Wirtinger's Inequality and Lyapunov-Based Sampled-Data Stabilization ⋆

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Abstract

Discontinuous Lyapunov functionals appeared to be very efficient for sampled-data systems (Naghshtabrizi et al., 2008; Fridman, 2010). In the present paper new discontinuous Lyapunov functionals are introduced for sampled-data control in the presence of a constant input delay. The construction of these functionals is based on the vector extension of Wirtinger's inequality. These functionals lead to simplified and efficient stability conditions in terms of Linear Matrix Inequalities (LMIs). The new stability analysis is applied to sampled-data state-feedback stabilization and to a novel sampled-data static output-feedback problem, where the delayed measurements are used for stabilization.

Key words: sampled-data systems, time-delay, Lyapunov-Krasovskii functional, LMI.

1 Introduction

Sampled-data systems have been studied extensively over the past decades (see e.g. Chen and Francis, 1995; Mirkin, 2007; Naghshtabrizi et al., 2008; Fujioka, 2009; Fridman, 2010 and the references therein). Three main approaches have been used to uncertain sampled-data systems leading to conditions in terms of LMIs: a discrete-time, a time-delay and an impulsive system approaches. Recently the impulsive approach was extended to uncertain and bounded sampling intervals, where a discontinuous Lyapunov function method was introduced (Naghshtabrizi et al., 2008). This method inspired a piecewise-continuous (in time) Lyapunov functional approach to sampled-data systems in the framework of time-delay approach (Fridman, 2010), which essentially improved the existing results based on time-independent Lyapunov functionals.

The input delay approach to sampled-data control has been revised by using the scaled small gain theorem and a tighter upper bound on the L_2-induced norm of the uncertain term (Mirkin, 2007). Recently the latter result was recovered via input-output approach by application of the vector extension of Wirtinger's inequality (Liu et al., 2010).

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Networked Control Systems (NCS), where the plant is controlled via communication network, became an active research area (Zhang et al., 2001; Zampieri, 2008). NCSs are usually modeled as sampled-data systems under variable sampling with an additional network-induced delay (Gao et al., 2008; Naghshtabrizi et al., 2007). Extensions of the above discontinuous Lyapunov constructions to sampled-data systems in the presence of input delay η lead to complicated conditions (Naghshtabrizi et al., 2007; Liu and Fridman, 2011). Moreover, these conditions become conservative if η is not small.

In the present paper we develop a direct Lyapunov approach via Wirtinger's inequality to sampled-data stabilization in the presence of a constant input delay η. In this approach, novel discontinuous terms are added to "nominal" Lyapunov functionals for the stability analysis of systems with the delay η (either to simple or to complete ones). Being applied to sampled-data systems with $\eta = 0$, the new method recovers the result of Mirkin (2007), but it is more conservative than the one of Fridman (2010). However, the *new analysis leads to simplified reduced-order LMIs* and improves the existing results for $\eta > 0$. Comparatively to the standard time-independent Lyapunov functional terms for interval time-varying delays, the Wirtinger-based terms take advantage of the sawtooth evolution of the delays induced by sampled-and-hold and, thus, improve the results (both via simple and via discretized Lyapunov functionals).

The new method is applied to the state-feedback sampled-data stabilization. Also, a novel sampled-data static output-feedback problem is studied via discontinuous discretized Lyapunov functionals, where the delayed measurements are used for stabilization. This is a sampled-data counterpart of using an artificial delay for continuous-time stabilization studied in (Kharitonov et al., 2005). Note that the observer-based sampled-data control of systems with uncertain coefficients may become complicated and may lead to conservative results. From the other side, a simple static output feedback using the previous measurements can be easily designed and implemented.

Notation: Throughout the paper R^n denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $R^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in R^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. The space of functions $\phi : [a, b] \to R^n$, which are absolutely continuous on $[a, b)$, have a finite $\lim_{\theta\to b^-} \phi(\theta)$ and have square integrable first order derivatives is denoted by $W[a, b]$ with the norm $\|\phi\|_W = \max_{\theta\in[a,b]} |\phi(\theta)| + \left[\int_a^b |\dot{\phi}(s)|^2 ds\right]^{\frac{1}{2}}$.

2 Problem Formulation and Useful Lemmas

Consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^{n_u}$ is the control input, A and B are system matrices with appropriate dimensions. Denote by t_k the updating instant time of the Zero-Order Hold (ZOH), and suppose that the updating signal (successfully transmitted signal from the sampler to the controller and to the ZOH) at the instant t_k has experienced a constant signal transmission delay η. We assume that the sampling intervals are bounded

$$t_{k+1} - t_k \le h_s, \ k = 0, 1, 2, ... \tag{2}$$

i.e. that

$$t_{k+1} - t_k + \eta \le h_s + \eta \stackrel{\Delta}{=} \tau_M, \ k = 0, 1, 2, ... \tag{3}$$

Here τ_M denotes the maximum time span between the time $t_k - \eta$ at which the state is sampled and the time t_{k+1} at which the next update arrives at the destination.

