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Brief paper

Control under quantization, saturation and delay: An LMI approach☆

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ABSTRACT

This paper studies quantized and delayed state-feedback control of linear systems with given constant bounds on the quantization error and on the time-varying delay. The quantizer is supposed to be *saturated*. We consider two types of quantizations: quantized control input and quantized state. The controller is designed with the following property: all the states of the closed-loop system starting from a neighborhood of the origin exponentially converge to some bounded region (both, in R^n and in some infinite-dimensional state space). Under suitable conditions the attractive region is inside the initial one. We propose decomposition of the quantization into a sum of a saturation and of a uniformly bounded (by the quantization error bound) disturbance. A Linear Matrix Inequalities (LMIs) approach via Lyapunov–Krasovskii method originating in the earlier work [Fridman, E., Dambrine, M., & Yeganefar, N. (2008). On input-to-state stability of systems with time-delay: A matrix inequalities approach. *Automatica*, 44, 2364–2369] is extended to the case of *saturated quantizer and of quantized state* and is based on the simplified and improved Lyapunov–Krasovskii technique.

1. Introduction

It is well known (Kalman, 1956), that quantization of a stabilizing controller may lead to limit cycles and chaotic behavior. Quantization in control systems has recently become an active research topic. The need for quantization arises when digital networks are part of the feedback loop. In this paper we study linear control systems with either quantized state or quantized control input. See e.g. (Brocket and Liberzon, 2000; Bullo and Liberzon, 2006; Corradini and Orlando, 2008; Fagnani and Zampieri, 2003; Fu and Xie, 2005; Ishii and Francis, 2003; Liberzon, 2003) and the references therein for control under different types of quantizations (in both, linear and nonlinear cases).

We think of a quantizer as a device that converts a real-valued signal into a piecewise constant one. In the present paper we consider a quantizer with an a priori given constant upper bound on the quantization error and, thus, asymptotic stability cannot be ensured. In the linear case the problem can be reduced to the analysis of the systems with bounded disturbances, where the ultimate bound can be derived via the quadratic Lyapunov function (see e.g. Liberzon (2003)). An alternative approach to ultimate bound computation is based on the componentwise analysis of disturbances (Haimovich, Kofman and Seron, 2007; Kofman, Seron and Haimovich, 2008).

Time-delay often appears in control systems and, in many cases, delay is a source of instability (Hale and Verduyn-Lunel, 1993). Delays often appear in networked control systems. Recently exponential convergence of linear state-delay systems with bounded non-delayed control and bounded disturbances was studied in Oucheriah (2006), where delay-independent conditions were derived via a quadratic Lyapunov function. We note that the delay-independent conditions are not applicable to systems with input delay, where the open-loop systems are unstable.

Delayed quantized control was studied in Liberzon (2006) by applying Input-to-State Stability (ISS) analysis (see Sontag and Wang (1995)) via Razumikhin approach (Teel, 1998). The Razumikhin approach leads usually to more conservative results than the Krasovskii method (see e.g. Example 2 in Fridman, Dambrine and Yeganefar (2008)). For systems with constant delays, ISS sufficient conditions were recently derived in terms of Lyapunov–Krasovskii functionals in Pepe and Jiang (2006). For systems with time-varying delays ISS sufficient delay-dependent conditions via Krasovskii method were obtained in Fridman et al. (2008) in terms of matrix inequalities, where quantized control input without saturation was considered.

LMI conditions in the case of the logarithmic quantizer of control feedback (where the asymptotic stability can be achieved) were derived in Fu and Xie (2005) by using the sector bound

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Then the solution of (8) *and* (7) *satisfies the inequality*

$$x^{\mathrm{T}}(t)Px(t) < \mathrm{e}^{-a(t-t_0)}x_0^{\mathrm{T}}Px_0 + [1-\mathrm{e}^{-a(t-t_0)}]\frac{b}{a}|w_{[t_0,t]}|_\infty^2 \tag{11}$$

for $t \geq t_0$ *and* $|x_0|^2 + |w_{[t_0,t]}|_\infty^2 > 0$.

