p)h\kappa}{\alpha} \ ;$$

hence again by (4.18)

$$x(t) \leq (3+p)h\kappa + (3\alpha + (1+p)h\kappa)(1 - e^{(t-h\kappa)/\alpha}) - (1-p)\int_{h\kappa}^{t}(1 - e^{(\tau-t)/\alpha})d\tau$$

$$= (3+p)h\kappa + (3\alpha + (1+p)h\kappa)(1 - e^{(h\kappa-t)/\alpha}) - (1-p)(t - h\kappa - \alpha(1 - e^{(h\kappa-t)/\alpha}))$$

$$\leq 6h\kappa + 4\alpha - (1-p)t \ .$$

Hence $x(t)$ has its first positive zero at some point $T_1 \leq (6h\kappa + 4\alpha)/(1-p)$, and consequently its second positive zero at some point $T_2 \leq (12h\kappa + 8\alpha)/(1-p)$.

The limit of the zero l-frequency solutions (the convergence is evident in view of the continuous dependence on κ and a, including the zero values) to the corresponding equations is a T^*-periodic function $x(t)$ such that, for some $T \in [0, T^*]$,

$$x(0) = x(T) = x(T^*) = 0, \quad x'(0) = x'(T^*) = a^*, \tag{4.25}$$

$$\alpha x''(t) = -x'(t) + F(x(t)) - 1, \quad x(t) \geq 0, \quad t \in [0, T], \tag{4.26}$$

$$\alpha x''(t) = -x'(t) + F(x(t)) + 1, \quad x(t) \leq 0, \quad t \in [T, T^*] \ .$$

Without loss of generality assume that $T > 0$. Then $a^* = x'(0) > 0$, since otherwise (4.26) would imply $x(t) < 0$ for small positive t. If $x'(T_0) = 0$ for some $T_0 \in [0, T]$ then $x''(T_0) = F(x(T_0)) - 1 \leq p - 1 < 0$, i.e., T_0 is a strong local maximum of $x(t)$. Hence such a point T_0 is unique, belongs to $(0, T)$ and is the global maximum of $x(t)$ on $[0, T]$. Moreover, $x'(t) > 0$, $t \in [0, T_0)$, and $x'(t) < 0$, $t \in (T_0, T]$. We shall show that

$$a^* = x'(0) > |x'(T)| \ . \tag{4.27}$$

Since $x(t)$ is strictly monotone in $[0, T_0]$ and in $[T_0, T]$, we can replace (4.26) by the equation

$$\frac{dy}{dx} = -\frac{1}{\alpha} - \frac{1 - F(x)}{\alpha y}, \quad 0 \leq x \leq x(T_0), \quad y(0) = a^*, \ y(x(T_0)) = 0,$$

corresponding to the interval $[0, T_0]$ with $y = x'$, and the equation

$$\frac{dz}{dx} = \frac{1}{\alpha} - \frac{1 - F(x)}{\alpha z}, \quad 0 \le x \le x(T_0), \quad z(0) = -x'(T),\ z(x(T_0)) = 0,$$

corresponding to the interval $[0, T_0]$ with $z = -x'$. This immediately implies $y(x) > z(x)$ for any $0 \le x < x(T_0)$, and hence (4.27). In the same manner one derives that $|x'(T)| > x(T^*)$, which together with (4.27) gives $a^* > x'(T^*)$, contradicting (4.25).

Step 3. Since $A \cap (\{\kappa\} \times [0,2]) \ne \emptyset$ for any $\kappa \in (0, \infty)$,

$$\lim_{\substack{\kappa \to 0 \\ (\kappa,a) \in A}} \Theta(\kappa, a) = 0, \qquad \lim_{\substack{\kappa \to \infty \\ (\kappa,a) \in A}} \Theta(\kappa, a) = \infty \tag{4.28}$$

uniformly on $a \in [0,2]$. Thus, to complete the proof, we have to establish the connectedness of $\Theta(A) \subset \mathbb{R}$. We argue on the contrary. Assume that $\Theta(A) \subset (0,\theta) \cup (\theta, \infty)$, $\theta > 0$. Then the sets

$$A_- = A \cap \Theta^{-1}(0, \theta), \quad A_+ = A \cap \Theta^{-1}(\theta, \infty)$$

are nonempty and closed in $(0, \infty) \times [0,2]$. Notice that

$$A \cap \{\kappa \in (0,\infty),\ a = 2\} = A \cap \{\kappa \in (0, \infty),\ a = 0\} = \emptyset\ .$$