The state-feedback controller has a form $u(t_k) = Kx(t_k - \eta)$, where K is the controller gain. Thus, considering the behavior of the ZOH, we have

$$u(t) = Kx(t_k - \eta), \ t_k \le t < t_{k+1}, \ k = 0, 1, 2, ... \tag{4}$$

with t_{k+1} being the next updating instant time of the ZOH after t_k. Defining $\tau(t) = t - t_k + \eta, \ t_k \le t < t_{k+1}$, we obtain the following closed-loop system (1), (4):

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)), \ t_k \le t < t_{k+1}, \tag{5}$$

where $k = 0, 1, 2, ...$ and $A_1 = BK$. Under (3), we have $\eta \le \tau(t) < t_{k+1} - t_k + \eta \le \tau_M$ and $\dot{\tau}(t) = 1$ for $t \ne t_k$. For the sake of brevity, further in the paper the notation τ stands for the time-varying delay $\tau(t)$.

The objective of the present paper is to derive efficient LMI (asymptotic and exponential) stability conditions for system (5). Moreover, we will consider the static output-feedback stabilization of (1) under the sampled-data measured output $y(t_k) = Cx(t_k), \ k = 0, 1, 2, ...,$ where $y(t_k) \in R^{n_l}$, C is a constant matrix. It is well-known, that using artificial delay in the (continuous-time) static output-feedback can stabilize some systems, which are not stabilizable without delay (Kharitonov et al., 2005). For such systems we will consider a sampled-data static output-feedback that uses the previous measurements and we will derive LMI conditions for stabilization.

We formulate next some useful lemmas. By using the standard arguments, the following can be proved:

Lemma 1 *Let there exist positive numbers α, β and a functional $V : R \times W[-\tau_M, 0] \times L_2[-\tau_M, 0] \to R$ such that*

$$\alpha|\phi(0)|^2 \le V(t, \phi, \dot{\phi}) \le \beta\|\phi\|_W^2. \tag{6}$$

Let the function $\bar{V}(t) = V(t, x_t, \dot{x}_t)$, where $x_t(\theta) = x(t+\theta)$ and $\dot{x}_t(\theta) = \dot{x}(t+\theta)$ with $\theta \in [-\tau_M, 0]$, is continuous from the right for $x(t)$ satisfying (5), absolutely continuous for $t \ne t_k$ and satisfies $\lim_{t\to t_k^-} \bar{V}(t) \ge \bar{V}(t_k)$.

If along (5) $\dot{\bar{V}}(t) \le -\gamma|x(t)|^2$ for $t \ne t_k$ and for some scalar $\gamma > 0$, then (5) is asymptotically stable.

A novel Lyapunov functional construction will be based on the extension of the Wirtinger inequality (Hardy et al., 1934) to the vector case:

Lemma 2 *(Liu et al., 2010) Let $z(t) \in W[a, b)$ and $z(a) = 0$. Then for any $n \times n$-matrix $R > 0$ the following inequality holds:*

$$\int_a^b z^T(\xi)Rz(\xi)d\xi \le \frac{4(b-a)^2}{\pi^2} \int_a^b \dot{z}^T(\xi)R\dot{z}(\xi)d\xi. \tag{7}$$

3 Stabilization via Novel Lyapunov Functionals

The stability of system (5) can be analyzed via time-independent functionals of the form (Fridman, 2006a):

$$V(x_t,\dot{x}_t) = V_n(x_t, \dot{x}_t) + V_Z(x_t,\dot{x}_t), \tag{8}$$

where V_n is a "nominal" functional for the "nominal" system with constant delay

$$\dot{x}(t) = Ax(t) + A_1 x(t-\eta) \tag{9}$$

and where (He et al., 2007)

$$\begin{array}{l} V_Z(x_t,\dot{x}_t) = \int_{t-\tau_M}^{t-\eta} x^T(s) Z_1 x(s) ds + V_{Z_2}(\dot{x}_t), \\ V_{Z_2}(\dot{x}_t) = (\tau_M - \eta) \int_{-\tau_M}^{-\eta} \int_{t+\theta}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds d\theta, \\ Z_1 > 0, Z_2 > 0. \end{array} \tag{10}$$

Remark 1 *The time-dependent term of Fridman (2010) can be modified to the case of $\eta > 0$ as follows:*

$$\begin{array}{l} V_U(t,\dot{x}_t) = (t_{k+1} - t) \int_{t_k-\eta}^{t-\eta} \dot{x}^T(s) U \dot{x}(s) ds, \\ U > 0, \ t \in [t_k, t_{k+1}). \end{array} \tag{11}$$

It is clear that V_U does not grow in the jumps since $V_{U|t=t_k} = 0$. Differentiation of V_U leads to

$$\begin{array}{l} \frac{d}{dt} V_U(t,\dot{x}_t) = -\int_{t_k-\eta}^{t-\eta} \dot{x}^T(s) U \dot{x}(s) ds \\ +(t_{k+1} - t) \dot{x}^T(t-\eta) U \dot{x}(t-\eta). \end{array} \tag{12}$$

Hence, the additional term

$$V_{0U}(\dot{x}_t) = (\tau_M - \eta) \int_{t-\eta}^{t} \dot{x}^T(s) U \dot{x}(s) ds$$

is needed with

$$\frac{d}{dt} V_{0U} = (\tau_M - \eta) \dot{x}^T(t) U \dot{x}(t) - (\tau_M - \eta) \dot{x}^T(t-\eta) U \dot{x}(t-\eta).$$

This leads to the same positive term and the same negative integral term (for $U = (\tau_M - \eta) Z_2$) as in

$$\begin{array}{l} \frac{d}{dt} V_{Z_2}(\dot{x}_t) = (\tau_M - \eta)^2 \dot{x}^T(t) Z_2 \dot{x}(t) \\ -(\tau_M - \eta) \int_{t_k-\eta}^{t-\eta} \dot{x}^T(s) Z_2 \dot{x}(s) ds \\ -(\tau_M - \eta) \int_{t-\tau_M}^{t_k-\eta} \dot{x}^T(s) Z_2 \dot{x}(s) ds. \end{array} \tag{13}$$

Therefore, $V_U + V_{0U}$ has no clear advantages over the standard double integral term V_{Z_2}.