Proof. Applying the comparison principle (Khalil, 2002), we have

$$x^{\mathrm{T}}(t)Px(t) \leq V(t,x_t,\dot{x}_t) < \mathrm{e}^{-a(t-t_0)}V(t_0,x_{t_0},\dot{x}_{t_0}) + \int_{t_0}^{t}\mathrm{e}^{-a(t-s)}b|w(s)|^2\mathrm{d}s,$$

that implies (11) (and so, the system is ISS). □

We will derive now LMI that guarantees $W < 0$. Differentiating V, we find

$$\begin{aligned} W \leq\ & 2x^{\mathrm{T}}(t)P\dot{x}(t) + ax^{\mathrm{T}}(t)Px(t) - bw^{\mathrm{T}}(t)w(t) \\ & + h^2\dot{x}^{\mathrm{T}}(t)R\dot{x}(t) - h\mathrm{e}^{-ah}\int_{t-h}^{t}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \\ & + x^{\mathrm{T}}(t)[S+E]x(t) - [x^{\mathrm{T}}(t-h)Sx(t-h) \\ & + (1-d)x^{\mathrm{T}}(t-\tau)Ex(t-\tau)]\mathrm{e}^{-ah}. \end{aligned}$$

Applying the standard arguments (see e.g. Ariba and Gouaisbaut (2007)), we obtain that

$$W \leq \eta^{\mathrm{T}}(t)\Phi\eta(t) < 0, \quad \forall \eta(t) \neq 0, \tag{12}$$

where $\eta(t) = \mathrm{col}\{x(t), \dot{x}(t), x(t-h), x(t-\tau(t)), w(t)\}$, if the matrix inequality

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & P_2^{\mathrm{T}}A_1 + R\mathrm{e}^{-ah} & P_2^{\mathrm{T}}B_1 \\ * & \Phi_{22} & 0 & P_3^{\mathrm{T}}A_1 & P_3^{\mathrm{T}}B_1 \\ * & * & -(S+R)\mathrm{e}^{-ah} & R\mathrm{e}^{-ah} & 0 \\ * & * & * & -(2R+(1-d)E)\mathrm{e}^{-ah} & 0 \\ * & * & * & * & -bI \end{bmatrix} < 0 \tag{13}$$

is feasible, where

$$\Phi_{11} = A^{\mathrm{T}}P_2 + P_2^{\mathrm{T}}A + aP + S + E - R\mathrm{e}^{-ah}, \tag{14}$$

$$\Phi_{12} = P - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3, \ \Phi_{22} = -P_3 - P_3^{\mathrm{T}} + h^2R.$$

Thus, the following result is obtained.

Lemma 2. *Given* $a > 0$ *and* $h > 0$, *let there exist* $n \times n$*-matrices* $P > 0, P_2, P_3, R > 0, S > 0, E \geq 0$ *and a scalar* $b > 0$ *such that the LMI* (13) *with notations given in* (14) *holds. Then the solution of* (8) *satisfies* (11) *for all delays* $0 \leq \tau(t) \leq h$. *Moreover, given* $\Delta > 0$ *and* $k > 0$, *the ellipsoid*

$$\mathcal{X}_\infty = \left\{x \in R^n : x^{\mathrm{T}}Px < \frac{b}{a}k\Delta^2\right\} \tag{15}$$

is exponentially attractive with the decay rate $a/2$ *for all* $x_0 \in R^n$ *and* $|w(t)|^2 \leq k\Delta^2$.

4. Quantized control input

Consider the saturated closed-loop system (5)

$$\dot{x}(t) = Ax(t) + Bsat(Kx(t-\tau(t)),\ \bar{q}) + Bw(t), \tag{16}$$

where $|w(t)|^2 \leq m\Delta^2$. We solve the problem by using *a linear system representation with polytopic type uncertainty* introduced in Hu and Lin (2001). Denoting the ith row of K by k_i, we define the polyhedron

$$\mathcal{L}(K,\bar{q}) = \{x \in R^n : |k_ix| \leq \bar{q}_i,\ i = 1,\ldots,m\}.$$

If the control and the disturbance are such that $x \in \mathcal{L}(K,\bar{q})$ then the system (16) admits the linear representation. Following Hu and Lin (2001), we denote the set of all diagonal matrices in $R^{m\times m}$ with diagonal elements that are either 1 or 0 by Υ, then there are 2^m elements D_i in Υ, and, for every $i = 1,\ldots,2^m$, $D_i^- \triangleq I_m - D_i$ is also in Υ.

Lemma 3 (Hu and Lin, 2001). *Given* K *and* H *in* $R^{m\times n}$. *Then, for all* $x \in \mathcal{L}(H,\bar{q})$,

$$sat(Kx(t),\ \bar{q}) \in \mathcal{C}o\{D_iKx + D_i^-Hx,\ i = 1,\ldots,2^m\}.$$

Let $\mathcal{X}_\beta$ be the ellipsoid $x^{\mathrm{T}}Px \leq \beta^{-1}$ for a given $\beta > 0$ and a $n\times n$-matrix $P > 0$. Assume that there exists H in $R^{m\times n}$ such that $\mathcal{X}_\beta \subset \mathcal{L}(H,\bar{q})$. Then, from Lemma 3, for $x(t) \in \mathcal{X}_\beta$, the system (16) admits the representation