Indeed, $\Psi(\kappa, 2) \le 1 + p < 2$ and $\Psi(\kappa, 0) > 0$. Here the equality $\Psi(\kappa, 0) = 0$ is impossible, because otherwise the corresponding periodic zero l-frequency solution x with $x(0) = x'(0) = 0$ would have a local minimum at 0 in view of $x''(0) = (1 + F(0))/\alpha > 0$, but x changes its sign at 0 from minus to plus by construction. The uniform convergence in (4.28) implies also that A_+ does not intersect some strip $(0, \varepsilon) \times [0,2]$. Take a sufficiently large $\kappa^* > 0$ such that $A_- \subset (0, \kappa^*) \times [0,2]$ and

$$A_+^* = A_+ \cap (0, \kappa^*] \times [0,2] \ne \emptyset\ .$$

Furthermore, A_+^* is compact and disjoint with the segments $\{0\} \times [0,2]$, $[0, \kappa^*] \times \{0\}$, $[0, \kappa^*] \times \{2\}$, and with the compact set $\overline{A}_-$. In particular, we can cover A_+^* by the union U of finitely many open discs such that the closure $\overline{U}$ is disjoint with the above three segments as well as with $\overline{A}_-$. take the segment $S = \{\kappa = \kappa^*,\ 0 \le a \le 2\}$. By construction, it does not meet A_-. If it crosses $\overline{U}$, then we construct a path in $(0, \kappa^*] \times [0,2]$, starting from the point $(\kappa^*, 0)$, then along S to the intersection point with $\overline{U}$, then along finitely many circle arcs of $\partial \overline{U}$ until we come to S again. After finitely many such moves along S and $\partial \overline{U}$, we end up at $(\kappa^*, 2)$. The continuous path constructed lies inside $(0, \kappa^*] \times [0,2] \backslash A$, but this is impossible, since the difference $\Psi(\kappa, a) - a$ changes its sign along this path from plus at $(\kappa^*, 0)$ to minus at $(\kappa^*, 2)$, so must be intersection point with A.

4.7. **Proof of Theorem 2.4.** The problem reduces to the study of the function Φ defined in section 2. We intend to show that

$$\Phi'(a) \ge 0, \quad a \in [0,2]\ . \tag{4.29}$$

If $F(x)$ is analytic then $\Phi(a)$ is analytic as well. Hence there are only finitely many roots of the equation $\Phi(a) = a$ in the segment $[0,2]$, because $\Phi(a) \not\equiv a$ in view of $\Phi(2) \le 1 + p < 2$. The latter inequality together with $\Phi(0) > 0$ (see section 4.6) says that there is $a_0 \in (0,2)$ such that $\Phi(a_0) = a_0$ and $\Phi(a) - a$ changes its sign from plus to minus at a_0. Assuming (4.29) to be true, this means that a_0 is an attractor of the dynamical system $a \mapsto \Phi(a)$ on $[0,2]$; hence the corresponding zero l-frequency periodic solution is quasi-stable.

Inequality (4.29) implies statement (2) of Theorem 2.4 as well. Indeed, the simplicity of the (unique) root a_0 of the function $\Phi(a) - a$ means that $\Phi'(a_0) < 1$. Together with (4.29) this implies that any sequence $a_1 \in [0,2]$, $a_{i+1} = \Phi(a_i)$, $i \geq 1$, satisfies

$$\varepsilon_{i+1} \leq q\varepsilon_i, \quad q = \text{const} \in (0,1), \quad \varepsilon_i = a_i - a_0,\ i \geq 1;$$

hence

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty \ ,$$

which in view of the boundedness of $|dT_a/da|$ in a neighborhood of a_0, gives the existence of the limit

$$\lim_{n\to\infty} (T_{a_1}^{(n)} - nT_{a_0}) \ ,$$

where $T_{a_1}^{(n)}$ is the $2n$-th positive zero of x_{a_1}.

Now we prove (4.29). Let $a > 0$. Denote by T'_a and T_a the first and the second positive zero of x_a and introduce the functions $\Psi_+, \Psi_- : [0,2] \to [0,2]$ by $\Psi_+(a) = -x'_a(T'_a)$, $\Psi_-(-x'_a(T'_a)) = x'_a(T_a)$. To prove (4.29) it is sufficient to show that $\Psi'_+(s) \geq 0$, $\Psi'_-(s) \geq 0$, $s \in [0,2]$, or, equivalently, that Ψ_+ and Ψ_- are increasing.