In the present paper we suggest a discontinuous Lyapunov functional

$$V_d(t,x_t,\dot{x}_t) = \bar{V}_1(t) = V_n(x_t,\dot{x}_t) + V_W(t,x_t,\dot{x}_t) \tag{14}$$

with a novel discontinuous term

$$\begin{array}{l} V_W(t,x_t,\dot{x}_t) = (\tau_M - \eta)^2 \int_{t_k-\eta}^{t} \dot{x}^T(s) W \dot{x}(s) ds \\ -\frac{\pi^2}{4} \int_{t_k-\eta}^{t-\eta} [x(s) - x(t_k-\eta)]^T W [x(s) - x(t_k-\eta)] ds, \\ W > 0, \ t_k \le t < t_{k+1}, \ k = 0,1,2,... \end{array} \tag{15}$$

We note that V_W can be represented as a sum of the continuous in time term $(\tau_M - \eta)^2 \int_{t-\eta}^{t} \dot{x}^T(s) W \dot{x}(s) ds \ge 0$ with the discontinuous one

$$\begin{array}{l} V_{W1} \triangleq (\tau_M - \eta)^2 \int_{t_k-\eta}^{t-\eta} \dot{x}^T(s) W \dot{x}(s) ds \\ -\frac{\pi^2}{4} \int_{t_k-\eta}^{t-\eta} [x(s) - x(t_k-\eta)]^T W [x(s) - x(t_k-\eta)] ds. \end{array}$$

Since $[x(s) - x(t_k-\eta)]_{|s=t_k-\eta} = 0$, by the extended Wirtinger's inequality (7) $V_{W1} \ge 0$. Moreover, V_{W1} vanishes at $t = t_k$. Hence, the condition $\lim_{t \to t_k^-} \bar{V}_1(t) \ge \bar{V}_1(t_k)$ holds.

Differentiating V_W, we have

$$\begin{array}{l} \frac{d}{dt} V_W = (\tau_M - \eta)^2 \dot{x}^T(t) W \dot{x}(t) - \frac{\pi^2}{4} v^T(t) W v(t), \\ v(t) = x(t_k-\eta) - x(t-\eta). \end{array} \tag{16}$$

Remark 2 *For $\eta = 0$, it is easily seen from (16) that application of the functional $V_0 = x^T(t) P_1 x(t) + V_W$ with $P_1 > 0$ to (5) recovers conditions of Mirkin (2007), which are based on the small-gain theorem. An advantage of the direct Lyapunov method considered in the present paper over the small-gain theorem-based results is in its wider applications: to exponential bounds on the solutions of the initial value problems, to finding domains of attraction of some nonlinear systems.*

3.1 Stabilization via the Simple Lyapunov Functional

We start with the stability conditions via $V_d = V_{n1} + V_W$, where V_{n1} is a simple functional of the form

$$\begin{array}{l} V_{n1}(t,x_t,\dot{x}_t) = x^T(t) P_1 x(t) + \int_{t-\eta}^{t} x^T(s) R_1 x(s) ds \\ +\eta \int_{-\eta}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds d\theta, \ P_1 > 0, R_1 > 0, R_2 > 0. \end{array} \tag{17}$$

Theorem 1 *(i) Given $\eta \ge 0, h_s > 0$ and K, if there exist $n \times n$ matrices $P_1 > 0$, $W > 0$, $R_i > 0$, $i = 1,2$ such* *that the following LMI is feasible:*

$$\begin{bmatrix} \Psi_1 & P_1A_1{+}R_2 & P_1A_1 & A^T(h_s^2W+\eta^2R_2) \\ * & -R_1{-}R_2 & 0 & A_1^T(h_s^2W+\eta^2R_2) \\ * & * & -\frac{\pi^2}{4}W & A_1^T(h_s^2W+\eta^2R_2) \\ * & * & * & -(h_s^2W+\eta^2R_2) \end{bmatrix} < 0, \quad (18)$$

$\Psi_1 = P_1A{+}A^TP_1{+}R_1{-}R_2.$

Then the system (5) is asymptotically stable.

(ii) Given $\eta \geq 0, h_s > 0$, if there exist $n \times n$ matrices $\bar{P}_1 > 0$, Q, $\bar{W} > 0$, $\bar{R}_i > 0$, $i = 1, 2$, an $n_u \times n$-matrix L and a tuning parameter $\epsilon > 0$ such that the following LMI is feasible:

$$\begin{bmatrix} Q^TA^T{+}AQ{+}\bar{R}_1{-}\bar{R}_2 & \bar{P}_1{-}Q{+}\epsilon Q^TA^T & \bar{R}_2{+}BL & BL \\ * & -\epsilon(Q{+}Q^T){+}\eta^2\bar{R}_2{+}h_s^2\bar{W} & \epsilon BL & \epsilon BL \\ * & * & -\bar{R}_1-\bar{R}_2 & 0 \\ * & * & * & -\frac{\pi^2}{4}\bar{W} \end{bmatrix} < 0. \quad (19)$$

Then the closed-loop system (1), (4) is asymptotically stable and the stabilizing gain is given by $K = LQ^{-1}$.