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{2^m}\lambda_j(t)A_jx(t-\tau(t)) + Bw(t) \tag{17}$$

where

$$A_j = B(D_jK + D_j^-H) \quad j = 1,\ldots,2^m, \qquad \sum_{j=1}^{2^m}\lambda_j(t) = 1, \quad 0 \leq \lambda_j(t),\ \forall t > 0. \tag{18}$$

The problem becomes one of finding $\mathcal{X}_\beta$ and a corresponding H such that $|h_ix| \leq \bar{q}_i$, $i = 1,\ldots 2^m$ for all $x \in \mathcal{X}_\beta$ and that the state of (17) remains in $\mathcal{X}_\beta$.

Theorem 4. *Consider the linear system* (1) *with the quantized constrained delayed control law* (3). *Given* $a > 0$ *and* $\epsilon \in R$, *let there exist* $n\times n$*-matrices* $\bar{P} > 0, Q, \bar{R} > 0, \bar{S} > 0, \bar{E} \geq 0$, $m\times n$*-matrices* Y, G *and scalars* $\bar{b} > 0$, $\beta > 0$ *such that the following LMIs hold:*

$$a\bar{b} - \beta m\Delta^2 > 0, \tag{19}$$

$$\begin{bmatrix} \beta & g_i \\ * & \bar{q}_i^2\bar{P} \end{bmatrix} \geq 0, \quad i = 1,\ldots,m, \tag{20}$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 & BZ_j + \bar{R}\mathrm{e}^{-ah} & B\bar{b} \\ * & \Psi_{22} & 0 & \epsilon BZ_j & \epsilon B\bar{b} \\ * & * & -(\bar{S}+\bar{R})\mathrm{e}^{-ah} & \bar{R}\mathrm{e}^{-ah} & 0 \\ * & * & * & -[2\bar{R}+(1-d)\bar{E}]\mathrm{e}^{-ah} & 0 \\ * & * & * & * & -\bar{b}I \end{bmatrix} < 0, \tag{21}$$

for $j = 1,\ldots,2^m$, *where* $Z_j = D_jY + D_j^-G$, *and*

$$\Psi_{11} = Q^{\mathrm{T}}A^{\mathrm{T}} + AQ + a\bar{P} + \bar{S} + \bar{E} - \bar{R}\mathrm{e}^{-ah}, \tag{22}$$

$$\Psi_{12} = \bar{P} - Q + \epsilon Q^{\mathrm{T}}A^{\mathrm{T}}, \quad \Psi_{22} = -\epsilon Q - \epsilon Q^{\mathrm{T}} + h^2\bar{R}.$$

Then, for all delays $\tau(t) \in [0,h]$, *and for all* x_0 *from the ellipsoid*

$$\mathcal{X}_0 = \left\{x_0 \in R^n : x_0^{\mathrm{T}}Px_0 \leq \beta^{-1} - \frac{m\Delta^2}{a\bar{b}} \triangleq \delta\right\} \tag{23}$$

the solutions of the closed-loop system (5) *satisfy* (11), *where* $K = YQ^{-1}$ *and* $P = Q^{-\mathrm{T}}\bar{P}Q^{-1}$. *Moreover, for* $T > t_0$, *the solutions of* (5) *starting from* $\mathcal{X}_0$ *enter the reachable ellipsoid* $x(t) \in \mathcal{X}_T$, $t \geq T$ *given by*

$$\mathcal{X}_T = \left\{x \in R^n : x^{\mathrm{T}}Px < \delta\mathrm{e}^{-a(T-t_0)} + (1-\mathrm{e}^{-a(T-t_0)})\frac{k\Delta^2b}{a}\right\}, \tag{24}$$

where $b = \bar{b}^{-1}$, $k = m$ *and the ellipsoid* (15) *is attractive from* $\mathcal{X}_0$. *If additionally*

$$bk\Delta^2/a < \beta^{-1}/2, \tag{25}$$

then the ellipsoids $\mathcal{X}_\infty$ *and* $\mathcal{X}_T$ *(for big enough* T*) are strictly smaller than* $\mathcal{X}_0$. *In the unsaturated case, if the LMI* (21) *holds with* $Z_j = Y$, *then for all* $x_0 \in R^n$ *the solutions of* (5) *satisfy* (11) *and the ellipsoid* (15) *is attractive.*

Proof. We apply conditions of Lemma 2 to (17), where we substitute $A_1 = \sum_{j=1}^{2^m}\lambda_j(t)A_j$ and $B_1 = B$. Since the resulting LMI