Let us consider Ψ_+. Put $b = a + \delta$, where $\delta > 0$ is a small parameter, and show that $\Psi_+(b) > \Psi_+(a)$. As seen in the proof of Theorem 1.3, section 4.3, any solution $x(t)$ of (2.6) with $x(0) = 0$, $x'(0) > 0$, is strictly increasing in $[0,h]$. Hence (2.6) reduces to the equation

$$\frac{dy}{dx} = -\frac{1}{\alpha} + \frac{1 + F(x)}{\alpha y}, \quad y(0) = x'(0),\ 0 \leq x \leq x(h) \ , \tag{4.30}$$

where $y(x) := x'(t)$. If $y_a(x), y_b(x)$ are solutions to (4.30) with $y_a(0) = a < y_b(0) = b$, then (due to the solution uniqueness theorem) $y_b(x) > y_a(x)$ on the whole interval $0 \leq x \leq x_a(h)$. In other words, $x'_b(\tau(t)) > x'_a(t)$ for any $t \in [0,h]$ and $\tau(t) \in [0,h]$ such that $x_b(\tau(t)) = x_a(t)$. It follows then that (see Figure 2)

$$x_b(t) > x_a(t) \quad \text{and} \quad \tau(t) < t \quad \text{as} \quad t \in (0,h] \ . \tag{4.31}$$

Smooth dependence with respect to initial data gives

$$\tau(h) = h - M_1\delta + O(\delta^2), \quad x'_b(\tau(h)) = x'_a(h) + M_2\delta + O(\delta^2) \ , \tag{4.32}$$

with positive numbers M_1, M_2 depending on a. Let $t_1 > h$ be the minimal positive such that $x_a(t_1) = x_b(h)$ (see Figure 2). Relations (4.32) imply that

$$t_1 - h = h - \tau(h) + O(\delta^2) \ .$$

Then

$$x'_b(h) = x'_b(\tau(h)) + \frac{1}{\alpha}(-x'_b(\tau(h)) + F(x_a(h)) + 1)(h - \tau(h)) + O(\delta^2)$$

$$= x'_a(h) + M_3\delta + O(\delta^2), \quad M_3 = M_2 + \frac{M_1}{\alpha}(-x'_a(h) + F(x_a(h)) + 1),$$

$$x'_a(t_1) = x'_a(h) + \frac{1}{\alpha}(-x'_a(h) + F(x_a(h)) - 1)(t_1 - h) + O(\delta^2)$$

$$= x'_a(h) + \frac{1}{\alpha}(-x'_a(h) + F(x_a(h)) - 1)(h - \tau(h)) + O(\delta^2)$$

$$= x'_a(h) + M_4\delta + O(\delta^2), \quad M_4 = \frac{M_1}{\alpha}(-x'_a(h) + F(x_a(h)) - 1).$$

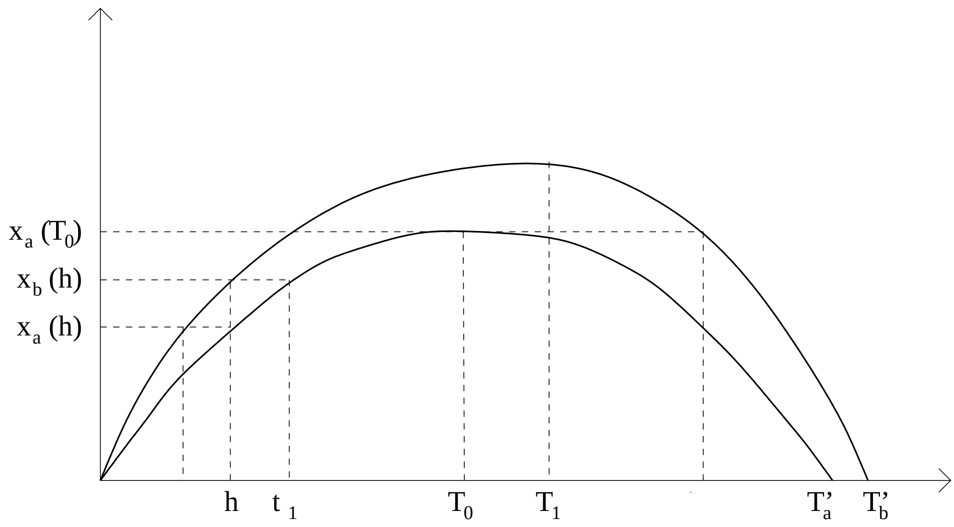

FIGURE 2. Comparison of x_a and x_b

Clearly, $M_3 > M_4$; hence, for a sufficiently small $\delta > 0$,

$$x_b'(h) > x_a'(t_1) \ . \tag{4.33}$$

As seen in the proof of Theorem 2.1, section 4.6, the solution $x_a(t)$ is positive in $[h, T_a')$ and has in $[h, T_a')$ a unique extremum, a global maximum, at some point $T_0 > h$. So, x_a satisfies the equation