Proof: (i)Differentiating $\bar{V}_1(t)$ along (5) and taking into account (16), we find

$$\begin{aligned} \dot{\bar{V}}_1(t) &= 2x^T(t)P_1\dot{x}(t) + x^T(t)R_1x(t) \\ &-x^T(t-\eta)R_1x(t-\eta) + \dot{x}^T(t)(\eta^2R_2 + h_s^2W)\dot{x}(t) \\ &-\tfrac{\pi^2}{4}v^T(t)Wv(t) - \eta\int_{t-\eta}^t \dot{x}^T(s)R_2\dot{x}(s)ds. \end{aligned} \quad (20)$$

By Jensen's inequality (Gu et al., 2003)

$$\begin{aligned} &\eta\int_{t-\eta}^t \dot{x}^T(s)R_2\dot{x}(s)ds \geq \int_{t-\eta}^t \dot{x}^T(s)dsR_2\int_{t-\eta}^t \dot{x}(s)ds \\ &= [x(t)-x(t-\eta)]^TR_2[x(t)-x(t-\eta)]. \end{aligned} \quad (21)$$

Then substitution of $Ax(t){+}A_1x(t{-}\eta){+}A_1v(t)$ for $\dot{x}(t)$ leads to

$$\begin{aligned} \dot{\bar{V}}_1(t) &\leq \zeta_1^T(t)\begin{bmatrix} \Psi_1 & P_1A_1{+}R_2 & P_1A_1 \\ * & -R_1{-}R_2 & 0 \\ * & * & -\frac{\pi^2}{4}W \end{bmatrix}\zeta_1(t) + [Ax(t){+}A_1x(t{-}\eta) \\ &+A_1v(t)]^T(\eta^2R_2 + h_s^2W)[Ax(t){+}A_1x(t{-}\eta){+}A_1v(t)], \end{aligned}$$

where $\zeta_1(t) = col\{x(t), x(t-\eta), v(t)\}$. Hence, by Schur complements, (18) guarantees that $\dot{\bar{V}}_1(t) \leq -\gamma|x(t)|^2$ for some $\gamma > 0$ which completes the proof of (i).

(ii) For the state feedback design, the descriptor method is used, where the right-hand side of the expression

$$\begin{aligned} &2[x^T(t)P_2^T{+}\dot{x}^T(t)P_3^T][Ax(t){+}A_1x(t{-}\eta){+}A_1v(t){-}\dot{x}(t)] \\ &= 0, \end{aligned}$$

with some $n \times n$-matrices P_2, P_3 is added to $\dot{\bar{V}}_1(t)$. Then (20), (21) lead to $\dot{\bar{V}}_1(t) \leq \zeta_2^T(t)\Xi_s\zeta_2(t) \leq -\gamma|x(t)|^2$ for some $\gamma > 0$, where $\zeta_2(t) = col\{x(t), \dot{x}(t), x(t-\eta), v(t)\}$, if

$$\Xi_s \triangleq \begin{bmatrix} P_2^TA{+}A^TP_2{+}R_1{-}R_2 & P_1{-}P_2^T{+}A^TP_3 & R_2{+}P_2^TA_1 & P_2^TA_1 \\ * & -P_3{-}P_3^T{+}\eta^2R_2{+}h_s^2W & P_3^TA_1 & P_3^TA_1 \\ * & * & -R_1{-}R_2 & 0 \\ * & * & * & -\frac{\pi^2}{4}W \end{bmatrix} < 0. \quad (22)$$

Following Fridman (2006b) and Suplin et al. (2007), we denote $P_3 = \varepsilon P_2$, where ε is a scalar, $Q = P_2^{-1}$, $\bar{P}_1 = Q^TP_1Q$, $\bar{W} = Q^TWQ$, $\bar{R}_i = Q^TR_iQ$ $(i = 1, 2)$ and $L = KQ$. Multiplication of (22) by $diag\{Q^T, Q^T, Q^T, Q^T\}$ and $diag\{Q, Q, Q, Q\}$, from the left and the right, completes the proof of (ii).

Remark 3 *The recent method of Park et al.(2011) for the stability of (5) (via functional (8) with $V_n = V_{n1}$, Jensen's inequality and convexity arguments) leads to the following (affine in A and A_1) LMIs:*

$$\begin{bmatrix} Z_2 & S_{12} \\ * & Z_2 \end{bmatrix} \geq 0, \quad (23)$$

$$\begin{bmatrix} \Psi_1 & R_2 & P_1A_1 & 0 & A^T(h_s^2Z_2+\eta^2R_2) \\ * & \Psi_2 & Z_2-S_{12} & S_{12} & 0 \\ * & * & -2Z_2{+}S_{12}{+}S_{12}^T & Z_2{-}S_{12} & A_1^T(h_s^2Z_2+\eta^2R_2) \\ * & * & * & -Z_1{-}Z_2 & 0 \\ * & * & * & * & -(h_s^2Z_2+\eta^2R_2) \end{bmatrix} < 0, \quad (24)$$

where S_{12} is $n \times n$ matrix and $\Psi_2 = -R_1{-}R_2{+}Z_1{-}Z_2$.

Comparing LMI (18) with LMIs (23), (24), it is seen that (18) is a lower order single LMI with a fewer decision variables (W in (18) instead of Z_1, Z_2, S_{12} in (23), (24)). Note that conditions in Liu and Fridman (2011) are essentially more complicated than those of Park et al.(2011). See Table 1 for numerical complexity of the above methods.