(13) is affine in $\sum_{j=1}^{2^m}\lambda_j(t)A_j$, one has to solve (13) simultaneously for all the 2^m vertices A_j, applying the same matrices P, P_2, P_3, S, E and R for all vertices. To find the unknown gain K we choose $P_3 = \epsilon P_2$, where ϵ is a tuning scalar parameter (which may be restrictive). Then P_2 is non-singular due to the fact that the only matrix which can be negative definite in Φ_{22} of (13) is $-\epsilon(P_2+P_2^{\mathrm{T}})$. Moreover, $\epsilon > 0$. Defining:

$$Q = P_2^{-1}, \quad \bar{P} = Q^{\mathrm{T}}PQ, \quad \bar{R} = Q^{\mathrm{T}}RQ, \\ \bar{S} = Q^{\mathrm{T}}SQ, \quad \bar{E} = Q^{\mathrm{T}}EQ, \quad Y = KQ, \tag{26}$$

we multiply (13), where $A_1 = B(D_jK+D_j^-H)$ by $\mathrm{diag}\{P_2^{-1}, P_2^{-1}, P_2^{-1}, P_2^{-1}, I\}$ and its transpose, from the right and the left, respectively. We obtain (21).

The ellipsoid $\mathcal{X}_\beta$ is contained in $\mathcal{L}(H, \bar{q})$, if

$$\bar{q}_i^2 \geq \bar{q}_i^2\beta x^{\mathrm{T}}Px \geq |h_ix|^2, \quad i = 1, \ldots, m$$

i.e. if $\begin{bmatrix}\beta & h_i \\ * & \bar{q}_i^2P\end{bmatrix} \geq 0$ or if (20) is feasible, where $g_i = h_iQ = h_iP_2^{-1}$ and $\bar{P} = P_2^{-T}PP_2^{-1}$. Thus, under (20) the polytopic system representation of (17) is valid for $x(t) \in \mathcal{X}_\beta$. From LMIs (21) it follows that solutions of (17) satisfy (11). Hence,

$$x^{\mathrm{T}}(t)Px(t) < x_0^{\mathrm{T}}Px_0 + \frac{m\Delta^2}{a\bar{b}} \leq \beta^{-1}, \tag{27}$$

if $x_0 \in \mathcal{X}_0$, where $\mathcal{X}_0$ is given by (23). Then LMI (19) is equivalent to $\delta > 0$ and for all x_0 from the ellipsoid (23), the trajectories $x(t)$ of (5) remain within $\mathcal{X}_\beta$ and satisfy the bound (11). Eq. (25) guarantees that $\delta > \frac{m\Delta^2}{a\bar{b}}$. □

5. Control under quantized State

Consider the saturated closed-loop system (5)

$$\dot{x}(t) = Ax(t) + BK(sat(x(t-\tau(t)), \bar{q}) + w(t)), \tag{28}$$

where $|w(t)|^2 \leq n\Delta^2$. We apply conditions of Lemma 2, where $A_1 = BK$ and $B_1 = BK$. Our main result (Theorem 5 below) studies the case of saturation avoidance: $|x_i(t)| \leq \bar{q}_i$. Next in Remark 1 we consider the case when the saturation is allowed. To find the unknown gain K we choose now $P_2 = \epsilon_2 I$ and $P_3 = \epsilon_3 I$, where ϵ_2 and ϵ_3 are tuning scalar parameters (which may be more restrictive than in the previous section). We obtain:

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & \epsilon_2BK + Re^{-ah} & \epsilon_2BK \\ * & \Xi_{22} & 0 & \epsilon_3BK & \epsilon_3BK \\ * & * & -(S+R)e^{-ah} & Re^{-ah} & 0 \\ * & * & * & -[2R+(1-d)E]e^{-ah} & 0 \\ * & * & * & * & -bI \end{bmatrix} < 0 \tag{29}$$

where

$$\Xi_{11} = \epsilon_2(A^{\mathrm{T}} + A) + aP + S + E - Re^{-ah}, \\ \Xi_{12} = P - \epsilon_2I + \epsilon_3A^{\mathrm{T}}, \quad \Xi_{22} = -2\epsilon_3I + h^2R. \tag{30}$$