$$\frac{dy}{dx} = -\frac{1}{\alpha} + \frac{-1+F(x)}{\alpha y}, \tag{4.34}$$

$$y(x_b(h)) = x_a'(t_1), \quad x_b(h) \le x \le x_a(T_0),$$

where $x = x_a$, $y = x_a'$. The function x_b satisfies equation (4.34) with the initial data

$$y(x_b(h)) = x_b'(h) \tag{4.35}$$

in a neighborhood of the point $x = x_b(h)$. Since $x_a'(t) > 0$ as $t \in [t_1, T_0)$, and $x_b'(\tau_+(t)) > x_a'(t)$ for t in a neighborhood of t_1 and $\tau_+(t)$ such that $x_b(\tau_+(t)) = x_a(t)$, according to (4.33), the initial value problem (4.34), (4.35) has a solution on the whole segment $[x_b(h), x_a(T_0)]$, and, moreover,

$$x_b'(\tau_+(T_0)) > x_a'(T_0) = 0 \ .$$

This implies that

$$x_b(T_1) \stackrel{\text{def}}{=} \max_{t\in[0,T_b']} x_b(t) > x_a(T_0) = \max_{t\in[0,T_a']} x_a(t),$$

and

$$x_b'(\tau_-(T_0)) < x_a'(T_0) = 0 \ ,$$

where $\tau_- : [T_0, T_a'] \to [T_1, T_b']$ is defined by $x_b(\tau_-(t)) = x_a(t)$. The latter inequality implies

$$\Psi_+(b) = -x_b'(T_b') > -x_a'(T_a') = \Psi_+(a),$$

since x_a and x_b induce solutions to (4.34) in $0 \le x \le x_a(T_0)$ with the initial data $y(x_a(T_0)) = 0$ and $y(x_a(T_0)) = x_b'(\tau_-(T_0))$, respectively.

In the same way one can show that Ψ_- is increasing.

4.8. **Proof of Theorem 2.3(1).** We assume that $F(x) \equiv q = \text{const}$, $0 \le q < 1$.

We have to show that the equation $\Phi(a) = a$ has a unique solution in the segment $[0, 1+q]$.

Step1. The case $q = 0$ has been treated in [17], section 3.6. Assume that $0 < q < 1$. Let $x(t)$ be a zero l-frequency solution of (2.6) with $x(0) = 0$, $x'(0) = a \ge 0$. It satisfies

$$\alpha x'' = -x' + q + 1, \quad 0 \le t \le h, \ u + h \le t \le u + v,$$

$$\alpha x'' = -x' = q - 1, \quad h \le t \le u + h,$$

where u and $u + v$ are the first and the second positive zeroes of $x(t)$. One can easily get that

$$x(t) = (1 + q - a)\alpha(e^{-t/\alpha} - 1) + (1 + q)t, \quad 0 \le t \le h,$$

$$x(t) = \alpha(1 + q - a - 2e^{h/\alpha})e^{-t/\alpha} + 2h + \alpha(1 + a - q) - (1 - q)t,$$

$$x'(t) = (2e^{h/\alpha} - 1 - q + a)e^{-t/\alpha} - 1 + q, \quad h \le t \le u.$$

Denoting $b = -x'(u)$, $c = x'(u + v)$, we obtain the equations

$$\alpha(1 + q - a - 2e^{h/\alpha})e^{-u/\alpha} + 2h + \alpha(1 + a - q) - (1 - q)u = 0, \tag{4.36}$$

$$b \overset{\text{def}}{=} \Psi_+(a) = 1 - q - (2e^{h/\alpha} - 1 - q + a)e^{-u/\alpha}, \tag{4.37}$$

$$\alpha(1 - q - b - 2e^{h/\alpha})e^{-v/\alpha} + 2h + \alpha(1 + b + q) - (1 + q)u = 0, \tag{4.38}$$

$$c \overset{\text{def}}{=} \Psi_-(b) = 1 + q - (2e^{h/\alpha} - 1 + q + b)e^{-v/\alpha}. \tag{4.39}$$