Table 1
The numerical complexity of different methods

Method	Decision variables	No. of LMIs	The maximum order of LMI
Liu & Fr (2011)	$12.5n^2 + 2.5n$	2	$7n$
Park et al.(2011)	$3.5n^2 + 2.5n$	2	$5n$
Theorem 1 (i)	$2n^2 + 2n$	1	$4n$

Remark 4 *Consider now the LMI conditions via $V_{n1} + V_{Z_2}$ and Jensen's inequality, which contain the*

same number of decision variables and LMIs as Theorem 1. From (13) and Jensen's inequality, we have $\frac{d}{dt}V_{Z_2}(\dot{x}_t) \le (\tau_M - \eta)^2 \dot{x}^T(t) Z_2 \dot{x}(t) - v^T(t) Z_2 v(t)$ *($v(t)$ is given in (16)), which leads to more restrictive LMI (18), where* $W = Z_2$ *and where the (3,3)-term* $-\frac{\pi^2}{4}W$ *is changed by the (more than twice) bigger term* $-Z_2$.

More complicated LMI conditions via time-independent Lyapunov functionals and Jensen's inequality sometimes can be less restrictive than the Wirtinger-based conditions. Thus, in Fridman (2010) for $\eta = 0$ *in one example out of three the results by Mirkin (2007) (which are equivalent to Theorem 1) are more conservative than the results by Park & Ko (2007). In order to get the less conservative LMI conditions, the functional* $V_{n1} + V_Z + V_W$ *can be applied by combining arguments of Park et al. (2011) and of Theorem 1.*

Remark 5 *For the exponential stability analysis we follow Seuret et al. (2005). By changing the variable* $\bar{x}(t) = x(t)e^{\lambda t}$, *(5) can be rewritten as*

$$\dot{\bar{x}}(t) = (A + \lambda I)\bar{x}(t) + e^{\lambda \tau} A_1 \bar{x}(t - \tau). \quad (25)$$

Asymptotic stability of (25) for some $\lambda > 0$ *implies the exponential stability with the decay rate* λ *of (5). Since* $e^{\lambda\tau} \in [\rho_1, \rho_2]$ *with* $\rho_1 = e^{\lambda\eta}$ *and* $\rho_2 = e^{\lambda\tau_M}$, *(25)can be represented in the following polytopic form:*

$$\dot{\bar{x}}(t) = \sum_{i=1}^{2} \mu_i(t)\{(A+\lambda I)\bar{x}(t) + \rho_i A_1 \bar{x}(t-\tau)\}, \quad (26)$$

where $\mu_1(t) = (\rho_2 - e^{\lambda\tau})/(\rho_2 - \rho_1)$ *and* $\mu_2(t) = (e^{\lambda\tau} - \rho_1)/(\rho_2 - \rho_1)$. *We note that the LMIs of Theorem 1 are affine in the system matrices. Therefore, one have to solve these LMIs simultaneously for the two vertices of system (26) given by* ${A_1}^{(i)} = \rho_i A_1 (i = 1, 2)$, *where the same decision matrices are applied.*

Example 1 *(Zhang et al, 2001) Consider the system*

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t). \quad (27)$$

We start with the analysis of the closed-loop system under the controller $u(t) = -[3.75\ 11.5]x(t_k - \eta)$, $t_k \le t < t_{k+1}$. *It was found in Naghshtabrizi et al. (2008) that the system remains stable for all constant samplings less than 1.72 and becomes unstable for samplings greater than 1.73. Moreover, the above system with the continuous control* $u(t) = -[3.75\ 11.5]x(t - \eta)$ *is asymptotically stable for* $\eta \le 1.16$ *and becomes unstable for* $\eta > 1.17$. *The latter means that all the existing methods, that are based on time-independent Lyapunov functionals, corresponding to stability analysis of systems with fast varying delays, cannot guarantee the stability for the samplings with the upper bound greater than* 1.17.

For the values of η *given in Table II, by applying (i) of Theorem 1, we obtain the maximum values of* $\tau_M = h_s + \eta$, *that preserve the stability (see Table II). For* $\eta = 0$, *the results of Mirkin (2007) and of Fujioka (2009) lead to* $\tau_M = 1.36$, *which coincides with our results.*

Table 2
Example 1: Max. value of τ_M for different η

$\tau_M \setminus \eta$	0.1	0.2	0.4	0.6
Park et al.(2011)	1.05	1.06	1.07	1.07
Liu & Fridman (2011)	1.33	1.26	1.18	1.14
Theorem 1 (i)	1.32	1.28	1.22	1.17

Choosing next $\tau_M = 1$, *by applying Remark 5 and either (i) of Theorem 1 or Park et al.(2011) in the affine form (23), (24), we obtain the maximum value of the decay rate* λ *given in Table III for different values of* η.