For $x \in \mathcal{X}_\beta$, we want to guarantee now that $\bar{q}_i^2 \geq \bar{q}_i^2\beta x^{\mathrm{T}}Px \geq x_i^2$, $i = 1, \ldots, n$. The latter inequality can be written as $x^{\mathrm{T}}(\bar{q}_i^2\beta P - F_i)x \geq 0$, where $F_i \in R^{n\times n}$ is a matrix with the only non-zero term (i, i), which is equal to 1. Hence, the following LMIs

$$\bar{q}_i^2\beta P - F_i \geq 0, \quad i = 1, \ldots, n \tag{31}$$

guarantee that $x_i^2 \leq \bar{q}_i^2$ if $x \in \mathcal{X}_\beta$. Denoting $\bar{\beta} = \beta^{-1}$, and

$$\delta \triangleq \beta^{-1} - \frac{b}{a}n\Delta^2 > 0 \tag{32}$$

we derive from (31) and (32) the following inequalities:

$$\bar{q}_i^2P - F_i\bar{\beta} \geq 0, \quad i = 1, \ldots, n, \qquad \bar{\beta} - \frac{b}{a}n\Delta^2 > 0. \tag{33}$$

We obtain

Theorem 5. *Consider the linear system (1) with the quantized constrained delayed control law (3). Given $a > 0$, $\Delta > 0$ and ϵ_2, $\epsilon_3 \in R$, let there exist $n \times n$-matrices $P > 0$, $R > 0$, $S > 0$, $E \geq 0$, an $m \times n$-matrix K, and scalars $b > 0$, $\bar{\beta} > 0$ such that the LMIs (33) and (29) with notations given in (30) are feasible.*

Then for all delays $\tau(t) \in [0, h]$ and for all initial conditions x_0 from the ellipsoid

$$\mathcal{X}_0 = \left\{x_0 \in R^n : x_0^{\mathrm{T}}Px_0 \leq \bar{\beta} - \frac{b}{a}n\Delta^2\right\},$$

the solutions of the closed-loop system (6) satisfy the inequality (11). Moreover, for $T > t_0$ the solutions of (5) starting from $\mathcal{X}_0$ enter the reachable ellipsoid $x(t) \in \mathcal{X}_T$, $t \geq T$ given by (24) with $k = n$ and the ellipsoid (15) is attractive from $\mathcal{X}_0$. If additionally (25) holds, then the ellipsoids $\mathcal{X}_\infty$ and $\mathcal{X}_T$ (for big enough T) are strictly smaller than $\mathcal{X}_0$. In the unsaturated case, if the LMI (29) holds, then for all $x_0 \in R^n$ the solutions of (5) satisfy (11) and the ellipsoid (15) is attractive.

Remark 1. To reduce the conservatism of Theorem 5 one could apply the following polytopic representation by using Lemma 3:

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{2^n}\lambda_j(t)A_jx(t-\tau(t)) + BKw(t), \\ A_j = BK(D_j + D_j^-H), \quad j = 1, \ldots, n, \tag{34}$$

where D_j, D_j^- and H are $n \times n$-matrices. However, this would complicate the design procedure leading to nonlinear in K and H term $A_j = BK(D_j + D_j^-H)$. Therefore, we propose a two stage design. First, we find K, a and b from Theorem 5. Next, similar to Theorem 5, we obtain

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & \epsilon_2BK(D_j+D_j^-H) + Re^{-ah} & \epsilon_2BK \\ * & \Xi_{22} & 0 & \epsilon_3BK(D_j+D_j^-H) & \epsilon_3BK \\ * & * & -(S+R)e^{-ah} & Re^{-ah} & 0 \\ * & * & * & -[2R+(1-d)E]e^{-ah} & 0 \\ * & * & * & * & -bI \end{bmatrix} < 0, \\ 1 - \beta\frac{b}{a}n\Delta^2 > 0, \qquad \begin{bmatrix}\beta\bar{q}_i & h_i \\ * & \bar{q}_iP\end{bmatrix} \geq 0, \tag{35}$$

for $i = 1, \ldots, n, j = 1, \ldots, 2^n$ and notations given in (30). Given $\Delta > 0$ and $\epsilon_2, \epsilon_3 \in R$, we solve the latter LMIs with the following decision variables: $n \times n$-matrices $P > 0$, $R > 0$, $S > 0$, $E \geq 0$, H, and scalar $\beta > 0$, trying to enlarge the ellipsoid of initial conditions (see Section 6.2 below).

6. Discussions and example

6.1. Bounds in the infinite-dimensional state space

Instead of (7) consider now a general piecewise-continuous initial functions x_{t_0} with square integrable $\dot{x}_{t_0}$ from the space W with the norm

$$\|x_{t_0}\|_W^2 = |x(t_0)|^2 + \int_{-h}^{0}[|x(t_0+s)|^2 + |\dot{x}(t_0+s)|^2]\mathrm{d}s.$$

From the proof of Proposition 1, it follows that

$$x^{\mathrm{T}}(t)Px(t) \leq V(t, x_t, \dot{x}_t) < e^{-a(t-t_0)}V(t_0, x_{t_0}, \dot{x}_{t_0}) \\ + [1 - e^{-a(t-t_0)}]\frac{b}{a}|w_{[t_0,t]}|_\infty^2. \tag{36}$$