For any α, q, we count solutions to (4.36)-(4.39) with $c = a$. Assuming the latter equality and introducing parameters

$$a' = 1 + q - a, \quad b' = 1 - q - b, \quad w = 2 - a' - b',$$

we rewrite (4.36)-(4.39) as

$$b' = (2e^{h/\alpha} - a')e^{-u/\alpha}, \quad a' = (2e^{h/\alpha} - b')e^{-v/\alpha}, \tag{4.40}$$

$$2h + \alpha w = (1 - q)u = (1 + q)v. \tag{4.41}$$

Note that $a \le 1 + q$, $b \le 1 - q$ by (4.37), (4.39). Hence $0 \le w \le 2$. Also, knowing the value of w, one can uniquely restore u, v, a', b', and hence a, b, from (4.40), (4.41). Solving (4.40), (4.41) with respect to a', b', u, v and substituting this into $w = 2 - a' - b'$, we obtain

$$G(w, \alpha, q) \overset{\text{def}}{=} (2 - w)\left(\exp\left(\tfrac{4h}{\alpha(1-q^2)} + \tfrac{2w}{1-q^2}\right) - 1\right)$$
$$-2e^{h/\alpha}\left(\exp\left(\tfrac{2h}{\alpha(1-q)} + \tfrac{w}{1-q}\right) + \exp\left(\tfrac{2h}{\alpha(1+q)} + \tfrac{w}{1+q}\right) - 2\right) = 0\ . \tag{4.42}$$

Hence our problem reduces to showing that, given $\alpha \in (0, \infty)$, $q \in (0, 1)$, there exists at most one (counting multiplicities) solution $w \in [0, 2]$ to (4.42). In [17] it was shown for $q = 0$. If, for some α, q there exist two solutions (counting multiplicities) to (4.42), then there exists a point $(w, \alpha, q) \in \Pi = [0, 2] \times (0, \infty) \times [0, 1)$ such that $G(w, \alpha, q) = G_w(w, \alpha, q) = 0$. We shall show that the set $\{G = G_w = 0\} \cap \Pi$ is empty.

Step 2. We shall now prove that there exists a function $\alpha_\infty : [0, 1) \to (0, \infty)$ such that $|G_w(w, \alpha, q)| + |G(w, \alpha, q)| > 0$ as $\alpha > \alpha_\infty(q)$.

Let us introduce the function $H(w,\beta,q) = G(w,h/\alpha,q)$. Then

$$H(w,0,q) = (2-w)\left(\exp\left(\frac{2w}{1-q^2}\right)-1\right)-2\left(\exp\left(\frac{w}{1-q}\right)+\exp\left(\frac{w}{1+q}\right)-2\right).$$

For any $q \in [0,1)$, the equation $H(w,0,q) = 0$ has the solution $w = 0$. We claim that this is the only solution. Indeed,

$$H_w(w,0,q) = 1 + \left(\frac{2(2-w)}{1-q^2}-1\right)\exp\left(\frac{2w}{1-q^2}\right)$$
$$-2\left(\frac{1}{1-q}\exp\left(\frac{w}{1-q}\right)+\frac{1}{1+q}\exp\left(\frac{w}{1+q}\right)\right),$$

which is zero at $w = 0$ and negative for $w > 0$. The latter statement follows from

$$\frac{\partial}{\partial w}\left(H_w(w,0,q)\exp\left(-\frac{2w}{1-q^2}\right)\right)$$
$$= -\frac{2}{1-q^2}\left(1-\exp\left(-\frac{w}{1-q}\right)\right)\left(1-\exp\left(-\frac{w}{1+q}\right)\right),$$

which is zero at $w = 0$ and negative for $w > 0$.

One also easily verifies that $H(0,0,q) = 0$, $q \in [0,1)$.

To complete the proof of the required statement, we have to show that no sequence (w_k,β_k,q_k) with $\beta_k > 0$, $H(w_k,\beta_k,q_k) = H_w(w_k,\beta_k,q_k) = 0$ can approach a point $(0,0,q)$, $q \in [0,1)$. Indeed, in a small neighborhood U of any point $(0,0,q)$, one has

$$H(w,\beta,q) = \frac{2}{1-q^2}\left(2\beta^3 + 3\beta^2 w + \beta w^2 - \frac{1}{12(1-q^2)^2}w^4\right) + \text{h.o.t.},$$
$$H_w(w,\beta,q) = \frac{2}{1-q^2}\left(3\beta^2 + 2\beta w - \frac{1}{3(1-q^2)^2}w^3\right) + \text{h.o.t.}.$$

Thus, the system $H = H_w = 0$ reduces in $U \cap \Pi$ to

$$\beta = \frac{1}{12(1-q^2)^2}w^2 + \sum_{k\ge 3} A_k(q)w^k, \quad \beta = \frac{1}{6(1-q^2)^2}w^2 + \sum_{k\ge 3} B_k(q)w^k,$$

which has no solutions in $U \cap \Pi$.