Table 3
Example 1: Max. value of λ for $\tau_M = 1$ and different η

$\lambda \setminus \eta$	0.1	0.2	0.4	0.6
Park et al.(2011))	0.04	0.05	0.05	0.05
Liu & Fridman (2011)	0.20	0.15	0.10	0.07
Theorem 1 (i)	0.26	0.23	0.17	0.12

We proceed next with the state-feedback design. Note that the poles of the open-loop system (27) have non-positive real parts. Therefore, by (ii) of Theorem 1 with $\varepsilon = 0.9$, *we obtain a low gain controller* $u(t) = -10^{-15} \times [0.1482\ \ 0.5412]x(t_k - \eta)$ *which stabilizes (27) preserving the stability for* $\tau_M \le 10^8$. *Choosing next* $\eta = 0.2, \tau_M = 0.8$ *and applying (ii) of Theorem 1 (as in Remark 5) with* $\varepsilon = 0.9$, *we find that the controller* $u(t) = -[4.8260\ \ 11.2343]x(t_k - \eta)$ *exponentially stabilizes the system with the decay rate* $\lambda = 0.50$. *Next, applying to the resulting closed-loop system the conditions of Theorem 1 (i), of Liu and Fridman (2011) and of Park et al.(2011) (as in Remark 5), the maximum decay rate is found to be* $0.52, 0.30$ *and* 0.23 *respectively. Hence, the method of Theorem 1 essentially simplifies the existing conditions and improves the results.*

3.2 Stability via Discretized Lyapunov Functionals

If (9) with some constant delay $\bar{\eta} \in [0, \eta)$ is not stable (and, thus, the simple Lyapunov functional V_{n_1} is not applicable), the nominal functional V_n can be chosen to be a complete one

$$\begin{aligned} V_{n2}(t, x_t, \dot{x}_t) &= x^T(t)P_1x(t) + 2x^T(t)\int_{-\eta}^{0} Q(s)x(t+s)ds \\ &+ \int_{-\eta}^{0}\int_{-\eta}^{0} x^T(t+s)R(s,\theta)ds x(t+\theta)d\theta \\ &+ \int_{-\eta}^{0} x^T(t+s)S(s)x(t+s)ds, \ P_1 > 0, \end{aligned} \quad (28)$$

with continuous and piecewise-linear functions $Q(s), S(s)$ and $R(s,\theta)$ (Gu, 1997). Following Gu (1997), we divide the delay interval $[-\eta, 0]$ into N segments $[\theta_p, \theta_{p-1}]$, $p = 1, ..., N$ of equal length $r = \eta/N$, where $\theta_p = -pr$. This divides the square $[-\eta, 0] \times [-\eta, 0]$ into $N \times N$ small squares $[\theta_p, \theta_{p-1}] \times [\theta_q, \theta_{q-1}]$. Each small square is further divided into two triangles. The continuous matrix functions $Q(s)$ and $S(s)$ are chosen to be linear within each segment and the continuous matrix function $R(s,\theta)$ is chosen to be linear within each triangular:

$$
\begin{aligned}
&Q(\theta_p + \alpha r) = (1-\alpha)Q_p + \alpha Q_{p-1},\\
&S(\theta_p + \alpha r) = (1-\alpha)S_p + \alpha S_{p-1},\ \alpha \in [0,1],\\
&R(\theta_p + \alpha r, \theta_q + \beta r) =\\
&\begin{cases}(1-\alpha)R_{pq} + \beta R_{p-1,q-1} + (\alpha-\beta)R_{p-1,q},\ \alpha \ge \beta,\\ (1-\beta)R_{pq} + \alpha R_{p-1,q-1} + (\beta-\alpha)R_{p,q-1},\ \alpha < \beta.\end{cases}
\end{aligned}
$$

We use $V_d = V_{n2} + V_W$. Then, following the descriptor method (see Fridman, 2006b) and the arguments of Theorem 1, we arrive to

Corollary 1 *Given $\eta \ge 0, h_s > 0$ and K, the system (5) is asymptotically stable, if there exist $n \times n$ matrices $P_1 > 0$, P_2, P_3, $S_p = S_p^T$, Q_p, $R_{pq} = R_{qp}^T$, $p = 0, 1, ..., N$, $q = 0, 1, ..., N$, $W > 0$, such that the following LMIs hold:*

$$
\begin{bmatrix} P_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0, \tag{29}
$$

$$
\Xi_d \triangleq \begin{bmatrix} \Omega_d & \begin{bmatrix} D^s \\ 0 \end{bmatrix} & \begin{bmatrix} D^a \\ 0 \end{bmatrix} \\ * & -R_d - S_d & 0 \\ * & * & -3S_d \end{bmatrix} < 0, \tag{30}
$$

where $r = \frac{\eta}{N}$ and

$$
\Omega_d = \begin{bmatrix} \Psi_{d11} & P_1 - P_2^T + A^T P_3 & -Q_N + P_2^T A_1 & P_2^T A_1 \\ * & -P_3 - P_3^T + h_s^2 W & P_3^T A_1 & P_3^T A_1 \\ * & * & -S_N & 0 \\ * & * & * & -\frac{\pi^2}{4} W \end{bmatrix}, \tag{31}
$$