Hence, the region of initial conditions in Theorems 4 and 5 will take the form

$$\bar{\mathcal{X}}_0 = \{x_{t_0} \in W : V(t_0, x_{t_0}, \dot{x}_{t_0}) \leq \delta\}, \tag{37}$$

Moreover, Theorems 4 and 5 guarantee the following bounds on reachable $\bar{\mathcal{X}}_T$ and attractive $\bar{\mathcal{X}}_\infty$ regions in W:

$$\bar{\mathcal{X}}_T = \Big\{x_t \in W : V(t, x_t, \dot{x}_t) < \delta e^{-a(T-t_0)} + (1 - e^{-a(T-t_0)})\frac{k\Delta^2 b}{a},\ t \geq T\Big\}, \quad (38)$$

$$\bar{\mathcal{X}}_\infty = \Big\{x_t \in W : V(t, x_t, \dot{x}_t) < \delta\frac{k\Delta^2 b}{a}\Big\}$$

and (25) guarantees that $\bar{\mathcal{X}}_\infty \subset \bar{\mathcal{X}}_T \subset \bar{\mathcal{X}}_0$ for big enough T. The ellipsoidal upper bounds in R^n on reachable and attractive regions are more conservative than the bounds in W because $x^T(t)Px(t) < V(t, x_t, \dot{x}_t)$ for $\|x_t\|^2_W > 0$.

Remark 2. If the attractive set is strictly inside the initial set in the same state space W and if the quantizer may have an adjustable zoom parameter, then a dynamic quantization strategy similar to Brocket and Liberzon (2000) and Liberzon (2003) can be extended for asymptotic stabilization of systems with quantized and delayed signals.

6.2. *On numerical and optimization issues*

Theorems 4 and 5 contain tuning parameters a, ϵ or ϵ_2 and ϵ_3. The parameter a gives a lower bound of the exponential rate of convergence of the closed-loop system. Increasing a (almost till the maximum achievable value a^*) leads to better convergence and smaller attractive ellipsoid. We note that, for a approaching very close to a^*, the attractive ellipsoid may grow due to numerical problems. In all the examples we treated, the choice of $\epsilon = 1$ gave satisfactory results. A simple method for finding the parameters is to constitute a grid of values around 1 for $\epsilon, \epsilon_2, \epsilon_3$ and of growing values for $a > 0$ and test the LMIs. The attractive and the initial ellipsoids can be optimized in the following way.

Consider first the case of the state quantization, where $\mathcal{X}_\infty$ is contained in the ball of center 0 and of radius r_M given by $r_M^2 = \frac{bn\Delta^2}{a\underline{\sigma}(P)}$, where $\underline{\sigma}(P)$ is the minimum eigenvalue of P. So, the smallest possible value of the radius r_M is then obtained by maximizing the quantity α under the LMIs of Theorem 5 and the additional constraint $P > \alpha bI$. This is a generalized eigenvalue minimization problem (see (Boyd, Ghaoui, Feron and Balakrishnan, 1994)) which can be solved efficiently by semidefinite optimization.

Once K, a, b and α are determined, the set $\mathcal{X}_0$ can be enlarged by solving LMIs (35) and maximizing the square of the semi-minor axis of $\mathcal{X}_0$, which is given by $r_{2m} = \frac{\bar{\beta} - bn\Delta^2/a}{\bar{\sigma}(P)}$. Since $\bar{\sigma}(P) > \alpha b$, we obtain that $r_{2m}\alpha b < \bar{\beta} - bn\Delta^2/a$. Finding the maximum value of r_{2m} satisfying this last inequality and the LMIs (35) is also a generalized eigenvalue minimization problem. Further improvement can be achieved by iterations in K, a, b, P, R, S, E and β in LMIs (35) with the initialization from Theorem 5.

In the input quantization case, we add the constraint $\begin{pmatrix} \bar{P} & Q^T \\ Q & \alpha I \end{pmatrix} > 0$, which is equivalent to $P > \alpha^{-1}I$ and implies that $r_M^2 < \alpha m\Delta^2/a$. In order to increase the size of the ellipsoid $\mathcal{X}_0$, we consider the minimization of $\beta + \alpha$.

6.3. *Example (Bullo and Liberzon, 2006)*

We consider (1) with $A = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

By applying (21) with $\epsilon = 10$ and $Z_j = Y$, we find that the system is input-to-state stabilizable for the maximum value of $h = 0.95$ (which appeared to be d-independent) and the resulting controller gain is given by $K = [-0.3491\ -0.7022]$. We will further assume that the delay is fast varying.