Step 3. We next claim that there exists a function $\alpha_0 : [0,1) \to (0,\infty)$ such that $|G_w(w,\alpha,q)| + |G(w,\alpha,q)| > 0$ as $0 < \alpha < \alpha_0(q)$.

Indeed, if not, one would have a sequence (w_k,α_k,q_k) approaching a point $(w,0,q)$, $q \in [0,1)$, such that $G(w_k,\alpha_k,q_k) = G_w(w_k,\alpha_k,q_k) = 0$. However, these equations can be written as

$$2-w = (2-w)\exp\left(-\frac{4h}{\alpha(1-q^2)}-\frac{2w}{1-q^2}\right)$$
$$+2\Bigg(\exp\left(-\frac{h(1-q)}{\alpha(1+q)}-\frac{w}{1+q}\right)+\exp\left(-\frac{h(1+q)}{\alpha(1-q)}-\frac{w}{1-q}\right)$$
$$-2\exp\left(-\frac{h(3+q^2)}{\alpha(1-q^2)}-\frac{2w}{1-q^2}\right)\Bigg),$$

$$\frac{3+q^2-2w}{1-q^2} = -\exp\left(-\frac{4h}{\alpha(1-q^2)}-\frac{2w}{1-q^2}\right)$$
$$+2\left(\frac{1}{1-q}\exp\left(-\frac{h(1-q)}{\alpha(1+q)}-\frac{w}{1+q}\right)+\frac{1}{1+q}\exp\left(-\frac{h(1+q)}{\alpha(1-q)}-\frac{w}{1-q}\right)\right),$$

which imply the inconsistent relations

$$\lim_{\alpha\to 0} w = 2 \quad \text{and} \quad \lim_{\alpha\to 0} w = \frac{3+q^2}{2} \ ,$$

respectively.

Step 4. Assume that $\{G = G_w = 0\} \cap \Pi \neq \emptyset$. Since this set is closed in Π, and in view of the results of Steps 1-4, there exists

$$q_0 = \min\{q \ : \ (w,\alpha,q) \in \Pi,\ G(w,\alpha,q) = G_w(w,\alpha,q) = 0\} > 0 \ .$$

At the corresponding point (w,α,q_0) we have

$$G = G_w = \begin{vmatrix} G_w & G_\alpha \\ G_{ww} & G_{w\alpha} \end{vmatrix} = 0 \ ,$$

which is equivalent to

$$G = G_w = G_\alpha G_{ww} = 0 \ .$$

Assuming that $G = G_w = G_\alpha = 0$, we obtain

$$-\frac{\alpha^2}{2h} G_\alpha = \frac{2(2-w)}{1-q^2} \exp\left(\frac{4h}{\alpha(1-q^2)} + \frac{2w}{1-q^2}\right)$$

$$-e^{h/\alpha}\left(\frac{3-q}{1-q}\exp\left(\frac{2h}{\alpha(1-q)} + \frac{w}{1-q}\right) + \frac{3+q}{1+q}\exp\left(\frac{2h}{\alpha(1+q)} + \frac{w}{1+q}\right) - 2\right)$$

$$= G_w + \frac{1}{2}G + \frac{w}{2}\left(\exp\left(\frac{4h}{\alpha(1-q^2)} + \frac{2w}{1-q^2}\right) - 1\right) = 0 \ ,$$

which together with $G = G_w = 0$ and $w \neq 0$ gives a contradiction.

The remaining case to consider is $G = G_w = G_{ww} = 0$. In the notation

$$E = e^{h/\alpha} - 1, \quad X = \exp\left(\frac{4h}{\alpha(1-q^2)} + \frac{2w}{1-q^2}\right) \ ,$$

$$Y = \exp\left(\frac{2h}{\alpha(1-q)} + \frac{w}{1-q}\right), \quad Z = \exp\left(\frac{2h}{\alpha(1+q)} + \frac{w}{1+q}\right) \ ,$$

the equations $G = G_w = G_{ww} = 0$ read as

$$(2-w)X - 2(1+E)Y - 2(1+E)Z = -4E - w - 2 \ ,$$
$$(3+q^2-2w)X - 2(1+E)(1+q)Y + 2(1+E)(1-q)Z = -1+q^2 \ ,$$
$$2(1+q^2-w)X - (1+E)(1+q)^2Y + (1+E)(1-q)^2Z = 0 \ ,$$

which we solve with respect to X, Y, Z:

$$X = \frac{4E+2w}{w(1+E)} \ ,$$

$$Y = \frac{4Eq + 2wq - w^2 - 4Ew + 2E + 2Eq^2}{2wq(1+E)} \ ,$$

$$Z = \frac{4Eq + 2wq + w^2 + 4Ew - 2E - 2Eq^2}{2wq(1+E)} \ .$$

We plug these formulae in the identity $X - YZ = 0$ and obtain

$$w^4 + 4q^2w^2E - 4w^2E + 4E^2 - 8q^2E^2 + 4q^4E^2 + 16q^2wE^2$$
$$+16q^2wE^3 + 4q^2w^2E^2 + 16w^2E^2 - 16wE^2 + 8w^3E = 0 \ ,$$

which transforms into an impossible equality

$$(w^2 + 2E(2w-1+q^2))^2 + 4q^2w^2E^2 + 16q^2wE^3 = 0 \ ,$$

and we are done.