$$
\begin{aligned}
&\Psi_{d11} = P_2^T A + A^T P_2 + Q_0 + Q_0^T + S_0,\\
&\tilde{Q} = [Q_0\ Q_1\ ...Q_N],\ \tilde{S} = diag\{1/rS_0, 1/rS_1, ..., 1/rS_N\},\\
&\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & \dots & R_{0N} \\ R_{10} & R_{11} & \dots & R_{1N} \\ \dots & \dots & \dots & \dots \\ R_{N0} & R_{N1} & \dots & R_{NN} \end{bmatrix}, R_d = \begin{bmatrix} R_{d11} & R_{d12} & \dots & R_{d1N} \\ R_{d21} & R_{d22} & \dots & R_{d2N} \\ \dots & \dots & \dots & \dots \\ R_{dN1} & R_{dN2} & \dots & R_{dNN} \end{bmatrix},\\
&R_{dpq} = r(R_{p-1,q-1} - R_{pq}),\\
&S_d = diag\{S_0 - S_1, S_1 - S_2, ..., S_{N-1} - S_N\},\\
&D^s = [D_1^s\ D_2^s\ ...\ D_N^s],\ D^a = [D_1^a\ D_2^a\ ...\ D_N^a],\\
&D_p^s = \begin{bmatrix} r/2(R_{0,p-1} + R_{0p}) - (Q_{p-1} - Q_p) \\ r/2(Q_{p-1} + Q_p) \\ -r/2(R_{N,p-1} + R_{Np}) \end{bmatrix},\\
&D_p^a = \begin{bmatrix} -r/2(R_{0,p-1} - R_{0p}) \\ -r/2(Q_{p-1} - Q_p) \\ r/2(R_{N,p-1} - R_{Np}) \end{bmatrix}.
\end{aligned} \tag{32}
$$

Remark 6 *Differently from Corollary 1, the results of Theorem 1 are convex in η: if LMIs of Theorem 1 are feasible for some $\bar{\eta} > 0$, then they are feasible for all $\eta \in [0, \bar{\eta}]$. Therefore, Theorem 1 gives sufficient conditions for the stability of (5) with the unknown but bounded constant delay $\eta \in [0, \bar{\eta}]$.*

Conditions of Corollary 1 are derived via the descriptor method and, thus, can be easily applied to the state-feedback design by choosing e.g. $P_3 = \varepsilon P_2$ (Fridman, 2006b).

Remark 7 *Following the method of Park et al. (2011), the stability of system (5) via the time-independent functional $V_{n2} + V_Z$ leads to LMIs (23), (29), (30), where Ω_d is changed by*

$$
\tilde{\Omega}_d =
\begin{bmatrix} \Psi_{d11} & P_1 - P_2^T + A^T P_3 & -Q_N & P_2^T A_1 & 0 \\ * & -P_3 - P_3^T + h_s^2 Z_2 & 0 & P_3^T A_1 & 0 \\ * & * & -S_N + Z_1 - Z_2 & Z_2 - S_{12} & S_{12} \\ * & * & * & -2Z_2 + S_{12} + S_{12}^T & Z_2 - S_{12} \\ * & * & * & * & -Z_1 - Z_2 \end{bmatrix}
$$

with Ψ_{d11} given by (32). It is seen that also in the case of complete V_{n2}, the discontinuous Lyapunov functional leads to numerically simpler conditions than the time-independent one.

Results of Corollary 1 and of Remark 7 can be applied to the exponential stability analysis by using the method of Remark 5.

Example 2 *Consider the system from Gu et al. (2003):*

$$
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{33}
$$

where $u(t) = [1\ \ 0]x(t_k - \eta)$, $t_k \le t < t_{k+1}$. This system with $x(t_k - \eta)$ changed by $x(t - \eta)$ is stable for $0.1003 < \eta < 1.72$ and unstable if $\eta \in [0, 0.1]$. Thus, the simple Lyapunov functional-based results of Park et al. (2011), Liu and Kun (2011) and Theorem 1 are not applicable.

This is an example of the system that can be stabilized by using an artificial delay. For the values of $\eta > 0$ given *in Table IV, by applying Corollary 1 and Remark 7 we obtain the maximum values of $\tau_M = h_s + \eta$ that preserve the stability.*

Table 4
Example 2: Max. value of τ_M for different η

$\tau_M \setminus \eta$		0.5	0.65	0.8
$N=1$	$V_{n2}+V_W$	1.03	1.27	1.36
	$V_{n2}+V_Z$	0.84	1.05	1.16
$N=2$	$V_{n2}+V_W$	1.07	1.39	1.65
	$V_{n2}+V_Z$	0.86	1.12	1.34

Choosing next $\tau_M = 0.81$, by applying Corollary 1 and Remark 7 with $N = 2$ via Remark 5, we obtain the maximum value of the decay rate λ given in Table V for different values of η. Also in this case the discontinuous discritized Lyapunov functional leads to reduced-order LMIs and improves the results via the time-independent one.

Table 5
Example 2: Max. value of λ for $\tau_M = 0.81$ and different η

$\lambda \setminus \eta$		0.5	0.65	0.8
$N=2$	$V_{n2}+V_W$	0.08	0.22	0.36
	$V_{n2}+V_Z$	0.02	0.18	0.35

4 Sampled-Data Stabilization by Using the Delayed Measurements

It is well-known, that using an artificial delay in the (continuous-time) static output-feedback can stabilize some systems, which are not stabilizable without delay (see e.g. Kharitonov et al., 2005 and Example 2 above). Thus, the double integrator

$$\ddot{x}(t) = u(t),\ y(t) = x(t) \tag{34}$$

can be stabilized by using a control action of the form $u(t) = -k_1 x(t-h_1) - k_2 x(t-h_2)$, where h_1 and h_2 are constant delays and $0 \le h_1 < h_2$. The main criticism of the above method, that it has no advantages over the dynamic output-feedback and that its implementation needs buffer for all the measurements $y(t+\theta),\ \theta \in [-h_2, 0]$.

For the sampled-data control of systems with uncertain coefficients, the observer-based design is complicated and may lead to conservative results. From the other side, a simple static output feedback using the previous measurements can be easily designed and implemented. Thus in the system of Example 2, one can insert an artificial delay η (as in Table IV) and apply the sampled-data controller with the sampling intervals satisfying $t_{k+1} - t_k \le \tau_M - \eta$.