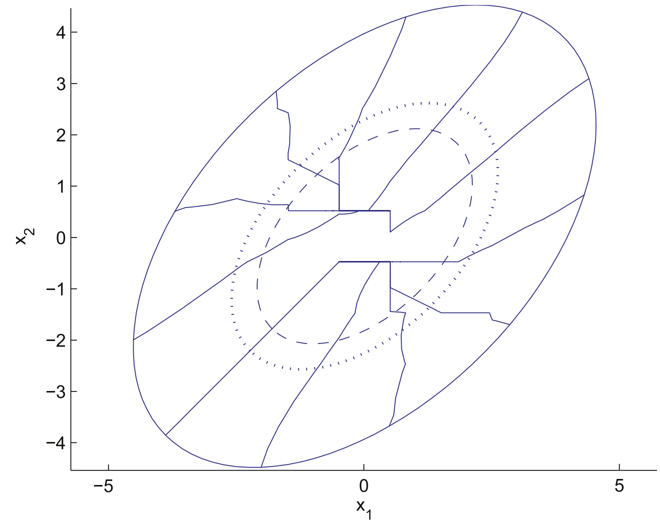

Fig. 1. Ellipsoids $\mathcal{X}_0$ (solid), $\mathcal{X}_\infty$ (dashed) and $\mathcal{X}_{T=2}$ (dotted): quantized state and $h = 0$.

(a) We consider first the case of *quantized state* with $\Delta = 1$ and $|x_i| \leq 5$. By Theorem 5 with $h = 0$ and $\epsilon_2 = 2.25$, $\epsilon_3 = 0.004$, $a = 0.98$ we find an attractive ball $|x| \leq 2.5$, where the resulting $K = [-1.2821\ -1.7791]$. By applying Lemma 2 of Bullo and Liberzon (2006) with the same K, we find a bigger attractive ball $|x| \leq 4.3202$, which is however less than the one $|x_i| \leq 4.472$ obtained in Bullo and Liberzon (2006) by choosing $K = [-0.5\ -1]$.

Proceeding as explained in Section 6.2, we find for $h = 0$, $\epsilon_2 = 2.26$, $\epsilon_3 = 0.69$, $a = 0.74$, $r_M = 3.38$ the following controller gain: $K = [-1.0348\ -1.5338]$. We depicted in Fig. 1 the resulting ellipses of initial conditions $\mathcal{X}_0$ (solid), the attractive ellipse $\mathcal{X}_\infty$ (dashed), the ellipse reachable from $\mathcal{X}_0$ in $T = 2$ (dotted) and some solutions for $t \in [0, 2]$ (which are *simulated in the case of a saturated uniform quantizer*). We see that in fact solutions reach an essentially smaller region than that predicted by Theorem 5, that illustrates the conservativeness of the method. We note only that Theorem 5 predicts the attractive ellipse for a wider class of *all quantizers* with the quantization error not greater than 1.

For $h > 0$, we find that conditions of Theorem 5, where $E = 0$, are feasible for the following maximum value of $h = 0.3923$, where $\epsilon_2 = 0.1033$, $\epsilon_3 = 0.1455$, $a = 0.5865$, $K = [-0.5540\ -1.0539]$. Hence, the saturated delayed state-feedback guarantees ISS for all $0 \leq \tau(t) \leq 0.3923$. For $h = 0.2$ the resulting initial, attractive and reachable in $T = 2$ ellipses are depicted in Fig. 2. The solutions are simulated in the case of a saturated uniform quantizer and a time-varying delay $\tau(t) = h|\sin t|$.

(b) Consider next the case of *quantized saturated feedback* with $\Delta = 1$ and $|Kx| \leq 5$. We find that conditions of Theorem 4 are feasible for the following maximum value of $h = 0.4745$. For $h = 0$, by applying Theorem 4 and taking $a = 1$ and $\epsilon = 1.9$, we obtain a gain $K = [-0.8185\ -1.4083]$. For $h = 0.2$, with $a = 1$, $\epsilon = 1.4$, we obtain the gain $K = [-0.7577\ -1.5155]$. We depicted in Fig. 3 (for $h = 0$) and Fig. 4 (for $h = 0.2$) the resulting ellipses of initial conditions $\mathcal{X}_0$, the attractive ellipse $\mathcal{X}_\infty$, the ellipse reachable from $\mathcal{X}_0$ in $T = 2$ and some solutions for $t \in [0, 2]$ (which are *simulated in the case of a saturated uniform quantizer* and a time-varying delay $\tau(t) = h|\sin t|$).

We note that Theorem 4 predicts the attractive ellipses for all quantizers with the quantization error not greater than 1 and for all delays not greater than h.