4.9. **Proof of Theorem 3.2.** We consider the equation

$$\alpha y''(t) = -y'(t) + ky(t) + p_0 - \operatorname{sign} y(t-h) \ , \tag{4.43}$$

corresponding to $F(x,t) \equiv F(x) = kx + p_0$. Introduce the roots $\lambda_1 > 0 > \lambda_2$ of the characteristic equation $\alpha\lambda^2 = -\lambda + k$.

Assuming

$$y(t) < 0,\ t \in [-h, 0), \quad y(0) = 0, \quad y'(0) = a \geq 0 \ ,$$

we obtain a solution to (4.43)

$$y(t) = \frac{\lambda_2(1+p_0) - ak}{k(\lambda_2 - \lambda_1)} e^{\lambda_1 t} + \frac{\lambda_1(1+p_0) - ak}{k(\lambda_1 - \lambda_2)} e^{\lambda_2 t} - \frac{1+p_0}{k}$$

as $t \in [0, h]$, and, respectively,

$$y(t) = \frac{\lambda_2(1 + p_0 - 2e^{-\lambda_1 h}) - ak}{k(\lambda_2 - \lambda_1)} e^{\lambda_1 t}$$

$$+ \frac{\lambda_1(1 + p_0 - 2e^{-\lambda_2 h}) - ak}{k(\lambda_1 - \lambda_2)} e^{\lambda_2 t} + \frac{1 - p_0}{k}$$

as $t \geq h$. Since $\lambda_1 > 0 > \lambda_2$, the solution $y(t)$ necessarily has a positive zero $t = T_1 > h$ if the coefficient of $e^{\lambda_1 t}$ is negative, i.e.,

$$a < \frac{2e^{\lambda_1 h} - 1 - p_0}{\alpha\lambda_1} \ ,$$

which is equivalent to (3.13).

Moreover, $\max\{y(t),\ t \in [0, T_1]\} < (1 - p_0)/k$, since, otherwise, attaining the value $(1 - p_0)/k$, the solution $y(t)$ would further be greater than $(1 - p_0)/k$. So, $F(y(t)) \leq p_1(a) < 1$ as $t \in [0, T_1]$.

Denote $b = -y'(T_1)$. It is, evidently, a nonnegative number. From the explicit formula for $y(t)$ it is not difficult to derive that

$$e^{\lambda_2 T_1}(2e^{-\lambda_2 h} - 1 - p_0 - \alpha\lambda_2 a) = 1 - p_0 + \alpha\lambda_2 b \ .$$

Since

$$2e^{-\lambda_2 h} - 1 - p_0 - \alpha\lambda_2 a \geq 2 - 1 - p_0 = 1 - p_0 > 0 \ ,$$

we obtain

$$b < -\frac{1 - p_0}{\alpha\lambda_2} = \eta(h, \alpha, k, -p_0) < \xi(h, \alpha, k, -p_0) \ , \tag{4.44}$$

where the latter inequality can be rewritten as

$$\exp\left(-h\frac{\sqrt{1 + 4\alpha k} - 1}{2\alpha}\right) > \frac{\sqrt{1 + 4\alpha k} + p_0}{\sqrt{1 + 4\alpha k} + 1} \ ,$$

which follows from (3.15).

In view of inequality (4.44), equivalent, in fact, to (3.14), the above argument (after switching the sign of $y(t)$) implies that $y(t)$ has the second positive zero $T_2 > T_1 + h$ and is negative for $t \in (T_1, T_2)$. Furthermore, $\min\{y(t),\ t \in [T_1, T_2]\} > -(1 + p_0)/k$; hence $F(y(t)) \geq -p_2(a) > -1$ as $t \in [T_1, T_2]$. Denoting $c = y'(T_2)$, we derive as above

$$e^{\lambda_2(T_2 - T_1)}(2e^{-\lambda_2 h} - 1 + p_0 - \alpha\lambda_2 b) = 1 + p_0 + \alpha\lambda_2 c \ .$$

Hence

$$c < -\frac{1+p_0}{\alpha\lambda_2} < \xi(h,\alpha,k,p_0) \ ,$$

the latter inequality being

$$\exp\left(-h\frac{\sqrt{1+4\alpha k}-1}{2\alpha}\right) > \frac{\sqrt{1+4\alpha k}-p_0}{\sqrt{1+4\alpha k}+1} \ ,$$

and hence coming from (3.11).