In this section we will extend sampled-data stabilization to the case, where (as in the double integrator) two sampled-data measurements are needed. Consider (1) and assume that the measured output $y(t_k) = Cx(t_k) \in R^{n_l}$ is available at the discrete time instants $0 = t_0 < t_1 < ... < t_k < ...$ with the constant sampling interval $t_{k+1} - t_k = h$. Consider the following static output-feedback controller, which uses the delayed measurement $y(t_{k-m})$:

$$\begin{aligned} u(t) &= K_1 y(t_k) + K_2 y(t_{k-m}) \\ &= K_1 C x(t_k) + K_2 C x(t_k - mh), \\ m &= 1, 2, ..., \quad t_k \le t < t_{k+1}. \end{aligned} \tag{35}$$

The closed-loop system (1), (35) has the form

$$\dot{x}(t) = Ax(t) + A_{c1} x(t_k) + A_{c2} x(t_k - \eta), \tag{36}$$

where $\eta = mh,\ A_{c1} = BK_1C,\ A_{c2} = BK_2C$.

We extend the analysis of section 3.2 to the system of (36) by adding the term (Fridman, 2010)

$$V_U(t, \dot{x}_t) = (h - t + t_k) \int_{t_k}^{t} \dot{x}^T(s) U \dot{x}(s) ds,\ U > 0.$$

to $V_d = V_{n2} + V_W$:

$$\begin{aligned} &V_{sam}(t, x_t, \dot{x}_t) = \bar{V}_2(t) = V_{n2}(x_t, \dot{x}_t) \\ &+ V_W(t, x_t, \dot{x}_t) + V_U(t, \dot{x}_t),\ t_k \le t < t_{k+1}, \end{aligned} \tag{37}$$

and where $V_W(t, x_t, \dot{x}_t)$ is given by (15) with $\tau_M = (m+1)h$. The term V_U vanishes before t_k and after t_k. By using arguments of Corollary 1 and of Fridman (2010) we arrive to the following:

Corollary 2 *Given $h > 0$ and K_1, K_2, the system (36) is asymptotically stable, if there exist $n \times n$ matrices $P_1 > 0$, P_2, P_3, $S_p = S_p^T$, Q_p, $R_{pq} = R_{qp}^T$, $p = 0, 1, ..., N$, $q = 0, 1, ..., N$, and $U > 0$, $W > 0$ such that LMIs (29) and*

$$\bar{\Xi}_{di} \triangleq \begin{bmatrix} \bar{\Omega}_{di} & \begin{bmatrix} D^s \\ 0 \end{bmatrix} & \begin{bmatrix} D^a \\ 0 \end{bmatrix} \\ * & -R_d - S_d & 0 \\ * & * & -3S_d \end{bmatrix} < 0,\ i = 1, 2, \tag{38}$$

hold, where $\tilde{Q}, \tilde{S}, \tilde{R},\ S_d, R_d, D^s$ and D^a are defined in (32). In (38)

$$\bar{\Omega}_{d1} = \Omega_d + diag\{0_{n\times n}, hU, 0\},\ \bar{\Omega}_{d2} = \begin{bmatrix} \Omega_d & \begin{bmatrix} -hP_2^T A_{c1} \\ -hP_3^T A_{c1} \\ 0 \end{bmatrix} \\ * & -hU \end{bmatrix},$$

with Ω_d given by (31), where A, A_1 and h_s are changed by $A + A_{c1}$, A_{c2} and h, respectively.

Remark 8 *LMIs of Corollary 2 are affine in A. Therefore, if A resides in the uncertain polytope*

$$A = \sum_{j=1}^{M} \mu_j(t) A^{(j)}, \quad 0 \le \mu_j(t) \le 1, \; \sum_{j=1}^{M} \mu_j(t) = 1,$$

one have to solve these LMIs simultaneously for all the M vertices $A^{(j)}$, applying the same decision matrices.

Example 3 *Consider the following system:*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ g(t) & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t_k) &= [1\ 0]x(t_k), \; t_k \le t < t_{k+1}, \; x(t) \in R^2, \end{aligned} \tag{39}$$

where $|g(t)| \le 0.1$. This system is not stabilizable by the non-delayed static output-feedback $u(t) = Ky(t_k)$, $t_k \le t < t_{k+1}$. We take $m = 3$ and choose

$$u(t) = -0.35y(t_k) + 0.1y(t_{k-3}), \; t_k \le t < t_{k+1}, \; t_{k+1} - t_k = h. \tag{40}$$

We treat the closed-loop system (39), (40) as a system with polytopic type uncertainty defined by the two vertices corresponding to $g(t) = \pm 0.1$. By applying Remark 8 to the closed-loop system (39), (40) we find the values of sampling period h that preserve the stability:

$$N = 1, \; h \in [10^{-5} \;\; 0.380], \; N = 2, \; h \in [10^{-5} \;\; 0.499].$$

5 Conclusions

Novel discontinuous Lyapunov functionals have been introduced for sampled-data systems in the presence of constant input delay. The construction of the functionals is based on the vector extension of the Wirtinger's inequality. The new method leads to numerically simplified LMIs for the stability analysis and it is applied to a novel problem of sampled-data stabilization by using the previous measurements. Numerical examples illustrate the efficiency of the method.

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