7. Conclusions

In this paper, a new methodology is proposed for the design of delayed controllers under saturated quantization of either the

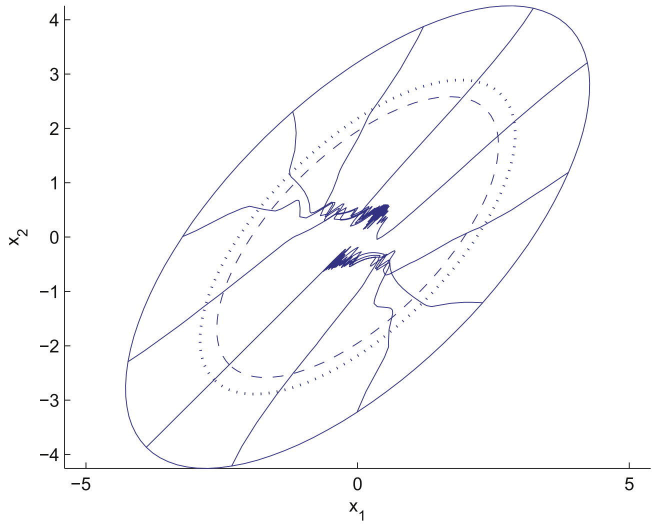

Fig. 2. Ellipsoids $\mathcal{X}_0$ (solid), $\mathcal{X}_\infty$ (dashed) and $\mathcal{X}_{T=2}$ (dotted): quantized state and $h = 0.2$.

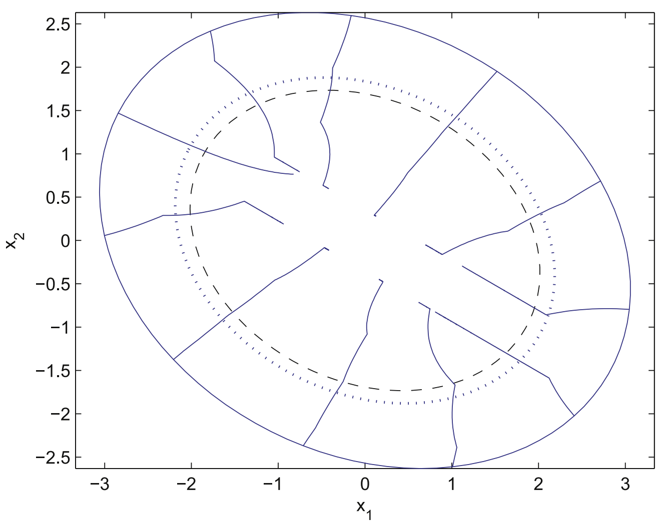

Fig. 3. Ellipsoids $\mathcal{X}_0$ (solid), $\mathcal{X}_\infty$ (dashed) and $\mathcal{X}_{T=2}$ (dotted): quantized input and $h = 0$.

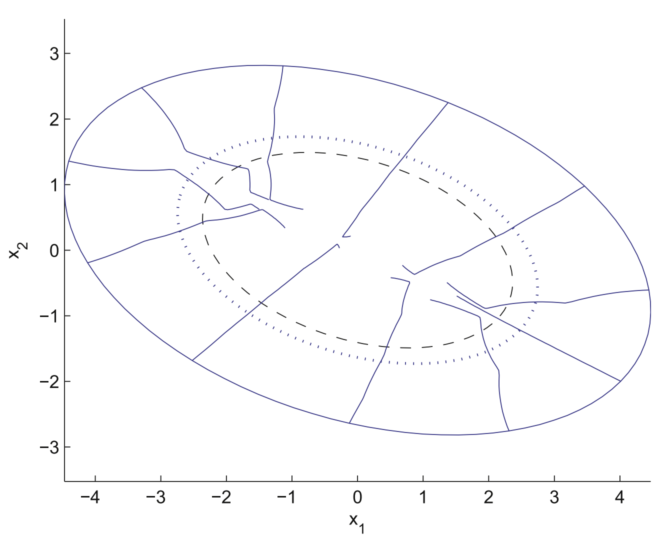

Fig. 4. Ellipsoids $\mathcal{X}_0$ (solid), $\mathcal{X}_\infty$ (dashed) and $\mathcal{X}_{T=2}$ (dotted): quantized input and $h = 0.2$.

control input or the state measurements, where the quantization error is supposed to be bounded by a given constant. The quantization is decomposed into a sum of a saturation and of a uniformly bounded disturbance. LMI solutions are derived via the comparison principle and the Lyapunov–Krasovskii method. The new method gives tools for the LMI approach to the dynamic quantization (originated by Brocket and Liberzon (2000)) of systems with quantized and delayed signals.

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