Extending $y(t)$ further we obtain in the same manner a bounded, oscillatory, zero l-frequency solution.

In addition, we have shown that $|F(y(t)| \le \max\{p_1(a), p_2(a)\} < 1$ confirming Remark 3.3(3).

4.10. **Proof of Theorem 3.1.** In the notation of the previous section, inequality (3.11) can be rewritten as

$$\frac{2e^{-\lambda_1 h}-1-p_0}{\alpha\lambda_1} \ge 1+p_0, \quad \frac{2e^{-\lambda_1 h}-1+p_0}{\alpha\lambda_1} \ge 1-p_0 \ .$$

Starting with the initial data (3.13), we obtain that the corresponding solution $x(t)$ of (0.1) satisfies

$$\alpha x''(t) = -x'(t) + F(x(t),t) - \operatorname{sign} x(t-h) \le -x'(t) + kx(t) + p_0 - \operatorname{sign} x(t-h) \ ,$$

as far as $x(t) \ge 0$. Hence $x(t) \le y(t)$ as far as $x(t) \ge 0$, where $y(t)$ is the solution to equation (4.43) with the same initial data (3.13). This immediately implies that $x(t)$ has a positive zero $T'_a > h$ in the interval $[0, T_1]$, and

$$\max_{t\in[0,T'_a]} |F(x(t),t)| \le \max_{t\in[0,T_1]} (ky(t)+p_0) \le p_1(a) < 1 \ .$$

Now we estimate $x'(T'_a)$. Note that $x(t)$ has a unique maximum point $T_0 \in [h, T'_a)$: if there were two maxima $T'_0 < T''_0$, one would have a point $t^* \in (T'_0, T''_0)$ such that $x'(t^*) = 0$, $x''(t^*) \ge 0$ which contradicts

$$\alpha x''(t^*) + x'(t^*) = -1 + F(x(t^*), t^*) \le -1 + p_1 < 0 \ .$$

In addition, $x'(T_0) = 0$, since $x'(T'_a) \le 0$ in view of $x(t) > 0$ as $t < T'_a$, and $x'(h) \ge 0$ because $x(t)$ strictly increases on $[0,h]$ as shown in section 4.3. Since $x(t)$ strictly decreases in the interval $[T_0, T'_a]$, we can write

$$F(x(t),t) = G(x(t)), \quad t \in [T_0, T'_a] \ ,$$

with some function $G(x)$. Hence equation (0.1) turns into

$$\alpha x''(t) = -x'(t) - 1 + G(x(t)), \quad t \in [T_0, T'_a] \ ,$$

which in the coordinates x, $z = x'$ reads as

$$\frac{dz}{dx} = -\frac{1}{\alpha} - \frac{1-G(x)}{\alpha z}, \quad 0 \le x \le x(T_0), \quad z(x(T_0)) = 0 \ .$$

Since $G(x) \ge p_0$, $x \in [0, x(T_0)]$, we have

$$x'(T'_a) = z(0) \ge w(0) \ ,$$

where

$$\frac{dw}{dx} = -\frac{1}{\alpha} - \frac{1-p_0}{\alpha w}, \quad 0 \le x \le x(T_0), \quad w(x(T_0)) = 0 \ .$$

A direct computation gives

$$x'(T'_a) \ge w(0) = -(1-p_0)(1-e^{-\tau'/\alpha}) \ ,$$

where τ' is the positive root of the equation

$$\alpha(1 - e^{-\tau'/\alpha}) = \tau' - x(T_0) \ .$$

Since $x(T_0) < (1 - p_0)/k$, we obtain that τ' is less that τ defined in Theorem 3.1, and hence

$$x'(T'_a) > -(1 - p_0)(1 - e^{-\tau/\alpha}) \geq -\frac{2e^{-\lambda_1 h} - 1 + p_0}{\alpha\lambda_1} \geq -\xi(h, \alpha, k, |p_0|) \ , \quad (4.45)$$

the middle inequality being (3.11) with $|p_0|$ replaced by $-p_0$. In fact, we get in the situation of initial value problem (3.14), and hence by the same reason, we can extend $x(t)$ to infinity as bounded, oscillatory, zero l-frequency solution.

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